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Nonlinear dynamics of composite laminated cantilever rectangular plate subject to third-order piston aerodynamics

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Abstract This paper presents the analysis of the nonlinear dynamics for a composite laminated cantilever rectangular plate subjected to the supersonic gas flows and the in-plane excitations. The aerodynamic pressure is modeled by using the third-order piston theory. Based on Reddy's third-order plate theory and the von Kármán-type equation for the geometric nonlinearity, the nonlinear partial differential equations of motion for the composite laminated cantilever rectangular plate under combined aerodynamic pressure and in-plane excitation are derived by using Hamilton's principle. The Galerkin's approach is used to transform the nonlinear partial differential equations of motion for the composite laminated cantilever rectangular plate to a two-degree-of-freedom nonlinear system under combined external and parametric excitations. The method of multiple scales is employed to obtain the four-dimensional averaged equation of the non-automatic nonlinear system. The case of 1:2 internal resonance and primary parametric resonance is taken into account. A numerical method is utilized to study the bifurcations and chaotic dynamics of the composite laminated cantilever rectangular plate. The frequency–response curves, bifurcation diagram, phase portrait and frequency spectra are obtained to analyze the nonlinear dynamic behavior of the composite laminated cantilever rectangular plate, which includes the periodic and chaotic motions.

1 Introduction

The expanding use of composite laminated structures in lightweight applications, for example, aircraft and astronautic engineering, indicates the significance of developing appropriate models in order to predict their nonlinear dynamic responses. Composite laminated cantilever rectangular plates are widely used as light-weight components in aeronautic, astronautic, naval and transportation engineering fields due to their excellent mechanical properties, such as less weight and more stiffness. However, a few research works deal with the complex nonlinear dynamics of composite laminated cantilever plates subjected to the supersonic gas flows and in-plane excitations, such as the bifurcations and multi-pulse chaotic dynamics. Therefore, research works on the complex nonlinear dynamics of composite laminated cantilever plates face the challenge. With the development of the theories for the nonlinear dynamics and chaos, the prediction, understanding and control become possible for more complicated nonlinear phenomena in laminated composite cantilever rectangular plates.

A lot of research works have been done about the flutter of plates subjected to a supersonic flow. Dowell [1,2] studied the nonlinear oscillations of simply supported fluttering plates by using Galerkin's method. He [3,4]

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also investigated the nonlinear flutter of doubly curved plates of shallow curvature by using a modified form of Donnell's nonlinear shallow shell theory. Recently, the limit cycle nonlinear oscillations of a cantilever plate were studied by Ye and Dowell [5]. They used a Rayleigh–Ritz approach and the direct numerical integration to prove that the length-to-width ratio of the cantilever plate is a significant factor on the flutter study. Shiau and Lu [6] investigated the nonlinear flutter behaviors of a composite laminated plate at high supersonic Mach number. The results showed that the anisotropic properties have significant effects on the behavior of both limit cycle nonlinear oscillations and chaotic motions. Chandiramani et al. [7] analyzed the nonlinear dynamic behavior of a uniformly compressed, composite panel subjected to nonlinear aerodynamic loading. Patil and Hodges [8] presented a theoretical basis for nonlinear aeroelastic analysis and flight dynamics of an aircraft with high-aspect-ratio wings. The results obtained from this paper illustrated the effects of the structural and aerodynamic nonlinearities on the flutter speed, amplitude of limit cycle nonlinear oscillations and flight dynamics.

Recently, Moon and Kim [9] proposed a new optimal active/passive hybrid control design with piezoceramic actuators to apply to the suppression of nonlinear panel flutter. Singha and Ganapathi [10] investigated the effect of the system parameters on supersonic panel flutter behaviors of laminated composite skew plates. Guo and Mei [11] studied the use of aeroelastic modes, which can reduce drastically the number of coupled nonlinear modal equations for the nonlinear panel flutter analysis. Haddadpour et al. [12] investigated the nonlinear aeroelastic behaviors of functionally graded material (FGM) plates in supersonic flow based on the von Kármán plate and piston theories. Singha and Mandal [13] used a 16-node isoparametric degenerated shell element to study the supersonic panel flutter behaviors of laminated composite plates and cylindrical panels. Haddadpour et al. [14] investigated the effect of internal pressure and temperature rise on the flutter boundaries of the simply supported FGM cylinder with different values of power-law index. Sohn and Kim [15] analyzed the effect of volume fraction distributions, boundary conditions, temperature changes and aerodynamic pressures on panel flutter characteristics in detail based on the von Kármán strain–displacement relations. Oh and Kim [16] investigated the nonlinear vibration characteristics and supersonic flutter of cylindrical laminated panels subjected to thermal loads by using geometrically nonlinear finite elements. Shin et al. [17] studied the nonlinear flutter of aerothermally buckled composite shells with damping treatments. Kuo [18] investigated the effect of variable fiber spacing on the supersonic flutter of rectangular composite plates and found that the flutter boundary may be increased or decreased due to variable fiber spacing.

Several research works focused on the nonlinear dynamic responses of laminated composite cantilever plates and shells. Oh and Nayfeh [19] used the experimental method to study the nonlinear combination resonances of cantilever composite laminated plates with a harmonic transverse excitation. Abe et al. [20] utilized the combination of Galerkin's procedure and the shooting method to investigate the nonlinear dynamic responses of clamped laminated shallow shells with 1:1 internal resonance. Hao et al. [21] investigated the nonlinear periodic and chaotic responses of a cantilever FGM rectangular plate subjected to the transverse excitation in the thermal environment. Nejad and Nazari [22] studied the nonlinear vibrations of an isotropic cantilever plate with viscoelastic laminate. Zhang and Zhao [23] investigated the nonlinear vibrations of a composite laminated cantilever rectangular plate with one-to-one internal resonance. Zhang et al. [24] studied the nonlinear responses of a symmetric cross-ply composite laminated cantilever rectangular plate under in-plane and moment excitations. Guo et al. [25] used the experimental and theoretical methods to analyze a new kind of energy transfer from high-frequency mode to low-frequency mode in a composite laminated plate.

Many new research works have been done on the nonlinear dynamics of plates. Lee and Reddy [26] analyzed the nonlinear responses of composite laminated plates under thermomechanical loading by using the third-order shear deformation plate theory. Onkar and Yadav [27] studied the random nonlinear vibrations of a simply supported cross-ply laminated composite plate and analyzed the characteristics of the random responses and sensitivity to the lamina thickness and plate aspect ratio. Zhang [28] studied the global bifurcations and chaotic dynamics of a parametrically excited, simply supported rectangular thin plate. Awrejcewicz et al. [29] used the Bubnov–Galerkin method to study the dynamics of flexible plates and shells and analyzed the chaotic behavior of the system. In addition, Awrejcewicz et al. [30] investigated the complex vibrations and bifurcations of plates and gave some examples of new nonlinear phenomena exhibited by the systems. Ye et al. [31] studied the nonlinear oscillations and chaotic dynamics of a simply supported anti-symmetric cross-ply laminated composite rectangular thin plate under parametric excitation. Zhang et al. [32] further investigated the nonlinear oscillations and chaotic dynamics of a parametrically excited simply supported symmetric cross-ply laminated composite rectangular thin plate. Zhang et al. [33] analyzed the nonlinear dynamics and chaos of a simply supported orthotropic FGM rectangular plate in thermal environment and subjected to parametric and external excitations. Hosseini and Fazelzadeh [34] analyzed the aero-thermoelastic post-critical and vibration characteristics of temperature-dependent functionally graded material panels under the excitation of supersonic airflow. Alijani et al. [35] investigated the nonlinear forced vibrations of FGM doubly curved shallow shells with a rectangular base. In addition, Alijani et al. [36] study the nonlinear vibrations of FGM rectangular plates in thermal environments. In their paper, the bifurcations and complex nonlinear dynamics are investigated by using bifurcation diagrams.

This paper focuses on studies of the bifurcations and chaotic dynamics of a composite laminated cantilever rectangular plate subjected to the aerodynamic pressure and the in-plane excitation. Based on the von Kármán-type plate equation and Reddy's third-order shear deformation plate theory, we employ Hamilton's principle to establish the nonlinear governing equations of motion for the composite laminated cantilever rectangular plate. The governing equations of motion can be reduced to a two-degree-freedom nonlinear system under combined parametric and external excitations by using Galerkin's method. The case of 1:2 internal resonance and primary parametric resonance is considered. The method of multiple scales is used to obtain the averaged equation of the original non-autonomous system. A numerical method is utilized to investigate the bifurcations, periodic and chaotic motions of the composite laminated cantilever rectangular plate. The bifurcation diagrams are also obtained by using numerical simulation. The frequency–response curves of the composite laminated cantilever rectangular plate are obtained to indicate the transfer of energy between different modes. It is found from the numerical results that there exist periodic and chaotic motions of the composite laminated cantilever rectangular plate under certain conditions.

2 Equations of motion

In this section, the governing equations of motion for the symmetric cross-ply composite laminated rectangular plate subjected to in-plane excitation and transversal aerodynamic pressure are established. The plate is clamped supported at edge ob, as shown in Fig. 1. The ply stacking sequence is $(0/90)_S$, and the number of layers of the plate is N, namely the ply angle of the composite laminated rectangular plate is 90° or 0°, which plays an important role in the stress–strain relationship. All layers are bonded perfectly. The edge width and length of the composite laminated cantilever rectangular plate in the x and y directions is, respectively, a and b, and the thickness is h. A Cartesian coordinate $Oxyz$ is located in the middle surface of the composite laminated cantilever rectangular plate. Assume that (u, v, w) and (u_0, v_0, w_0) represent the displacements of an arbitrary point and a point in the middle surface of the composite laminated cantilever rectangular plate in the x, y and z directions, respectively. It is also assumed that ϕ_x and ϕ_y represent the rotations of two transverse normals on the mid-plane about the x- and y- axes, respectively. The in-plane excitation is distributed along the y direction at $y = 0$ and b and is of the form $F = F_0 + F_1 \cos \Omega_1 t$, where Ω_1 is the frequency of the in-plane excitation.

The third-order piston aerodynamic load, namely transverse excitation, is given by [37]

$$\Delta P = \frac{4q_d\gamma}{M_\infty}\left[\frac{1}{v}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial y} + \frac{\kappa+1}{12}\gamma^2 M_\infty^2\left(\frac{1}{v}\frac{\partial w}{\partial t} + \frac{\partial w}{\partial y}\right)^3\right], \tag{1}$$

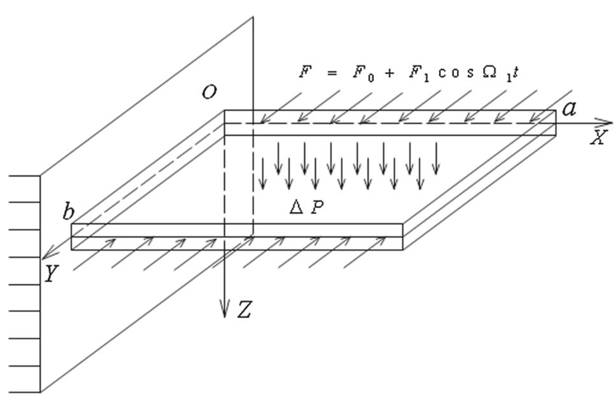

Fig. 1 The model of a composite laminated cantilever rectangular plate and the coordinate system

where $\gamma = \frac{M_\infty}{\sqrt{M_\infty^2 - 1}}$, M_∞ represents free stream Mach number, υ is air flow velocity, κ is the adiabatic exponent, and q_d is dynamic pressure.

The stress–strain relationship is obtained as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{Bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix}, \tag{2}$$

where the elastic stiffness coefficients of the composite laminated cantilever plate are given by

$$\begin{gathered} Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \\ Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{31}. \end{gathered} \tag{3}$$

According to Reddy's third-order shear deformation theory (TSDT) given in reference [38], the displacement field of the composite laminated cantilever rectangular plate is obtained as

$$u(x,\ y,\ z,\ t) = u_0(x,\ y,\ t) + z\varphi_x(x,\ y,\ t) - z^3 \frac{4}{3h^2} \left(\varphi_x + \frac{\partial w_0}{\partial x} \right), \tag{4.1}$$

$$v(x,\ y,\ z,\ t) = v_0(x,\ y,\ t) + z\varphi_y(x,\ y,\ t) - z^3 \frac{4}{3h^2} \left(\varphi_y + \frac{\partial w_0}{\partial y} \right), \tag{4.2}$$

$$w(x,\ y,\ t) = w_0(x,\ y,\ t). \tag{4.3}$$

Using the nonlinear strain–displacement relation and the aforementioned displacement field yields

$$\begin{gathered} \varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{gathered} \tag{5.1}$$

and

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}, \tag{5.2}$$

where $c_2 = 3c_1$, $c_1 = \frac{4}{3h^2}$, and

$$\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix}, \tag{6}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}.$$

According to Hamilton's principle, the nonlinear governing equations of motion for the composite laminated cantilever rectangular plate are obtained as

$$N_{xx,x} + N_{xy,y} = I_0\ddot{u}_0 + (I_1 - c_1 I_3)\,\ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x}, \tag{7.1}$$

$$N_{yy,y} + N_{xy,x} = I_0\ddot{v}_0 + (I_1 - c_1 I_3)\,\ddot{\phi}_y - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y}, \tag{7.2}$$

$$\begin{aligned} &N_{yy,y}\frac{\partial w_0}{\partial y} + N_{yy}\frac{\partial^2 w_0}{\partial y^2} + N_{xy,x}\frac{\partial w_0}{\partial y} + N_{xy,y}\frac{\partial w_0}{\partial x} + 2N_{xy}\frac{\partial^2 w_0}{\partial x \partial y} + N_{xx,x}\frac{\partial w_0}{\partial x} + N_{xx}\frac{\partial^2 w_0}{\partial x^2} \\ &\quad + c_1\left(P_{xx,xx} + 2P_{xy,xy} + P_{yy,yy}\right) + \left(Q_{x,x} - c_2 R_{x,x}\right) + \left(Q_{y,y} - c_2 R_{y,y}\right) + \Delta P - \gamma \dot{w}_0 \\ &\quad = c_1 I_3\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) + c_1\,(I_4 - c_1 I_6)\left(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}\right) + I_0\ddot{w}_0 - c_1^2 I_6\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right), \end{aligned} \tag{7.3}$$

$$\begin{aligned} &M_{xx,x} + M_{xy,y} - c_1 P_{xx,x} - c_1 P_{xy,y} - (Q_x - c_2 R_x) \\ &\quad = (I_1 - c_1 I_3)\,\ddot{u}_0 + \left(I_2 - 2c_1 I_4 + c_1^2 I_6\right)\ddot{\phi}_x - c_1\,(I_4 - c_1 I_6)\,\frac{\partial \ddot{w}_0}{\partial x}, \end{aligned} \tag{7.4}$$

$$\begin{aligned} &M_{yy,y} + M_{xy,x} - c_1 P_{yy,y} - c_1 P_{xy,x} - \left(Q_y - c_2 R_y\right) \\ &\quad = (I_1 - c_1 I_3)\,\ddot{v}_0 + \left(I_2 - 2c_1 I_4 + c_1^2 I_6\right)\ddot{\phi}_y - c_1\,(I_4 - c_1 I_6)\,\frac{\partial \ddot{w}_0}{\partial y}, \end{aligned} \tag{7.5}$$

where γ is the damping coefficient, a comma denotes partial differentiation with respect to a specified coordinate, a overdot implies partial differentiation with respect to time, and the stress resultants are represented as follows:

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} \mathrm{d}z, \tag{8.1}$$

$$\begin{Bmatrix} Q_{\alpha} \\ R_{\alpha} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \mathrm{d}z, \tag{8.2}$$

where α and β, respectively, denote x and y.

Substituting the stress resultants of Eq. (8) into Eq. (7), we can write Eq. (7) in terms of generalized displacements (u_0; v_0; w_0; ϕ_x; ϕ_y),

$$\begin{aligned} &A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66})\,\frac{\partial^2 v_0}{\partial x \partial y} + A_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} \\ &\quad + (A_{12} + A_{66})\,\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x \partial y} = I_0\ddot{u}_0 + J\ddot{\phi}_x - c_1 I_3\frac{\partial \ddot{w}_0}{\partial x}, \end{aligned} \tag{9.1}$$

$$\begin{aligned} &A_{66}\frac{\partial^2 v_0}{\partial x^2} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + (A_{21} + A_{66})\,\frac{\partial^2 u_0}{\partial x \partial y} + A_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2} + A_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2} \\ &\quad + (A_{21} + A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x \partial y} = I_0\ddot{v}_0 + J_1\ddot{\phi}_y - c_1 I_3\frac{\partial \ddot{w}_0}{\partial y}, \end{aligned} \tag{9.2}$$

$$\begin{aligned} &A_{11}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + A_{21}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} + 2A_{66}\frac{\partial u_0}{\partial y}\frac{\partial^2 w_0}{\partial x \partial y} + A_{11}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial w_0}{\partial x} + (A_{21} + A_{66})\,\frac{\partial^2 u_0}{\partial x \partial y}\frac{\partial w_0}{\partial y} \\ &\quad + A_{66}\frac{\partial^2 v_0}{\partial x^2}\frac{\partial w_0}{\partial y} + (A_{12} + A_{66})\,\frac{\partial^2 v_0}{\partial x \partial y}\frac{\partial w_0}{\partial x} + A_{22}\frac{\partial^2 v_0}{\partial y^2}\frac{\partial w_0}{\partial y} + \frac{3}{2}A_{11}\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial x^2} \\ &\quad + \left(\frac{1}{2}A_{21} + A_{66}\right)\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial y^2} + (A_{12} + A_{21} + 4A_{66})\,\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x \partial y} + \frac{3}{2}A_{22}\left(\frac{\partial w_0}{\partial y}\right)^2\frac{\partial^2 w_0}{\partial y^2} \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}A_{12}+A_{66}\right)\left(\frac{\partial w_0}{\partial y}\right)^2\frac{\partial^2 w_0}{\partial x^2}+\left(A_{55}-2c_2D_{55}+c_2^2F_{55}\right)\frac{\partial^2 w_0}{\partial x^2}+\left(F_0+F_1\cos\Omega_1 t\right)\frac{\partial^2 w_0}{\partial y^2}\\
&+A_{22}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 u_0}{\partial y^2}+2A_{66}\frac{\partial v_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}+A_{12}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}\\
&+\left(A_{44}-2c_2D_{44}+c_2^2F_{44}\right)\frac{\partial^2 w_0}{\partial y^2}-c_1^2H_{11}\frac{\partial^4 w_0}{\partial x^4}-c_1^2\left(H_{12}+H_{21}+4H_{66}\right)\frac{\partial^4 w_0}{\partial x^2\partial y^2}\\
&+\left(A_{55}-2c_2D_{55}+c_2^2F_{55}\right)\frac{\partial \phi_x}{\partial x}-c_1^2H_{22}\frac{\partial^4 w_0}{\partial y_4}+c_1\left(F_{11}-c_1H_{11}\right)\frac{\partial^3 \phi_x}{\partial x^3}\\
&+c_1\left(F_{21}+2F_{66}-c_1H_{21}-2c_1H_{66}\right)\frac{\partial^3 \phi_x}{\partial x\partial y^2}+\left(A_{44}-2c_2D_{44}+c_2^2F_{44}\right)\frac{\partial \phi_y}{\partial y}\\
&+c_1\left(F_{12}+2F_{66}-c_1H_{12}-2c_1H_{66}\right)\frac{\partial^3 \phi_x}{\partial x^2\partial y}+c_1\left(F_{22}-c_1H_{22}\right)\frac{\partial^3 \phi_y}{\partial y^3}-\gamma\dot{w}_0\\
&+\frac{4q_d\gamma}{M_\infty}\left[\frac{1}{\nu}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial y}+\frac{\kappa+1}{12}\gamma^2M_\infty^2\left(\frac{1}{\nu}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial y}\right)^3\right]\\
&=I_0\ddot{w}_0-c_1^2I_6\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right)+c_1I_3\left(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial y}\right)+c_1J_4\left(\frac{\partial\ddot{\phi}_x}{\partial x}+\frac{\partial\ddot{\phi}_y}{\partial y}\right),
\end{aligned}\tag{9.3}
$$

$$
\begin{aligned}
&\left(D_{11}-2c_1F_{11}+c_1^2H_{11}\right)\frac{\partial^2\phi_x}{\partial x^2}+\left(D_{66}-2c_1F_{66}+c_1^2H_{66}\right)\frac{\partial^2\phi_x}{\partial y^2}-c_1\left(F_{11}-c_1H_{11}\right)\frac{\partial^3 w_0}{\partial x^3}\\
&-\left(A_{55}-2c_2D_{55}+c_2^2F_{55}\right)\frac{\partial w_0}{\partial x}+\left(D_{12}+D_{66}+c_1^2H_{66}-2c_1F_{66}+c_1^2H_{12}-2c_1F_{12}\right)\frac{\partial^2\phi_y}{\partial x\partial y}\\
&-c_1\left(F_{12}+2F_{66}-c_1H_{12}-2c_1H_{66}\right)\frac{\partial^3 w_0}{\partial x\partial y^2}+\left(2c_2D_{55}-A_{55}-c_2^2F_{55}\right)\phi_x\\
&=J_1\ddot{u}_0+K_2\ddot{\phi}_x-c_1J_4\frac{\partial\ddot{w}_0}{\partial x},
\end{aligned}\tag{9.4}
$$

$$
\begin{aligned}
&\left(D_{66}-2c_1F_{66}+c_1^2H_{66}\right)\frac{\partial^2\phi_y}{\partial x^2}-c_1\left(F_{21}+2F_{66}-c_1H_{21}-2c_1H_{66}\right)\frac{\partial^3 w_0}{\partial x^2\partial y}\\
&-c_1\left(F_{22}-c_1H_{22}\right)\frac{\partial^3 w_0}{\partial y^3}+\left(D_{21}+D_{66}+c_1^2H_{21}-2c_1F_{21}+c_1^2H_{66}-2c_1F_{66}\right)\frac{\partial^2\phi_x}{\partial x\partial y}\\
&+\left(c_1^2H_{22}+D_{22}-2c_1F_{22}\right)\frac{\partial^2\phi_y}{\partial x^2}-\left(F_{44}c_2^2-2c_2D_{44}+A_{44}\right)\frac{\partial w_0}{\partial y}+\left(2c_2D_{44}-A_{44}-c_2^2F_{44}\right)\phi_y\\
&=J_1\ddot{v}_0+K_2\ddot{\phi}_y-c_1J_4\frac{\partial\ddot{w}_0}{\partial y}.
\end{aligned}\tag{9.5}
$$

where

$$
\left(A_{ij},\ B_{ij},\ D_{ij},\ E_{ij},\ F_{ij},H_{ij}\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}Q_{ij}^k\ \left(1,z,z^2,z^3,z^4,z^6\right)\ \mathrm{d}z,\quad (i,j=1,\ 2,\ 6),\tag{10.1}
$$

$$
\left(A_{ij},D_{ij},F_{ij}\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}Q_{ij}^k\ \left(1,z^2,z^4\right)\ \mathrm{d}z,\quad (i,\ j=4,\ 5)\,,\tag{10.2}
$$

$$
I_i=\sum_{k=1}^{3}\int_{z_k}^{z_k+1}\rho_k\,(z)^i\,\mathrm{d}z,\quad (i=0,\ 1,\ 2,\ \ldots,\ 6)\,,\tag{10.3}
$$

$$
J_i=I_i-c_1I_{i+2},\quad K_2=I_2-2c_1I_4+c_1^2I_6.\tag{10.4}
$$

The boundary conditions of the composite laminated cantilever rectangular plate can be expressed as

$$x = 0 : w = v = u = \phi_y = \phi_x = 0, \tag{11.1}$$
$$x = a : N_{xx} = N_{xy} = M_{xx} = M_{xy} - c_1 P_{xy} = \bar{Q}_x = 0, \tag{11.2}$$
$$y = 0 : N_{xy} = M_{yy} = M_{xy} - c_1 P_{xy} = \bar{Q}_y = 0, \tag{11.3}$$
$$y = b : N_{xy} = M_{yy} = M_{xy} - c_1 P_{xy} = \bar{Q}_y = 0, \tag{11.4}$$
$$\int_{-h/2}^{h/2} N_{yy} \mathrm{d}z = \pm \int_{-h/2}^{h/2} (F_0 + F_1 coc\Omega_1 t) \mathrm{d}z, \quad (y = 0,\ b), \tag{11.5}$$

where

$$\bar{Q}_x = Q_x + \frac{\partial M_{xy}}{\partial y} - c_2 R_x + c_1 \left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right), \tag{12.1}$$
$$\bar{Q}_y = Q_y + \frac{\partial M_{xy}}{\partial x} - c_2 R_y + c_1 \left(\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} \right). \tag{12.2}$$

In order to obtain the dimensionless equations, we introduce the transformations of the variables and parameters

$$\begin{gathered} w_o = \vec{w} h, \quad t = \frac{\bar{t}}{\pi^2} \left(\frac{E}{ab\rho} \right)^{-\frac{1}{2}}, \quad A_{ij} = \frac{Eh^2}{(ab)^{\frac{1}{2}}} \bar{A}_{ij}, \quad B_{ij} = \frac{Eh^3}{(ab)^{\frac{1}{2}}} \bar{B}_{ij}, \quad I_i = (ab)^{\frac{i+1}{2}} \rho \bar{I}_i, \\ \gamma = \frac{\pi^2 h^4}{(ab)^2} (\rho E)^{\frac{1}{2}} \bar{\gamma}, \quad D_{ij} = \frac{Eh^4}{(ab)^{\frac{1}{2}}} \bar{D}_{ij}, \quad E_{ij} = \frac{Eh^2}{(ab)^{\frac{1}{2}}} \bar{E}_{ij}, \quad F_{ij} = \frac{Eh^6}{(ab)^{\frac{1}{2}}} \bar{F}_{ij}, \quad x = a\bar{x}, \\ H_{ij} = \frac{Eh^8}{(ab)^{\frac{1}{2}}} \bar{H}_{ij}, \quad \Omega_i = \pi^2 \left(\frac{E}{ab\rho} \right)^{-\frac{1}{2}} \bar{\Omega}_i, \quad q_d = \frac{Eh^7}{(ab)^{\frac{7}{2}}} \bar{q}_d, \quad f = \frac{Eh^3}{b^2} \bar{f}, \quad y = b\bar{y}. \end{gathered} \tag{13}$$

It is our aim to choose a suitable mode function to satisfy the first two modes of transverse nonlinear oscillations and the boundary conditions for the composite laminated cantilever rectangular plate. Thus, according to the mode approximation functions given in reference [38], we represent w_0 as

$$w_0 = w(t)_1 X_1(x) Y_1(y) + w(t)_2 X_2(x) Y_2(y), \tag{14}$$

where

$$X_i(x) = \sin \frac{\lambda_i}{a} x - \sinh \frac{\lambda_i}{a} x + \alpha_i \left(\cosh \frac{\lambda_i}{a} x - \cos \frac{\lambda_i}{a} x \right), \tag{15.1}$$
$$Y_j(y) = \sin \frac{\beta_m}{b} y + \sinh \frac{\beta_m}{b} y - \alpha_m \left(\cosh \frac{\beta_m}{b} y + \cos \frac{\beta_m}{b} y \right), \tag{15.2}$$

where λ_i and μ_j $(i,\ j = 1,\ 2)$ are the roots of the characteristic equations

$$\cos \lambda_i a \cosh \lambda_i a - 1 = 0, \quad \cos \beta_m b \cosh \beta_m b - 1 = 0, \tag{16.1}$$
$$\alpha_i = \frac{\cosh \lambda_i - \cos \lambda_i}{\sinh \lambda_i + \sin \lambda_i}, \quad \alpha_m = -\frac{\cosh \beta_m - \cos \beta_m}{\sinh \beta_m - \sin \beta_m}, \tag{16.2}$$

and u, v, φ_x and φ_y in Eq. (9) are supposed to be of the form

$$\begin{aligned} u_0 = u(t) &\left[\sin \frac{\lambda_1}{a} x - \sinh \frac{\lambda_1}{a} x + \alpha_i \left(\cosh \frac{\lambda_1}{a} x - \cos \frac{\lambda_1}{a} x \right) \right] \\ &\times \left[\left(\frac{\beta_1}{b} \right)^2 \sinh \frac{\beta_1}{b} y - \left(\frac{\beta_1}{b} \right)^2 \sin \frac{\beta_1}{b} y - \alpha_m \left(\frac{\beta_1}{b} \right)^2 \left(\cosh \frac{\beta_1}{b} y - \cos \frac{\beta_1}{b} y \right) \right], \end{aligned}$$

$$v_0 = v\,(t)\left[\sin\frac{\lambda_1}{a}x - \sinh\frac{\lambda_1}{a}x + \alpha_i\left(\cosh\frac{\lambda_1}{a}x - \cos\frac{\lambda_1}{a}x\right)\right]$$
$$\times\left[\left(\frac{\beta_2}{b}\right)^2\sinh\frac{\beta_2}{b}y - \left(\frac{\beta_2}{b}\right)^2\sin\frac{\beta_2}{b}y - \alpha_m\left(\frac{\beta_2}{b}\right)^2\left(\cosh\frac{\beta_2}{b}y - \cos\frac{\beta_2}{b}y\right)\right],$$
$$\varphi_x = \varphi_x\,(t)\left[\sin\frac{\lambda_1}{a}x - \sinh\frac{\lambda_1}{a}x + \alpha_i\left(\cosh\frac{\lambda_1}{a}x - \cos\frac{\lambda_1}{a}x\right)\right]$$
$$\times\left[\left(\frac{\beta_2}{b}\right)^2\sinh\frac{\beta_2}{b}y - \left(\frac{\beta_2}{b}\right)^2\sin\frac{\beta_2}{b}y - \alpha_m\left(\frac{\beta_2}{b}\right)^2\left(\cosh\frac{\beta_2}{b}y - \cos\frac{\beta_2}{b}y\right)\right],$$
$$\varphi_y = \varphi_y\,(t)\left[\sin\frac{\lambda_1}{a}x - \sinh\frac{\lambda_1}{a}x + \alpha_i\left(\cosh\frac{\lambda_1}{a}x - \cos\frac{\lambda_1}{a}x\right)\right]$$
$$\times\left[\left(\frac{\beta_1}{b}\right)^2\sinh\frac{\beta_1}{b}y - \left(\frac{\beta_1}{b}\right)^2\sin\frac{\beta_1}{b}y - \alpha_m\left(\frac{\beta_1}{b}\right)^2\left(\cosh\frac{\beta_1}{b}y - \cos\frac{\beta_1}{b}y\right)\right], \tag{17}$$

where $w_1(t)$, $w_2(t)$, $u(t)$, $v(t)$, $\varphi_x(t)$ and $\varphi_y(t)$ are the amplitudes of the modes, respectively.

Based on research given by Reddy [38], we neglect all the inertia terms on u_0, v_0, φ_x and φ_y in (9.1), (9.2), (9.4) and (9.5) since the deformations in these directions are smaller than that in the transverse direction. Then, substituting equations (14) and (17) into equations (9.1), (9.2), (9.4) and (9.5), the displacement components $u(t)$, $v(t)$, $\varphi_x(t)$ and $\varphi_y(t)$ can be expressed in terms of $w(t)$. Applying the Galerkin procedure yields the governing equations of transverse motion for the composite laminated cantilever rectangular plate under combined aerodynamic pressure and in-plane excitation in dimensionless form,

$$\ddot{w}_1 + \mu_1\dot{w}_1 + \omega_1^2 w_1 + (\gamma_7\dot{w}_1 + \gamma_8 w_1 + \gamma_9 w_2 + \gamma_{10}\dot{w}_2)\dot{w}_1^2 + (\gamma_{12}\dot{w}_2 + \gamma_{15}w_1 + \gamma_{16}w_2)\dot{w}_2\dot{w}_1$$
$$+ (\gamma_{11}\dot{w}_2 + \gamma_{13}w_1 + \gamma_{14}w_2)\dot{w}_2^2 + (\gamma_{17}w_1^2 + \gamma_{18}w_2^2 + \gamma_{19}w_1w_2)\dot{w}_1 + (\gamma_{20}w_1w_2 + \gamma_{21}w_1^2$$
$$+ \gamma_{22}w_2^2)\dot{w}_2 + \gamma_3 w_1^3 + \gamma_4 w_2 w_1^2 + \gamma_5 w_1 w_2^2 + \gamma_6 w_2^3 + \gamma_2 w_2 + \gamma_1 f\cos\Omega_1 t w_1 = 0, \tag{18.1}$$
$$\ddot{w}_2 + \mu_2\dot{w}_2 + \omega_2^2 w_2 + (\delta_7\dot{w}_2 + \delta_8 w_2 + \delta_9 w_1 + \delta_{10}\dot{w}_1)\dot{w}_2^2 + (\delta_{12}\dot{w}_1 + \delta_{15}w_2 + \delta_{16}w_1)\dot{w}_1\dot{w}_2$$
$$+ (\delta_{11}\dot{w}_1 + \delta_{13}w_2 + \delta_{14}w_1)\dot{w}_1^2 + (\delta_{17}w_2^2 + \delta_{18}w_1^2 + \delta_{19}w_2w_1)\dot{w}_2$$
$$+ (\delta_{20}w_2w_1 + \delta_{21}w_2^2 + \delta_{22}w_1^2)\dot{w}_1 + \delta_3 w_2^3 + \delta_4 w_1 w_2^2 + \delta_5 w_2 w_1^2$$
$$+ \delta_6 w_1^3 + \delta_2 w_1 + \delta_1 f\cos\Omega_1 t w_2 = 0, \tag{18.2}$$

where all coefficients can be found in the "Appendix", and the f_1 is the magnitude of the in-plane excitation.

3 Perturbation analysis

To guarantee the validity of the perturbation analysis, we use the method of multiple scales [39] to obtain the averaged equation of system (18). We only consider the case of 1:2 internal resonance and primary parametric resonance for the composite laminated cantilever rectangular plate,

$$\omega_1^2 = \frac{1}{4}\Omega^2 + \varepsilon\sigma_1, \quad \omega_2^2 = \Omega^2 + \varepsilon\sigma_2, \quad \Omega_1 = \Omega_2 = 1, \tag{19}$$

where ω_1 and ω_2 are two linear natural frequencies, and σ_1 and σ_2 are the two detuning parameters.

The scale transformations may be introduced as

$$\mu_1 \to \varepsilon\mu_1,\quad \mu_2 \to \varepsilon\mu_2,\quad \gamma_1 \to \varepsilon\gamma_1,\quad \gamma_2 \to \varepsilon\gamma_2,\quad \gamma_3 \to \varepsilon\gamma_3,\quad \gamma_4 \to \varepsilon\gamma_4,\quad \gamma_5 \to \varepsilon\gamma_5,$$
$$\gamma_6 \to \varepsilon\gamma_6,\quad \gamma_7 \to \varepsilon\gamma_7,\quad \gamma_8 \to \varepsilon\gamma_8,\quad \gamma_9 \to \varepsilon\gamma_9,\quad \gamma_{10} \to \varepsilon\gamma_{10},\quad \gamma_{11} \to \varepsilon\gamma_{11},\quad \gamma_{12} \to \varepsilon\gamma_{12},$$
$$\gamma_{13} \to \varepsilon\gamma_{13},\quad \gamma_{14} \to \varepsilon\gamma_{14},\quad \gamma_{15} \to \varepsilon\gamma_{15},\quad \gamma_{16} \to \varepsilon\gamma_{16},\quad \gamma_{17} \to \varepsilon\gamma_{17},\quad \gamma_{18} \to \varepsilon\gamma_{18},$$
$$\gamma_{19} \to \varepsilon\gamma_{19},\quad \gamma_{20} \to \varepsilon\gamma_{20},\quad \gamma_{21} \to \varepsilon\gamma_{21},\quad \gamma_{22} \to \varepsilon\gamma_{22},\quad \delta_1 \to \varepsilon\delta_1,\quad \delta_2 \to \varepsilon\delta_2,\quad \delta_3 \to \varepsilon\delta_3,$$
$$\delta_4 \to \varepsilon\delta_4,\quad \delta_5 \to \varepsilon\delta_5,\quad \delta_6 \to \varepsilon\delta_6,\quad \delta_7 \to \varepsilon\delta_7,\quad \delta_8 \to \varepsilon\delta_8,\quad \delta_9 \to \varepsilon\delta_9,\quad \delta_{10} \to \varepsilon\delta_{10},$$
$$\delta_{11} \to \varepsilon\delta_{11},\quad \delta_{12} \to \varepsilon\delta_{12},\quad \delta_{13} \to \varepsilon\delta_{13},\quad \delta_{14} \to \varepsilon\delta_{14},\quad \delta_{15} \to \varepsilon\delta_{15},\quad \delta_{16} \to \varepsilon\delta_{16},$$
$$\delta_{17} \to \varepsilon\delta_{17},\quad \delta_{18} \to \varepsilon\delta_{18},\quad \delta_{19} \to \varepsilon\delta_{19},\quad \delta_{20} \to \varepsilon\delta_{20},\quad \delta_{21} \to \varepsilon\delta_{21},\quad \delta_{22} \to \varepsilon\delta_{22}. \tag{20}$$

Substituting Eq. (20) into Eq. (18) yields

$$\begin{aligned}
&\ddot{w}_1 + \varepsilon\mu_1\dot{w}_1 + \omega_1^2 w_1 + \varepsilon\gamma_7\dot{w}_1^3 + \varepsilon\gamma_8 w_1\dot{w}_1^2 + \varepsilon\gamma_9\dot{w}_1^2 w_2 + \varepsilon\gamma_{10}\dot{w}_1^2\dot{w}_2 + \varepsilon\gamma_{12}\dot{w}_2^2\dot{w}_1 + \varepsilon\gamma_{15} w_1\dot{w}_2\dot{w}_1\\
&\quad + \varepsilon\gamma_{16} w_2\dot{w}_2\dot{w}_1 + \varepsilon\gamma_{17} w_1^2\dot{w}_1 + \varepsilon\gamma_{18} w_2^2\dot{w}_1 + \varepsilon\gamma_{19} w_1 w_2\dot{w}_1 + \varepsilon\gamma_1 f\cos\Omega_1 t w_1\\
&\quad + \varepsilon\gamma_{11}\dot{w}_2^3 + \varepsilon\gamma_{13} w_1\dot{w}_2^2 + \varepsilon\gamma_{14} w_2\dot{w}_2^2 + \varepsilon\gamma_{20} w_1 w_2\dot{w}_2 + \varepsilon\gamma_{21} w_1^2\dot{w}_2 + \varepsilon\gamma_{22} w_2^2\dot{w}_2\\
&\quad + \varepsilon\gamma_2 w_2 + \varepsilon\gamma_3 w_1^3 + \varepsilon\gamma_4 w_2 w_1^2 + \varepsilon\gamma_5 w_1 w_2^2 + \varepsilon\gamma_6 w_2^3 = 0, \qquad (21.1)\\
&\ddot{w}_2 + \varepsilon\mu_2\dot{w}_2 + \omega_2^2 w_2 + \varepsilon\delta_7\dot{w}_2^3 + \varepsilon\delta_8 w_2\dot{w}_2^2 + \varepsilon\delta_9\dot{w}_2^2 w_1 + \varepsilon\delta_{10}\dot{w}_1\dot{w}_2^2 + \varepsilon\delta_{12}\dot{w}_1^2\dot{w}_2 + \varepsilon\delta_{15} w_2\dot{w}_1\dot{w}_2\\
&\quad + \varepsilon\delta_{16} w_1\dot{w}_1\dot{w}_2 + \varepsilon\delta_{17} w_2^2\dot{w}_2 + \varepsilon\delta_{18} w_1^2\dot{w}_2 + \varepsilon\delta_{19} w_2 w_1\dot{w}_2 + \varepsilon\delta_1 f\cos\Omega_1 t w_2\\
&\quad + \varepsilon\delta_{11}\dot{w}_1^3 + \varepsilon\delta_{13} w_2\dot{w}_1^2 + \varepsilon\delta_{14} w_1\dot{w}_1^2 + \varepsilon\delta_{20} w_2 w_1\dot{w}_1^{+}\varepsilon\delta_{21} w_2^2\dot{w}_1 + \varepsilon\delta_{22} w_1^2\dot{w}_1\\
&\quad + \varepsilon\delta_2 w_1 + \varepsilon\delta_3 w_2^3 + \varepsilon\delta_4 w_1 w_2^2 + \varepsilon\delta_5 w_2 w_1^2 + \varepsilon\delta_6 w_1^3 = 0. \qquad (21.2)
\end{aligned}$$

We use the method of multiple scales to find the uniform solutions of Eq. (21) in the following form:

$$w(t,\ \varepsilon) = w_0(T_0,\ T_1) + \varepsilon w_1(T_0,\ T_1) + \cdots, \qquad (22)$$

where $T_0 = t$, $T_1 = \varepsilon t$.

The time derivatives used in the method of multiple scales are obtained as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0}\frac{\mathrm{d}T_0}{\mathrm{d}t} + \frac{\partial}{\partial T_1}\frac{\mathrm{d}T_1}{\mathrm{d}t} + \cdots = D_0 + \varepsilon D_1 + \cdots, \qquad (23.1)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = (D_0 + \varepsilon D_1 + \cdots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \qquad (23.2)$$

where $D_n = \dfrac{\partial}{\partial T_n}$, $n = 1,\ 2,\ \ldots$.

Introducing Eqs. (22) and (23) into Eq. (21) and eliminating secular terms, we obtain the averaged equation in the complex form,

$$\begin{aligned}
D_1 A_1 &= -\frac{1}{2}\mu_1 A_1 + i\sigma_1 A_1 + \left(3i\gamma_3 + \frac{1}{4}i\gamma_8 - \frac{3}{8}\gamma_7 - \frac{1}{2}\gamma_{17}\right) A_1^2\bar{A}_1\\
&\quad + (2i\gamma_5 + 2i\gamma_{13} - \gamma_{12} - \gamma_{18}) A_1 A_2\bar{A}_2 + \frac{1}{2}i\gamma_1 f\bar{A}_1, \qquad (24.1)\\
D_1 A_2 &= -\frac{1}{2}\mu_2 A_2 + \frac{1}{2}i\sigma_2 A_2 + \left(\frac{3}{2}i\delta_3 + \frac{1}{2}i\delta_8 - \frac{3}{2}\delta_7 - \frac{1}{2}\delta_{17}\right) A_2^2\bar{A}_2\\
&\quad + \left(\frac{1}{4}i\delta_{13} + i\delta_5 - \frac{1}{4}\delta_{12} - \delta_{18}\right) A_1\bar{A}_1 A_2. \qquad (24.2)
\end{aligned}$$

Functions A_1 and A_2 may be denoted in the Cartesian form

$$A_1(T_1) = x_1(T_1) + i x_2(T_1), \qquad (25.1)$$

$$A_2(T_1) = x_3(T_1) + i x_4(T_1). \qquad (25.2)$$

Therefore, we achieve the four-dimensional averaged equation in the Cartesian form

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{2}\mu_1 x_1 - \sigma_1 x_2 - \left(\frac{3}{8}\gamma_7 + \frac{1}{2}\gamma_{17}\right) x_1^3 - \left(\frac{3}{8}\gamma_7 + \frac{1}{2}\gamma_{17}\right) x_1 x_2^2 - \left(3\gamma_3 + \frac{1}{4}\gamma_8\right) x_1^2 x_2\\
&\quad - \left(3\gamma_3 + \frac{1}{4}\gamma_8\right) x_2^3 - (\gamma_{12} + \gamma_{18})\left(x_3^2 + x_4^2\right) x_1 - 2(\gamma_{13} + \gamma_5) x_2\left(x_3^2 + x_4^2\right) + \frac{1}{2}\gamma_1 f x_2, \qquad (26.1)\\
\dot{x}_2 &= -\frac{1}{2}\mu_1 x_2 + \sigma_1 x_1 - \left(\frac{3}{8}\gamma_7 + \frac{1}{2}\gamma_{17}\right) x_1^2 x_2 - \left(\frac{3}{8}\gamma_7 + \frac{1}{2}\gamma_{17}\right) x_2^3 + \left(3\gamma_3 + \frac{1}{4}\gamma_8\right) x_1^3\\
&\quad + \left(3\gamma_3 + \frac{1}{4}\gamma_8\right) x_1 x_2^2 - (\gamma_{12} + \gamma_{18})\left(x_3^2 + x_4^2\right) x_2 + 2(\gamma_{13} + \gamma_5)\left(x_3^2 + x_4^2\right) x_1 + \frac{1}{2}\gamma_1 f x_1, \qquad (26.2)
\end{aligned}$$

$$\dot{x}_3 = -\frac{1}{2}\mu_2 x_3 - \frac{1}{2}\sigma_2 x_4 - \left(\frac{3}{2}\delta_7 + \frac{1}{2}\delta_{17}\right) x_3^3 - \left(\frac{3}{2}\delta_7 + \frac{1}{2}\delta_{17}\right) x_3 x_4^2 - \left(\frac{1}{2}\delta_8 + \frac{3}{2}\delta_3\right) x_4^3$$
$$- \left(\frac{1}{2}\delta_8 + \frac{3}{2}\delta_3\right) x_3^2 x_4 - \left(\frac{1}{4}\delta_{12} + \delta_{18}\right) \left(x_1^2 + x_2^2\right) x_3 - \left(\frac{1}{4}\delta_{13} + \delta_5\right) \left(x_1^2 + x_2^2\right) x_4, \quad (26.3)$$
$$\dot{x}_4 = -\frac{1}{2}\mu_2 x_4 + \frac{1}{2}\sigma_2 x_3 - \left(\frac{3}{2}\delta_7 + \frac{1}{2}\delta_{17}\right) x_4^3 - \left(\frac{3}{2}\delta_7 + \frac{1}{2}\delta_{17}\right) x_3^2 x_4 + \left(\frac{1}{2}\delta_8 + \frac{3}{2}\delta_3\right) x_3^3$$
$$+ \left(\frac{1}{2}\delta_8 + \frac{3}{2}\delta_3\right) x_3 x_4^2 - \left(\frac{1}{4}\delta_{12} + \delta_{18}\right) \left(x_1^2 + x_2^2\right) x_4 + \left(\frac{1}{4}\delta_{13} + \delta_5\right) \left(x_1^2 + x_2^2\right) x_3. \quad (26.4)$$

Based on the average equation, numerical simulations can be done to investigate the characteristic of the nonlinear dynamics for the composite laminated cantilever rectangular plate.

4 Numerical simulations and discussion

In the following investigation, the fourth-order Runge–Kutta algorithm is utilized to numerically analyze the periodic and chaotic motions of the symmetric cross-ply composite laminated cantilever rectangular plate subjected to aerodynamic pressure for the case of 1:2 internal resonance and primary parametric resonance. The ply stacking sequence is $(0/90)_s$, and the number of players of the plate is N. We consider the averaged equation (26) to carry out numerical simulations. We choose δ_{12} and γ_{12}, which are relative to the flow velocity q_d, as the controlling parameters when the periodic and chaotic responses of the composite laminated cantilever rectangular plate are investigated. By analyzing the bifurcation diagrams, the complicated nonlinear dynamics, including periodic and chaotic motions, may be observed globally from a range of parameter values. The two-dimensional phase portrait, waveform, three-dimensional phase portrait and frequency spectrum are plotted to demonstrate the nonlinear dynamic behaviors of the composite laminated cantilever rectangular plate. It can be clearly found from the numerical results that the periodic and chaotic motions occur for this system.

It is found from Fig. 2 that multiple solutions of Eq. (26) exist for the first two modes. Figure 2 represents the frequency–response curves of the first-order and the second-order modes, in which the vertical coordinate is the amplitude. We note that the detuning parameter σ_2 must be between 0 and 3 if the first-order and second-order modes are excited simultaneously. It is found from Fig. 2 that the phenomena of energy transfer from the first-order mode to the second-order mode can occur in the nonlinear oscillations of the composite laminated cantilever rectangular plate. In Fig. 2, the parameters and initial conditions are chosen as: $\mu_2 =$

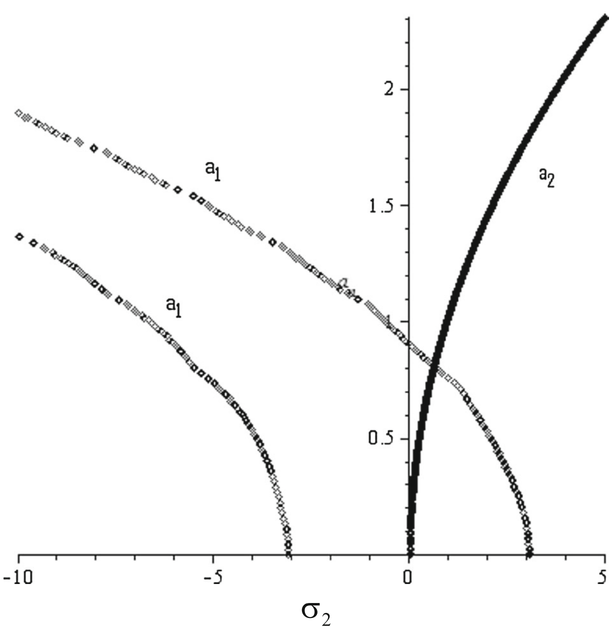

Fig. 2 Nonlinear frequency–response curves of system

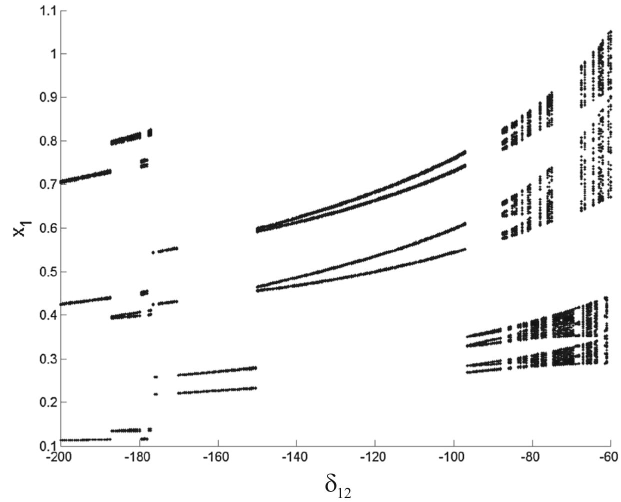

Fig. 3 The bifurcation diagram of the composite laminated cantilever rectangular plate for x_1 via the forcing excitation δ_{12}

$0.15, \sigma_1 = 2, \gamma_1 = 3, \gamma_3 = 0.08, \gamma_5 = 46, \gamma_7 = 0.5, \gamma_8 = 0.2, f = 90, \gamma_{12} = 9, \gamma_{13} = 1.5, \gamma_{17} = 4.5, \gamma_{18} = 5, \delta_3 = 0.05, \delta_5 = 4.2, \delta_7 = 0.05, \delta_8 = 1.2, \delta_{12} = -185, \delta_{17} = 30, \delta_{13} = 12.5, \delta_{18} = 2.8. x_1 = -0.5, x_2 = 0.798, x_3 = 0.5, x_4 = 0.8.$

Figure 3 illustrates the bifurcation diagram of the composite laminated cantilever rectangular plate when δ_{12} is chosen as the control parameter. From the "Appendix", we obtain the expression function of δ_{12},

$$\delta_{12} = \frac{\dfrac{(k+1)\, q_d \gamma^3 h^5 M_\infty B_{66}}{a^2 b^3 \bar{A}_{11} V^2}}{\dfrac{a^2\pi^4}{h^2 \bar{A}_{11}}\bar{I} - \dfrac{16a^3b^3}{9h^6}\dfrac{\bar{I}_6\pi^4}{\bar{A}_{11}} - \dfrac{16a^5 b}{9h^6}\dfrac{\bar{I}_6\pi^4}{\bar{A}_{11}}}. \tag{27}$$

It is seen from Eq. (27) that the parameter δ_{12} can be negative. In this case, we may chose that the interval of δ_{12} is $-200 \sim -60$. The other parameters and the initial conditions are, respectively, chosen as $\mu_1 = 0.9, \mu_2 = 0.15, \sigma_1 = 2, \sigma_2 = 5.3, \gamma_1 = 3, \gamma_3 = 0.08, \gamma_5 = 46, \gamma_7 = 0.5, \gamma_8 = 0.2, f = 90, \gamma_{12} = 11.8, \gamma_{13} = 1.5, \gamma_{17} = 4.5, \gamma_{18} = 5, \delta_3 = 0.05, \delta_5 = 4.2, \delta_7 = 0.05, \delta_8 = 1.2, \delta_{13} = 12.5, \delta_{17} = 30, \delta_{18} = 2.8, x_1 = -0.5, x_2 = 0.798, x_3 = 0.5, x_4 = 0.8$. It is observed from Fig. 3 that the parameter δ_{12} has significant effect on the nonlinear dynamic responses of the composite laminated cantilever rectangular plate. In Fig. 3, the longitudinal coordinate denotes the deflection of the plate, while the abscissa denotes the parameter δ_{12}. It is seen from Fig. 3 that the motions of the composite laminated cantilever rectangular plate change from the period-2 motion to the multiple period motion, and then from the multiple period motion to chaotic motions.

In the following investigation, we change the parameter δ_{12} to find the periodic and chaotic motions of the composite laminated cantilever rectangular plate based on Fig. 3. Figure 4 indicates the existence of the periodic motion for the composite laminated cantilever rectangular plate when the parameter δ_{12} is –160. Figure 4b, c represents the phase portraits on the planes (x_1, x_2) and (x_3, x_4), respectively. Figure 4d, e, respectively, denotes the waveforms on the planes (t, x_1) and (t, x_3). Figure 4a, f represents the three-dimensional phase portrait in space (x_1, x_2, x_3) and the frequency spectrum (f, x_3), respectively. Figure 5 illustrates that the period-4 motion of the composite laminated cantilever rectangular plate occurs when the parameter δ_{12} is -120. Figure 6 demonstrates that the quasi-periodic motion of the composite laminated cantilever rectangular plate occurs when the parameter δ_{12} is -90. Figure 7 shows that the chaotic motion of the composite laminated cantilever rectangular plate occurs when the parameter δ_{12} changes to -60.

Figure 8 demonstrates the other kind of bifurcation of the system when the control parameter is γ_{12}. The other parameters and the initial conditions are, respectively, chosen as $\mu_1 = 0.9, \mu_2 = 0.15, \sigma_1 = 2, \sigma_2 = 5.3, \gamma_1 = 3, \gamma_3 = 0.08, \gamma_5 = 46, \gamma_7 = 0.5, \gamma_8 = 0.2, f = 90, \gamma_{13} = 1.5, \gamma_{17} = 4.5, \gamma_{18} = 5, \delta_3 = 0.05, \delta_5 = 4.2, \delta_7 = 0.05, \delta_8 = 1.2, \delta_{12} = -185, \delta_{17} = 30, \delta_{13} = 12.5, \delta_{18} = 2.8, x_1 = -0.5, x_2 =$

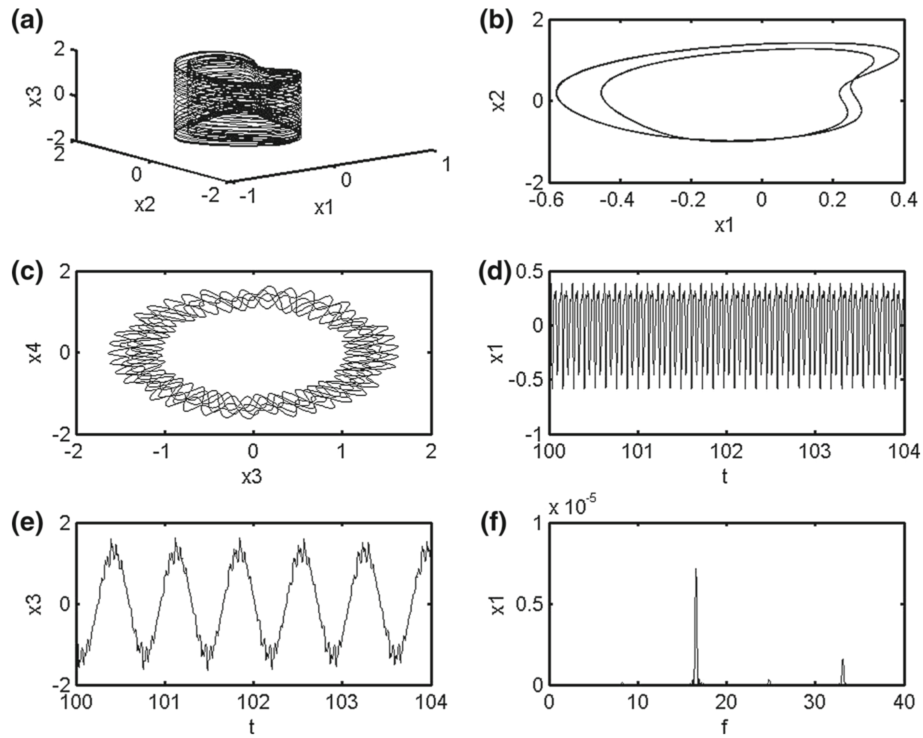

Fig. 4 The period-2 motion of the composite laminated cantilever rectangular plate exists when $\delta_{12} = -160$, **a** three-dimensional phase portrait in space $(x_1,\ x_2,\ x_3)$; **b** the phase portrait on plane $(x_1,\ x_2)$; **c** the phase portrait on plane $(x_3,\ x_4)$; **d** the waveforms on the planes $(t,\ x_1)$; **e** the waveforms on the planes $(t,\ x_3)$; **f** the frequency spectrum $(f,\ x_3)$

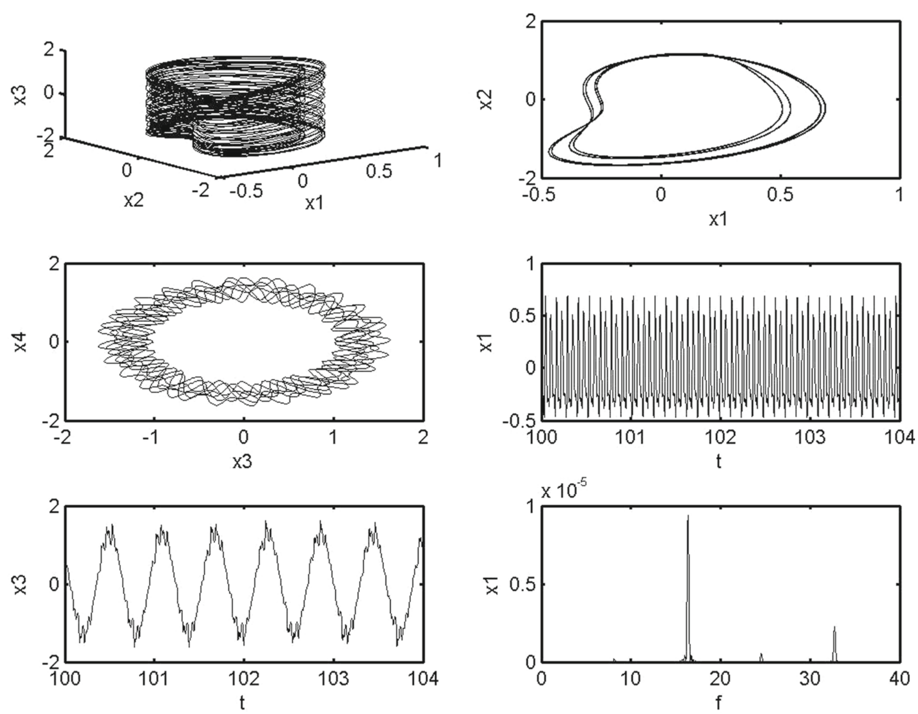

Fig. 5 The period-4 motion of the composite laminated cantilever rectangular plate exists when $\delta_{12} = -120$

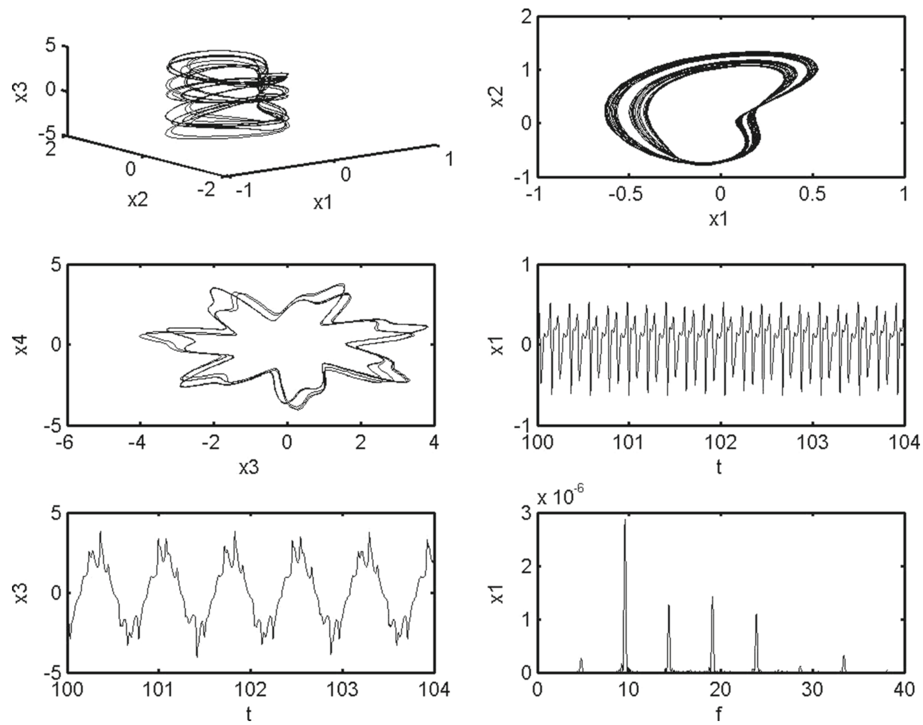

Fig. 6 The quasi-periodic motion of the composite laminated cantilever rectangular plate exists when $\delta_{12} = -90$

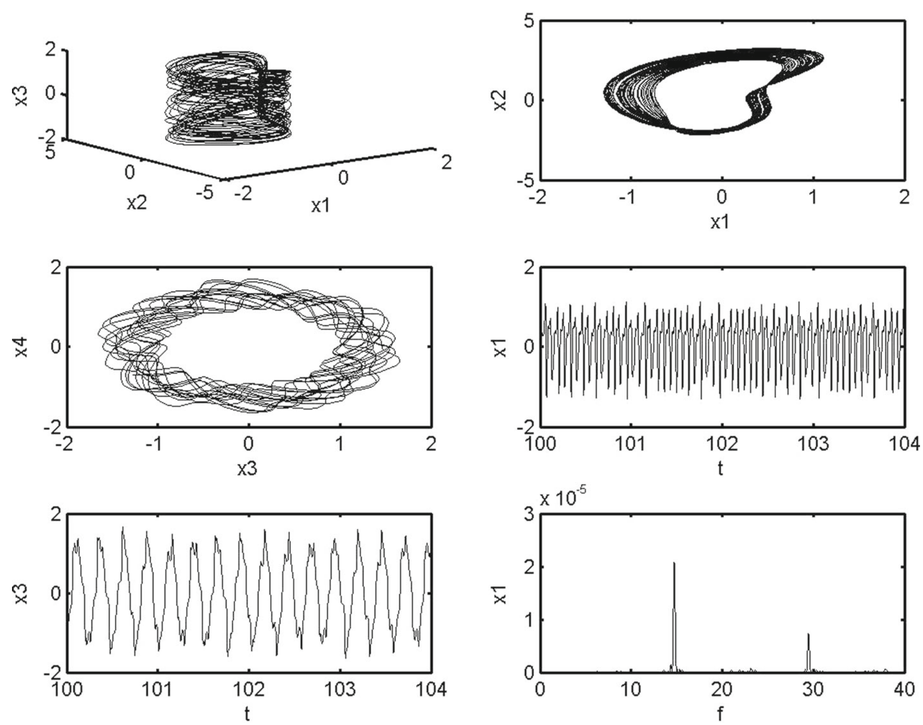

Fig. 7 The chaotic motion of the composite laminated cantilever rectangular plate exists when $\delta_{12} = -60$

0.798, $x_3 = 0.5$, $x_4 = 0.8$. It is observed from Fig. 8 that the motions of the composite laminated cantilever rectangular plate change from the periodic motion to the multiple periodic motion, and then to the chaotic motion with the increase of the parameter γ_{12}.

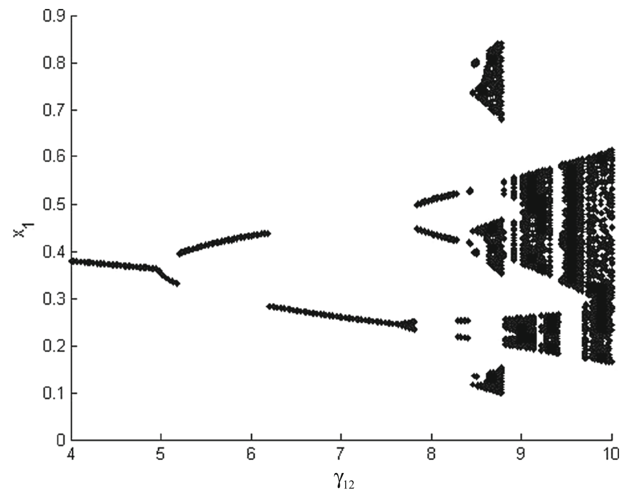

Fig. 8 The bifurcation diagram of the composite laminated cantilever rectangular plate for x_1 via the forcing excitation γ_{12}

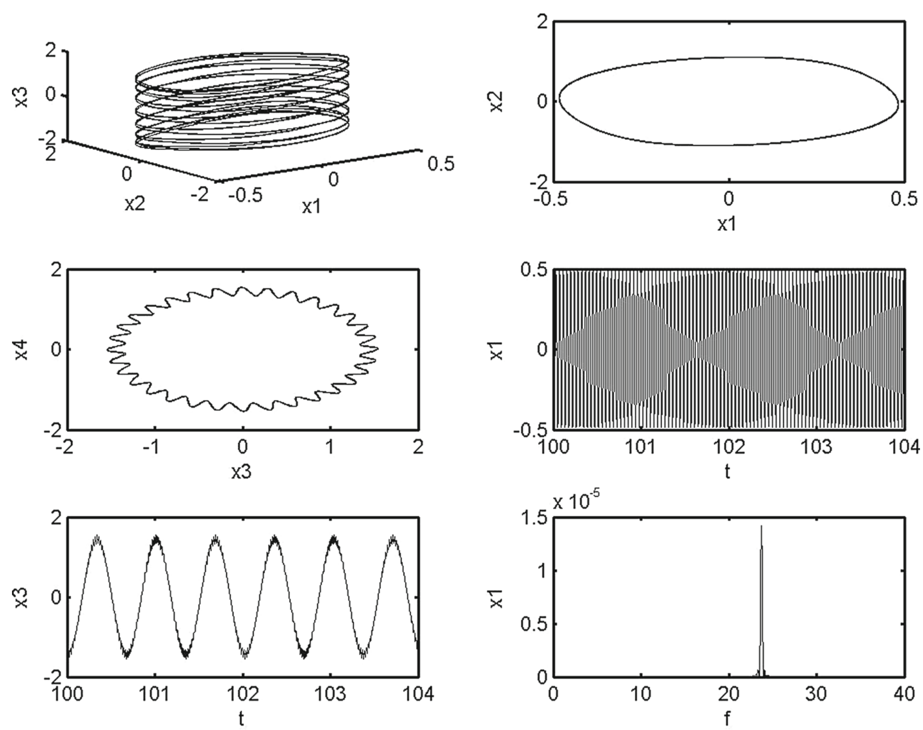

Fig. 9 The period-1 motion of the composite laminated cantilever rectangular plate exists when $\gamma_{12} = 4$

Figures 9, 10, 11 and 12 demonstrate that period-1 motion of the system is altered to period-2 motion, to the multiple periodic motion and to the chaotic motion, gradually.

5 Conclusions

The bifurcations, periodic and chaotic dynamics of the composite laminated cantilever rectangular plate under the aerodynamic pressure and the in-plane excitation are investigated. Based on the von Kármán-type equations and Reddy's third-order shear deformation plate theory, the governing equations of motion for the composite laminated cantilever rectangular plate are derived by using Hamilton's principle. The resonant case considered

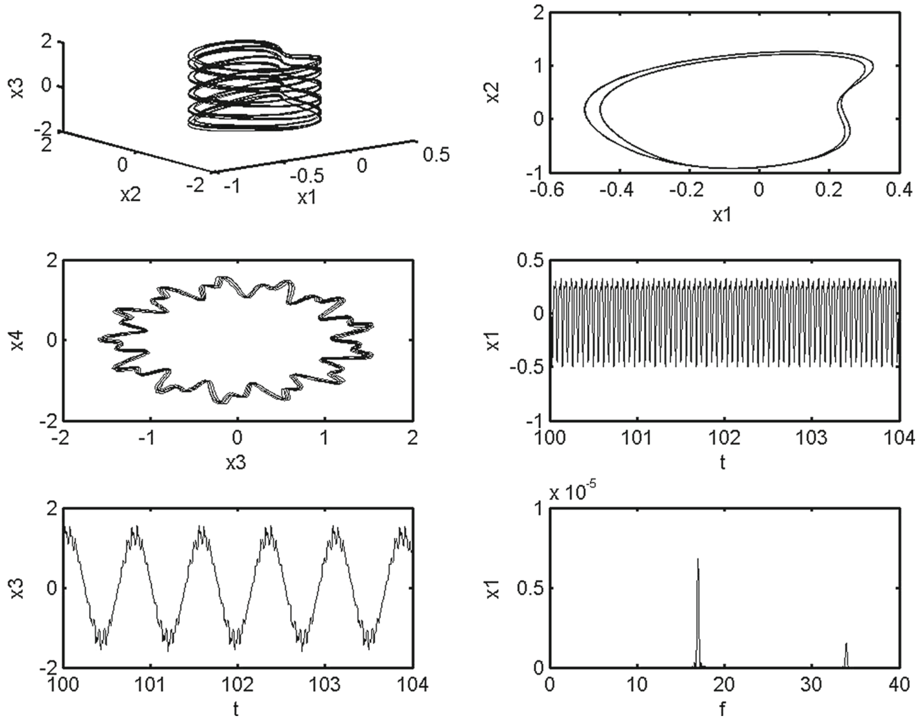

Fig. 10 The period-2 motion of the composite laminated cantilever rectangular plate exists when $\gamma_{12} = 7.8$

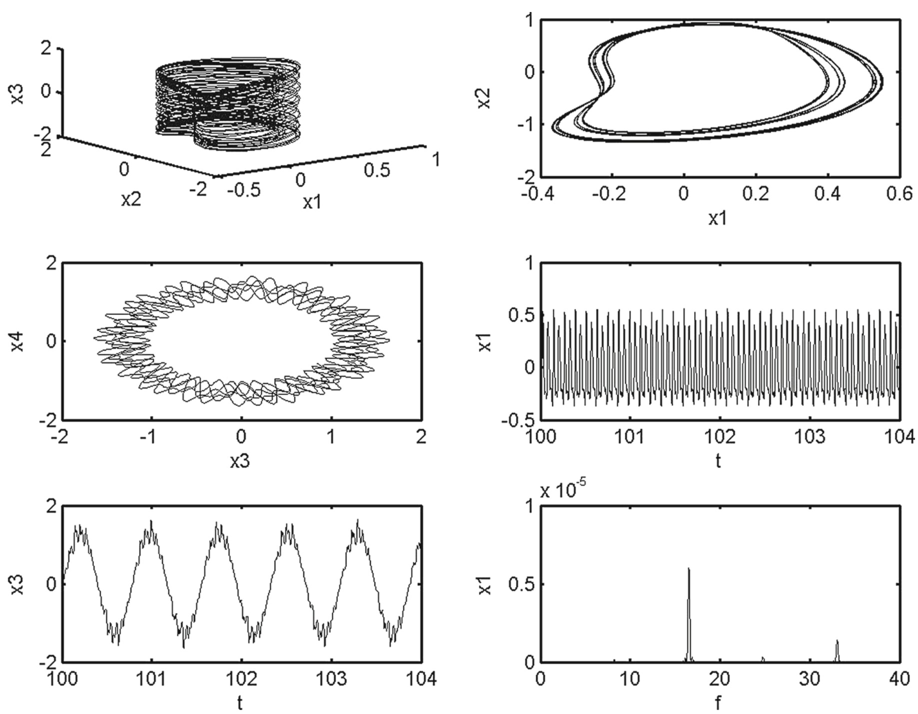

Fig. 11 The multi-periodic motion of the composite laminated cantilever rectangular plate exists when $\gamma_{12} = 8.8$

here is 1:2 internal resonance and primary parametric resonance. A numerical method is used to investigate the bifurcations, periodic and chaotic motions of the composite laminated cantilever rectangular plate.

The periodic, quasi-periodic and chaotic motions of the composite laminated cantilever rectangular plate are found in the numerical results. The parameters are changed to obtain two types of bifurcation diagrams of

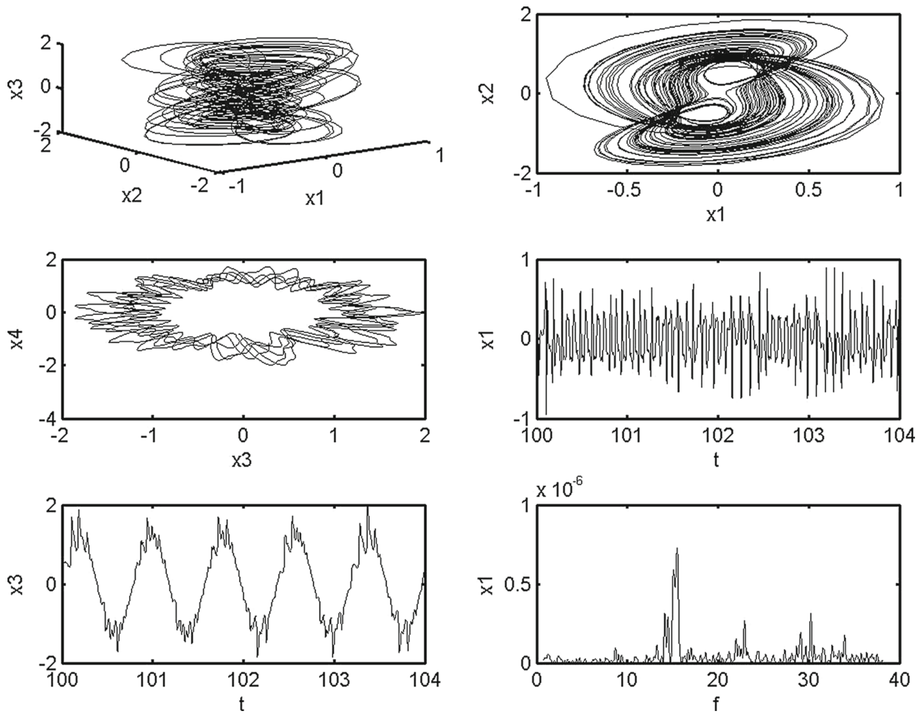

Fig. 12 The chaotic motion of the composite laminated cantilever rectangular plate exists when $\gamma_{12} = 10$

the composite laminated cantilever rectangular plate. The influence of the parameters δ_{12} and γ_{12}, which are relative to the flow velocity q_d, on the nonlinear dynamic behaviors of the composite laminated cantilever rectangular plate is investigated. Two parameters can control the responses of the composite laminated cantilever rectangular plate from the period n or quasi-periodic motions to the chaotic motions. The frequency–response curves are obtained by using numerical simulation. We observe that the energy transfer between the first-order and second-order modes occurs with the change of the detuning parameter σ_2.

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Appendix

All coefficients given in Eq. (18) are presented as follows:

$$
\begin{aligned}
&\omega_1^2 = \frac{i_5}{i_{25}}, \quad \dot{\mu}_1 = \frac{i_{24}}{i_{25}}, \quad \gamma_1 = \frac{i_6}{i_{25}}, \quad \gamma_2 = \frac{i_7}{i_{25}}, \quad \gamma_3 = \frac{i_1}{i_{25}}, \quad \gamma_4 = \frac{i_2}{i_{25}}, \quad \gamma_5 = \frac{i_3}{i_{25}}, \quad \gamma_6 = \frac{i_4}{i_{25}},\\
&\gamma_7 = \frac{i_4}{i_{25}}, \quad \gamma_8 = \frac{i_8}{i_{25}}, \quad \gamma_9 = \frac{i_{11}}{i_{25}}, \quad \gamma_{10} = \frac{i_{15}}{i_{25}}, \quad \gamma_{11} = \frac{i_{17}}{i_{25}}, \quad \gamma_{12} = \frac{i_9}{i_{25}}, \quad \gamma_{13} = \frac{i_{12}}{i_{25}}, \quad \gamma_{14} = \frac{i_{10}}{i_{25}},\\
&\gamma_{15} = \frac{i_{13}}{i_{25}}, \quad \gamma_{16} = \frac{i_{18}}{i_{25}}, \quad \gamma_{17} = \frac{i_{19}}{i_{25}}, \quad \gamma_{18} = \frac{i_{20}}{i_{25}}, \quad \gamma_{19} = \frac{i_{23}}{i_{25}}, \quad \gamma_{20} = \frac{i_{21}}{i_{25}}, \quad \gamma_{21} = \frac{i_{22}}{i_{25}},\\
&\omega_2^2 = \frac{x_6}{x_{25}}, \quad \mu_2 = \frac{x_{24}}{x_{25}}, \quad \delta_1 = \frac{x_7}{x_{25}}, \quad \delta_2 = \frac{x_5}{x_{25}}, \quad \delta_3 = \frac{x_4}{x_{25}}, \quad \delta_4 = \frac{x_3}{x_{25}}, \quad \delta_5 = \frac{x_2}{x_{25}}, \quad \delta_6 = \frac{x_1}{x_{25}},\\
&\delta_7 = \frac{x_{12}}{x_{25}}, \quad \delta_8 = \frac{x_{15}}{x_{25}}, \quad \delta_9 = \frac{x_9}{x_{25}}, \quad \delta_{10} = \frac{x_{14}}{x_{25}}, \quad \delta_{11} = \frac{x_{16}}{x_{25}}, \quad \delta_{12} = \frac{x_{11}}{x_{25}}, \quad \delta_{13} = \frac{x_8}{x_{25}}, \quad \delta_{14} = \frac{x_{13}}{x_{25}},\\
&\delta_{15} = \frac{x_{10}}{x_{25}}, \quad \delta_{16} = \frac{x_{22}}{x_{25}}, \quad \delta_{17} = \frac{x_{21}}{x_{25}}, \quad \delta_{18} = \frac{x_{23}}{x_{25}}, \quad \delta_{19} = \frac{x_{20}}{x_{25}}, \quad \delta_{20} = \frac{x_{21}}{x_{25}}, \quad \delta_{21} = \frac{x_{18}}{x_{25}},
\end{aligned} \tag{A1}
$$

where

$$
\begin{aligned}
& i_1 = A_1\alpha_1 + C_1A_3\alpha_1 + C_2A_5\alpha_1 + C_3A_7\alpha_1 + C_4A_9\alpha_1 + C_5A_{11}\beta_1 + C_6A_{13}\beta_1 + C_7A_{15}\beta_1 + C_8A_{17} \\
& \quad + C_9A_{21} + C_{10}A_{25} + C_{11}A_{29} + C_{12}A_{33} + C_{24}A_{51}\beta_1 + C_{25}A_{53}\beta_1 + C_{26}A_{55}\alpha_1 + C_{27}A_{57}\beta_1 + C_{30}A_{59}, \\
& i_2 = A_1\alpha_2 + C_1A_3\alpha_2 + C_2A_5\alpha_2 + C_3A_7\alpha_2 + C_4A_9\alpha_2 + C_5A_{11}\beta_2 + A_2\alpha_1 + C_1A_4\alpha_1 + C_2A_6\alpha_1 \\
& \quad + C_{10}A_{26} + C_{11}A_{31} + C_{12}A_{35} + C_{24}A_{51}\beta_2 + C_{24}A_{52}\beta_1 + C_{25}A_{53}\beta_2 + C_{25}A_{54}\beta_1 + C_{26}A_{55}\alpha_2 \\
& \quad + C_3A_8\alpha_1 + C_4A_{10}\alpha_1 + C_5A_{12}\beta_1 + C_6A_{13}\beta_2 + C_6A_{14}\beta_1 + C_7A_{15}\beta_2 + C_7A_{16}\beta_1 + C_8A_{19} + C_9A_{23} \\
& \quad + C_{26}A_{56}\alpha_1 + C_{27}A_{57}\beta_2 + C_{27}A_{58}\beta_1 + C_{30}A_{61}, \\
& i_3 = A_1\alpha_2 + A_2\alpha_2 + C_1A_3\alpha_3 + C_1A_4\alpha_2 + C_2A_5\alpha_3 + C_2A_6\alpha_2 + C_3A_7\alpha_3 + C_3A_8\alpha_2 + C_4A_9\alpha_3 \\
& \quad + C_4A_{10}\alpha_2 + C_5A_{11}\beta_3 + C_5A_{12}\beta_2 + C_6A_{13}\beta_3 + C_6A_{14}\beta_2 + C_7A_{15}\beta_3 + C_7A_{16}\beta_2 + C_8A_{18} + C_9A_{22} \\
& \quad + C_{10}A_{27} + C_{11}A_{30} + C_{12}A_{34} + C_{24}A_{51}\beta_3 + C_{24}A_{52}\beta_2 + C_{25}A_{53}\beta_3 + C_{25}A_{54}\beta_2 + C_{26}A_{55}\alpha_3 \\
& \quad + C_{26}A_{56}\alpha_2 + C_{27}A_{57}\beta_3 + C_{27}A_{58}\beta_2 + C_{30}A_{62}, \\
& i_4 = A_2\alpha_3 + C_1A_4\alpha_3 + C_2A_6\alpha_3 + C_3A_8\alpha_3 + C_4A_{10}\alpha_3 + C_5A_{12}\beta_3 + C_6A_{14}\beta_3 + C_7A_{16}\beta_3 + C_8A_{20} \\
& \quad + C_9A_{24} + C_{10}A_{28} + C_{11}A_{32} + C_{12}A_{36} + C_{24}A_{52}\beta_3 + C_{25}A_{54}\beta_3 + C_{26}A_{56}\alpha_3 + C_{27}A_{58}\beta_3 + C_{30}A_{60}, \\
& i_5 = C_{13}A_{37} + C_{14}A_{39} + C_{15}A_{41} + C_{16}A_{43} + C_{17}A_{45}\varepsilon + C_{18}A_{47} + C_{19}A_{28}\varepsilon_1 + C_{19}A_{49}\varepsilon_1 \\
& \quad + C_{22}A_{50}\varepsilon_1 + C_{29}, \\
& i_6 = C_{28}, \quad i_{25} = C_{35} + C_{36} + C_{37}, \\
& i_7 = C_{13}A_{38} + C_{14}A_{40} + C_{15}A_{42} + C_{16}A_{44} + C_{17}A_{46}\varepsilon_2 + C_{19}A_{48}\varepsilon_2 + C_{20}A_{49}\varepsilon_2 + C_{22}A_{50}\varepsilon_2, \\
& i_8 = C_{31}A_{63}, \quad i_9 = C_{31}A_{64}, \quad i_{10} = C_{31}A_{65}, \quad i_{11} = C_{31}A_{66}, \quad i_{12} = C_{31}A_{67}, \quad i_{13} = C_{31}A_{68}, \\
& i_{14} = C_{33}A_{69}, \quad i_{15} = C_{33}A_{70}, \quad i_{16} = C_{33}A_{71}, \quad i_{17} = C_{33}A_{72}, \quad i_{18} = C_{32}A_{73}, \quad i_{19} = C_{32}A_{74}, \\
& i_{20} = C_{32}A_{75}, \quad i_{21} = C_{32}A_{76}, \quad i_{22} = C_{32}A_{77}, \quad i_{23} = C_{32}A_{78}, \quad i_{24} = C_{34}\mu, \\
& x_1 = B_1\alpha_1 + C_1B_3\alpha_1 + C_2B_5\alpha_1 + C_3B_7\alpha_1 + C_4B_9\alpha_1 + C_5B_{11}\beta_1 + C_6B_{13}\beta_1 + C_7B_{15}\beta_1 + C_8B_{17} \\
& \quad + C_9B_{21} + C_{10}B_{25} + C_{11}B_{29} + C_{12}B_{33} + C_{24}B_{51}\beta_1 + C_{25}B_{53}\beta_1 + C_{26}B_{55}\alpha_1 + C_{27}B_{57}\beta_1 + C_{30}B_{59}, \\
& x_2 = B_1\alpha_2 + B_2\alpha_1 + C_1B_3\alpha_2 + C_1B_4\alpha_1 + C_2B_5\alpha_2 + C_2B_6\alpha_1 + C_3B_7\alpha_2 + C_3B_8\alpha_1 + C_4B_9\alpha_2 \\
& \quad + C_4B_{10}\alpha_1 + C_5B_{11}\beta_2 + C_5B_{12}\beta_1 + C_6B_{13}\beta_2 + C_6B_{14}\beta_1 + C_7B_{15}\beta_2 + C_7B_{16}\beta_1 + C_8B_{19} + C_9B_{23} \\
& \quad + C_{10}B_{26} + C_{11}B_{31} + C_{12}B_{35} + C_{24}B_{51}\beta_2 + C_{24}B_{52}\beta_1 + C_{25}B_{53}\beta_2 + C_{25}B_{54}\beta_1 + C_{26}B_{55}\alpha_2 \\
& \quad + C_{26}B_{56}\alpha_1 + C_{27}B_{57}\beta_2 + C_{27}B_{58}\beta_1 + C_{30}B_{61}, \\
& x_3 = B_1\alpha_3 + B_2\alpha_2 + C_1B_3\alpha_3 + C_1B_4\alpha_2 + C_2B_5\alpha_3 + C_2B_6\alpha_2 + C_3B_7\alpha_3 + C_3B_8\alpha_2 + C_4B_9\alpha_3 \\
& \quad + C_4B_{10}\alpha_2 + C_5B_{11}\beta_3 + C_5B_{12}\beta_2 + C_6B_{13}\beta_3 + C_6B_{14}\beta_2 + C_7B_{15}\beta_3 + C_7B_{16}\beta_2 + C_8B_{18} + C_9B_{22} \\
& \quad + C_{10}B_{27} + C_{11}B_{30} + C_{12}B_{34} + C_{24}B_{51}\beta_3 + C_{24}B_{52}\beta_2 + C_{25}B_{53}\beta_3 + C_{25}B_{54}\beta_2 + C_{26}B_{55}\alpha_3 \\
& \quad + C_{26}B_{56}\alpha_2 + C_{27}B_{57}\beta_3 + C_{27}B_{58}\beta_2 + C_{30}B_{62}, \\
& x_4 = B_2\alpha_3 + C_1\alpha_3B_4 + C_2B_6\alpha_3 + C_3B_8\alpha_3 + C_4B_{10}\alpha_3 + C_5B_{12}\beta_3 + C_6B_{14}\beta_3 + C_7B_{16}\beta_3 + C_8B_{20} \\
& \quad + C_9B_{24} + C_{10}B_{28} + C_{11}B_{32} + C_{12}B_{36} + C_{24}B_{52}\beta_3 + C_{25}B_{54}\beta_3 + C_{26}B_{56}\alpha_3 \\
& \quad + C_{27}B_{58}\beta_3 + C_{30}B_{60}, \\
& x_5 = C_{13}B_{37} + C_{14}B_{39} + C_{15}B_{41} + C_{16}B_{43} + C_{17}B_{45}\varepsilon_1 + C_{19}B_{48}\varepsilon_1 + C_{20}B_{49}\varepsilon_1 + C_{21}\theta_1 \\
& x_6 = C_{13}B_{38} + C_{14}B_{40} + C_{15}B_{42} + C_{16}B_{44} + C_{17}B_{46}\varepsilon_2 + C_{18}B_{47} + C_{19}B_{48}\varepsilon_2 + C_{20}B_{49}\varepsilon_2 \\
& \quad + C_{21}\theta_2 + C_{22}B_{50}\varepsilon_2 + C_{23}\theta_2 + C_{29}, \\
& x_7 = C_{28}, \quad x_8 = C_{31}B_{63}, \quad x_9 = C_{31}B_{64}, \quad x_{10} = C_{31}B_{65}, \quad x_{11} = C_{31}B_{66}, \quad x_{12} = C_{31}B_{67}, \\
& x_{13} = C_{31}B_{68}, \quad x_{14} = C_{33}B_{69}, \quad x_{15} = C_{33}B_{70}, \quad x_{16} = C_{33}B_{71}, \quad x_{17} = C_{33}B_{72}, \quad x_{18} = C_{32}B_{73}, \\
& x_{19} = C_{32}B_{74}, \quad x_{20} = C_{32}B_{75}, \quad x_{21} = C_{32}B_{76}, \quad x_{22} = C_{32}B_{77}, \quad x_{23} = C_{32}B_{78}, \\
& x_{24} = C_{34}\mu, \quad x_{25} = C_{35} + C_{36} + C_{37},
\end{aligned} \tag{A2}
$$

where

$$
a_{11} = \frac{a^2}{b^2}\frac{\bar{A}_{66}}{\bar{A}_{11}}, \quad a_{12} = \frac{\bar{A}_{12} + \bar{A}_{66}}{\bar{A}_{11}}, \quad a_{13} = \frac{h^2}{a^2}, \quad a_{14} = \frac{h^2}{b^2}\frac{\bar{A}_{66}}{\bar{A}_{11}}, \quad a_{15} = \frac{h^2}{b^2}\frac{\bar{A}_{12} + \bar{A}_{66}}{\bar{A}_{11}},
$$

$$b_{21}=\frac{a^2}{b^2}\frac{\bar{A}_{22}}{\bar{A}_{66}},\quad b_{22}=\frac{a^2}{b^2}\frac{\bar{A}_{21}+\bar{A}_{66}}{\bar{A}_{66}},\quad b_{23}=\frac{h^2}{b^2},\quad b_{24}=\frac{h^2a^2}{b^4}\frac{\bar{A}_{22}}{\bar{A}_{66}},\quad b_{25}=\frac{h^2}{b^2}\frac{\bar{A}_{21}+\bar{A}_{66}}{\bar{A}_{66}},$$

$$m_1=\bar{D}_{11}-\frac{8}{3}\bar{F}_{11}+\frac{16}{9}\bar{H}_{11},\quad d_{41}=\frac{a^2}{b^2}\frac{1}{m_1}\left(-\frac{8}{3}\bar{F}_{66}+\frac{16}{9}\bar{H}_{66}\right),\quad d_{42}=\frac{4h}{3am_1}\left(\bar{F}_{11}-\frac{4}{3}\bar{H}_{11}\right),$$

$$d_{43}=\frac{a}{h}\frac{1}{m_1}\left(16\bar{F}_{55}-8\bar{D}_{55}+\bar{A}_{55}\right),\quad d_{44}=\frac{a}{bm_1}\left(\bar{D}_{12}-\frac{8}{3}\bar{F}_{12}+\frac{16}{9}\bar{H}_{12}+\bar{D}_{66}-\frac{8}{3}\bar{F}_{66}+\frac{16}{9}\bar{H}_{66}\right),$$

$$d_{45}=\frac{4ha}{3b^2m_1}\left(\bar{F}_{12}-\frac{4}{3}\bar{H}_{12}+2\bar{F}_{66}-\frac{8}{3}\bar{H}_{66}\right),\quad d_{46}=\frac{a^2}{h^2}\frac{1}{m_1}\left(-16\bar{F}_{55}+8\bar{D}_{55}-\bar{A}_{55}\right),$$

$$m_2=\bar{D}_{22}-\frac{8}{3}\bar{F}_{22}+\frac{16}{9}\bar{H}_{22},\quad e_{51}=\frac{b^2}{a^2}\frac{1}{m_2}\left(\bar{D}_{66}-\frac{8}{3}\bar{F}_{66}+\frac{16}{9}\bar{H}_{66}\right),$$

$$e_{52}=\frac{4hb}{3a^2m_2}\left(\bar{F}_{21}-\frac{4}{3}\bar{H}_{21}+2\bar{F}_{66}-\frac{8}{3}\bar{H}_{66}\right),$$

$$e_{53}=\frac{4h}{3bm_2}\left(\bar{F}_{22}-\frac{4}{3}\bar{H}_{22}\right),\quad e_{54}=\frac{b}{am_2}\left(\bar{D}_{21}-\frac{8}{3}\bar{F}_{21}+\frac{16}{9}\bar{H}_{21}+\bar{D}_{66}-\frac{8}{3}\bar{F}_{66}+\frac{16}{9}\bar{H}_{66}\right),$$

$$e_{55}=\frac{b^2}{h^2m_2}\left(16\bar{F}_{44}-8\bar{D}_{44}+\bar{A}_{44}\right),\quad e_{56}=\frac{b}{hm_2}\left(16\bar{F}_{44}+8\bar{D}_{44}-\bar{A}_{44}\right),$$

$$C_1=\frac{a^2}{b^2}\frac{\bar{A}_{21}}{\bar{A}_{11}},\quad C_2=\frac{2a^2}{b^2}\frac{\bar{A}_{66}}{\bar{A}_{11}},\quad C_3=1,\quad C_4=\frac{a^2}{b^2}\frac{\bar{A}_{21}+\bar{A}_{66}}{\bar{A}_{11}},\quad C_5=\frac{\bar{A}_{66}}{\bar{A}_{11}},\quad C_6=\frac{\bar{A}_{66}+\bar{A}_{12}}{\bar{A}_{11}},$$

$$C_7=\frac{a^2}{b^2}\frac{\bar{A}_{22}}{\bar{A}_{11}},\quad C_8=\frac{3h^2}{2a^2},\quad C_9=\frac{h^2}{b^2}\frac{\bar{A}_{12}+2\bar{A}_{66}}{2\bar{A}_{11}},\quad C_{10}=\frac{h^2}{b^2}\frac{\bar{A}_{12}+\bar{A}_{21}+4\bar{A}_{66}}{\bar{A}_{11}},$$

$$C_{11}=\frac{3a^2h^2}{2b^4}\frac{\bar{A}_{22}}{\bar{A}_{11}},\quad C_{12}=\frac{h^2}{b^2}\frac{\bar{A}_{21}+2\bar{A}_{66}}{2\bar{A}_{11}},\quad C_{13}=\frac{\bar{A}_{55}-8\bar{D}_{55}+16\bar{F}_{55}}{\bar{A}_{11}},$$

$$C_{14}=\frac{a^2}{b^2}\frac{\bar{A}_{44}-8\bar{D}_{44}+16\bar{F}_{44}}{\bar{A}_{11}},\quad C_{15}=-\frac{16h^2}{9a^2}\frac{\bar{H}_{11}}{\bar{A}_{11}},\quad C_{16}=-\frac{16h^2}{9b^2}\frac{\bar{H}_{12}+\bar{H}_{21}+4\bar{H}_{66}}{\bar{A}_{11}},$$

$$C_{17}=\frac{a}{h}\frac{\bar{A}_{55}-8\bar{D}_{55}+16\bar{F}_{55}}{\bar{A}_{11}},\quad C_{18}=-\frac{16a^2h^2}{9b^4}\frac{\bar{H}_{22}}{\bar{A}_{11}},\quad C_{19}=\frac{4h}{3a}\frac{3\bar{F}_{11}-4\bar{H}_{11}}{3\bar{A}_{11}},$$

$$C_{20}=\frac{4ah}{3b^2}\frac{3\bar{F}_{21}+6\bar{F}_{66}-4\bar{H}_{21}+8\bar{H}_{66}}{3\bar{A}_{11}},\quad C_{21}=\frac{a^2}{bh}\frac{\bar{A}_{44}-8\bar{D}_{44}+16\bar{F}_{44}}{\bar{A}_{11}},$$

$$C_{22}=\frac{4h}{3b}\frac{3\bar{F}_{12}+6\bar{F}_{66}-4\bar{H}_{12}-8\bar{H}_{66}}{3\bar{A}_{11}},\quad C_{23}=\frac{4ha^2}{3b^3}\frac{3\bar{F}_{22}-4\bar{H}_{22}}{3\bar{A}_{11}},$$

$$C_{24}=\frac{\bar{A}_{12}}{\bar{A}_{11}},\quad C_{25}=\frac{2\bar{A}_{66}}{\bar{A}_{11}},\quad C_{26}=\frac{a^2}{b^2}\frac{\bar{A}_{66}}{\bar{A}_{11}},\quad C_{27}=\frac{a^2}{b^2}\frac{\bar{A}_{22}}{\bar{A}_{11}},\quad C_{28}=\frac{a^{\frac{5}{2}}}{b^{\frac{7}{2}}}\frac{h}{\bar{A}_{11}},$$

$$C_{31}=\frac{(k+1)\,q_d\gamma^3h^5M_\infty}{a^2b^3\bar{A}_{11}V^2},\quad C_{32}=\frac{(k+1)\,q_d\gamma^3h^6M_\infty}{a^3b^3\bar{A}_{11}V},\quad C_{33}=\frac{(k+1)\,q_d\gamma^3h^4M_\infty}{3ab^3\bar{A}_{11}V^3},$$

$$C_{34}=\frac{4h^2}{lb^2}\frac{\gamma}{\bar{A}_{11}},\quad C_{35}=\frac{a^2\pi^4}{h^2\bar{A}_{11}}\bar{I}_0,\quad C_{36}=-\frac{16a^3b^3}{9h^6}\frac{\bar{I}_6\pi^4}{\bar{A}_{11}},\quad C_{37}=-\frac{16a^5b}{9h^6}\frac{\bar{I}_6\pi^4}{\bar{A}_{11}},$$

$$C_{38}=\frac{4a^{\frac{7}{2}}b^{\frac{3}{2}}\pi^4}{3h^5\bar{A}_{11}}\bar{I}^3,\quad C_{39}=\frac{4a\pi^4}{3h^5\bar{A}_{11}}\left[(ab)^2\,\bar{I}_4-\frac{4}{3h^2}\,(ab)^3\,\bar{I}_6\right],$$

$$C_{40}=\frac{4a^2\pi^4}{3h^5\bar{A}_{11}b}\left[(ab)^2\,\bar{I}_4-\frac{4}{3h^2}\,(ab)^3\,\bar{I}_6\right],\quad C_{29}=\frac{4q_d\gamma h^5}{M_\infty a^2b^3\bar{A}_{11}},\quad C_{30}=\frac{(k+1)\,q_d\gamma^3h^7M_\infty}{3a^4b^3\bar{A}_{11}},$$

$$\alpha_1=\frac{0.056a_{12}k_1-(0.114+0.113b_{21})\,y_1}{\Lambda_2},\quad \alpha_2=\frac{0.0056a_{12}k_2-(0.114+0.113b_{21})\,y_2}{\Lambda_2},$$

$$\alpha_3=\frac{0.056a_{12}k_3-(0.114+0.113b_{21})\,y_3}{\Lambda_2},\quad \beta_1=\frac{0.000006b_{22}y_1+(0.05+0.052a_{11})\,k_1}{\Lambda_1},$$

$$
\begin{aligned}
\beta_2 &= \frac{0.000006 b_{22} y_2 + (0.05 + 0.052 a_{11})\, k_2}{\Lambda_1}, \quad \beta_3 = \frac{0.000006 b_{22} y_3 + (0.05 + 0.052 a_{11})\, k_3}{\Lambda_1},\\
\varepsilon_1 &= \frac{(-0.11 + 9.08 e_{51} - 1.15 e_{55})\, \tau_1 + 0.00036 d_{44} \tau_2}{\Lambda_3},\\
\varepsilon_2 &= \frac{(-0.11 + 9.08 e_{51} - 1.15 e_{55})\, \tau_2 + 0.00036 d_{44} \tau_1}{\Lambda_3},\\
\theta_1 &= \frac{(0.00064 + 25.19 d_{41} + 0.232 d_{46})\, \tau_2 + 0.001 e_{54} \tau_1}{\Lambda_3},\\
\theta_2 &= \frac{(0.00064 + 25.19 d_{41} + 0.232 d_{46})\, \tau_1 + 0.001 e_{54} \tau_2}{\Lambda_3},\\
\Lambda_1 &= 0.0057 - 0.006 b_{21} - 0.006 a_{11} - 0.006 a_{11} b_{21} - 3.36 \times 10^{-7} b_{22} a_{12},\\
\Lambda_2 &= 0.0057 + 0.006 a_{11} + 0.006 b_{21} + 0.006 a_{11} b_{21} + 3.36 \times 10^{-7} b_{22} a_{12},\\
\Lambda_3 &= (-0.11 + 9.08 e_{51} - 1.15 e_{55}) \times (0.00064 + 25.19 d_{41} + 0.232 d_{46}) - 0.36 \times 10^{-6} e_{54} d_{44},\\
k_1 &= -0.64 \times 10^{-10} b_{23} - 174.17 b_{24} + 0.036 \times 10^{-8} b_{25},\\
k_2 &= -0.04 b_{23} - 0.27 b_{24} + 0.000001 b_{25},\\
k_3 &= -0.000061 b_{23} - 0.000014 b_{24} - 0.000028 b_{25},\\
y_1 &= -1.944 \times 10^{-14} a_{13} - 96.492 \times 10^{-8} a_{14} + 0.065 a_{15},\\
y_2 &= -3.025 \times 10^{-10} a_{13} - 0.0037 a_{14} - 1.3 a_{15},\\
y_3 &= -8.74 \times 10^{-7} a_{13} - 2.07 \times 10^{-7} a_{14} - 0.00192 a_{15},\\
\tau_1 &= -0.000049 d_{42} + 6 \times 10^{-8} d_{43} - 0.0095 d_{45}, \quad \tau_2 = 1.93 e_{52} + 0.007 e_{53} + 0.015 e_{56}. \qquad \text{(A3)}
\end{aligned}
$$

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