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The inverse problem of Lagrangian mechanics for Meshchersky's equation

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Abstract Variable-mass systems are not included in the conventional domain of the analytical and variational methods of classical mechanics. This is due to the fact that the fundamental principles of mechanics were primarily conceived for constant-mass systems. In the present article, an analytical and variational formulation for variable-mass systems will be proposed. This will be done from the solution of the here called 'inverse problem of Lagrangian mechanics for Meshchersky's equation'. The first problem of this nature was posed in 1887, by Helmholtz (J. reine angew. Math. 100:137–166, 1887). Investigations on the matter are far from being exhausted. Within mechanics, it means the construction of a Lagrangian from a given equation of motion. To the authors' best knowledge, aiming at general results, the inverse problem of Lagrangian mechanics has not been properly connected to Meshchersky's equation yet. This is the main goal of this article. We will address the issue by assuming that mass depends on generalized coordinate, generalized velocity and on time. After the construction of a Lagrangian from Meshchersky's equation, a general and unifying mathematical formulation will emerge in accordance. Therefore, variable-mass systems will be accommodated at the level of analytical mechanics. A variational formulation, which will be written via a principle of stationary action, and a Hamiltonian formulation will be both stated. The latter could be read as the 'Hamiltonization' of variable-mass systems from the solution of the inverse problem of Lagrangian mechanics. An energy-like conservation law will naturally appear from the simplification of the general theory to the case of a system with mass solely dependent on a generalized coordinate.

1 Introduction

Galileo (1564–1642), Descartes (1596–1650), Fermat (1601–1665), Newton (1643–1727) and Leibnitz (1646–1716) are some of the remarkable names who are responsible for one of the most important steps toward the comprehension of the nature, namely (see [1, p. 3]) "the idea that the observable events are extreme in their character and that the general principles sought are variational (...)". The mathematical organization of this concept gives rise to the analytical and variational approach of mechanics, an elegant and general method which is well featured by the Lagrangian and Hamiltonian formalism (see, e.g., [2,3]). However, if the significance of the involved quantities and the mathematical structure of the equations are both simultaneously preserved as in their original forms, the fundamental principles of mechanics may acquire a restrictive applicability. As commented by Riewe [4, p. 1890], "it is a strange paradox that the most advanced methods of classical

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mechanics deal only with conservative systems, while almost all classical processes observed in the physical world are nonconservative". Santilli [5, p. 7], citing [2,6], exposes that "in essence, Lagrange and Hamilton were fully aware that the Newtonian forces are generally not derivable from a potential". From a different angle, Irschik and Holl [7, p. 145] alert that "the fundamental equations of classical mechanics were originally formulated for the case of an invariant mass contained in a material volume", as well as Eke and Mao [8, p. 124] advise on the difficulties to write an adequate formalism for variable-mass systems, since the basic concepts of mechanics were first enounced under the assumption of constant mass.

Attempts to elaborate analytical and variational formulations that are appropriate either to nonconservative systems (see, e.g., [4,9–17]) or to variable-mass systems (see, e.g., [7,8,18–25]) have been figuring as two particular and active research fields of modern mechanics. We aim at originally contributing toward the latter field.

Nevertheless, it is important to mention that this is not the only way to be followed. Mechanics can be treated from different methodologies or perspectives.[1] Santilli [5, p. 2] argues that "one reason for constructing such a variety of formulations is that a sufficient depth in studying a given system is reached only when a sufficient number of aspects are taken into consideration. Physical reality is polyhedric, to say the least, in relation to our capability to represent it. Therefore, the level of our knowledge depends on how many aspects are considered and how deeply each of them is analyzed. This does not imply, however, that theoretical formulations are compartmentalized. Actually, all the above-mentioned formulations are so deeply interrelated that they form a single articulated body of methodological tools".

In this article, we specifically propose an analytical and variational formulation for variable-mass systems from the solution of the here called 'inverse problem of Lagrangian mechanics for Meshchersky's equation'. This type of problem is a very traditional issue within mathematical physics. In general terms, it means to write a principle of stationary action that leads to a given differential equation. Fundamental investigations on this matter are due to Helmholtz [26], Darboux [27], and others. An extensive treatment of the problem and some historical perspective can be found, for instance, in [5] and references therein. Yet, investigations on the subject are far from being exhausted (see, e.g., [28,29]).

From the perspective of Newtonian mechanics, solving such inverse problem means to find a Lagrangian that, when inserted into the original form of Hamilton's principle, yields the corresponding equation of motion. We thus could name it as 'the inverse problem of Lagrangian mechanics', whose solution is the mathematical construction of a Lagrangian from the previous knowledge of the Newtonian equation of motion. This appears as offering a valuable advantage: the technique is essentially based on mathematical arguments rather than on physical arguments. For this reason, the inverse problem of Lagrangian mechanics can be considered as an important step to have the applicability of the analytical and variational methods of mechanics coherently expanded beyond the classical and original standards.

Within the variable-mass context, investigations on the inverse problem of Lagrangian mechanics are not encountered very often. We can cite, for instance, the isolated work of Leubner and Krumm [30], who demonstrated some results of restrictive character. These authors addressed only the case in which the mass of the system is purely dependent on time. In truth, to our best knowledge, aiming at general results in the perspective of analytical mechanics, the inverse problem of Lagrangian mechanics has not been properly connected to Meshchersky's equation yet. There is still a necessity of investigating variable-mass systems in terms of a $(q, \dot{q}, t)$-description (see earlier works in, e.g., [19–21,25]). This is one of the goals of the present article.

Important advantages to have a variable-mass problem accordingly formulated at the level of analytical mechanics can be read from the testimonies of some authors. Havas [9, p. 363–364] notices that "it has been found in many branches of physics that the solution of a variety of problems can be greatly simplified if the basic equations can be expressed in the form of a variational principle (...) a knowledge of the Lagrangian and of its invariance properties enables one to obtain all the conservation laws of the system (...)." Riewe [4, p. 1890] justifies that, "by using a variational principle, one can directly obtain Newtonian equations of motion, definitions of the momenta, and the Hamiltonian function. Once the Hamiltonian is known, the system becomes amenable to the techniques of quantum mechanics. Because of the importance of the variational

[1] As in [5, p. 1, 2 and references therein], one may have, at least for conservative systems, the following aspects to be considered: "1. analytic formulations, e.g., Lagrange's and Hamilton's equations and Hamilton–Jacobi theory; 2. variational formulations, e.g., variational problems and variational principles; 3. algebraic formulations, e.g., infinitesimal and finite canonical transformations, Lie algebras and Lie groups, and symmetries and conservation laws; 4. geometric formulations, e.g., symplectic geometry and canonical structure; 5. statistical formulations, e.g., Liouville's theorem, equilibrium and nonequilibrium statistical mechanics; 6. thermodynamic formulations, e.g., irreversible processes and entropy; 7. many-body formulations, e.g., stability of orbits and quadrature problems; and so on".

approach, it has become the starting point for both specific calculation and general theory". Venturi [31] argues that, with a principle of stationary action at hand, besides the very elegant formulation, it is possible to establish an immediate connection between symmetry principles, conservation laws and Noether's theorem. Van Brunt [32, p. 70] also explains that "these and other features (e.g., Rayleigh–Ritz numerical methods) make it attractive to identify a given differential equation as the Euler-Lagrange equation of some functional". In conclusion, Anderson [33, p. 344] states that "today most physicists would be not only willing to accept as axiomatic the existence of a variational principle but would be also loath to accept any dynamical equations that were not derivable from such a principle".

The crux of the matter is that when the conventional mathematical structure of Lagrange's and Hamilton's equations can be used for the representation of more general systems, one has that, see [5, p. 8], "the methodological profile is basically that for systems with forces derivable from a potential, in the sense that the analytic equations, the time evolution law, the underlying algebraic and geometric structure, etc. remain formally unchanged". Following these particular motivations, the inverse problem of Lagrangian mechanics for Meshchersky's equation is the cornerstone for the construction of an analytical and variational formulation in the variable-mass scenario. Hence, after having a Lagrangian, a principle of stationary action will be written for variable-mass systems. In consequence, the corresponding canonical momentum and Hamiltonian will both appear to fit the conventional set of canonical equations. Special attention will be dedicated to the position-dependent case, in which explicit calculations of the so-called 'Jacobi last multiplier' will become feasible. Some novelty will be also found in the demonstration of an energy-like conservation law for this case.

2 Outline and preliminary assumptions

In the following discussion, variable-mass systems that obey Meshchersky's equation will be taken into account. The one degree of freedom case will be assumed. The absolute velocity at which mass is expelled or accreted, w, will be written as a linear function of the velocity, $\dot{q}$, that is, $w = k(t)\dot{q}$.[2] The classical method of Darboux [27], which is originally conceived for the solution of the inverse problem of calculus of variations for second-order ordinary differential equations, will be adopted for the solution of the inverse problem of Lagrangian mechanics for Meshchersky's equation.

In fact, it is known that all the second-order differential equations of the form $\ddot{q} + G(q, \dot{q}, t) = 0$ admit a variational formulation. Under certain continuity and regularity conditions, namely by assuming that $G = G(q, \dot{q}, t)$ is differentiable and integrable to the degree required, the theory of partial differential equations guarantees the existence of a solution for the corresponding inverse problem of calculus of variations (see the discussion in [5, p. 12 and 139], [6], [9, p. 367, footnote (*)], [32, p. 72]).

Within the variable-mass scenery, the applicability of the method of Darboux is based on, at least, two main assumptions. First, the concept of Cayley [34] on 'continuous variation of mass' is to be taken into account, that is to say (see [34, p. 506]), systems which are "continually taking into connexion with itself particles of infinitesimal mass (i.e., of a mass containing the increment of time dt as a factor), so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion. For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system, and in each element of time dt, the system takes into connexion with itself, and sets in motion with a finite velocity (...)". As noted by Irschik [23, p. 718], Arthur Cayley was one of the leading mathematicians of the nineteenth century, and his contribution [34] represents a fundamental step in the theory of variable-mass systems. Indeed, historically speaking, the concept of continually varying-mass systems is involved in most of the equivalent derivations of the corresponding equation of motion (see [35]). Following this sense, the formulation which is here derived is devoted to those systems whose mass can be expressed by a continuous, differentiable and integrable function in terms of the variables $(q, \dot{q}, t)$. Second, in the same manner, only acting forces equating continuous, differentiable and integrable functions are to be considered.[3]

In order to avoid unnecessary difficulties for the presentation of the analysis, other simplifying assumptions will be adopted along the text.

[2] Within a different approach, an equivalent assumption was adopted by us in the construction of a Hamiltonian formulation for time-dependent mass systems (see [24]). This is not the most general expression for $w = w(q, \dot{q}, t)$. However, aiming at a better enlightenment of the essential aspects of the upcoming analysis, $w = k(t)\dot{q}$ can be considered as a reasonable assumption.

[3] This means that impulsive forces, for example, are to be excluded (see the comment in [5, p. 240, 241]).

The superscript '∼' will be used to differently label the $\tilde{L}$-Lagrangian, which will be derived from the method of Darboux, with respect to the conventional L-Lagrangian, which is defined as kinetic energy minus potential energy. Canonical momenta and Hamiltonians will be labeled in the same manner, that is, $\tilde{p}$, p and $\tilde{H}, H$.

This article is organized as follows: in Sect. 3, the method of Darboux will be outlined by following the procedure as in [30,36], and so the solution of the inverse problem of Lagrangian mechanics for $\ddot{q} + G(q, \dot{q}, t) = 0$ will be recovered. Thereupon, the theoretical basis that is here required will be found in Sect. 3. In Sect. 4, the mathematical results of Sect. 3 will be applied on Meshchersky's equation. A system with mass depending on generalized coordinate, generalized velocity and on time, that is, $m = m(q, \dot{q}, t)$, will be considered as the most general case. The $\tilde{L}$-Lagrangian, the corresponding $\tilde{p}$-canonical momentum and the following $\tilde{H}$-Hamiltonian will be derived then. In Sect. 4, the reader will find a proper analytical and variational formulation for variable-mass systems, which will be stated by a principle of stationary action and by the following Lagrange's and Hamilton's equations. This is intended to be the main theoretical contribution of this article. In Sect. 5, the results of Sect. 4 will be particularly driven to achieve a more detailed analysis of the position-dependent case, that is, $m = m(q)$. This will emerge as a special case of the general theory. The expressions of Sect. 4 will be greatly simplified in Sect. 5, and so we expect to give the reader some insights concerning the applicability of the formulation here proposed. The reader will also find some novelty in Sect. 5 with the derivation of an energy-like conservation law for $m = m(q)$. For the sake of an illustration, in Sect. 5.1, the particular formulation for $m = m(q)$ will be applied on the classical Cayley's [34] falling-chain problem. In Sect. 5.2, a comparison between the formulation of the time-dependent case, $m = m(t)$, and the formulation of the position-dependent case, $m = m(q)$, will be presented to some extent. In Sect. 6, we will briefly comment on the difference between the formulation for variable-mass systems that is here proposed and that starting from D'Alembert's principle.

3 A background on the method of Darboux

Within second-order theory with one degree of freedom, that is to say,

$$\ddot{q} + G(q, \dot{q}, t) = 0, \tag{1}$$

the inverse problem of calculus of variations corresponds to the search of a function $\tilde{L} = \tilde{L}(q, \dot{q}, t)$ that, when inserted into the conventional form of Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = 0, \tag{2}$$

equivalently recovers Eq. (1) by means of the general expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \Lambda(q, \dot{q}, t)\,(\ddot{q} + G(q, \dot{q}, t)), \tag{3}$$

where $\Lambda = \Lambda(q, \dot{q}, t)$ is called a 'Jacobi last multiplier' of Eq. (1) and plays a fundamental role in the solvability of Eq. (3) for $\tilde{L}$ (see, e.g., [9, p. 371], [32, p. 72], [37], and references therein).

From the historical literature, Eq. (3) was apparently first solved by Darboux [27] (see comment in [5, p. 12], [9, p. 365, footnote ($^{\mathrm{x}}$)]). The solution technique can be outlined by following the systematic procedure of Leubner and Krumm [30] and Yan [36]. This will be shown hereafter.

Once $\tilde{L} = \tilde{L}(q, \dot{q}, t)$, from Eq. (2),

$$\dot{q}\frac{\partial}{\partial q}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) + \ddot{q}\frac{\partial}{\partial \dot{q}}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) - \frac{\partial \tilde{L}}{\partial q} = 0. \tag{4}$$

From Eq. (1) into (4),

$$\dot{q}\frac{\partial}{\partial q}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) - G\frac{\partial}{\partial \dot{q}}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) - \frac{\partial \tilde{L}}{\partial q} = 0. \tag{5}$$

Writing $\Lambda = \Lambda(q, \dot{q}, t)$ as

$$\Lambda = \partial^2 \tilde{L}(q, \dot{q}, t)/\partial \dot{q}^2, \tag{6}$$

one has that Eq. (5), which is a partial differential equation for $\tilde{L}$, can be conveniently transformed into a first-order ordinary differential equation for Λ. Note that, by differentiating Eq. (5) with respect to $\dot{q}$, it is found that

$$\frac{\partial^2 \tilde{L}}{\partial q \partial \dot{q}} + \dot{q}\frac{\partial}{\partial q}\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^2}\right) - \frac{\partial G}{\partial \dot{q}}\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^2}\right) - G\frac{\partial}{\partial \dot{q}}\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^2}\right) + \frac{\partial}{\partial t}\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^2}\right) - \frac{\partial^2 \tilde{L}}{\partial q \partial \dot{q}} = 0. \tag{7}$$

The first term and the last term of Eq. (7) cancel out each other, and from Eq. (6),

$$\dot{q}\frac{\partial \Lambda}{\partial q} - \frac{\partial G}{\partial \dot{q}}\Lambda - G\frac{\partial \Lambda}{\partial \dot{q}} + \frac{\partial \Lambda}{\partial t} = 0. \tag{8}$$

Recalling Eq. (1) one more time, Eq. (8) changes to

$$\dot{q}\frac{\partial \Lambda}{\partial q} - \frac{\partial G}{\partial \dot{q}}\Lambda + \ddot{q}\frac{\partial \Lambda}{\partial \dot{q}} + \frac{\partial \Lambda}{\partial t} = 0. \tag{9}$$

Once

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \dot{q}\frac{\partial \Lambda}{\partial q} + \ddot{q}\frac{\partial \Lambda}{\partial \dot{q}} + \frac{\partial \Lambda}{\partial t}, \tag{10}$$

Equation (9) becomes

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} - \frac{\partial G}{\partial \dot{q}}\Lambda = 0. \tag{11}$$

Equation (11) is a first-order ordinary differential equation for Λ, and so

$$\Lambda = \exp \int \frac{\partial G(q, \dot{q}, t)}{\partial \dot{q}}\mathrm{d}t. \tag{12}$$

From Eq. (6) into (12),

$$\frac{\partial^2 \tilde{L}}{\partial \dot{q}^2} = \exp \int \frac{\partial G(q, \dot{q}, t)}{\partial \dot{q}}\mathrm{d}t. \tag{13}$$

The function $\tilde{L} = \tilde{L}(q, \dot{q}, t)$ can be finally obtained by integrating Eq. (13) twice with respect to $\dot{q}$, what furnishes

$$\tilde{L} = \int_a^{\dot{q}} (\dot{q} - \omega)\Lambda(q, \omega, t)\mathrm{d}\omega - \int_b^{q} G(\xi, a, t)\Lambda(\xi, a, t)\mathrm{d}\xi + \frac{\mathrm{d}F(q, t)}{\mathrm{d}t} \tag{14}$$

(see [36, p. 672, Eq. (2.26)]), where $F = F(q, t)$ is an arbitrary function.

For the sake of simplicity, we will assume $F = a = b = 0$ as discussed in [30,36], namely

$$\tilde{L} = \int_0^{\dot{q}} (\dot{q} - \omega)\Lambda(q, \omega, t)\mathrm{d}\omega - \int_0^{q} G(\xi, 0, t)\Lambda(\xi, 0, t)\mathrm{d}\xi. \tag{15}$$

The definition of the $\tilde{p}$-canonical momentum is accordingly understood,

$$\tilde{p} = \frac{\partial \tilde{L}}{\partial \dot{q}}, \tag{16}$$

and, from Eq. (15),

$$\tilde{p} = \int_0^{\dot{q}} \Lambda(q, \omega, t)\mathrm{d}\omega, \tag{17}$$

(see [36, p. 672, Eq. (2.28)]).

The $\tilde{H}$-Hamiltonian follows from the definitions given by Eqs. (15) and (17) into the classical identity

$$\tilde{H} = \tilde{p}\dot{q} - \tilde{L}, \tag{18}$$

what yields

$$\tilde{H} = \int_0^{\dot{q}} \omega\Lambda(q, \omega, t)\mathrm{d}\omega + \int_0^{q} G(\xi, 0, t)\Lambda(\xi, 0, t)\mathrm{d}\xi \tag{19}$$

(see [36, p. 672, Eq. (2.27)]).

Equations (15), (17) and (19) transcribe a differential equation as in Eq. (1) into quantities of analytical mechanics. On account of the fact that no specific physical meaning was initially attached to $G = G(q, \dot{q}, t)$, these identities can be accepted to be applied on any system whose associated equation of motion is reducible to the form of Eq. (1).

4 An analytical and variational formulation for variable-mass systems

Within mechanics, Meshchersky's equation corresponds to

$$m\ddot{q} - Q - (w - \dot{q})\frac{\mathrm{d}m}{\mathrm{d}t} = 0, \tag{20}$$

where $m, \dot{q}$ and $\ddot{q}$ are, respectively, mass, absolute velocity and acceleration, Q is the acting force, $(w - \dot{q})(\mathrm{d}m/\mathrm{d}t)$ is Meshchersky's reactive force, and w is the absolute velocity at which mass is expelled or accreted. Assuming that

$$w = k(t)\dot{q}, \tag{21}$$

$$Q = Q(q, \dot{q}, t), \tag{22}$$

and that mass depends on generalized coordinate, generalized velocity and on time, viz., $m = m(q, \dot{q}, t)$ (see, e.g., [21]), Eq. (20) can be read as

$$m(q, \dot{q}, t)\ddot{q} - Q(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\mathrm{d}m(q, \dot{q}, t)}{\mathrm{d}t} = 0, \tag{23}$$

or, splitting the third term, as

$$m(q, \dot{q}, t)\ddot{q} - Q(q, \dot{q}, t) - \alpha(t)\dot{q}\left(\dot{q}\frac{\partial m(q, \dot{q}, t)}{\partial q} + \ddot{q}\frac{\partial m(q, \dot{q}, t)}{\partial \dot{q}} + \frac{\partial m(q, \dot{q}, t)}{\partial t}\right) = 0, \tag{24}$$

where

$$\alpha(t) = k(t) - 1. \tag{25}$$

An algebraic manipulation allows Meshchersky's equation as in Eq. (24) to be put in the form of Eq. (1),

$$\ddot{q} - \frac{Q(q, \dot{q}, t) + \alpha(t)\dot{q}\left(\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial q} + \frac{\partial m(q,\dot{q},t)}{\partial t}\right)}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}} = 0. \tag{26}$$

In the classical physical world, the amount of mass of an existing system cannot be zero. For this reason, singularities in Eq. (26) may occur only under quite specific conditions. In this article, we assume that such singularities do not happen.

By comparing Eqs. (1) and (26), the function $G = G(q, \dot{q}, t)$ is recognized as

$$G = -\frac{Q(q, \dot{q}, t) + \alpha(t)\dot{q}\left(\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial q} + \frac{\partial m(q,\dot{q},t)}{\partial t}\right)}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}}. \tag{27}$$

This means that, when considering Meshchersky's equation, the inverse problem of Lagrangian mechanics equals to find a $\tilde{L}$-Lagrangian that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \Lambda(q, \dot{q}, t)\left(\ddot{q} - \frac{Q(q, \dot{q}, t) + \alpha(t)\dot{q}\left(\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial q} + \frac{\partial m(q,\dot{q},t)}{\partial t}\right)}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}}\right) \tag{28}$$

(see Eq. (27) into (3)), with $\Lambda = \Lambda(q, \dot{q}, t)$ given as

$$\begin{aligned}\Lambda = \exp\int &\frac{1}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}}\Bigg\{-\frac{\partial Q(q, \dot{q}, t)}{\partial \dot{q}} + \frac{Q(q, \dot{q}, t)}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}}\\
&\times\left((1-\alpha(t))\frac{\partial m(q, \dot{q}, t)}{\partial \dot{q}} - \alpha(t)\dot{q}\frac{\partial^2 m(q, \dot{q}, t)}{\partial \dot{q}^2}\right) - \alpha(t)\Bigg[\dot{q}\Bigg(2\frac{\partial m(q, \dot{q}, t)}{\partial q} + \dot{q}\frac{\partial^2 m(q, \dot{q}, t)}{\partial q\partial \dot{q}}\\
&+\frac{\partial^2 m(q, \dot{q}, t)}{\partial t\partial \dot{q}} - \frac{\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial q} + \frac{\partial m(q,\dot{q},t)}{\partial t}}{m(q, \dot{q}, t) - \alpha(t)\dot{q}\frac{\partial m(q,\dot{q},t)}{\partial \dot{q}}}\left((1-\alpha(t))\frac{\partial m(q, \dot{q}, t)}{\partial \dot{q}} - \alpha(t)\dot{q}\frac{\partial^2 m(q, \dot{q}, t)}{\partial \dot{q}^2}\right)\Bigg)\\
&+\frac{\partial m(q, \dot{q}, t)}{\partial t}\Bigg]\Bigg\}\,\mathrm{d}t\end{aligned} \tag{29}$$

(see Eq. (27) into (12)).

Then, from Eqs. (27) and (29) into Eqs. (15), (17) and (19),

$$\begin{aligned}\tilde{L} = \int_0^{\dot{q}}(\dot{q}-\omega)\exp\int &\frac{1}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\Bigg\{-\frac{\partial Q(q, \omega, t)}{\partial \omega} + \frac{Q(q, \omega, t)}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\\
&\times\left((1-\alpha(t))\frac{\partial m(q, \omega, t)}{\partial \omega} - \alpha(t)\omega\frac{\partial^2 m(q, \omega, t)}{\partial \omega^2}\right) - \alpha(t)\Bigg[\omega\Bigg(2\frac{\partial m(q, \omega, t)}{\partial q} + \omega\frac{\partial^2 m(q, \omega, t)}{\partial q\partial \omega}\\
&+\frac{\partial^2 m(q, \omega, t)}{\partial t\partial \omega} - \frac{\omega\frac{\partial m(q,\omega,t)}{\partial q} + \frac{\partial m(q,\omega,t)}{\partial t}}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\left((1-\alpha(t))\frac{\partial m(q, \omega, t)}{\partial \omega} - \alpha(t)\omega\frac{\partial^2 m(q, \omega, t)}{\partial \omega^2}\right)\Bigg)\\
&+\frac{\partial m(q, \omega, t)}{\partial t}\Bigg]\Bigg\}\,\mathrm{d}t\mathrm{d}\omega + \int_0^{q}\frac{Q(\xi, 0, t)}{m(\xi, 0, t)}\exp\int\frac{1}{m(\xi, 0, t)}\Bigg(-\left.\frac{\partial Q(\xi, \dot{q}, t)}{\partial \dot{q}}\right|_{\dot{q}=0}\\
&+(1-\alpha(t))\frac{Q(\xi, 0, t)}{m(\xi, 0, t)}\left.\frac{\partial m(\xi, \dot{q}, t)}{\partial \dot{q}}\right|_{\dot{q}=0} - \alpha(t)\frac{\partial m(\xi, 0, t)}{\partial t}\Bigg)\mathrm{d}t\mathrm{d}\xi,\end{aligned} \tag{30}$$

$$\begin{aligned}\tilde{p} = \int_0^{\dot{q}}\exp\int &\frac{1}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\Bigg\{-\frac{\partial Q(q, \omega, t)}{\partial \omega} + \frac{Q(q, \omega, t)}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\\
&\times\left((1-\alpha(t))\frac{\partial m(q, \omega, t)}{\partial \omega} - \alpha(t)\omega\frac{\partial^2 m(q, \omega, t)}{\partial \omega^2}\right) - \alpha(t)\Bigg[\omega\Bigg(2\frac{\partial m(q, \omega, t)}{\partial q} + \omega\frac{\partial^2 m(q, \omega, t)}{\partial q\partial \omega} + \frac{\partial^2 m(q, \omega, t)}{\partial t\partial \omega}\\
&-\frac{\omega\frac{\partial m(q,\omega,t)}{\partial q} + \frac{\partial m(q,\omega,t)}{\partial t}}{m(q, \omega, t) - \alpha(t)\omega\frac{\partial m(q,\omega,t)}{\partial \omega}}\left((1-\alpha(t))\frac{\partial m(q, \omega, t)}{\partial \omega} - \alpha(t)\omega\frac{\partial^2 m(q, \omega, t)}{\partial \omega^2}\right)\Bigg) + \frac{\partial m(q, \omega, t)}{\partial t}\Bigg]\Bigg\}\,\mathrm{d}t\mathrm{d}\omega,\end{aligned} \tag{31}$$

$$
\begin{aligned}
\tilde{H} = & \int_0^{\dot{q}} \omega \exp \int \frac{1}{m(q,\omega,t) - \alpha(t)\omega \frac{\partial m(q,\omega,t)}{\partial \omega}} \left\{ -\frac{\partial Q(q,\omega,t)}{\partial \omega} + \frac{Q(q,\omega,t)}{m(q,\omega,t) - \alpha(t)\omega \frac{\partial m(q,\omega,t)}{\partial \omega}} \right. \\
& \times \left((1-\alpha(t)) \frac{\partial m(q,\omega,t)}{\partial \omega} - \alpha(t)\omega \frac{\partial^2 m(q,\omega,t)}{\partial \omega^2} \right) - \alpha(t) \left[\omega \left(2 \frac{\partial m(q,\omega,t)}{\partial q} + \omega \frac{\partial^2 m(q,\omega,t)}{\partial q \partial \omega} + \frac{\partial^2 m(q,\omega,t)}{\partial t \partial \omega} \right. \right. \\
& \left. \left. \left. - \frac{\omega \frac{\partial m(q,\omega,t)}{\partial q} + \frac{\partial m(q,\omega,t)}{\partial t}}{m(q,\omega,t) - \alpha(t)\omega \frac{\partial m(q,\omega,t)}{\partial \omega}} \left((1-\alpha(t)) \frac{\partial m(q,\omega,t)}{\partial \omega} - \alpha(t)\omega \frac{\partial^2 m(q,\omega,t)}{\partial \omega^2} \right) \right) + \frac{\partial m(q,\omega,t)}{\partial t} \right] \right\} \mathrm{d}t \mathrm{d}\omega \\
& - \int_0^{q} \frac{Q(\xi,0,t)}{m(\xi,0,t)} \exp \int \frac{1}{m(\xi,0,t)} \left(- \left. \frac{\partial Q(\xi,\dot{q},t)}{\partial \dot{q}} \right|_{\dot{q}=0} + (1-\alpha(t)) \frac{Q(\xi,0,t)}{m(\xi,0,t)} \left. \frac{\partial m(\xi,\dot{q},t)}{\partial \dot{q}} \right|_{\dot{q}=0} \right. \\
& \left. - \alpha(t) \frac{\partial m(\xi,0,t)}{\partial t} \right) \mathrm{d}t \mathrm{d}\xi. \qquad (32)
\end{aligned}
$$

Equations (30), (31) and (32) appropriately state the $\tilde{L}$-Lagrangian, which so figures as the solution of the inverse problem of Lagrangian mechanics for Meshchersky's equation considering $m = m(q, \dot{q}, t)$, the corresponding $\tilde{p}$-canonical momentum and the following $\tilde{H}$-Hamiltonian. In consequence, at least three fundamental results can be derived.

First, it is possible to formulate a general principle of stationary action in the realm of variable-mass systems mechanics, namely

$$
\delta \int_{t_1}^{t_2} \tilde{L} \mathrm{d}t = 0, \qquad (33)
$$

where the adequate $\tilde{L}$-Lagrangian is that described by Eq. (30). That is to say, from the usual identity

$$
\delta \tilde{L} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \tilde{L}}{\partial \dot{q}} \delta q \right) + \left(-\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \tilde{L}}{\partial \dot{q}} + \frac{\partial \tilde{L}}{\partial q} \right) \delta q, \qquad (34)
$$

and from the assumption that the variation δq is an arbitrary virtual change, which is required to vanish at the limiting instants t_1 and t_2, the variational principle as in Eq. (33) is able to recover Eq. (28).

Second, the conventional set of canonical equations follows, that is,

$$
\dot{q} = \frac{\partial \tilde{H}}{\partial \tilde{p}}, \qquad (35)
$$

$$
\dot{\tilde{p}} = -\frac{\partial \tilde{H}}{\partial q}, \qquad (36)
$$

with the $\tilde{p}$-canonical momentum as in Eq. (31) and the $\tilde{H}$-Hamiltonian as in Eq. (32) (see [36, p. 671, Eq. (2.7)]).

Third, energy-like conservation laws can be thought to be written in terms of the $\tilde{L}$-Lagrangian. Thus, by following the procedure as in [38, p. 61], one has that

$$
\frac{\mathrm{d}\tilde{L}}{\mathrm{d}t} = \frac{\partial \tilde{L}}{\partial q} \dot{q} + \frac{\partial \tilde{L}}{\partial \dot{q}} \ddot{q} + \frac{\partial \tilde{L}}{\partial t}, \qquad (37)
$$

which is obtained after taking the total time derivative of $\tilde{L} = \tilde{L}(q, \dot{q}, t)$.

Since Eq. (2) holds, Eq. (37) can be rewritten as

$$
\frac{\mathrm{d}\tilde{L}}{\mathrm{d}t} = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \tilde{L}}{\partial \dot{q}} \right) \dot{q} + \frac{\partial \tilde{L}}{\partial \dot{q}} \ddot{q} + \frac{\partial \tilde{L}}{\partial t}, \qquad (38)
$$

or, by rearranging the first term and the second term of the right-hand side of Eq. (38), as

$$\frac{\mathrm{d}\tilde{L}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\dot{q}\right)+\frac{\partial \tilde{L}}{\partial t}, \tag{39}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\dot{q}-\tilde{L}\right)=-\frac{\partial \tilde{L}}{\partial t}. \tag{40}$$

This means that, from Eq. (40), one is able to consider energy-like conservation laws within the variable-mass context, having a $\tilde{L}$-Lagrangian (see Eq. (30)) at hand.

5 Example: mass solely dependent on a generalized coordinate and *k* = const.

Systems with mass solely dependent on a generalized coordinate, or, by the simplification of the terminology, position-dependent mass systems, obey

$$m=m(q). \tag{41}$$

This concept can arise from assumptions, or even modelings, and it has been in the focus of a good number of applied problems (see, e.g., [21,39,40]). The function $m=m(q)$ is supposed to be continuous, differentiable and integrable (see Sect. 2). For the sake of an exemplification of the applicability of the formulation in Sect. 4, the absolute velocity at which mass is expelled or accreted is taken as the following particular case of Eq. (21):

$$w=k\dot{q}, \tag{42}$$

where

$$k=\text{const.}, \tag{43}$$

and so, from Eq. (25),

$$\alpha=k-1=\text{const.} \tag{44}$$

The hypothesis given by Eqs. (42), (43) and (44) certainly fails to describe the flux of momentum within a more general context. On the other hand, it can be considered to be a reasonable assumption to deal with some situations as, for instance, falling-chain-like problems (see also examples discussed in [21]).

The customary position-dependent potential

$$V=V(q), \tag{45}$$

from which

$$Q=-\frac{\partial V(q)}{\partial q}, \tag{46}$$

is also adopted for the same purpose.

From Eqs. (41), (44) and (46), Eq. (26) is simplified as

$$\ddot{q}-\frac{-\frac{\partial V(q)}{\partial q}+\alpha\dot{q}^2\frac{\partial m(q)}{\partial q}}{m(q)}=0, \tag{47}$$

and the function G (see Eq. (27)) as

$$G=-\frac{-\frac{\partial V(q)}{\partial q}+\alpha\dot{q}^2\frac{\partial m(q)}{\partial q}}{m(q)}. \tag{48}$$

Thereupon, the inverse problem of Lagrangian mechanics becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \Lambda\left(\ddot{q} - \frac{-\frac{\partial V(q)}{\partial q} + \alpha \dot{q}^2 \frac{\partial m(q)}{\partial q}}{m(q)}\right) \tag{49}$$

(see Eq. (48) into (3)).

Now, there is an important aspect to be noticed. In the right-hand side of Eq. (29), if one changes the variation of integration from t to m, the Jacobi last multiplier Λ in Eq. (49) can be explicitly calculated in terms of m and α, namely

$$\Lambda = \exp\int\left(-\frac{2\alpha\dot{q}}{m(q)}\frac{\partial m(q)}{\partial q}\right)\mathrm{d}t = \exp\int\left(-\frac{2\alpha}{m(q)}\frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\right)\mathrm{d}t = \exp\int\left(-\frac{2\alpha}{m}\right)\mathrm{d}m = m(q)^{-2\alpha}. \tag{50}$$

Under these circumstances, this appears as a particular case of the general theory.

The associated $\tilde{L}$-Lagrangian is written from substituting Eqs. (48) and (50) into Eq. (15) as

$$\tilde{L} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 - \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q. \tag{51}$$

The $\tilde{p}$-canonical momentum follows from inserting Eq. (50) into (17), or from inserting Eq. (51) into (16),

$$\tilde{p} = m(q)^{-2\alpha}\dot{q}. \tag{52}$$

The $\tilde{H}$-Hamiltonian comes from inserting Eqs. (48) and (50) into (19), or Eqs. (51) and (52) into (18),

$$\tilde{H} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q. \tag{53}$$

Note that, from Eq. (52), the $\tilde{H}$-Hamiltonian can be written in terms of the canonical variables q and $\tilde{p}$ as

$$\tilde{H}(q,\tilde{p}) = \frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q. \tag{54}$$

Inserting Eq. (54) into (35) and (36) yields the set of canonical equations

$$\dot{q} = \frac{\tilde{p}}{m(q)^{-2\alpha}}, \tag{55}$$

$$\dot{\tilde{p}} = -\alpha\frac{\partial m(q)}{\partial q}\frac{\tilde{p}^2}{m(q)^{1-2\alpha}} - m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}. \tag{56}$$

Since the $\tilde{L}$-Lagrangian as in Eq. (51) does not depend on time explicitly, viz., $\partial\tilde{L}/\partial t = 0$, from Eq. (40),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\dot{q} - \tilde{L}\right) = 0. \tag{57}$$

From Eqs. (16) and (18), Eq. (57) can be converted into

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}t} = 0, \tag{58}$$

and one immediately has that

$$\tilde{H} = \tilde{E} = \text{const}. \tag{59}$$

Equation (59) assumes the nature of an $\tilde{E}$-energy theorem, in the sense it equates a conserved quantity. Or, in terms of Eq. (54),

$$\frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q = \text{const}. \tag{60}$$

This signifies that, with respect to the case in which $m = m(q)$ and $\alpha = \text{const.}$, the method of Darboux offers a mathematical organization that attributes to this class of systems the 'pseudoconservative'[4] character of Mušicki [16]. From such a perspective, Eq. (59) (or (60)) not only defines a constant of motion, which is given by the own $\tilde{H}$-Hamiltonian, but, following the elegant depiction of Lanczos [3, p. 177], asserts that the system remains constantly on the energy-like surface $\tilde{E} = \text{const.}$ of the $(q, \tilde{p})$-space. Fundamental contributions on conservation laws for variable-mass systems can also be found in [19,41].

5.1 Illustrative case: Cayley's falling-chain problem

Let us consider the classical falling-chain problem in the form that is stated by Cayley [34, p. 506], that is, "(...) a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system (...)". If m represents the mass of the moving part of the chain, then

$$m = \rho z, \tag{61}$$

where z is the length of the moving part and ρ is the linear mass density of the chain. Equation (61) states a continuous, differentiable and integrable function.

From the fundamentals involved in the classical derivation of Meshchersky's equation, w is the absolute velocity of an infinitesimal element of mass that is to be accreted by the system. In Cayley's [34] falling-chain problem, particles to be accreted are originally at rest, which yields

$$w = 0, \tag{62}$$

and so,

$$k = 0, \tag{63}$$

(see Eqs. (42) and (43)), and, from Eq. (44),

$$\alpha = -1. \tag{64}$$

If measured vertically downward, z is the coordinate of the lower extremity of the moving part of the chain, which can be taken as the generalized coordinate $z = q$ of the problem. The potential energy $V = V(z)$ of the moving part of the chain is accordingly written as

$$V = -\int m(z) g \mathrm{d}z, \tag{65}$$

and from Eq. (61),

$$V = -g\rho \int z \mathrm{d}z = -\frac{1}{2} g\rho z^2, \tag{66}$$

which, as a consequence, is also a continuous, differentiable and integrable function.

The associated differential equation is found from substituting Eqs. (61), (64) and (66) into (47),

$$\ddot{z} - g + \frac{\dot{z}^2}{z} = 0. \tag{67}$$

The function G appears from inserting Eqs. (61), (64) and (66) into (48),

$$G = -g + \frac{\dot{z}^2}{z}. \tag{68}$$

[4] It refers to a nonconservative system in nature which, via a $\tilde{L}$-like Lagrangian, is fit to be mathematically read as a conservative system.

The inverse problem is written from inserting Eqs. (61), (64) and (66) into (49),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{z}} - \frac{\partial \tilde{L}}{\partial z} = \Lambda\left(\ddot{z} - g + \frac{\dot{z}^2}{z}\right). \tag{69}$$

The Jacobi last multiplier Λ is obtained from inserting Eqs. (61) and (64) into (50),

$$\Lambda = (\rho z)^2. \tag{70}$$

Thus, from inserting Eq. (70) into (69),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{z}} - \frac{\partial \tilde{L}}{\partial z} = (\rho z)^2\left(\ddot{z} - g + \frac{\dot{z}^2}{z}\right). \tag{71}$$

The solution of Eq. (71) for $\tilde{L}$ is given from substituting Eqs. (61), (64) and (66) into (51),

$$\tilde{L} = \frac{1}{2}(\rho z)^2\dot{z}^2 + \frac{1}{3}g\rho^2 z^3. \tag{72}$$

In fact, since

$$\frac{\partial \tilde{L}}{\partial z} = \rho^2 z\dot{z}^2 + g(\rho z)^2, \tag{73}$$

$$\frac{\partial \tilde{L}}{\partial \dot{z}} = (\rho z)^2\dot{z}, \tag{74}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{z}} = 2\rho^2 z\dot{z}^2 + (\rho z)^2\ddot{z}, \tag{75}$$

the $\tilde{L}$-Lagrangian as in Eq. (72) does satisfy Eq. (71).

The corresponding variational formulation follows from inserting Eq. (72) into (33):

$$\delta\int_{t_1}^{t_2}\left(\frac{1}{2}(\rho z)^2\dot{z}^2 + \frac{1}{3}g\rho^2 z^3\right)\mathrm{d}t = 0. \tag{76}$$

From the assumption that the variation δz is an arbitrary virtual change, which is required to vanish at the limiting instants t_1 and t_2, Eq. (76) is able to alternatively recover Eq. (71).

Using Eqs. (61) and (64) in (52) gives the $\tilde{p}$-canonical momentum:

$$\tilde{p} = (\rho z)^2\dot{z}, \tag{77}$$

and substituting Eqs. (61), (64) and (66) into (53) gives the $\tilde{H}$-Hamiltonian:

$$\tilde{H} = \frac{1}{2}(\rho z)^2\dot{z}^2 - \frac{1}{3}g\rho^2 z^3. \tag{78}$$

In terms of canonical variables, that is, from inserting Eq. (77) into (78), the $\tilde{H}$-Hamiltonian is

$$\tilde{H}(z, \tilde{p}) = \frac{1}{2}\frac{\tilde{p}^2}{(\rho z)^2} - \frac{1}{3}g\rho^2 z^3. \tag{79}$$

The canonical equations follow from inserting Eqs. (61), (64) and (66) into (55) and (56):

$$\dot{z} = \frac{\tilde{p}}{(\rho z)^2}, \tag{80}$$

$$\dot{\tilde{p}} = \frac{\tilde{p}^2}{\rho^2 z^3} + g(\rho z)^2. \tag{81}$$

Once the $\tilde{L}$-Lagrangian as in Eq. (72) does not explicitly depend on time, Eq. (78) states a constant of motion for the problem, namely

$$\frac{1}{2}(\rho z)^2\dot{z}^2 - \frac{1}{3}g\rho^2 z^3 = \text{const.}, \tag{82}$$

or, arbitrarily multiplying Eq. (82) by $2/\rho^2$,

$$z^2\left(\dot{z}^2 - \frac{2}{3}gz\right) = \text{const.} \tag{83}$$

5.2 On the 'factorization': time-dependent case and k = const. versus position-dependent case and k = const.

In our previous work [24],[5] we showed that, in the case in which $m = m(t)$ and $k = \text{const.}$, the $\tilde{H}$-Hamiltonian and the $\tilde{p}$-canonical momentum can be expressed in the following form:

$$\tilde{H} = m(t)^{-k}H, \tag{84}$$
$$\tilde{p} = m(t)^{-k}p. \tag{85}$$

From inserting Eqs. (84) and (85) into (18), one immediately finds the corresponding $\tilde{L}$-Lagrangian,

$$\tilde{L} = m(t)^{-k}L. \tag{86}$$

To derive Eq. (86), Eq. (18) was considered in terms of the conventional quantities H, p and L, that is,

$$H = p\dot{q} - L, \tag{87}$$

where

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{88}$$
$$L = T - V, \tag{89}$$

with

$$T = \frac{1}{2}m(t)\dot{q}^2 \tag{90}$$

as the kinetic energy and V as the potential energy.

Therefore, from Eqs. (84), (85) and (86),

$$\frac{\tilde{H}}{H} = \frac{\tilde{p}}{p} = \frac{\tilde{L}}{L} = m(t)^{-k}. \tag{91}$$

In the case in which $m = m(q)$ and $k = const.$, only the $\tilde{p}$-canonical momentum can be expressed as in Eq. (91). Note that, since in this case

$$T = \frac{1}{2}m(q)\dot{q}^2, \tag{92}$$

one has that

$$p = \frac{\partial L}{\partial \dot{q}} = m(q)\dot{q}. \tag{93}$$

Hence, from substituting Eq. (93) into (52),

$$\frac{\tilde{p}}{p} = m(q)^{-2\alpha-1}, \tag{94}$$

[5] The same results can be alternatively obtained from the formulation proposed in this article. Notwithstanding, the method as in [24] has a different range of applicability.

or from Eq. (44),

$$\frac{\tilde{p}}{p} = m(q)^{1-2k}. \tag{95}$$

From Eq. (92), Eqs. (51) and (53) turn to be[6]

$$\tilde{L} = m(q)^{-2\alpha-1}T - \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q, \tag{96}$$

$$\tilde{H} = m(q)^{-2\alpha-1}T + \int m(q)^{-2\alpha-1}\frac{\partial V(q)}{\partial q}\mathrm{d}q, \tag{97}$$

or from integration by parts,

$$\tilde{L} = m(q)^{-2\alpha-1}L - (2\alpha+1)\int m(q)^{-2\alpha-2}\frac{\partial m(q)}{\partial q}V(q)\mathrm{d}q, \tag{98}$$

$$\tilde{H} = m(q)^{-2\alpha-1}H + (2\alpha+1)\int m(q)^{-2\alpha-2}\frac{\partial m(q)}{\partial q}V(q)\mathrm{d}q. \tag{99}$$

Finally, from Eq. (44), Eqs. (98) and (99) appear as

$$\tilde{L} = m(q)^{1-2k}L + (1-2k)\int m(q)^{-2k}\frac{\partial m(q)}{\partial q}V(q)\mathrm{d}q, \tag{100}$$

$$\tilde{H} = m(q)^{1-2k}H - (1-2k)\int m(q)^{-2k}\frac{\partial m(q)}{\partial q}V(q)\mathrm{d}q. \tag{101}$$

That is to say, in the case in which $m = m(q)$ and $k = \text{const.}$, $\tilde{L}/L \neq m(q)^{1-2k}$ and $\tilde{H}/H \neq m(q)^{1-2k}$.

6 A brief comparison with the analytical formulation for variable-mass systems that follows from D'Alembert's principle

Lagrange's equation is considered by many as the basic equation of classical mechanics. It can be said that, throughout history, the formal derivation and comprehension of this fundamental has been following the development of the mathematical tooling. According to Casey [42, p. 836 and ending notes], "several different derivations of Lagrange's equation can be found in the literature (...) the derivations may be classified as follows (although some overlapping of the categories occurs): (a) derivations leading from Newton's second law by a formal manipulation of partial derivatives; (b) derivations resting on D'Alembert's principle and the principle of virtual work; (c) derivations originating in variational principles; (d) derivations employing differential geometry and tensor calculus". In a modern geometrical approach, Casey [42] shows that Lagrange's equation appears as being equivalent to the covariant components of Newton's law, and no appeal to the principle of virtual work is made for that. This can be considered as an efficient method to deal with a finite set of constant-mass particles, and it can also be extended to the continuum case (see [43,44]).

The merited discussion on the application of different methodologies to obtain Lagrange's equation under the variable-mass condition is out of the scope of the present article. In this section, we only intend to give the reader a brief and contrastive analysis between the formulation for variable-mass systems starting from the inverse problem of Lagrangian mechanics and that following from D'Alembert's differential variational principle, namely

$$\sum_i \left(m_i \ddot{\boldsymbol{x}}_i - \boldsymbol{F}_i - (\boldsymbol{w}_i - \dot{\boldsymbol{x}}_i)\frac{\mathrm{d}m_i}{\mathrm{d}t} \right) \cdot \delta \boldsymbol{x}_i = 0, \tag{102}$$

where, in Cartesian coordinates, m_i, $\boldsymbol{x}_i$, $\dot{\boldsymbol{x}}_i$ and $\ddot{\boldsymbol{x}}_i$ are, respectively, the mass, the position, the absolute velocity and the acceleration of the ith particle, $\boldsymbol{F}_i$ is the corresponding acting force, $(\boldsymbol{w}_i - \dot{\boldsymbol{x}}_i)(\mathrm{d}m_i/\mathrm{d}t)$ is the associated Meshchersky's reactive force, and $\boldsymbol{w}_i$ is the absolute velocity at which mass is expelled or accreted with respect to the ith particle.

[6] The representation of $\tilde{L}$-Lagrangians in terms of kinetic energy with both multiplicative and additive interaction terms is discussed in [5, Sect. 3.7].

In the case in which $m_i = m_i(q_k, \dot{q}_k, t)$, it can be demonstrated from Eq. (102) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \hat{Q}_k, \tag{103}$$

where

$$\hat{Q}_k = Q_k + \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m_i}{\partial \dot{q}_k}\dot{\boldsymbol{x}}_i^2\right) - \frac{1}{2}\frac{\partial m_i}{\partial q_k}\dot{\boldsymbol{x}}_i^2 + \boldsymbol{w}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q_k}\frac{\mathrm{d}m_i}{\mathrm{d}t} \right], \tag{104}$$

Q_k being the kth nonpotential generalized force and L the conventional Lagrangian (see the discussion in [21]).

From the usual identity

$$H = \sum_k p_k \dot{q}_k - L \tag{105}$$

and from Eq. (103), the following set of canonical equations follows (see the discussion in [5, p. 6]):

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \tag{106}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} + \hat{Q}_k, \tag{107}$$

or from Eq. (104),

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} + Q_k + \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m_i}{\partial \dot{q}_k}\dot{\boldsymbol{x}}_i^2\right) - \frac{1}{2}\frac{\partial m_i}{\partial q_k}\dot{\boldsymbol{x}}_i^2 + \boldsymbol{w}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q_k}\frac{\mathrm{d}m_i}{\mathrm{d}t} \right], \tag{108}$$

where

$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \tag{109}$$

The canonical equations as in Eqs. (106) and (108) are in correspondence with the Lagrange equation for a variable-mass system as in Eqs. (103) and (104). In the one degree of freedom case, and assuming Eq. (21), the formulation simplifies as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m}{\partial \dot{q}}\dot{q}^2\right) - \frac{1}{2}\frac{\partial m}{\partial q}\dot{q}^2 + k\dot{q}\frac{\mathrm{d}m}{\mathrm{d}t}, \tag{110}$$

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{111}$$

$$\dot{p} = -\frac{\partial H}{\partial q} + Q + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial m}{\partial \dot{q}}\dot{q}^2\right) - \frac{1}{2}\frac{\partial m}{\partial q}\dot{q}^2 + k\dot{q}\frac{\mathrm{d}m}{\mathrm{d}t}. \tag{112}$$

The essential difference between the formulation from D'Alembert's principle and that outlined in Sects. 4 and 5 can be explained by the following general terms: In the former, the fundamental quantities L, p and H preserve their original physical significance. The inconvenience is that 'external terms' appear in the equations. As named by Santilli [5], external terms are those terms that do not fit the classical mathematical structure as in Eqs. (2), (35) and (36). The right-hand side of Eq. (110), which also appears in the right-hand side of Eq. (112), is an external term. In the latter formulation, the opposite occurs, that is to say, Eqs. (2), (35) and (36) obey the conventional mathematical structure, and the inconvenience is that the fundamental quantities $\tilde{L}$, $\tilde{p}$ and $\tilde{H}$ do not exhibit direct physical meanings.

In truth, both methodologies are complementary tools in the proposition of a consistent mathematical formalism for variable-mass systems. According to Santilli [5, p. 9], "it is hoped that a judicious interplay between these two complementary approaches to the same systems will be effective on methodological as well as physical grounds. On the former grounds, certain aspects which are difficult to treat within the context of one approach could be more manageable within the context of the other approach, and vice versa. On the latter grounds, the two complementary approaches could be useful for the identification of the physical significance of the algorithms at hand (...)".

7 Conclusions

This article addressed the inverse problem of Lagrangian mechanics for Meshchersky's equation. The method of Darboux as discussed by Leubner and Krumm [30] and Yan [36] was taken as the solution technique. It was shown that Meshchersky's equation could be algebraically manipulated to the form $\ddot{q} + G(q, \dot{q}, t) = 0$. This explains the adherence to the method of Darboux. The $\tilde{L}$-Lagrangian, the $\tilde{p}$-canonical momentum and the $\tilde{H}$-Hamiltonian were derived in accordance with the assumption that mass is dependent on generalized coordinate, generalized velocity and on time, and that the absolute velocity at which mass is expelled or accreted is given as a linear function of the generalized velocity. In harmony, a principle of stationary action and the corresponding Lagrange's and Hamilton's equations could be connected to Meshchersky's equation. This means the proposition of an analytical and variational formulation that obeys the conventional mathematical structure of classical mechanics, and it can be applied to variable-mass systems.

The situation in which mass is solely dependent on a generalized coordinate was discussed in a more detailed manner. This could be noticed as a special case of the general theory, seeing that an explicit calculation of the Jacobi last multiplier was feasible. As a consequence, the $\tilde{L}$-Lagrangian, the $\tilde{p}$-canonical momentum and the $\tilde{H}$-Hamiltonian emerged as simpler mathematical expressions. Moreover, the $\tilde{H}$-Hamiltonian could be written in terms of the canonical variables q and $\tilde{p}$. Within this case, it was also possible to derive an energy-like conservation law, from which a constant of motion appeared as given by the own $\tilde{H}$-Hamiltonian. This allowed one to interpret, at least to a convenient extent, a position-dependent mass system as a pseudoconservative system in the sense of Mušicki [16].

When coming out with the $(q, \dot{q}, t)$-description of analytical mechanics, the so-called extended formulations are normally required in order to have variable-mass systems properly considered within the matter. Hence, one observes the appearance of external terms, in Santilli's [5] sense, with respect to the original mathematical structure of the equations (see, e.g., [20,21,25]). In the present article, it was shown that, in the domain of the inverse problem of Lagrangian mechanics for Meshchersky's equation, physical effects related to mass variation appear as being 'mathematically compacted', and therefore, one could identify multiplicative factors in the original physical quantities as, for example, m^{1-2k} in the position-dependent case and m^{-k} in the time-dependent case. This seems to be somehow necessary, if one intends to preserve the conventional mathematical structure of the equations.

We finish by arguing that, in the scenario of variable-mass systems, the solution of the inverse problem of Lagrangian mechanics comes to offer a unifying mathematical tool not only for the proper construction of a general principle of stationary action, but also for a consistent Hamiltonization and following investigations on energy-like conservation laws. This work is thus expected to have somehow contributed to the elaboration of a proper mathematical formalism for variable-mass systems, a pervasive process that began in the early years of the nineteenth century (see [35]), and is found to be currently in progress.

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