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Rate formulations in nonlinear continuum mechanics

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Abstract In solving problems of geometrically nonlinear structural mechanics, a prominent role is played by formulation of rate equilibrium conditions. In the computational machinery, the evaluation of the stiffness operator provides the trial incremental displacement field as fixed point of an iterative algorithm. The issue is investigated by a new geometric approach to continuum mechanics. Kinematics is described by the motion along a trajectory manifold embedded in the affine four-dimensional space-time. Variational conditions of equilibrium and rate equilibrium are formulated in terms of natural time rates of stress and stretching. The rate elastostatic problem is formulated in the full nonlinear context by adopting a newly contributed rate-elastic constitutive model. The geometric stiffness and forcing operators are expressed in terms of an arbitrary linear spatial connection. It is shown that the adoption of a LEVI- CIVITA connection provides a linear expression of the geometric stiffness involving a curvature term. For bodies in motion in the flat EUCLID space with parallel transport by translation, a symmetric expression of the geometric stiffness is obtained, thus extending the standard formula to bodies of any dimensionality.

1 Introduction

Nonlinear continuum mechanics (NLCM) is a natural gym for differential geometry (DG), and in fact, the recourse to concepts and methods of this mathematical discipline is compelling when dealing with nonlinear problems.

The shining success of the geometrically linearized theory of elasticity, based on linear algebra and calculus in linear spaces (LAC), and of elastoviscoplasticity theory, based on convex analysis (CA) in linear spaces, has led to an underestimate of the need for DG in getting a satisfactory mathematical modeling.

As a consequence, in most treatments of NLCM contributed in the last decades, mainly notions proper of LAC and CA are referred to, without significant recourse to DG.

On the other hand, it is well known that basic questions of NLCM can only be answered by relying on the results of DG, see e.g. [1–11]. A classical example is provided by the differential condition of kinematical compatibility of a finite strain measure, consisting in the vanishing of the relevant RIEMANN curvature.

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The role played by DG is, however, definitely more basic, and a knowledge of the elements of DG is definitely needed to set up the very foundations of NLCM.

Instances of application of notions from DG to solve longly debated issues in NLCM are provided by the proper geometric definition of stress time rate (*stressing*), by the formulation of a consistent rate-elastic behavior and by the statement of the principle of constitutive frame-invariance, recently clarified in [12–14].

The elements of NLCM are illustrated in this paper with a geometric approach developed in the space-time framework. The rules governing time differentiation of involved vector and tensor fields along the motion are brought to full evidence, leading to a consistent theoretical and computational formulation of NLCM.

A main point concerns the proper definition of material and spatial fields on the trajectory manifold and of the relevant transformation rules, dictated by a geometric paradigm. These are, respectively, push–pull operations and parallel transports, as illustrated in Sects. 3–9. In this way, the longly debated issue about the way time rates of stress tensors ought to be defined is finally resolved with a unique well-definite answer.

Under the guidelines of the geometric formulation, the variational equilibrium and rate equilibrium conditions along the trajectory are expressed by the virtual power principle (VPP) and by a rate virtual power principle (RVPP) involving time rates of stress, stretching and force system, defined in a natural way, and time-independent virtual velocity fields extended by parallel transport according to an arbitrary linear connection.

The RVPP is then *not* a natural notion since its formulation depends on the choice of the linear spatial connection to perform the parallel transport of virtual velocities along the motion. Anyway, all formulations provide equivalent conditions of rate equilibrium.

The geometric analysis leads to an expression of the time rate of the virtual stretching along the motion, which in general depends in a nonlinear way on the spatial velocity field. Thus, a nonlinear geometric stiffness is generated. A nonlinear dependence on the velocity is shared also by the forcing (force time rate) so that the adoption of an iterative algorithm for the iterative solution of nonlinear rate equilibrium problems is compelling.

This feature is not evidenced in previous treatments investigating 3D bodies in the framework of the standard EUCLIDIAN space with connection by translation [15–20].

For bodies of any dimensionality, the adoption of a LEVI- CIVITA connection (torsion-free and metric-preserving) yields a linear geometric stiffness whose expression involves a curvature term.

For bodies in the context of the flat EUCLID ambient space with parallel transport by translation, a symmetric geometric stiffness is found. This expression extends to bodies of any dimensionality the known formula reported in the literature for 3D bodies. A comparison with standard treatments is provided in Sect. 13.

The rate elastostatic problem is formulated in Sects. 14, 15 on the basis of the contributed new model of rate-elasticity [12–14].

Nonsymmetric linear connections induced by mobile frames in curvilinear coordinate systems are briefly illustrated in Sect. 16. Computational issues are briefly outlined in Sect. 17.

2 Preliminary geometric notions

All notions listed below are illustrated in detail in [21–23]. In the following, a circle $\circ$ means the composition of maps and an interposed dot $\cdot$ denotes linear dependence on subsequent arguments.

At each point $\mathbf{x} \in \mathbb{M}$ of a manifold, there is a corresponding tangent linear space $T_{\mathbf{x}}\mathbb{M}$ made of the velocities of curves through that point.

To a smooth transformation $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$, there corresponds at each point $\mathbf{x} \in \mathbb{M}$ a linear infinitesimal transformation $T_{\mathbf{x}}\boldsymbol{\phi} : T_{\mathbf{x}}\mathbb{M} \mapsto T_{\boldsymbol{\phi}(\mathbf{x})}\mathbb{N}$ between the tangent spaces, called the *differential*, whose action on the tangent vector $\mathbf{u}_{\mathbf{x}} := \partial_{s=0}\, \mathbf{c}(s) \in T_{\mathbf{x}}\mathbb{M}$ to a curve $\mathbf{c} : \mathcal{R} \mapsto \mathbb{M}$, at the point $\mathbf{x} = \mathbf{c}(0)$, is defined by

$$T_{\mathbf{x}}\boldsymbol{\phi} \cdot \mathbf{u}_{\mathbf{x}} = \partial_{s=0}\, (\boldsymbol{\phi} \circ \mathbf{c})(s). \tag{1}$$

A chochét $\langle , \rangle$ denotes the bilinear, nondegenerate duality between pairs of dual linear spaces $(T_{\mathbf{x}}\mathbb{M}\,, T^*_{\mathbf{x}}\mathbb{M})$ or $(T_{\boldsymbol{\phi}(\mathbf{x})}\mathbb{N}\,, T^*_{\boldsymbol{\phi}(\mathbf{x})}\mathbb{N})$. The dual linear map

$$(T_{\mathbf{x}}\boldsymbol{\phi})^* : T^*_{\boldsymbol{\phi}(\mathbf{x})}\mathbb{N} \mapsto T^*_{\mathbf{x}}\mathbb{M}$$

is defined for any $\mathbf{u}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbb{M}$ and $\mathbf{w}_{\boldsymbol{\phi}(\mathbf{x})} \in T_{\boldsymbol{\phi}(\mathbf{x})}\mathbb{N}$ by the identity

$$\langle T_{\mathbf{x}}\boldsymbol{\phi} \cdot \mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\boldsymbol{\phi}(\mathbf{x})} \rangle = \langle \mathbf{u}_{\mathbf{x}}, (T_{\mathbf{x}}\boldsymbol{\phi})^* \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{x})} \rangle. \tag{2}$$

Zeroth-order tensors are just real-valued functions. Second-order tensors at $\mathbf{x} \in \mathbb{M}$ are bilinear maps on pairs of vectors or covectors based at that point, named covariant, contravariant or mixed depending on whether the arguments are both vectors, both covectors, or a vector and a covector. Corresponding tensor spaces at $\mathbf{x} \in \mathbb{M}$ are denoted by $\mathrm{FUN}(T_{\mathbf{x}}\mathbb{M})$, $\mathrm{COV}(T_{\mathbf{x}}\mathbb{M})$, $\mathrm{CON}(T_{\mathbf{x}}\mathbb{M})$ and $\mathrm{MIX}(T_{\mathbf{x}}\mathbb{M})$, and a generic tensor space by $\mathrm{TENS}(T_{\mathbf{x}}\mathbb{M})$.

First-order covariant tensors are covectors and first-order contravariant tensors are tangent vectors. Second-order tensors at $\mathbf{x} \in \mathbb{M}$ are equivalently defined as linear operators from a or cotangent space to another such space at that point:

$$(\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{M} \mapsto T^*_{\mathbf{x}}\mathbb{M} \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M}), \tag{3}$$

$$(\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} : T^*_{\mathbf{x}}\mathbb{M} \mapsto T_{\mathbf{x}}\mathbb{M} \in \mathrm{CON}(T_{\mathbf{x}}\mathbb{M}), \tag{4}$$

$$(\mathbf{s}_{\mathrm{MIX}})_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{M} \mapsto T_{\mathbf{x}}\mathbb{M} \in \mathrm{MIX}(T_{\mathbf{x}}\mathbb{M}). \tag{5}$$

Maximal alternating multilinear tensors of order $n = \dim T_{\mathbf{x}}\mathbb{M}$ are called *volume forms* and denoted by $(\mathbf{s}_{\mathrm{VOL}})_{\mathbf{x}} \in \mathrm{VOL}(T_{\mathbf{x}}\mathbb{M})$.

Covariant, contravariant and mixed tensors may be altered, one to another, by means of (pre or post) composition with an invertible covariant tensor that provides a one-to-one correspondence between the tangent and the cotangent space. The covariant metric tensor $\mathbf{g}$, being positive definite, is the natural candidate.

2.1 Tensor bundles

The domains of the *tangent bundle* $T\mathbb{M}$ and the *cotangent bundle* $T^*\mathbb{M}$ are the disjoint union of all linear tangent spaces and dual cotangent spaces, respectively, based at points of the manifold. The bundles are defined by the projections $\boldsymbol{\pi}_{\mathbb{M}} \in \mathrm{C}^1(T\mathbb{M}\,;\mathbb{M})$ and $\boldsymbol{\pi}^*_{\mathbb{M}} \in \mathrm{C}^1(T^*\mathbb{M}\,;\mathbb{M})$ associating the base points corresponding to tangent and cotangent vectors in the bundles.

The global transformation between tangent bundles $T\boldsymbol{\phi} : T\mathbb{M} \mapsto T\mathbb{N}$ is called the *tangent transformation*. The operator T, acting on manifolds and on maps between them, is named the *tangent functor* [21,23].

More general, a *fibration* of a manifold $\mathbb{F}$ over a base manifold $\mathbb{B}$ is characterized by a surjective submersion[1]

$$\boldsymbol{\pi}_{\mathbb{B}} \in \mathrm{C}^1(\mathbb{F}\,;\mathbb{B}), \tag{6}$$

called the *projection*. The *fiber* $\mathbb{F}(\mathbf{x})$ is the inverse image of $\mathbf{x} \in \mathbb{B}$ by the projection. A *fiber-bundle* is a fibration whose fibers are diffeomorphic manifolds. A *linear-bundle* has linear fibers.

A *field* in a fiber-bundle $\boldsymbol{\pi}_{\mathbb{B}} \in \mathrm{C}^1(\mathbb{F}\,;\mathbb{B})$ is a map $\mathbf{s} \in \mathrm{C}^1(\mathbb{B}\,;\mathbb{F})$ such that $\boldsymbol{\pi}_{\mathbb{B}} \circ \mathbf{s} = \mathrm{ID}_{\mathbb{B}}$, the identity over $\mathbb{B}$, that is such that $\mathbf{s}(\mathbf{x}) \in \mathbb{F}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{B}$. In the geometric terminology, *fields* are called *sections* of the fiber-bundle.

Tensor bundles are linear bundles defined on disjoint unions of linear tensor spaces at points of a manifold and are characterized by the projection $\boldsymbol{\pi}_{\mathbb{M}} \in \mathrm{C}^1(\mathrm{TENS}(T\mathbb{M})\,;\mathbb{M})$ that associates the base point corresponding to each tensor in the bundle.

A tensor field $\mathbf{s}_{\mathrm{TENS}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{TENS}(T\mathbb{M}))$ is a *section* of a tensor bundle. This means that $\mathbf{s}_{\mathrm{TENS}}(\mathbf{x}) \in \mathrm{TENS}(T_{\mathbf{x}}\mathbb{M})$ for all $\mathbf{x} \in \mathbb{M}$.

By (3)–(5), a (second-order) tensor field is equivalently described as a homomorphism[2] between tangent and/or cotangent bundles.

2.2 Push–pull, flows and Lie derivatives

A transformation $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$ maps a curve on $\mathbb{M}$ into a curve in $\mathbb{N}$ and, under suitable assumptions, scalar, vector and covector fields from $\mathbb{M}$ onto $\boldsymbol{\phi}(\mathbb{M}) \subset \mathbb{N}$ (push forward $\uparrow$) and vice versa (pull back $\downarrow$)

[1] *Immersion* (*submersion*) is a map with injective (surjective) tangent map.

[2] A *morphism* is a fiber preserving map between fiber-bundles. A *homomorphism* is a fiberwise linear morphism between vector-bundles.

[23].[3] Assumptions of differentiability and of invertibility of the differential are claimed whenever needed. The defining formulae for push–pull are the following.

Push forward from $\mathbb{M}$ to $\boldsymbol{\phi}(\mathbb{M})$, $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$ injective.

$$\psi : \mathbb{M} \mapsto \mathcal{R}\,, \qquad (\boldsymbol{\phi}\!\uparrow\!\psi)_{\boldsymbol{\phi}(\mathbf{x})} = \psi_{\mathbf{x}}\,, \tag{7}$$

$$\mathbf{v} : \mathbb{M} \mapsto T\mathbb{M}\,, \qquad (\boldsymbol{\phi}\!\uparrow\!\mathbf{v})_{\boldsymbol{\phi}(\mathbf{x})} = T_{\mathbf{x}}\boldsymbol{\phi} \cdot \mathbf{v}_{\mathbf{x}}\,, \tag{8}$$

$$\mathbf{v}^* : \mathbb{M} \mapsto T^*\mathbb{M}\,, \qquad \langle \boldsymbol{\phi}\!\uparrow\!\mathbf{v}^*, \mathbf{w} \rangle_{\boldsymbol{\phi}(\mathbf{x})} = \langle \mathbf{v}^*_{\mathbf{x}}, (T_{\mathbf{x}}\boldsymbol{\phi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{x})} \rangle. \tag{9}$$

Pull back from $\boldsymbol{\phi}(\mathbb{M})$ to $\mathbb{M}$.

$$\phi : \mathbb{N} \mapsto \mathcal{R}\,, \qquad (\boldsymbol{\phi}\!\downarrow\!\phi)_{\mathbf{x}} = \phi_{\boldsymbol{\phi}(\mathbf{x})}\,, \tag{10}$$

$$\mathbf{w} : \mathbb{N} \mapsto T\mathbb{N}\,, \qquad (\boldsymbol{\phi}\!\downarrow\!\mathbf{w})_{\mathbf{x}} = (T_{\mathbf{x}}\boldsymbol{\phi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{x})}\,, \tag{11}$$

$$\mathbf{w}^* : \mathbb{N} \mapsto T^*\mathbb{N}\,, \qquad \langle \boldsymbol{\phi}\!\downarrow\!\mathbf{w}^*, \mathbf{v} \rangle_{\mathbf{x}} = \langle \mathbf{w}^*_{\boldsymbol{\phi}(\mathbf{x})}, T_{\mathbf{x}}\boldsymbol{\phi} \cdot \mathbf{v}_{\mathbf{x}} \rangle. \tag{12}$$

Push–pull operations for second-order covariant, contravariant and mixed tensors are defined so that their scalar values are invariant by the following formulae:

$$(\boldsymbol{\phi}\!\downarrow\!\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} = (T_{\mathbf{x}}\boldsymbol{\phi})^* \cdot (\mathbf{s}_{\mathrm{COV}})_{\boldsymbol{\phi}(\mathbf{x})} \cdot T_{\mathbf{x}}\boldsymbol{\phi}\,, \tag{13}$$

$$(\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{CON}})_{\boldsymbol{\phi}(\mathbf{x})} = T_{\mathbf{x}}\boldsymbol{\phi} \cdot (\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} \cdot (T_{\mathbf{x}}\boldsymbol{\phi})^*\,, \tag{14}$$

$$(\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{MIX}})_{\boldsymbol{\phi}(\mathbf{x})} = T_{\mathbf{x}}\boldsymbol{\phi} \cdot (\mathbf{s}_{\mathrm{MIX}})_{\mathbf{x}} \cdot (T_{\mathbf{x}}\boldsymbol{\phi})^{-1}. \tag{15}$$

From (13)–(15) and the chain rule of linear calculus, it follows that push–pull operations by diffeomorphisms enjoy the following commutativity properties with composition:

$$\boldsymbol{\phi}\!\uparrow\!(\mathbf{s}_{\mathrm{CON}} \cdot \mathbf{s}_{\mathrm{COV}}) = (\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{CON}}) \cdot (\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{COV}})\,, \tag{16}$$

$$\boldsymbol{\phi}\!\uparrow\!(\mathbf{s}_{\mathrm{COV}} \cdot \mathbf{s}_{\mathrm{MIX}}) = (\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{COV}}) \cdot (\boldsymbol{\phi}\!\uparrow\!\mathbf{s}_{\mathrm{MIX}}). \tag{17}$$

The pull-back of a tensor field is the tensor field defined by

$$\boldsymbol{\phi}\!\downarrow\!\mathbf{s}_{\mathrm{TENS}} = \boldsymbol{\phi}\!\downarrow \circ\, \mathbf{s}_{\mathrm{TENS}} \circ \boldsymbol{\phi}. \tag{18}$$

The flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ is generated by solutions of the differential equation $\mathbf{v} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}$, and the LIE derivative $\mathcal{L}_{\mathbf{v}}\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ of a tangent vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ according to a tangent vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ is the derivative of the pullback by the relevant flow:

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{v}}_{\lambda}\!\downarrow\!\mathbf{w}) = \partial_{\lambda=0}\,T\mathbf{Fl}^{\mathbf{v}}_{-\lambda} \cdot \mathbf{w} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda}.$$

LIE derivatives of tensor fields are defined in the same way. Push–pull of scalar fields is just a change of base points so that LIE derivatives coincide with directional derivatives.

Adopting the notation $\mathbf{v}f := \mathcal{L}_{\mathbf{v}}\,f$, with $f \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{R})$ any scalar field, the *commutator* of tangent vector fields $\mathbf{v}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ is the skew-symmetric tangent-vector valued differential operator defined by

$$[\mathbf{v}\,,\mathbf{w}]f := (\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})f = (\mathcal{L}_{\mathbf{v}}\mathcal{L}_{\mathbf{w}} - \mathcal{L}_{\mathbf{w}}\mathcal{L}_{\mathbf{v}})f. \tag{19}$$

A basic theorem concerning LIE derivatives states that

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v}\,,\mathbf{w}], \tag{20}$$

and hence, the commutator of tangent vector fields is called the LIE *bracket*. For any injective morphism $\boldsymbol{\phi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, the following push naturality property holds

$$[\boldsymbol{\phi}\!\uparrow\!\mathbf{v}\,,\boldsymbol{\phi}\!\uparrow\!\mathbf{w}] = \boldsymbol{\phi}\!\uparrow\![\mathbf{v}\,,\mathbf{w}]. \tag{21}$$

[3] In differential geometry, push and pull are, respectively, denoted by low and high asterisks $_*;^*$ [21,22]. This standard notation leads, however, to consider too many similar stars in the geometric sky, i.e., push, pull, duality, HODGE operator.

2.3 Connections and parallel derivatives

A linear connection ∇ in a manifold $\mathbb{M}$ can be described by a differentiation fulfilling the characteristic properties of a point derivation

$$\nabla_{\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2} \mathbf{v} = \alpha_1 \nabla_{\mathbf{w}_1} \mathbf{v} + \alpha_2 \nabla_{\mathbf{w}_2} \mathbf{v} \,, \tag{22}$$
$$\nabla_{\mathbf{w}}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 \nabla_{\mathbf{w}} \mathbf{v}_1 + \alpha_2 \nabla_{\mathbf{w}} \mathbf{v}_2 \,, \tag{23}$$
$$\nabla_{\mathbf{w}}(f \mathbf{v}) = f \, \nabla_{\mathbf{w}} \mathbf{v} + (\nabla_{\mathbf{w}} f) \mathbf{v} \,, \tag{24}$$

with $\alpha_1, \alpha_2 \in \mathrm{C}^1(\mathbb{M}\,; \mathcal{R})$ scalar fields and $\mathbf{v}, \mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ tangent vector fields.

In terms of *parallel transport* $\Uparrow$ ($\Downarrow$ denotes backward parallel transport) along a curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,; \mathbb{M})$ with $\mathbf{h}_{\mathbf{x}} = \partial_{\lambda=0}\, \mathbf{c}(\lambda)$ and $\mathbf{x} = \mathbf{c}(0)$, the *parallel derivative*[4] of a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ according to a connection is defined by

$$\nabla_{\mathbf{h}} \mathbf{v} := \partial_{\lambda=0}\, \mathbf{c}(\lambda)\Downarrow (\mathbf{v} \circ \mathbf{c})(\lambda).$$

Parallel transported vector fields $(\mathbf{v} \circ \mathbf{c})(\lambda) = \mathbf{c}(\lambda)\Uparrow \mathbf{v}_{\mathbf{x}}$ are characterized by a null parallel derivative, because

$$\nabla_{\mathbf{h}_{\mathbf{x}}} \mathbf{v} := \partial_{\lambda=0}\, \mathbf{c}(\lambda)\Downarrow (\mathbf{v} \circ \mathbf{c})(\lambda) = \partial_{\lambda=0}\, (\mathbf{c}(\lambda)\Downarrow \circ\, \mathbf{c}(\lambda)\Uparrow) \mathbf{v}_{\mathbf{x}} = \partial_{\lambda=0}\, \mathbf{v}_{\mathbf{x}} = 0.$$

Parallel transport of a tensor field is defined by invariance and the parallel derivative along a flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M})$ is given by

$$\nabla_{\mathbf{v}} \mathbf{s}_{\mathrm{TENS}} := \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow \mathbf{s}_{\mathrm{TENS}}) = \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow \circ\, \mathbf{s}_{\mathrm{TENS}} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda}). \tag{25}$$

The parallel derivative fulfills a LEIBNIZ rule, which for a covector field $\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{M}\,; T^*\mathbb{M})$ and a vector field $\mathbf{h} \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ writes

$$\nabla_{\mathbf{h}} \boldsymbol{\omega}(\mathbf{v}) = \nabla_{\mathbf{h}}(\boldsymbol{\omega}(\mathbf{v})) - \boldsymbol{\omega}(\nabla_{\mathbf{h}} \mathbf{v}).$$

The curvature of the connection is the operator CURV that maps tensorially a tangent vector field $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ to a tangent-vector valued two-form[5] $\mathrm{CURV}(\mathbf{s})$ defined by

$$\mathrm{CURV}(\mathbf{s})(\mathbf{v}\,, \mathbf{w}) := ([\nabla_{\mathbf{v}}\,, \nabla_{\mathbf{w}}] - \nabla_{[\mathbf{v}\,,\mathbf{w}]})(\mathbf{s}) \,, \tag{26}$$

and the torsion TORS is the tangent-vector valued two-form defined by

$$\mathrm{TORS}(\mathbf{v}\,, \mathbf{w}) := \nabla_{\mathbf{v}} \mathbf{w} - \nabla_{\mathbf{w}} \mathbf{v} - [\mathbf{v}\,, \mathbf{w}]. \tag{27}$$

Mixed tensor fields $\mathrm{T}\text{ORS}(\mathbf{v})$ and $\mathrm{C}\text{URV}(\mathbf{s}\,, \mathbf{v})$ are defined by the identities

$$\mathrm{T}\text{ORS}(\mathbf{v}) \cdot \mathbf{w} := \mathrm{TORS}(\mathbf{v}\,, \mathbf{w}) = -\mathrm{TORS}(\mathbf{w}\,, \mathbf{v}) \,, \tag{28}$$
$$\mathrm{C}\text{URV}(\mathbf{s}\,, \mathbf{v}) \cdot \mathbf{w} := \mathrm{CURV}(\mathbf{s})(\mathbf{v}\,, \mathbf{w}) = -\mathrm{CURV}(\mathbf{s})(\mathbf{w}\,, \mathbf{v}). \tag{29}$$

The second parallel derivative, defined by

$$\nabla^2_{\mathbf{v},\mathbf{w}} := \nabla_{\mathbf{v}} \nabla_{\mathbf{w}} - \nabla_{\nabla_{\mathbf{v}} \mathbf{w}}, \tag{30}$$

is tensorial in the vector fields $\mathbf{v}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ and also tensorial are the expressions (26), (27) defining the curvature and the torsion forms [23]. The symmetry gap of the second derivative is provided by

$$\nabla^2_{\mathbf{v},\mathbf{w}} - \nabla^2_{\mathbf{w},\mathbf{v}} = \mathrm{C}\text{URV}(\bullet\,, \mathbf{v}) \cdot \mathbf{w} - \nabla_{\mathrm{TORS}(\mathbf{v}\,,\mathbf{w})} \,, \tag{31}$$

which follows from (30) and definitions (26) and (27).

A connection with vanishing torsion is named *torsion-free* or *symmetric*, and a connection with vanishing curvature is said to be *curvature-free* or *flat*. In a RIEMANN manifold $(\mathbb{M}\,, \mathbf{g})$, a linear connection ∇ is *metric preserving* (or simply *metric*) if the metric tensor is invariant under parallel transport,

$$\mathbf{g}_{\mathbf{x}}(\mathbf{v}_{\mathbf{x}}\,, \mathbf{w}_{\mathbf{x}}) = \mathbf{g}_{\mathbf{c}(\lambda)}(\mathbf{c}(\lambda)\Uparrow \mathbf{v}_{\mathbf{x}}\,, \mathbf{c}(\lambda)\Uparrow \mathbf{w}_{\mathbf{x}}) \,,$$

so that its parallel derivative vanishes: $\nabla \mathbf{g} = 0$. The LEVI-CIVITA connection in $(\mathbb{M}\,, \mathbf{g})$ is the unique connection which is *metric* and *symmetric*.

[4] Commonly named *covariant derivative*, despite there is no clear relation with the homonymic property of covariance under diffeomorphic transformations.

[5] Tensoriality of a multilinear map on vector fields and generating a vector field means that point values of the image field depend only on the values of source fields at the same point. A form is a vector-valued, tensorial, alternating multilinear map.

Remark 1 (Path independence) Connections whose associated parallel transport from one point to another is independent of the path connecting the end points are flat. Indeed, by tensoriality, in computing the curvature at a point by definition (26), the tangent vector field $\mathbf{s} \in C^1(\mathbb{M}\,; T\mathbb{M})$ may be generated by path-independent parallel transport. Hence, all parallel derivatives are zero and the curvature operator vanishes identically. Special instances are the EUCLID connection by translation or connections defined by natural frames generated by coordinate systems, considered in Sect. 16.

On a manifold $\mathbb{M}$, setting $\mathbf{Y}(\mathbf{v}) := \nabla(\mathbf{v}) + \mathrm{TORS}(\mathbf{v})$, the LIE derivative of a covariant tensor field $\mathbf{s}_{\mathrm{COV}} : T\mathbb{M} \mapsto T^*\mathbb{M}$ along the flow associated with a vector field $\mathbf{v} \in C^1(\mathbb{M}\,; T\mathbb{M})$, is expressed in terms of the parallel derivative $\nabla^{\mathbb{M}}$ by the formula

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{COV}} = \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{COV}} + \mathbf{s}_{\mathrm{COV}} \cdot \mathbf{Y}(\mathbf{v}) + \mathbf{Y}(\mathbf{v})^* \cdot \mathbf{s}_{\mathrm{COV}}. \tag{32}$$

This result, and the analogous ones for contravariant and mixed tensor fields, can be inferred from a comparison between the expressions obtained by applying LEIBNIZ rule to the LIE derivative and to the parallel derivative [23].

The following Lemma will be recalled to get formula (96).

Lemma 1 (Parallel derivative of a composition) *If* $\mathbf{s}_{\mathrm{COV}} = \boldsymbol{\alpha}_{\mathrm{COV}} \cdot \boldsymbol{\beta}_{\mathrm{MIX}}$ *is the composition of two tensor fields, then the following* LEIBNIZ *rule holds:*

$$\nabla_{\mathbf{v}}(\boldsymbol{\alpha}_{\mathrm{COV}} \cdot \boldsymbol{\beta}_{\mathrm{MIX}}) = (\nabla_{\mathbf{v}}\boldsymbol{\alpha}_{\mathrm{COV}}) \cdot \boldsymbol{\beta}_{\mathrm{MIX}} + \boldsymbol{\alpha}_{\mathrm{COV}} \cdot (\nabla_{\mathbf{v}}\boldsymbol{\beta}_{\mathrm{MIX}}). \tag{33}$$

Proof. Observing that, for any $\mathbf{a}, \mathbf{b} \in T\mathbb{M}$

$$\mathbf{s}_{\mathrm{COV}}(\mathbf{a}\,, \mathbf{b}) = \langle \mathbf{s}_{\mathrm{COV}} \cdot \mathbf{a}, \mathbf{b} \rangle = \langle \boldsymbol{\alpha}_{\mathrm{COV}} \cdot \boldsymbol{\beta}_{\mathrm{MIX}} \cdot \mathbf{a}, \mathbf{b} \rangle = \boldsymbol{\alpha}_{\mathrm{COV}}(\boldsymbol{\beta}_{\mathrm{MIX}} \cdot \mathbf{a}\,, \mathbf{b})\,, \tag{34}$$

from the definition of parallel transport of a tensor field, it follows that

$$\begin{aligned}
(\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \mathbf{s}_{\mathrm{COV}})(\mathbf{a}\,, \mathbf{b}) &= \mathbf{s}_{\mathrm{COV}}(\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{a}\,, \mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{b}) \\
&= \boldsymbol{\alpha}_{\mathrm{COV}}(\boldsymbol{\beta}_{\mathrm{MIX}} \cdot (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{a})\,, \mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{b}) \\
&= \boldsymbol{\alpha}_{\mathrm{COV}}(\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow ((\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\beta}_{\mathrm{MIX}}) \cdot \mathbf{a})\,, \mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{b}) \\
&= (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\alpha}_{\mathrm{COV}})((\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\beta}_{\mathrm{MIX}}) \cdot \mathbf{a}\,, \mathbf{b})\,,
\end{aligned} \tag{35}$$

that is,

$$\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow (\boldsymbol{\alpha}_{\mathrm{COV}} \cdot \boldsymbol{\beta}_{\mathrm{MIX}}) = (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\alpha}_{\mathrm{COV}}) \cdot (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\beta}_{\mathrm{MIX}}). \tag{36}$$

Then, applying the LEIBNIZ rule of linear calculus to the definition (25) of parallel derivative in terms of parallel transport

$$\nabla_{\mathbf{v}}\mathbf{s}_{\mathrm{COV}} := \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \mathbf{s}_{\mathrm{COV}}) = \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\alpha}_{\mathrm{COV}}) \cdot (\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow \boldsymbol{\beta}_{\mathrm{MIX}})\,, \tag{37}$$

the rule (33) follows. □

3 Continuum kinematics

Classical continuum kinematics (CK) deals with material bodies in motion in a four-dimensional events space $\mathcal{E}$ during an *interval* $I \subset \mathcal{Z}$ of instants in the time line.[6]

An EUCLID observer $\boldsymbol{\gamma} \in C^1(\mathcal{E}, \mathcal{S} \times \mathcal{Z})$ detects the event manifold as space-time product between the ambient space manifold $\mathcal{S}$ with $\dim \mathcal{S} = m = 3$ and the time line $\mathcal{Z}$.

Both $\mathcal{S}$ and $\mathcal{Z}$ are affine manifolds endowed with metric tensor fields

$$\mathbf{g}_{\mathcal{S}} \in C^1(\mathcal{S}\,; \mathrm{POS}(T\mathcal{S}))\,, \quad \mathbf{g}_{\mathcal{Z}} \in C^1(I\,; \mathrm{POS}(T\mathcal{Z}))\,, \tag{38}$$

which are twice-covariant, symmetric and positive definite tensor fields, invariant by translation.

[6] The symbol $\mathcal{Z}$ is the initial of the German word *Zeit* (Time).

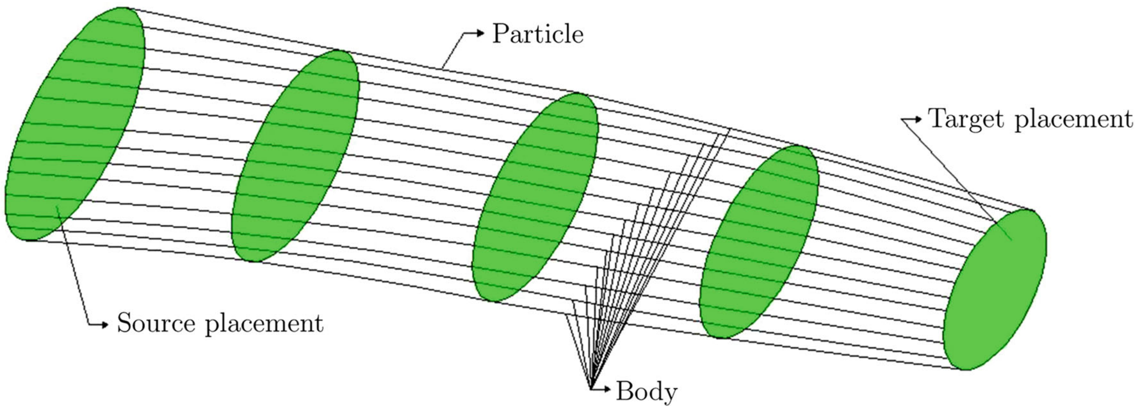

Fig. 1 Trajectory tube in space-time

Space and time manifolds are assumed to be endowed with linear connections $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{Z}}$, and the associated parallel transports are denoted by $\Uparrow^{\mathcal{S}}$ and $\Uparrow^{\mathcal{Z}}$. In the one-dimensional time line $\mathcal{Z}$ standard metric tensor and connection are induced by the adopted coordinate axis.

The object under investigation is a body in motion which describes a nonlinear trajectory manifold $\mathcal{T}$ embedded in the four-dimensional affine event manifold $\mathcal{E}$. The image $\mathcal{T}_{\mathcal{E}} := \mathbf{i}_{\mathcal{E},\mathcal{T}}(\mathcal{T}) \subset \mathcal{E}$ of the trajectory by the embedding[7]

$$\mathbf{i}_{\mathcal{E},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{E})\,, \tag{39}$$

is a submanifold of the event manifold.[8]

The event manifold is fibrated by two complementary projections $\boldsymbol{\pi}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{S})$ and $t_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{Z})$, the latter being independent of the observer, since in classical mechanics, *time is absolute*.

The fibers $\mathcal{E}(t) \equiv \mathcal{S} \times \{\,t\,\}$ of *simultaneous* events, with $t \in I$, are called *spatial slices* and are isomorphic to the ambient space $\mathcal{S}$.

The fibers $\mathcal{E}(\mathbf{x}) \equiv \{\,\mathbf{x}\,\} \times \mathcal{Z}$ which are made of *isotopic* events, that is, events with the same spatial location $\mathbf{x} \in \mathcal{S}$, are isomorphic to time line $\mathcal{Z}$.

The theory illustrated hereafter is constructed by a procedure whose starting basis is the information directly available from physical experience. The mathematical treatment is naturally led to resorts to notions and tools provided by elementary differential geometry.

The starting point is the consideration that the mechanical behavior of a body is investigated by the experimental observation of its motion along the space-time trajectory and is formulated as an abstract theory by the detection of the essential aspects involved in the phenomena.

This is a distinctive feature of the theory with respect to common treatments in the literature where the notion of a body manifold is assumed *a priori*, see Remark 2, and plays an essential role in the formulation of constitutive properties.

The main issues to be concerned with, according to the new geometric formulation, are the ones enunciated hereafter and sketched in Fig. 1.

The motion is a one-parameter family of automorphisms[9] of the trajectory time-bundle $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ over the time shift $\mathrm{SH}_\alpha \in \mathrm{C}^1(\mathcal{Z}\,;\mathcal{Z})$ defined by $\mathrm{SH}_\alpha(t) := t + \alpha$ with $t, \alpha \in \mathcal{Z}$, as described by the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[d, "t_{\mathcal{T}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{SH}_\alpha"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad t_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha = \mathrm{SH}_\alpha \circ t_{\mathcal{T}}\,.$$

The diagram provides a formal expression of the property of conservation of simultaneity; that is, the motion maps simultaneous events into time-shifted simultaneous events. A map $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ of the family is the displacement of the body along the motion during the time lapse α.

[7] *Embedding*: injective immersion with co-restriction continuous with inverse.
[8] The immersed trajectory is of dimension less than or equal to the dimension of the event manifold.
[9] *Automorphism*: invertible morphism from a fiber-bundle onto itself.

Events related by the space-time motion along the trajectory, i.e.,

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E} \mid \exists\, \alpha \in \mathcal{R} \,:\, \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha(\mathbf{e}_1),$$

form a class of equivalence, and the equivalence relation foliates the trajectory manifold. The ensuing natural physical objects are depicted in Fig. 1.

A material particle is a line (a one-dimensional manifold) whose elements are motion-related events in the trajectory.

The body is the union of the disjoint material particles, i.e., the quotient manifold induced by the foliation of the trajectory manifold.

A placement is a spatial slice of the trajectory manifold.

Space and time projections of the trajectory manifold are defined by

$$\boldsymbol{\pi}_{\mathcal{T}} := \boldsymbol{\pi}_{\mathcal{S}} \circ \mathbf{i}_{\mathcal{E},\mathcal{T}} \,, \quad t_{\mathcal{T}} := t_{\mathcal{E}} \circ \mathbf{i}_{\mathcal{E},\mathcal{T}} \,.$$

The time projection $t_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{Z})$ induces a time fibration of the trajectory manifold $\mathcal{T}$ over the base $\mathcal{Z}$. The body *placement* $\boldsymbol{\Omega}_t$ at time $t \in I$, with $\dim \boldsymbol{\Omega}_t = n$ is the spatial projection of the fiber of simultaneous events in the trajectory, a compact connected submanifold defined by

$$\mathcal{T}(t) = \{\, \mathbf{e} \in \mathcal{T} \mid t_{\mathcal{T}}(\mathbf{e}) = t \,\}.$$

The time fibration makes the trajectory manifold $\mathcal{T}$ a fiber-bundle.

The space projection $\boldsymbol{\pi}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{S})$ induces a fibration of the trajectory manifold. The fiber based at $\mathbf{x} \in \mathcal{S}$ is a set of isotopic events in the trajectory

$$\mathcal{T}(\mathbf{x}) = \{\, \mathbf{e} \in \mathcal{T} \mid \boldsymbol{\pi}_{\mathcal{T}}(\mathbf{e}) = \mathbf{x} \,\},$$

which in general is not a manifold. Indeed, a set of isotopic events in the trajectory is a collection of trajectory events all having the same spatial projection $\mathbf{x} \in \mathcal{S}$, and this collection can well be made of isolated points or even by a singleton. It follows that the space fibration fails to make the trajectory manifold $\mathcal{T}$ a fiber-bundle, unless some quite special assumption concerning the trajectory can be made.[10]

The *trajectory space-time velocity* $\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,; T\mathcal{T})$ has a unit time projection since

$$\partial_{\alpha=0}\,(t_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha) = T t_{\mathcal{T}} \cdot \mathbf{v}_{\mathcal{T}} = \partial_{\alpha=0}\, \mathrm{SH}_\alpha \circ t_{\mathcal{T}} \,, \tag{40}$$

so that $t_{\mathcal{T}} \uparrow \mathbf{v}_{\mathcal{T}} = \partial_{\alpha=0}\, \mathrm{SH}_\alpha = 1$. The space-time immersion of the motion is $\boldsymbol{\varphi}^{\mathcal{E}}_\alpha = \mathbf{i}_{\mathcal{E},\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathcal{T}_{\mathcal{E}})$ with velocity $\mathbf{v}_{\mathcal{E}} = \mathbf{i}_{\mathcal{E},\mathcal{T}} \uparrow \mathbf{v}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; T\mathcal{T}_{\mathcal{E}})$.

Although space-time motion is the natural notion, the time shift $\mathrm{SH}_\alpha \in \mathrm{C}^1(\mathcal{Z}\,;\mathcal{Z})$ can be eliminated because a complete kinematical information is provided by the *spatial motion* $\boldsymbol{\varphi}^{\mathcal{S}}_\alpha$, defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathcal{E}} & \xrightarrow{\boldsymbol{\varphi}^{\mathcal{E}}_\alpha} & \mathcal{T}_{\mathcal{E}} \\ {\scriptstyle \boldsymbol{\pi}_{\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_{\mathcal{T}}} \\ \mathcal{S} & \xrightarrow{\boldsymbol{\varphi}^{\mathcal{S}}_\alpha} & \mathcal{S} \end{array} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{\mathcal{T}} \circ \boldsymbol{\varphi}^{\mathcal{E}}_\alpha = \boldsymbol{\varphi}^{\mathcal{S}}_\alpha \circ \boldsymbol{\pi}_{\mathcal{T}} \,.$$

Given a manifold $\boldsymbol{\Omega}$ diffeomorphic to a body placement, with tangent bundle $\boldsymbol{\tau}_{\boldsymbol{\Omega}} : T\boldsymbol{\Omega} \mapsto \boldsymbol{\Omega}$, the spatial motion is conveniently described by a one-parameter family of injective immersions of the manifold $\boldsymbol{\Omega}$ into the ambient space $\mathcal{S}$

$$\boldsymbol{\varphi}^{\mathcal{S}}_{\boldsymbol{\Omega}}(\alpha) : \boldsymbol{\Omega} \mapsto \mathcal{S} \,, \tag{41}$$

with $\boldsymbol{\varphi}^{\mathcal{S}}_{\boldsymbol{\Omega}}(0) = \boldsymbol{\Omega}$. The tangent map $T\boldsymbol{\varphi}^{\mathcal{S}}_{\boldsymbol{\Omega}}(\alpha) : T\boldsymbol{\Omega} \mapsto T\mathcal{S}$ has a matrix representation with respect to a referential frame $\{\, \mathbf{e}_A \,\}$, $A = 1, \ldots, n$, in $T\boldsymbol{\Omega}$ and a spatial frame $\{\, \mathbf{a}_i \,\}$, $i = 1, \ldots, m$, in $T\mathcal{S}$, denoted $F_A^{\cdot i}$ and given by

$$F_A^{\cdot i}\, \mathbf{a}_i = T\boldsymbol{\varphi}^{\mathcal{S}}_{\boldsymbol{\Omega}}(\alpha) \cdot \mathbf{e}_A \,, \tag{42}$$

with $\boldsymbol{\pi}_{\mathcal{S}}(\mathbf{a}_i) = (\boldsymbol{\varphi}^{\mathcal{S}}_\alpha \circ \boldsymbol{\tau}_{\boldsymbol{\Omega}})(\mathbf{e}_A)$.

[10] Fiber regularity in space fibration is an exception in mechanics and is not a reasonable request for lower dimensional bodies. This fact has important consequences, for instance, EULER formulae for material time derivatives in fluid-dynamics hold only under a special assumption concerning the trajectory, see Remark 3.

Let $\{\mathbf{e}_A\}$ and $\{\mathbf{a}_i\}$ be natural frames generated by the velocities of coordinate maps $\boldsymbol{\xi}_{\boldsymbol{\Omega}} : \mathcal{R}^n \mapsto \boldsymbol{\Omega}$ and $\boldsymbol{\xi}_{\mathcal{S}}(\alpha) : \mathcal{R}^m \mapsto \mathcal{S}$ so that

$$\mathbf{e}_A = T\boldsymbol{\xi}_{\boldsymbol{\Omega}} \cdot 1_A \,, \quad \mathbf{a}_i = T\boldsymbol{\xi}_{\mathcal{S}} \cdot 1_i \,. \tag{43}$$

The coordinate motion $\boldsymbol{\chi}_\alpha : \mathcal{R}^n \mapsto \mathcal{R}^m$ is defined by the commutative diagram

$$\begin{tikzcd} \boldsymbol{\Omega} \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"] & \mathcal{S} \\ \mathcal{R}^n \arrow[u, "\boldsymbol{\xi}_{\boldsymbol{\Omega}}"] \arrow[r, "\boldsymbol{\chi}_\alpha"] & \mathcal{R}^m \arrow[u, "\boldsymbol{\xi}_{\mathcal{S}}"] \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\chi}_\alpha := \boldsymbol{\xi}_{\mathcal{S}}^{-1} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{S}} \circ \boldsymbol{\xi}_{\boldsymbol{\Omega}} \,,$$

and the tangent map $T\boldsymbol{\varphi}_{\boldsymbol{\Omega}}^{\mathcal{S}}(\alpha) : T\boldsymbol{\Omega} \mapsto T\mathcal{S}$ is expressed by

$$T\boldsymbol{\varphi}_{\boldsymbol{\Omega}}^{\mathcal{S}}(\alpha) = T\boldsymbol{\xi}_{\mathcal{S}}(\alpha) \cdot T\boldsymbol{\chi}_\alpha \cdot (T\boldsymbol{\xi}_{\boldsymbol{\Omega}})^{-1}. \tag{44}$$

The rank n JACOBI matrix $T\boldsymbol{\chi}_\alpha : \mathcal{R}^n \mapsto \mathcal{R}^m$ of the coordinate motion is not square unless 3D bodies are considered. The tools provided by a univocal polar decomposition are then not available in a general treatment intended to be applicable to bodies of any dimensionality.

4 Time and space verticality

Time-vertical tangent vectors $\mathbf{u} \in V_{\mathbf{e}}\mathcal{E} \subset T_{\mathbf{e}}\mathcal{E}$ are tangent to the fiber $\mathcal{E}(t)$ based at $t = t_{\mathcal{E}}(\mathbf{e})$, that is, $\mathbf{u} \in T_{\mathbf{e}}\mathcal{E}(t)$.

The characteristic property is vanishing of time projection $T_{\mathbf{e}}t_{\mathcal{E}} \cdot \mathbf{u} = 0$. Time-vertical vectors are the elements of a sub-bundle $V\mathcal{E} \subset T\mathcal{E}$ of the tangent fibration $Tt_{\mathcal{E}} \in \mathrm{C}^0(T\mathcal{E}\,; T\mathcal{Z})$.

The restriction of the time-vertical bundle $V\mathcal{E}$ to the immersed trajectory $\mathcal{T}_{\mathcal{E}}$ is called the *spatial bundle* and is denoted by $(V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}$.

There are natural diffeomorphisms $(\mathbf{x}\,, t) \in \mathcal{E}(t) \mapsto \mathbf{x} \in \mathcal{S}$ and $(\mathbf{x}\,, t) \in \mathcal{E}(\mathbf{x}) \mapsto t \in \mathcal{Z}$, respectively, between a fiber $\mathcal{E}(t)$ of simultaneous events and the space manifold $\mathcal{S}$ and between a fiber $\mathcal{E}(\mathbf{x})$ of isotopic events and the time line $\mathcal{Z}$.

Applying the tangent functor, isomorphisms between the tangent bundles $T\mathcal{E}(t)$ and $T\mathcal{S}$ and between the tangent bundles $T\mathcal{E}(\mathbf{x})$ and $T\mathcal{Z}$ are induced.

In the tangent bundle to the trajectory, with the fibration induced by the projection $t_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{Z})$, the *material bundle* is made of time-vertical tangent vectors $T_{\mathbf{e}}t_{\mathcal{T}} \cdot \mathbf{u}_{\mathcal{T}} = 0$, i.e., $\mathbf{u}_{\mathcal{T}} \in T_{\mathbf{e}}\mathcal{T}(t)$.

5 Spatial and material fields

It is definitely important to carefully distinguish three kinds of vector fields and related tensor fields in CK, which have peculiar physical meanings.

Space-time vector fields $\mathbf{u} \in \mathrm{C}^1(\mathcal{E}\,; T\mathcal{E})$ are tangent vector fields to the event manifold.

Spatial vector fields $\mathbf{u}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; V\mathcal{E})$ are defined on the immersed trajectory and are time-vertical tangent to the event manifold, characterized by $dt_{\mathcal{E}} \cdot \mathbf{u}_{\mathcal{E}} = 0$.

Material vector fields $\mathbf{u}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; V\mathcal{T})$ are time-vertical tangent to the trajectory manifold, characterized by $dt_{\mathcal{T}} \cdot \mathbf{u}_{\mathcal{T}} = 0$.

Since the immersion preserves simultaneity, so that $V\mathcal{T}_{\mathcal{E}} = V(\mathbf{i}_{\mathcal{E},\mathcal{T}}\!\uparrow\!\mathcal{T}) = \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\uparrow\!(V\mathcal{T})$, sections $\mathbf{s} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{TENS}(V\mathcal{T}_{\mathcal{E}}))$ of the immersed material bundle $\boldsymbol{\pi}_{\mathcal{T}_{\mathcal{E}}} \in \mathrm{C}^1(V\mathcal{T}_{\mathcal{E}}\,; \mathcal{T}_{\mathcal{E}})$ will still be called *material* tensor fields.

Remark 2 Up to now, the treatment of nonlinear continuum mechanics (NLCM) developed in [24,25] has been taken as a standard reference in the literature. There the notions of material and spatial descriptions are introduced to refer, respectively, to tensor fields over the body manifold and over the current body placement [26,27]. Accordingly, due to the diffeomorphic correspondence between these two manifolds, no basic physical distinction is made. The physical motivation for the new geometric classification is instead basic and clear.

Material vector fields are tangent to a current body placement, while spatial vector fields are tangent to the space manifold. In general, no natural diffeomorphic correspondence exists between the material and the spatial bundle, and only material tensor fields can be involved in constitutive relations. This distinction is decisive in resolving debated issues and in developing a well-posed constitutive theory [12–14].

6 Space-time, spatial and material metric

The *space-time metric* $\mathbf{g}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{E}\,;\mathrm{Pos}(T\mathcal{E}))$ is defined by the pullback

$$\mathbf{g}_{\mathcal{E}} := \boldsymbol{\pi}_{\mathcal{S}}\!\downarrow\!\mathbf{g}_{\mathcal{S}} + t_{\mathcal{E}}\!\downarrow\!\mathbf{g}_{\mathcal{Z}}\,,$$

or explicitly $\mathbf{g}_{\mathcal{E}}(\mathbf{u}\,,\mathbf{w}) := \mathbf{g}_{\mathcal{S}}(T\boldsymbol{\pi}_{\mathcal{S}}\cdot\mathbf{u}\,, T\boldsymbol{\pi}_{\mathcal{S}}\cdot\mathbf{w}) + \mathbf{g}_{\mathcal{Z}}(Tt_{\mathcal{E}}\cdot\mathbf{u}\,, Tt_{\mathcal{E}}\cdot\mathbf{w})$ for all vector fields $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathcal{E}\,;T\mathcal{E})$ tangent to the event manifold.

The *spatial metric* field $\mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{Pos}(V\mathcal{E}))$ is the restriction of the space-time metric to spatial vector fields. The time line metric plays then no role, so that

$$\mathbf{g} := \boldsymbol{\pi}_{\mathcal{S}}\!\downarrow\!\mathbf{g}_{\mathcal{S}}. \tag{45}$$

The *material metric* $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{Pos}(V\mathcal{T}))$ is the pullback of the spatial metric to the material bundle

$$\mathbf{g}_{\mathcal{T}} := \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\downarrow\!\mathbf{g}. \tag{46}$$

The dual $\mathbf{L}^* : (V\mathcal{E})^*_{\mathcal{T}_{\mathcal{E}}} \mapsto (V\mathcal{E})^*_{\mathcal{T}_{\mathcal{E}}}$ and the $\mathbf{g}$-adjoint $\mathbf{L}^A : (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}} \mapsto (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}$ of a mixed spatial tensor $\mathbf{L} : (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}} \mapsto (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}$ are defined by

$$\langle \mathbf{L}^*\mathbf{w}^*, \mathbf{u}\rangle = \langle \mathbf{w}^*, \mathbf{L}\mathbf{u}\rangle\,, \tag{47}$$

$$\mathbf{g}(\mathbf{L}^A\mathbf{w}\,,\mathbf{u}) = \mathbf{g}(\mathbf{L}\mathbf{u}\,,\mathbf{w})\,, \tag{48}$$

for all $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;(V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}})$ and $\mathbf{w}^* \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;(V\mathcal{E})^*_{\mathcal{T}_{\mathcal{E}}})$. Dual and adjoint linear operators are related by

$$\mathbf{L}^* \cdot \mathbf{g} = \mathbf{g} \cdot \mathbf{L}^A. \tag{49}$$

Hence, the $\mathbf{g}$-symmetric part of $\mathbf{L}$ is defined by

$$\mathrm{sym}_{\mathbf{g}}\mathbf{L} := \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^A) \quad\Longleftrightarrow\quad \mathrm{sym}\,(\mathbf{g}\cdot\mathbf{L}) = \mathbf{g}\cdot\mathrm{sym}_{\mathbf{g}}\mathbf{L}. \tag{50}$$

7 Spatial and material projections

The *spatial projection* $\mathbf{P}_{\mathcal{S}} \in \mathrm{C}^1(T\mathcal{E}\,;V\mathcal{E})$ is a homomorphism from the tangent space-time bundle $T\mathcal{E}$ onto the time-vertical bundle $V\mathcal{E}$, characterized by the fiberwise $\mathbf{g}_{\mathcal{E}}$-orthogonality property

$$\mathbf{g}_{\mathcal{E}}(\mathbf{u} - \mathbf{P}_{\mathcal{S}}\cdot\mathbf{u}\,,\mathbf{w}) = 0, \quad \mathbf{u} \in T\mathcal{E}\,, \quad \forall\,\mathbf{u} \in V\mathcal{E}.$$

The spatial projection is $\mathbf{g}_{\mathcal{E}}$-symmetric and idempotent $\mathbf{P}_{\mathcal{S}} \circ \mathbf{P}_{\mathcal{S}} = \mathbf{P}_{\mathcal{S}}$ by

$$\mathbf{g}_{\mathcal{E}}(\mathbf{P}_{\mathcal{S}}\cdot\mathbf{u}\,,\mathbf{w}) = \mathbf{g}_{\mathcal{E}}(\mathbf{P}_{\mathcal{S}}\cdot\mathbf{w}\,,\mathbf{u}) = \mathbf{g}_{\mathcal{E}}(\mathbf{P}_{\mathcal{S}}\cdot\mathbf{u}\,,\mathbf{P}_{\mathcal{S}}\cdot\mathbf{w})\,, \quad \forall\,\mathbf{u},\mathbf{w} \in T\mathcal{E}.$$

The complementary *temporal projector* $\mathbf{P}_{\mathcal{Z}} := \mathbf{I} - \mathbf{P}_{\mathcal{S}} \in \mathrm{C}^1(T\mathcal{E}\,;H\mathcal{E})$ is the fiberwise $\mathbf{g}_{\mathcal{E}}$-orthogonal projector on the space-vertical bundle.

The *material projection* $\boldsymbol{\Pi} \in \mathrm{C}^1((V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}\,;V\mathcal{T})$, from the spatial bundle onto the material bundle, is defined by the variational condition

$$\mathbf{g}_{\mathcal{T}}(\boldsymbol{\Pi}\cdot\mathbf{u}\,,\mathbf{w}) = \mathbf{g}(\mathbf{u}\,,\mathbf{i}_{\mathcal{E},\mathcal{T}}\!\uparrow\!\mathbf{w})\,, \quad \mathbf{u} \in (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}\,, \quad \forall\,\mathbf{w} \in V\mathcal{T}. \tag{51}$$

This means that

$$\boldsymbol{\Pi} \in \mathrm{C}^1((V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}\,;V\mathcal{T})\,, \quad \boldsymbol{\Pi}^A = \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\uparrow\, \in \mathrm{C}^1(V\mathcal{T}\,;(V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}})$$

are $(\mathbf{g}_{\mathcal{T}}, \mathbf{g})$-adjoint homomorphisms. The image of $\boldsymbol{\Pi}^A$ is the immersed material bundle $V\mathcal{T}_{\mathcal{E}} = V\mathcal{E} \cap T\mathcal{T}_{\mathcal{E}}$.

The composition $\boldsymbol{\Pi} \cdot \boldsymbol{\Pi}^A : V\mathcal{T} \mapsto V\mathcal{T}$ is the identity on $V\mathcal{T}$.

The converse composition $\mathbf{P} := \boldsymbol{\Pi}^A \cdot \boldsymbol{\Pi} \in C^1((V\mathcal{E})_{\mathcal{T}_\mathcal{E}} ; V\mathcal{T}_\mathcal{E})$ is the $\mathbf{g}$-orthogonal projector of the spatial bundle on the immersed material bundle, characterized by the variational condition

$$\mathbf{g}(\mathbf{u} - \mathbf{P} \cdot \mathbf{u}, \mathbf{w}) = 0, \quad \mathbf{u} \in (V\mathcal{E})_{\mathcal{T}_\mathcal{E}}, \quad \forall\, \mathbf{w} \in V\mathcal{T}_\mathcal{E}, \tag{52}$$

which is equivalent to (51) by injectivity of $\boldsymbol{\Pi}^A$.

For a mixed spatial tensor field $\mathbf{L} \in C^1(\mathcal{T}_\mathcal{E} ; \mathrm{MIX}(V\mathcal{E}))$ and a pair of material vectors $\mathbf{u}_\mathcal{T}, \mathbf{w}_\mathcal{T} \in V\mathcal{T}$, we have the equality

$$\begin{aligned} \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\downarrow\!(\mathbf{g} \cdot \mathbf{L})(\mathbf{u}_\mathcal{T}, \mathbf{w}_\mathcal{T}) &= \mathbf{g}((\mathbf{L} \cdot \boldsymbol{\Pi}^A \cdot \mathbf{u}_\mathcal{T}, \boldsymbol{\Pi}^A \cdot \mathbf{w}_\mathcal{T}) \\ &= \mathbf{g}_\mathcal{T}((\boldsymbol{\Pi} \cdot \mathbf{L} \cdot \boldsymbol{\Pi}^A) \cdot \mathbf{u}_\mathcal{T}, \mathbf{w}_\mathcal{T}), \end{aligned} \tag{53}$$

that is,

$$\mathbf{i}_{\mathcal{E},\mathcal{T}}\!\downarrow\!(\mathbf{g} \cdot \mathbf{L}) = \mathbf{g}_\mathcal{T} \cdot \mathbf{L}. \tag{54}$$

The mixed material tensor field,

$$\mathbf{L}_\mathcal{T} := \boldsymbol{\Pi} \cdot \mathbf{L} \cdot \boldsymbol{\Pi}^A \in C^1(\mathcal{T} ; \mathrm{MIX}(V\mathcal{T})), \tag{55}$$

is material pullback of the spatial tensor field $\mathbf{L} \in C^1(\mathcal{T}_\mathcal{E} ; \mathrm{MIX}(V\mathcal{E}))$. The immersion of (55) is $\mathbf{P} \cdot \mathbf{L} \in C^1(\mathcal{T}_\mathcal{E} ; \mathrm{MIX}(V\mathcal{T}_\mathcal{E}))$. These formulae will be resorted to in Sect. 12.

8 Space-time connection

By assigning a connection $\nabla^\mathcal{S}$ in the space manifold $\mathcal{S}$ and a connection $\nabla^\mathcal{Z}$ in the time line $\mathcal{Z}$, an observer-dependent connection ∇ in the space-time manifold $\mathcal{E}$ is induced by defining the transports of spatial and temporal projections of space-time vectors.

Definition 1 (*Space-time parallel transport*) The parallel transport of a tangent vector $\mathbf{u}(\mathbf{e}) \in T_\mathbf{e}\mathcal{E}$, along a space-time curve $\mathbf{c} \in C^1(\mathcal{R} ; \mathcal{E})$ through $\mathbf{e} = \mathbf{c}(0)$, is performed componentwise by considering the spatial (time) projection $\mathbf{P}_\mathcal{S}$ ($\mathbf{P}_\mathcal{Z}$), in transporting the corresponding vector in the ambient space (in the time line) along the projected curve and transforming back the result.

The parallel transport in space-time of a time-vertical vector is depicted in Fig. 2. The parallel derivative, along the tangent vector $\mathbf{v}_\mathcal{E}(\mathbf{e}) = \partial_{\lambda=0}\, \mathbf{c}(\lambda) \in T_\mathbf{e}\mathcal{E}$ based at $\mathbf{e} = \mathbf{c}(0)$, of a tangent vector field $\mathbf{u} \in C^1(\mathcal{E} ; T\mathcal{E})$, defined on a segment of the space-time curve $\mathbf{c} \in C^1(\mathcal{R} ; \mathcal{E})$ around $\mathbf{e} \in \mathcal{E}$, is then

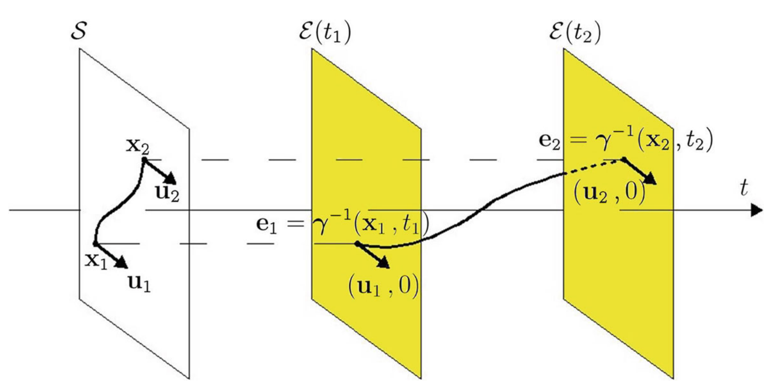

Fig. 2 Parallel transport in the spatial bundle

$$\nabla_{\mathbf{v}(\mathbf{e})}\mathbf{u} := \partial_{\lambda=0}\,(\mathbf{c}(\lambda)\Downarrow \mathbf{u}) = \partial_{\lambda=0}\,(\mathbf{c}(\lambda)\Downarrow \circ\, \mathbf{u} \circ \mathbf{c})(\lambda)\,,$$

where $\mathbf{c}(\lambda)\Downarrow$ denotes the parallel transport from $\mathbf{c}(\lambda)$ to $\mathbf{c}(0)$ along the curve $\mathbf{c} \in C^1(\mathcal{R}\,;\mathcal{E})$.

The space-time parallel derivative $\nabla_{\mathbf{h}(\mathbf{e})}\mathbf{u}$ of a space-time vector field $\mathbf{u} \in C^1(\mathcal{E}\,;T\mathcal{E})$ is time-vertical if either the time projection $t_{\mathcal{E}}\uparrow\mathbf{u} \in C^1(\mathcal{E}\,;T\mathcal{Z})$ is constant along the curve $\mathbf{c} \in C^1(\mathcal{R}\,;\mathcal{E})$ or the vector $\mathbf{h}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$ is time-vertical, i.e., $T_{\mathbf{e}}t_{\mathcal{E}} \cdot \mathbf{h}(\mathbf{e}) = 0$.

The spatial and the temporal projections of the immersed trajectory velocity are denoted by

$$\mathbf{v}_{\mathcal{S}} = \mathbf{P}_{\mathcal{S}} \cdot \mathbf{v}_{\mathcal{E}}\,, \quad \mathbf{v}_{\mathcal{Z}} = \mathbf{P}_{\mathcal{Z}} \cdot \mathbf{v}_{\mathcal{E}}.$$

It will be tacitly assumed that parallel derivatives of spatial or material fields are performed along vectors tangent to the trajectory manifold. In the following, we will often set $\mathbf{v} = \mathbf{v}_{\mathcal{S}}$ to simplify the notation.

Being $t_{\mathcal{E}}\uparrow\mathbf{v}_{\mathcal{E}} = 1$ and hence $\nabla\mathbf{v}_{\mathcal{Z}} = 0$, the parallel derivative of the immersed trajectory velocity $\mathbf{v}_{\mathcal{E}} = \mathbf{v}_{\mathcal{S}} + \mathbf{v}_{\mathcal{Z}}$ is given by

$$\nabla\mathbf{v}_{\mathcal{E}} = \nabla\mathbf{v} \in C^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{MIX}(V\mathcal{E})). \tag{56}$$

Remark 3 (Maximal dimensionality) The issue of induced connections in a product manifold is considered in the literature with reference to the simplest case in which the vector field $\mathbf{u}_{\mathcal{E}} \in C^1(\mathcal{E}\,;T\mathcal{E})$ to be differentiated is defined on the whole manifold $\mathcal{E}$ or at least on an open subset of it. For spatial fields, which are defined on the immersed trajectory, this circumstance is verified only if the continuous body manifold has the same dimensionality of the ambient space $\mathcal{S}$, so that the immersed trajectory $\mathcal{T}_{\mathcal{E}}$ is an open subset of the space-time manifold $\mathcal{E}$. If the *maximal dimensionality* (MDP) holds true, the parallel derivative can be performed along any vector tangent to space-time at a given event. A paradigmatic example is the acceleration field $\mathbf{a} \in C^1(\mathcal{T}_{\mathcal{E}}\,;V\mathcal{E})$ which is the spatial vector field to be defined in general by

$$\mathbf{a} := \nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{v}_{\mathcal{E}} = \nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{v}_{\mathcal{S}} := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_{\alpha}\Downarrow \mathbf{v}). \tag{57}$$

Only under validity of MDP, the acceleration $\mathbf{a} \in C^1(\mathcal{T}_{\mathcal{E}}\,;V\mathcal{E})$ can be evaluated by the EULER split formula

$$\mathbf{a} = \nabla_{\mathbf{v}_{\mathcal{Z}}}\mathbf{v} + \nabla_{\mathbf{v}}\mathbf{v}. \tag{58}$$

This is, however, the definition of acceleration adopted in treatments dealing with 3D bodies [24–27].

Remark 4 Assuming that the maximal dimensionality property (MDP) holds true, the parallel derivative of a spatial field in the time direction and the parallel derivative of a temporal field in the direction of a spatial vector both vanish. An analogous result holds true for the LIE bracket [28]. Then, torsion and curvature of the induced connection in the space-time manifold have spatial (temporal) projections equal to the torsion and curvature evaluated on the corresponding projected arguments. Moreover, the induced space-time connection is LEVI- CIVITA for the induced metric if and only if the spatial and the temporal connections are LEVI- CIVITA.

9 Time independence, time rates and time-invariance

Time independence is a property pertaining to the space-time metric field. Time rates of *material vector fields* and of *spatial vector fields*, and of tensor fields constructed upon them, are evaluated according to the following transformation rule [12,14].

The geometric paradigm states that, in comparing the values taken along a particle in the trajectory, the tensor fields should be acted upon in two distinct ways. Material fields may be acted upon only by the push along the motion, while spatial fields may be acted upon only by a parallel transport along the motion.

It follows that time rates of material tensors are evaluated, in a natural way, by LIE derivatives along the motion. Time rates of spatial tensors are instead evaluated by parallel derivatives along the motion with a *not* natural dependence on the choice of a connection.

Definition 2 (*Time independence in space-time*) A tangent vector field on the event manifold $\mathbf{u} \in C^1(\mathcal{E}\,;T\mathcal{E})$ is time-independent if it is spatially projectable; i.e., there exists a tangent vector field, on the ambient space manifold $\mathbf{u}_{\mathcal{S}} \in C^1(\mathcal{S}\,;T\mathcal{S})$, that completes the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\mathbf{u}} & T\mathcal{E} \\ {\scriptstyle \boldsymbol{\pi}_{\mathcal{S}}}\downarrow & & \downarrow{\scriptstyle T\boldsymbol{\pi}_{\mathcal{S}}} \\ \mathcal{S} & \xrightarrow{\mathbf{u}_{\mathcal{S}}} & T\mathcal{S} \end{array} \quad \begin{array}{l} \iff \quad \mathbf{u}_{\mathcal{S}} \circ \boldsymbol{\pi}_{\mathcal{S}} = T\boldsymbol{\pi}_{\mathcal{S}} \cdot \mathbf{u}\,, \\ \iff \quad \mathbf{u}_{\mathcal{S}} = \boldsymbol{\pi}_{\mathcal{S}}\uparrow\mathbf{u}. \end{array}$$

Space-projectability means that the value of $T\boldsymbol{\pi}_{\mathcal{S}} \cdot \mathbf{u} \in \mathrm{C}^1(\mathcal{E}\,;T\mathcal{S})$ is invariant over the fibers $\mathcal{E}(\mathbf{x})$, i.e., that for any pair of events $\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E}$, the following implication holds

$$\boldsymbol{\pi}_{\mathcal{S}}(\mathbf{e}_1) = \boldsymbol{\pi}_{\mathcal{S}}(\mathbf{e}_2) \quad \Longrightarrow \quad T_{\mathbf{e}_1}\boldsymbol{\pi}_{\mathcal{S}} \cdot \mathbf{u}_{\mathcal{E}}(\mathbf{e}_1) = T_{\mathbf{e}_2}\boldsymbol{\pi}_{\mathcal{S}} \cdot \mathbf{u}_{\mathcal{E}}(\mathbf{e}_2).$$

For covariant tensor fields over the event manifold $\mathbf{s}_{\mathrm{COV}} \in \mathrm{C}^1(\mathcal{E}\,;\mathrm{Cov}(T\mathcal{E}))$, *time-independence* means existence of a tensor field $\mathbf{s}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{Cov}(T\mathcal{S}))$ such that

$$\mathbf{s}_{\mathrm{COV}} = \boldsymbol{\pi}_{\mathcal{S}}{\downarrow}\mathbf{s}_{\mathcal{S}}. \tag{59}$$

The metric tensor field over the spatial vector bundle is time-independent since by definition $\mathbf{g} := \boldsymbol{\pi}_{\mathcal{S}}{\downarrow}\mathbf{g}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{E}\,;\mathrm{Cov}(V\mathcal{E}))$.

Definition 3 (*Time-invariance of material fields*) The time rate of a material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{Tens}(V\mathcal{T}))$ is given by the Lie derivative along the motion

$$\dot{\mathbf{s}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}} = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_{\alpha}{\downarrow}\mathbf{s}_{\mathcal{T}}). \tag{60}$$

Accordingly, time-invariance of a material tensor field along the motion is expressed by the differential condition $\dot{\mathbf{s}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}} = 0$, equivalent to the pullback property

$$\mathbf{s}_{\mathcal{T}} = \boldsymbol{\varphi}_{\alpha}{\downarrow}\mathbf{s}_{\mathcal{T}} := \boldsymbol{\varphi}_{\alpha}{\downarrow} \circ \mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_{\alpha}. \tag{61}$$

Definition 4 (*Time-invariance of spatial fields*) The time rate of a spatial tensor field $\mathbf{s}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{Tens}((V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}))$ along the motion is given by the parallel derivative

$$\dot{\mathbf{s}}_{\mathcal{E}} := \nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{s}_{\mathcal{E}} = \partial_{\alpha=0}\,(\boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}{\Downarrow}\,\mathbf{s}_{\mathcal{E}}). \tag{62}$$

Accordingly, invariance of a spatial tensor field along the motion is expressed by the differential condition $\dot{\mathbf{s}}_{\mathcal{E}} := \nabla_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{E}} = 0$, equivalent to the integral property

$$\mathbf{s}_{\mathcal{E}} = \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}{\Downarrow}\,\mathbf{s}_{\mathcal{E}} := \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}{\Downarrow} \circ \mathbf{s}_{\mathcal{E}} \circ \boldsymbol{\varphi}^{\mathcal{E}}_{\alpha}. \tag{63}$$

We underline that, resulting from the definitions above, invariance of a material field along the motion is a *natural* notion, being determined only by the motion itself. On the contrary, the notion of invariance of a spatial field along the motion depends on the choice of a linear connection in the space-time manifold and therefore is *not* natural.

10 Virtual motion

To perform variations of the trajectory manifold, an enlargement of the event manifold is considered.

The virtual event manifold is the cartesian product[11]

$$\delta\mathcal{E} := \mathcal{E} \times \Lambda\,, \quad \text{with} \quad \boldsymbol{\gamma}(\delta\mathcal{E}) = \mathcal{S} \times I \times \Lambda\,,$$

and $\Lambda = [0, 1]$ called the set of *virtual-time instants*.

Virtual events are denoted by $\delta\mathbf{e} \in \delta\mathcal{E}$. The fiber at $\lambda \in \Lambda$, according to the projection $\delta\boldsymbol{\pi} : \mathcal{E} \times \Lambda \mapsto \Lambda$, is the product $\delta\mathcal{E}(\lambda) = \mathcal{E} \times \{\lambda\}$ with the identification $\delta\mathcal{E}(0) = \mathcal{E} \times \{0\} = \mathcal{E}$.

A *virtual trajectory* $\delta\mathcal{T}_{\mathcal{E}}$ is a submanifold of the virtual event manifold $\delta\mathcal{E}$, such that $\delta\mathcal{T}_{\mathcal{E}}(0) = \mathcal{T}_{\mathcal{E}}$.

A *virtual motion* is an automorphism of the virtual trajectory over the virtual-time shift, as depicted by the commutative diagram

$$\begin{tikzcd} \delta\mathcal{T}_{\mathcal{E}} \arrow[r, "\delta\boldsymbol{\varphi}_{\lambda}"] \arrow[d, "\delta\boldsymbol{\pi}"'] & \delta\mathcal{T}_{\mathcal{E}} \arrow[d, "\delta\boldsymbol{\pi}"] \\ \Lambda \arrow[r, "\mathrm{SH}_{\lambda}"] & \Lambda \end{tikzcd} \quad \Longleftrightarrow \quad \delta\boldsymbol{\pi} \circ \delta\boldsymbol{\varphi}_{\lambda} = \mathrm{SH}_{\lambda} \circ \delta\boldsymbol{\pi}\,.$$

[11] The prefix δ is adopted to denote geometric objects related to the virtual motion. The symbol δ by itself has no meaning.

The λ-shifted images $\delta\boldsymbol{\varphi}_\lambda(\mathcal{T}_\mathcal{E})$ are all diffeomorphic to the immersed trajectory. The *velocity* of the virtual motion $\delta\mathbf{v} := \partial_{\lambda=0}\,\delta\boldsymbol{\varphi}_\lambda$ has unit virtual-time projection:

$$T\delta\boldsymbol{\pi}\cdot\delta\mathbf{v} = \partial_{\lambda=0}\,\mathrm{SH}_\lambda \circ \delta\boldsymbol{\pi} = 1 \circ \delta\boldsymbol{\pi}\,.$$

A *synchronous* virtual motion fulfills the commutative diagram

$$\begin{tikzcd} \delta\mathcal{T}_\mathcal{E} \arrow[r, "\delta\boldsymbol{\varphi}_\lambda"] \arrow[d, "t_\mathcal{E}"'] & \delta\mathcal{T}_\mathcal{E} \arrow[d, "t_\mathcal{E}"] \\ I \arrow[r, "\mathrm{ID}_I"] & I \end{tikzcd} \quad \Longleftrightarrow \quad t_\mathcal{E} \circ \delta\boldsymbol{\varphi}_\lambda = t_\mathcal{E}\,.$$

Accordingly, the tangent map $T\delta\boldsymbol{\varphi}_\lambda$ to a synchronous virtual motion will transform time-vertical tangent vectors into time-vertical tangent vectors, and the *velocity* field of a synchronous virtual motion will be time-vertical:

$$T t_\mathcal{E}\cdot\delta\mathbf{v}(\delta\mathbf{e}) := \partial_{\lambda=0}\,(t_\mathcal{E}\circ\delta\boldsymbol{\varphi}_\lambda)(\delta\mathbf{e}) = \partial_{\lambda=0}\,t_\mathcal{E}(\delta\mathbf{e}) = 0.$$

The restriction $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;(V\mathcal{E})_{\mathcal{T}_\mathcal{E}})$ to the immersed trajectory, of the velocity of a synchronous virtual motion, is a *virtual velocity* field.

11 Equilibrium and rate equilibrium

The sub-bundle of time and virtual-time vertical tangent vectors to the virtual trajectory manifold is the natural extension $V\delta\mathcal{E}$ of the sub-bundle $V\mathcal{E}$ of time-vertical tangent vectors to the event manifold, defined by

$$V\delta\mathcal{E} := \{\,\delta\mathbf{u}\in T\delta\mathcal{E}\;:\;T t_\mathcal{E}\cdot\delta\mathbf{u} = 0\,,\,T\delta\boldsymbol{\pi}\cdot\delta\mathbf{u} = 0\,\}.$$

The bundle $V\delta\mathcal{T}$ of time and virtual-time vertical vectors tangent to the virtual trajectory is defined in an analogous way. The spatial bundle connection ∇ has a natural extension to the bundle $V\delta\mathcal{E}$.

The spatial metric $\mathbf{g}\in\mathrm{C}^1(\mathcal{E}\,;\mathrm{Pos}((V\mathcal{E})_{\mathcal{T}_\mathcal{E}}))$ is extended to a metric tensor field $\mathbf{g}\in\mathrm{C}^1(\delta\mathcal{E}\,;\mathrm{Pos}(V\delta\mathcal{E}))$ in the virtual spatial bundle.

The stretching due to the *trajectory velocity* field $\mathbf{v}_\mathcal{E}\in\mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;V\mathcal{T}_\mathcal{E})$ is the covariant symmetric material tensor field $\boldsymbol{\varepsilon}(\mathbf{v})\in\mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;\mathrm{Sym}(V\mathcal{T}_\mathcal{E}))$ defined as one-half the Lie derivative of the material metric tensor field $\mathbf{g}\in\mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;\mathrm{Pos}(V\mathcal{T}_\mathcal{E}))$, according to the following formula:

$$\boldsymbol{\varepsilon}(\mathbf{v}) := \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_\mathcal{E}}\,\mathbf{g}. \tag{64}$$

Pulling back to the trajectory manifold gives

$$\boldsymbol{\varepsilon}_\mathcal{T}(\mathbf{v}) := \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\downarrow\!(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_\mathcal{E}}\,\mathbf{g}) = \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_\mathcal{T}}\,\mathbf{g}_\mathcal{T}\in\mathrm{C}^1(\mathcal{T}\,;\mathrm{Sym}(V\mathcal{T}))\,, \tag{65}$$

with the last equality following from the naturality property of the Lie derivative with respect to push by an injective morphism, see (21).

The virtual stretching due to a *virtual velocity* $\delta\mathbf{v}\in\mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;(V\mathcal{E})_{\mathcal{T}_\mathcal{E}})$ is the covariant symmetric material tensor field $\boldsymbol{\varepsilon}(\delta\mathbf{v})\in\mathrm{C}^1(\mathcal{T}_\mathcal{E}\,;\mathrm{Sym}(V\mathcal{T}_\mathcal{E}))$ defined as one-half the Lie derivative of the extended metric tensor field $\mathbf{g}\in\mathrm{C}^1(\delta\mathcal{E}\,;\mathrm{Pos}(V\delta\mathcal{E}))$, according to the following formula:

$$\boldsymbol{\varepsilon}(\delta\mathbf{v}) := \tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g} := \tfrac{1}{2}\partial_{\lambda=0}\,\delta\boldsymbol{\varphi}_\lambda\!\downarrow\!\mathbf{g}. \tag{66}$$

Pulling back to the trajectory manifold gives

$$\boldsymbol{\varepsilon}_\mathcal{T}(\delta\mathbf{v}) := \mathbf{i}_{\mathcal{E},\mathcal{T}}\!\downarrow\!(\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g})\in\mathrm{C}^1(\mathcal{T}\,;\mathrm{Sym}(V\mathcal{T})). \tag{67}$$

Stretching and virtual stretching can be expressed in terms of parallel derivatives according to the spatial connection.

A treatment in Riemann spaces endowed with an arbitrary linear connection, introduced in [30], will be fully developed in Proposition 1. In the standard Euclid context, the result yields the celebrated Euler formula [29].

The virtual power principle (VPP) is the variational formulation of equilibrium along the trajectory that amounts in the condition

$$\langle \mathbf{f}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}} = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle \, \mathbf{m} \tag{68}$$

for any virtual velocity field $\delta\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Omega}\,; V\mathcal{E})$.

Here above,

- $\mathbf{f} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; (V\mathcal{E})^*)$ is the spatial force system,
- $\mathbf{m} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{VOL}(V\mathcal{T}))$ is the material mass-form,
- $\boldsymbol{\sigma} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{CON}(V\mathcal{T}))$ is the contravariant symmetric KIRCHHOFF material stress tensor field, whose duality pairing with the stretching yields the mechanical power per unit mass on the trajectory [31–33].

The rate virtual power principle (RVPP) is stated by requiring that the equality

$$\partial_{\alpha=0} \langle \mathbf{f}, \delta\mathbf{v} \rangle_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \tag{69}$$

holds for any virtual velocity field $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; V\mathcal{E})$ generated by extending a virtual velocity field $\delta\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Omega}\,; V\mathcal{E})$ on the placement $\boldsymbol{\Omega}$ by parallel transport along the motion that is by time-invariance expressed by (63),

$$\delta\mathbf{v} = \boldsymbol{\varphi}_\alpha\Uparrow \delta\mathbf{v} := \boldsymbol{\varphi}_\alpha\Uparrow \circ \delta\mathbf{v} \circ \boldsymbol{\varphi}_{-\alpha}\,, \tag{70}$$

equivalent to the differential condition $(\nabla\delta\mathbf{v}) \cdot \mathbf{v}_{\mathcal{E}} = 0$.

Fulfillment of the RVPP is implied by fulfillment of the VPP at any time instant $t \in I$. Vice versa, fulfillment of the VPP at a placement $\boldsymbol{\Omega}$ and of the RVPP at any placement implies fulfillment of the VPP at any placement.[12]

Forcing is the time rate of variation $\dot{\mathbf{f}}(\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; (V\mathcal{E})^*)$ of the spatial one-form *force*, expressed by the parallel derivative of force, along the space-time motion:

$$\dot{\mathbf{f}}(\mathbf{v}) := \nabla_{\mathbf{v}_{\mathcal{T}}} \mathbf{f} := \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\Downarrow \mathbf{f}) = \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\Downarrow \circ \mathbf{f} \circ \boldsymbol{\varphi}_\alpha). \tag{71}$$

By time-invariance of the spatial virtual velocity field, the time rate of the external virtual power may be written as

$$\begin{aligned} \partial_{\alpha=0} \langle \mathbf{f}, \delta\mathbf{v} \rangle &= \partial_{\alpha=0} \langle \mathbf{f}, \boldsymbol{\varphi}_\alpha\Uparrow \delta\mathbf{v} \rangle = \partial_{\alpha=0} \langle \boldsymbol{\varphi}_\alpha\Downarrow \mathbf{f}, \delta\mathbf{v} \rangle \\ &= \langle \nabla_{\mathbf{v}_{\mathcal{T}}} \mathbf{f}, \delta\mathbf{v} \rangle = \langle \dot{\mathbf{f}}(\mathbf{v}), \delta\mathbf{v} \rangle. \end{aligned} \tag{72}$$

Dependence of the forcing on the spatial velocity field is described by the response to a guiding process, with feedback, which controls the variation of the force system as a function of the motion. In general, a nonlinear dependence is to be expected in engineering applications, and this leads to the formulation of nonlinear rate equilibrium problems and to the need for iterative algorithms of solution.

Stressing is the time rate of variation $\dot{\boldsymbol{\sigma}} : \mathcal{T} \mapsto \mathrm{CON}(V\mathcal{T})$ of the stress, expressed by the LIE derivative along the motion:

$$\dot{\boldsymbol{\sigma}} = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \boldsymbol{\sigma} = \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\downarrow \boldsymbol{\sigma}) = \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\downarrow \circ \boldsymbol{\sigma} \circ \boldsymbol{\varphi}_\alpha). \tag{73}$$

Massing is the time rate of variation $\dot{\mathbf{m}} : \mathcal{T} \mapsto \mathrm{VOL}(V\mathcal{T})$ of the mass, expressed by the LIE derivative of the mass-form $\mathbf{m} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{VOL}(V\mathcal{T}))$, along the motion

$$\dot{\mathbf{m}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{m} := \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\downarrow \mathbf{m}) = \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha\downarrow \circ \mathbf{m} \circ \boldsymbol{\varphi}_\alpha). \tag{74}$$

[12] The VPP is independent of the special choice of a spatial connection and is therefore a natural condition of equilibrium. Although the RVPP depends on the choice of a spatial connection to perform the extension of virtual velocities by parallel transport, equivalence to the VPP holds for any chosen connection.

Conservation of mass is the property stating time-invariance of the mass pertaining to infinitesimal material parallelepipeds convected by motion. It is expressed by the condition of vanishing massing, i.e., $\dot{\mathbf{m}} = 0$ or

$$\mathbf{m} = \boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{m} := \boldsymbol{\varphi}_\alpha{\downarrow} \circ \mathbf{m} \circ \boldsymbol{\varphi}_\alpha . \tag{75}$$

By conservation of mass, recalling the pullback formula for changes in the integration domain and applying the LEIBNIZ rule, the time rate of internal virtual power at a placement $\boldsymbol{\Omega}$ may be evaluated as

$$\begin{aligned} \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})} \langle \boldsymbol{\sigma} , \boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} &= \int_{\boldsymbol{\Omega}} \partial_{\alpha=0} \left(\langle \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\sigma} , \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} \right) \\ &= \int_{\boldsymbol{\Omega}} \langle \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\sigma} , \boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma} , \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} \\ &= \int_{\boldsymbol{\Omega}} \langle \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\boldsymbol{\sigma} , \boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma} , \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m}. \end{aligned} \tag{76}$$

The last term in (76) is independent of constitutive relations since the integrand is the interaction per unit mass between the current state of stress $\boldsymbol{\sigma}$ and the LIE *derivative of the virtual stretching* along the motion, defined by

$$\dot{\boldsymbol{\varepsilon}}_{\mathcal{T}}(\mathbf{v} , \delta\mathbf{v}) := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}}) = \mathbf{i}_{\mathcal{E},\mathcal{T}}{\downarrow}(\dot{\boldsymbol{\varepsilon}}(\mathbf{v} , \delta\mathbf{v})) , \tag{77}$$

where, by the naturality property of the LIE derivative with respect to push by injective morphisms, see (21),

$$\dot{\boldsymbol{\varepsilon}}(\mathbf{v} , \delta\mathbf{v}) := \mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}\mathbf{g}). \tag{78}$$

The RVPP may then be written as [13]

$$\langle \dot{\mathbf{f}}(\mathbf{v}), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \dot{\boldsymbol{\sigma}} , \boldsymbol{\varepsilon}_{\mathcal{T}}(\delta\mathbf{v}) \rangle\, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma} , \dot{\boldsymbol{\varepsilon}}_{\mathcal{T}}(\mathbf{v} , \delta\mathbf{v}) \rangle\, \mathbf{m}. \tag{79}$$

The expression of the material tensor fields $\boldsymbol{\varepsilon}(\mathbf{v})$, $\boldsymbol{\varepsilon}(\delta\mathbf{v})$ and $\dot{\boldsymbol{\varepsilon}}(\mathbf{v} , \delta\mathbf{v})$ in terms of parallel derivatives will be provided by Prop. 1.

12 Euler formula and rate Euler formula

The linear operator TORS is tensorial in the vector field $\mathbf{u} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}} ; V\mathcal{E})$, while the linear operator $\mathbf{Y} = \nabla + \mathrm{TORS}$ is *not* tensorial, due to the differential character of the linear *connection* ∇ introduced in Sect. 8. Time independency of $\mathbf{g} \in \mathrm{C}^1(\mathcal{E} ; \mathrm{COV}(V\mathcal{E}))$ implies that $\nabla_{\mathbf{v}_{\mathcal{Z}}}\, \mathbf{g} = 0$ and hence that

$$\nabla_{\mathbf{v}_{\mathcal{T}}}\, \mathbf{g} = \nabla_{\mathbf{v}_{\mathcal{Z}}}\, \mathbf{g} + \nabla_{\mathbf{v}_{\mathcal{S}}}\, \mathbf{g} = \nabla_{\mathbf{v}_{\mathcal{S}}}\, \mathbf{g}. \tag{80}$$

The properties enunciated in the next Lemma will be resorted to in the proof of formula (90) in Proposition 1 and in the proof of Lemma 3.

Lemma 2 *The connection and the forms of torsion and curvature fulfill the identities*

$$\nabla(\mathbf{u}) \cdot \mathbf{w} = \nabla(\mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}) \cdot \mathbf{w} , \tag{81}$$

$$\mathrm{TORS}(\mathbf{u}) \cdot \mathbf{w} = \mathrm{TORS}(\mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}) \cdot \mathbf{w} , \tag{82}$$

$$\mathrm{CURV}(\mathbf{s} , \mathbf{u}) \cdot \mathbf{w} = \mathrm{CURV}(\mathbf{s} , \mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}) \cdot \mathbf{w}, \tag{83}$$

for all spatial fields $\mathbf{s}, \mathbf{w} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}} ; V\mathcal{E})$ *and space-time field* $\mathbf{u} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}} ; T\mathcal{E})$.

[13] A superposed dot denotes time rates with different definitions depending on whether involved tensor fields are material or spatial, in accordance with the geometric paradigm enunciated in Sect. 9.

Proof. By definition, we have that

$$\mathrm{T\small{ORS}}(\mathbf{u}) \cdot \mathbf{w} = \mathrm{\small{TORS}}(\mathbf{u}\,,\mathbf{w}) = \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}\,,\mathbf{w}].$$

By tensoriality of the torsion form, explicated in footnote 5, the point values of the involved vector fields at a given event may be extended in any way to regular vector fields $\mathbf{u} \in \mathrm{C}^1(\mathcal{E}\,; T\mathcal{E})$ and $\mathbf{w} \in \mathrm{C}^1(\mathcal{E}\,; V\mathcal{E})$ defined in a neighborhood of that event. Remark 4 implies that $\nabla_{(\mathbf{P}_{\mathcal{Z}}\cdot\mathbf{u})}\mathbf{w} = 0$ and $\nabla_{\mathbf{w}}(\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u}) = 0$ so that

$$\nabla_{\mathbf{u}}(\mathbf{w}) = \nabla_{(\mathbf{P}_{\mathcal{S}}\cdot\mathbf{u})}\mathbf{w}, \qquad \nabla_{\mathbf{w}}(\mathbf{u}) = \nabla_{\mathbf{w}}(\mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}),$$

and formula (81) follows.

Moreover, being $t_{\mathcal{E}}{\uparrow}\mathbf{w} = 0$ and $\boldsymbol{\pi}_{\mathcal{S}}{\uparrow}(\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u}) = 0$, by naturality of the LIE-bracket with respect to push, we get

$$\begin{aligned} t_{\mathcal{E}}{\uparrow}[\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u}\,, \mathbf{w}] &= [t_{\mathcal{E}}{\uparrow}\mathbf{u}\,, t_{\mathcal{E}}{\uparrow}\mathbf{w}] = 0, \\ \boldsymbol{\pi}_{\mathcal{S}}{\uparrow}[\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u}\,, \mathbf{w}] &= [\boldsymbol{\pi}_{\mathcal{S}}{\uparrow}(\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u})\,, \boldsymbol{\pi}_{\mathcal{S}}{\uparrow}\mathbf{w}] = 0, \end{aligned}$$

so that $[\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{u}\,, \mathbf{w}] = 0$ and hence $[\mathbf{u}\,, \mathbf{w}] = [\mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}\,, \mathbf{w}]$. Then, formula (82) holds and formula (83) is obtained in an analogous way. □

From Lemma 2, being $\mathbf{Y} := \nabla + \mathrm{T\small{ORS}}$, it follows that

$$\mathbf{Y}(\mathbf{u}) \cdot \mathbf{w} = \mathbf{Y}(\mathbf{P}_{\mathcal{S}} \cdot \mathbf{u}) \cdot \mathbf{w}. \tag{84}$$

The material symmetric covariant tensor fields (64), (66), (77) can be transformed to $\mathbf{g}$-symmetric mixed tensor fields on the immersed trajectory by composition with the metric tensor field $\mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{P\small{OS}}(V\mathcal{T}_{\mathcal{E}}))$, according to the following formulae:

$$\begin{aligned} \boldsymbol{\varepsilon}(\mathbf{v}) &= \mathbf{g} \cdot \mathbf{D}(\mathbf{v})\,, & (85)\\ \boldsymbol{\varepsilon}(\delta\mathbf{v}) &= \mathbf{g} \cdot \mathbf{D}(\delta\mathbf{v})\,, & (86)\\ \dot{\boldsymbol{\varepsilon}}(\mathbf{v}\,, \delta\mathbf{v}) &= \mathbf{g} \cdot \dot{\mathbf{D}}(\mathbf{v}\,, \delta\mathbf{v}). & (87) \end{aligned}$$

The next result provides a generalized version of the celebrated EULER formula [29] for the stretching and the formula for the time derivative of the virtual stretching along the motion.

Proposition 1 *The* LIE *derivative* $\frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{C\small{OV}}(V\mathcal{T}_{\mathcal{E}}))$ *along the trajectory velocity field* $\mathbf{v}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; T\mathcal{T}_{\mathcal{E}})$ *is expressed in terms of the linear connection* ∇ *in the spatial bundle by the following formula:*

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} = \tfrac{1}{2}\mathbf{g} \cdot \mathbf{G}(\mathbf{v}) + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\mathbf{v})), \tag{88}$$

The LIE *derivative* $\frac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{C\small{OV}}(V\mathcal{T}_{\mathcal{E}}))$ *along a virtual velocity field* $\delta\mathbf{v} \in \mathrm{C}^1(\delta\mathcal{T}_{\mathcal{E}}\,; V\delta\mathcal{E})$ *is expressed in terms of the linear connection* ∇ *in the spatial bundle by the following formula:*

$$\boldsymbol{\varepsilon}(\delta\mathbf{v}) = \tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g} = \tfrac{1}{2}\mathbf{g} \cdot \mathbf{G}(\delta\mathbf{v}) + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\delta\mathbf{v})). \tag{89}$$

The LIE *derivative* $\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\frac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{C\small{OV}}(V\mathcal{T}_{\mathcal{E}}))$ *is the covariant material tensor field given by*

$$\dot{\boldsymbol{\varepsilon}}(\mathbf{v}\,, \delta\mathbf{v}) = \mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g}) \;= \nabla_{\mathbf{v}_{\mathcal{E}}}(\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g}) + \mathrm{sym}\,((\mathcal{L}_{\delta\mathbf{v}}\mathbf{g}) \cdot \mathbf{Y}(\mathbf{v})). \tag{90}$$

Proof. Setting $\mathbb{M} = \mathcal{T}_{\mathcal{E}}$, $\mathbf{u} = \mathbf{v}_{\mathcal{E}}$ and $\mathbf{s}_{\mathrm{COV}} = \mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{P\small{OS}}(V\mathcal{T}_{\mathcal{E}}))$, Lemma 2 yields (88) since

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} &= \tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\mathbf{v}_{\mathcal{E}})) \\ &= \tfrac{1}{2}\mathbf{g} \cdot \mathbf{G}(\mathbf{v}) + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\mathbf{v}))\,, \end{aligned} \tag{91}$$

with the mixed tensor field $\mathbf{G}(\mathbf{v}_{\mathcal{E}}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; \mathrm{M\small{IX}}(V\mathcal{T}_{\mathcal{E}}))$ defined by

$$\mathbf{g} \cdot \mathbf{G}(\mathbf{v}) := \tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} = \tfrac{1}{2}\nabla_{\mathbf{v}}\mathbf{g}. \tag{92}$$

Setting $\mathbb{M} = \delta\mathcal{E}$, $\mathbf{u} = \delta\mathbf{v}$, $\mathbf{s}_{\mathrm{COV}} = \mathbf{g}$ in (32), we get (89) since

$$\begin{aligned} \tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\mathbf{g} &= \tfrac{1}{2}\nabla_{\delta\mathbf{v}}\mathbf{g} + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\delta\mathbf{v})) \\ &= \tfrac{1}{2}\mathbf{g} \cdot \mathbf{G}(\delta\mathbf{v}) + \mathrm{sym}\,(\mathbf{g} \cdot \mathbf{Y}(\delta\mathbf{v}))\,, \end{aligned} \tag{93}$$

with the mixed tensor field $\mathbf{G}(\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{MIX}(V\mathcal{T}_{\mathcal{E}}))$ defined by

$$\mathbf{g}\cdot\mathbf{G}(\delta\mathbf{v}) := \tfrac{1}{2}\nabla_{\delta\mathbf{v}}\,\mathbf{g}. \tag{94}$$

The LIE derivative $\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\frac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g})$ is expressed in terms of parallel derivatives by setting $\mathbb{M} = \mathcal{T}_{\mathcal{E}}$, $\mathbf{u} = \mathbf{v}_{\mathcal{E}}$ and $\mathbf{s}_{\mathrm{COV}} = \frac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g}$ in (32), to get the covariant tensor

$$\mathcal{L}_{\mathbf{v}_{\mathcal{E}}}(\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g}) = \nabla_{\mathbf{v}_{\mathcal{E}}}(\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g}) + 2\,\mathrm{sym}\,((\tfrac{1}{2}\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g})\cdot\mathbf{Y}(\mathbf{v}_{\mathcal{E}})). \tag{95}$$

Hence, by Lemma 2, we get (90). □

From Lemma 1, being $\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{g} = \nabla_{\mathbf{v}}\mathbf{g}$, we infer that

$$\begin{aligned}\nabla_{\mathbf{v}_{\mathcal{E}}}(\mathbf{g}\cdot\mathbf{D}(\delta\mathbf{v})) &= \mathbf{g}\cdot(\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{D}(\delta\mathbf{v})) + (\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{g})\cdot\mathbf{D}(\delta\mathbf{v})\\ &= \mathbf{g}\cdot(\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{D}(\delta\mathbf{v})) + 2\,\mathbf{g}\cdot\mathbf{G}(\mathbf{v})\cdot\mathbf{D}(\delta\mathbf{v}).\end{aligned} \tag{96}$$

From the pairs (85)–(88), (86)–(89), (87)–(90) it follows that

$$\mathbf{D}(\mathbf{v}) = \tfrac{1}{2}\mathbf{G}(\mathbf{v}) + \mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\mathbf{v})\,, \tag{97}$$

$$\mathbf{D}(\delta\mathbf{v}) = \tfrac{1}{2}\mathbf{G}(\delta\mathbf{v}) + \mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\delta\mathbf{v})\,, \tag{98}$$

$$\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}(\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{D}(\delta\mathbf{v}) + 2\,\mathbf{G}(\mathbf{v})\cdot\mathbf{D}(\delta\mathbf{v}) + 2\,\mathbf{D}(\delta\mathbf{v})\cdot\mathbf{Y}(\mathbf{v})). \tag{99}$$

Changing the linear connection ∇ in the event manifold will change the formal expressions in terms of parallel derivatives, but the mixed tensor fields $\mathbf{D}(\mathbf{v}), \mathbf{D}(\delta\mathbf{v}), \dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{MIX}(V\mathcal{T}_{\mathcal{E}}))$ remain unaffected being natural objects defined by LIE derivatives.

Lemma 3 *Time-invariance of the virtual velocity field* $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;V\mathcal{E})$ *yields for the material second derivative* $\nabla_{\mathbf{v}_{\mathcal{E}}}(\nabla\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{MIX}(V\mathcal{T}_{\mathcal{E}}))$*; the following expression depending only on the restriction of the fields* $\mathbf{v}, \delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;V\mathcal{E})$ *on the current body placement*

$$\nabla_{\mathbf{v}_{\mathcal{E}}}(\nabla\delta\mathbf{v}) = \mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}) - \nabla\delta\mathbf{v}\cdot(\nabla\mathbf{v} + \mathrm{TORS}(\mathbf{v})). \tag{100}$$

Proof. For a pair made of the space-time velocity $\mathbf{v}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;T\mathcal{T}_{\mathcal{E}})$ and of any material vector field $\mathbf{h} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;V\mathcal{T}_{\mathcal{E}})$, a formal LEIBNIZ rule defines the second parallel derivatives

$$(\nabla\nabla\delta\mathbf{v})\cdot\mathbf{h}\cdot\mathbf{v}_{\mathcal{E}} := \nabla_{\mathbf{v}_{\mathcal{E}}}\cdot\nabla_{\mathbf{h}}\delta\mathbf{v} - (\nabla\delta\mathbf{v})\cdot\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{h}\,, \tag{101}$$

$$(\nabla\nabla\delta\mathbf{v})\cdot\mathbf{v}_{\mathcal{E}}\cdot\mathbf{h} := \nabla_{\mathbf{h}}\cdot\nabla_{\mathbf{v}_{\mathcal{E}}}\delta\mathbf{v} - (\nabla\delta\mathbf{v})\cdot\nabla_{\mathbf{h}}\mathbf{v}_{\mathcal{E}}. \tag{102}$$

The symmetry gap is expressed by the formula (31)

$$(\nabla\nabla\delta\mathbf{v})\cdot\mathbf{h}\cdot\mathbf{v}_{\mathcal{E}} - (\nabla\nabla\delta\mathbf{v})\cdot\mathbf{v}_{\mathcal{E}}\cdot\mathbf{h} = (\mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}_{\mathcal{E}}) - (\nabla\delta\mathbf{v})\cdot\mathrm{TORS}(\mathbf{v}_{\mathcal{E}}))\cdot\mathbf{h}. \tag{103}$$

By assumption (70) of time-invariance of the virtual velocity field, we have that $\nabla_{\mathbf{v}_{\mathcal{E}}}\delta\mathbf{v} = 0$. Hence, from (102), it follows that

$$(\nabla\nabla\delta\mathbf{v})\cdot\mathbf{v}_{\mathcal{E}}\cdot\mathbf{h} = -(\nabla\delta\mathbf{v})\cdot\nabla\mathbf{v}_{\mathcal{E}}\cdot\mathbf{h}. \tag{104}$$

Then, from (103) and (104), we get

$$(\nabla\nabla\delta\mathbf{v})\cdot\mathbf{h}\cdot\mathbf{v}_{\mathcal{E}} = (\mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}_{\mathcal{E}}) - (\nabla\delta\mathbf{v})\cdot(\nabla\mathbf{v}_{\mathcal{E}} + \mathrm{TORS}(\mathbf{v}_{\mathcal{E}})))\cdot\mathbf{h}. \tag{105}$$

The result then follows recalling Lemma 2. □

Lemma 4 *Adopting the spatial* LEVI-CIVITA *connection, the mixed material tensor fields* $\mathbf{D}(\mathbf{v}), \mathbf{D}(\delta\mathbf{v}), \dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,;\mathrm{MIX}(V\mathcal{T}_{\mathcal{E}}))$ *take the simple expressions*

$$\mathbf{D}(\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}\nabla\mathbf{v}\,, \tag{106}$$

$$\mathbf{D}(\delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}\nabla\delta\mathbf{v}\,, \tag{107}$$

$$\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}(\mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}) + (\nabla\delta\mathbf{v})^{A}\cdot\nabla\mathbf{v}). \tag{108}$$

Proof. Being $\mathbf{G}(\mathbf{v}) = 0$ by (92) and $\mathrm{TORS} = 0$, the Euler formulae (106) and (107) follow immediately from (97) and (98). On the other hand, (100) simplifies into

$$\nabla_{\mathbf{v}_{\mathcal{E}}}(\nabla\delta\mathbf{v}) = \mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}) - \nabla\delta\mathbf{v}\cdot\nabla\mathbf{v}. \tag{109}$$

Substituting (107) and (109) into (99) gives

$$\begin{aligned}\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) &= \mathrm{sym}_{\mathbf{g}}(\nabla_{\mathbf{v}_{\mathcal{E}}}\nabla\delta\mathbf{v} + (\nabla\delta\mathbf{v} + (\nabla\delta\mathbf{v})^A)\cdot\nabla\mathbf{v})\\ &= \mathrm{sym}_{\mathbf{g}}(\mathrm{CURV}(\delta\mathbf{v}\,,\mathbf{v}) - \nabla\delta\mathbf{v}\cdot\nabla\mathbf{v} + (\nabla\delta\mathbf{v} + (\nabla\delta\mathbf{v})^A)\cdot\nabla\mathbf{v}),\end{aligned}$$

which, simplified, yields (108). □

As proven in Remark 1, if the parallel transport corresponding to the spatial connection ∇ is path independent, then the curvature tensor CURV vanishes identically. For continuous bodies in motion in the flat EUCLID space with path-independent parallel transport by translation, we have $\mathrm{CURV} = 0$, and from Lemma 4, we get

$$\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}((\nabla\delta\mathbf{v})^A\cdot\nabla\mathbf{v}). \tag{110}$$

The material covariant field $\dot{\boldsymbol{\varepsilon}}_{\mathcal{T}}(\mathbf{v}\,,\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{SYM}(V\mathcal{T}))$ defined by (77) is associated with the material mixed field $\dot{\mathbf{D}}_{\mathcal{T}}(\mathbf{v}\,,\delta\mathbf{v}) := \mathbf{i}_{\mathcal{E},\mathcal{T}}{\downarrow}\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v}) \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{MIX}(V\mathcal{T}))$, which, by naturality of the pull back with respect to composition stated in (17), is given by

$$\mathbf{g}_{\mathcal{T}}\cdot\dot{\mathbf{D}}_{\mathcal{T}}(\mathbf{v}\,,\delta\mathbf{v}) = \dot{\boldsymbol{\varepsilon}}_{\mathcal{T}}(\mathbf{v}\,,\delta\mathbf{v}). \tag{111}$$

Hence, recalling (55), we get the $\mathbf{g}_{\mathcal{T}}$-symmetric expression

$$\dot{\mathbf{D}}_{\mathcal{T}}(\mathbf{v}\,,\delta\mathbf{v}) = \boldsymbol{\Pi}\cdot\dot{\mathbf{D}}(\mathbf{v}\,,\delta\mathbf{v})\cdot\boldsymbol{\Pi}^A\,, \tag{112}$$

which by (110) is also symmetric in the pair $(\mathbf{v}\,,\delta\mathbf{v})$. The formula (112) extends to bodies of any dimensionality a known result in the literature concerning the geometric stiffness of a 3D body, see (115) of Sect. 13.

13 Standard approach to rate formulations

In the literature, treatments of rate equilibrium are developed in the framework of three-dimensional EUCLID space and with reference to a three-dimensional continuous body [15–20].

We provide here a precise description of the path of reasoning followed in evaluating the time rate of the virtual stretching, with a specific attention to definitions of notion and symbols, so that a clearer connection with the geometric treatment developed in the previous sections is provided.

According to a common notation, the symbol δ is the derivative along the virtual motion [14] and the dot symbol $(\,)^{\cdot}$ denotes the time derivative along the motion.[15]

Denoting by $\mathbf{F}$ the tangent displacement map (the *deformation gradient* in [24,25]), explicit definitions of the involved issues are

$$\begin{aligned}&\mathbf{F}_{\tau,t} = T\boldsymbol{\varphi}_{\tau,t}\,, &&\mathbf{F}_{\lambda,t} = T\boldsymbol{\varphi}_{\lambda,t}\,,\\ &(\,)^{\cdot} = \partial_{\tau=t}\,, &&\delta = \partial_{\lambda=0}\,,\\ &\dot{\mathbf{F}}_t = \partial_{\tau=t}\,T\boldsymbol{\varphi}_{\tau,t}\,, &&\delta\mathbf{F}_t = \partial_{\lambda=0}\,T\boldsymbol{\varphi}_{\lambda,t}\,.\end{aligned}$$

Here, $\boldsymbol{\varphi}_{\tau,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_\tau)$ is the displacement along the motion and $\boldsymbol{\varphi}_{\lambda,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_{\lambda,t})$ is a virtual displacement at time $t \in I$.

Thus, $\mathbf{F}_{\tau,t} \in \mathrm{C}^1(T\boldsymbol{\Omega}_t\,;T\boldsymbol{\Omega}_\tau)$, $\mathbf{F}_{\lambda,t} \in \mathrm{C}^1(T\boldsymbol{\Omega}_t\,;T\boldsymbol{\Omega}_{\lambda,t})$ with $\mathbf{F}_{t,t}$, $\mathbf{F}_{0,t}$ identity maps. By composition, we define $\mathbf{F}_{\lambda,\tau,t} := \mathbf{F}_{\lambda,\tau}\circ\mathbf{F}_{\tau,t}$.

In precise geometric terms, the derivatives, along the motion and along the virtual motion, should be defined as

$$\dot{\mathbf{F}}_t := \partial_{\tau=t}\,(\boldsymbol{\varphi}_{\tau,t}{\Downarrow}^{\mathcal{S}}\circ T\boldsymbol{\varphi}_{\tau,t})\,, \qquad \delta\mathbf{F}_t := \partial_{\lambda=0}\,(\boldsymbol{\varphi}_{\lambda,t}{\Downarrow}^{\mathcal{S}}\circ T\boldsymbol{\varphi}_{\lambda,t})\,, \tag{113}$$

with $\boldsymbol{\varphi}_{\tau,t}{\Downarrow}^{\mathcal{S}}$, $\boldsymbol{\varphi}_{\lambda,t}{\Downarrow}^{\mathcal{S}}$ backward parallel transports by translation in space.

[14] On the contrary, according to the notation adopted in the present paper, the symbol δ by itself has no meaning, but appended before other symbols denotes virtual objects.

[15] In [16], the symbol δ denotes also the time derivative along the motion and a subscript $_{\mathrm{V}}$ is adopted to mark virtual objects.

By linearity of the adjoint involution $()^A$, it results in

$$\dot{\mathbf{F}}_t^A := (\mathbf{F}_t^A)^{\cdot} := (\dot{\mathbf{F}}_t)^A , \qquad \delta\mathbf{F}_\tau^A := (\delta\mathbf{F}_\tau)^A = \delta(\mathbf{F}_\tau^A).$$

To evaluate the time rate of the virtual stretching, we first consider the expression of the finite strain associated with the displacement $\boldsymbol{\varphi}_{\lambda,\tau} \circ \boldsymbol{\varphi}_{\tau,t}$,

$$\begin{aligned}\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\lambda,\tau,t} &:= \tfrac{1}{2}(T\boldsymbol{\varphi}_{\tau,t})^A \cdot (T\boldsymbol{\varphi}_{\lambda,\tau})^A \cdot T\boldsymbol{\varphi}_{\lambda,\tau} \cdot T\boldsymbol{\varphi}_{\tau,t}\\ &= \tfrac{1}{2}\mathbf{F}_{\tau,t}^A \cdot \mathbf{F}_{\lambda,\tau}^A \cdot \mathbf{F}_{\lambda,\tau} \cdot \mathbf{F}_{\tau,t}.\end{aligned}$$

Second, the operator δ is applied and the LEIBNIZ rule is invoked to get [16]

$$\begin{aligned}\delta\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\tau,t} &:= \partial_{\lambda=0}\,\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\lambda,\tau,t} = \partial_{\lambda=0}\,\tfrac{1}{2}(\mathbf{F}_{\tau,t}^A \cdot \mathbf{F}_{\lambda,\tau}^A \cdot \mathbf{F}_{\lambda,\tau} \cdot \mathbf{F}_{\tau,t})\\ &= \tfrac{1}{2}\mathbf{F}_{\tau,t}^A \cdot \partial_{\lambda=0}\,(\mathbf{F}_{\lambda,\tau}^A \cdot \mathbf{F}_{\lambda,\tau}) \cdot \mathbf{F}_{\tau,t}\\ &= \tfrac{1}{2}\mathbf{F}_{\tau,t}^A \cdot \delta\mathbf{F}_\tau^A \cdot \mathbf{F}_{\tau,t} + \tfrac{1}{2}\mathbf{F}_{\tau,t}^A \cdot \delta\mathbf{F}_\tau \cdot \mathbf{F}_{\tau,t}\\ &= \mathrm{sym}_{\mathbf{g}}(\mathbf{F}_{\tau,t}^A \cdot \delta\mathbf{F}_\tau \cdot \mathbf{F}_{\tau,t}).\end{aligned}$$

Third, we prove that $\delta\mathbf{F}_\tau \cdot \mathbf{F}_{\tau,t} = \delta\mathbf{F}_t$. To this end, since virtual velocities $\delta\mathbf{v}_t := \partial_{\lambda=0}\,\boldsymbol{\varphi}_{\lambda,t}$ are translated along the motion, we write

$$\begin{aligned}\delta\mathbf{v}_t = (\boldsymbol{\varphi}_{\tau,t}\Downarrow^{\mathcal{S}}\delta\mathbf{v}_\tau) &= \boldsymbol{\varphi}_{\tau,t}\Downarrow^{\mathcal{S}} \circ\, \delta\mathbf{v}_\tau \circ \boldsymbol{\varphi}_{\tau,t} = \boldsymbol{\varphi}_{\tau,t}\Downarrow^{\mathcal{S}} \circ\, (\partial_{\lambda=0}\,\boldsymbol{\varphi}_{\lambda,\tau}) \circ \boldsymbol{\varphi}_{\tau,t}\\ &= (\partial_{\lambda=0}\,\boldsymbol{\varphi}_{\lambda,\tau}) \circ \boldsymbol{\varphi}_{\tau,t}.\end{aligned}$$

Applying the tangent functor T and exchanging the derivative $\partial_{\lambda=0}$ with the tangent functor, we get the result

$$\delta\mathbf{F}_t = T\delta\mathbf{v}_t = \partial_{\lambda=0}\,T\boldsymbol{\varphi}_{\lambda,\tau} \cdot T\boldsymbol{\varphi}_{\tau,t} = \delta\mathbf{F}_\tau \cdot \mathbf{F}_{\tau,t}.$$

Then, the time derivative of $\delta\frac{1}{2}(\mathbf{F}^A\mathbf{F})_{\tau,t}$ can be evaluated as

$$\begin{aligned}\partial_{\tau=t}\,\delta\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\tau,t} &= \partial_{\tau=t}\,\partial_{\lambda=0}\,\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\lambda,\tau,t}\\ &= \partial_{\tau=t}\,\mathrm{sym}_{\mathbf{g}}(\mathbf{F}_{\tau,t}^A \cdot \delta\mathbf{F}_\tau \cdot \mathbf{F}_{\tau,t})\\ &= \partial_{\tau=t}\,\mathrm{sym}_{\mathbf{g}}(\mathbf{F}_{\tau,t}^A \cdot \delta\mathbf{F}_t) = \mathrm{sym}_{\mathbf{g}}(\dot{\mathbf{F}}_t^A \cdot \delta\mathbf{F}_t)\,,\end{aligned} \tag{114}$$

which is symmetric with respect to an exchange of $\dot{\mathbf{F}}_t$ and $\delta\mathbf{F}_t$. The notation in (114) should be compared with the one adopted in the formula (4.81) of [16].

By relying on the special properties of EUCLID connection by translation, it is allowed to exchange the derivatives $\partial_{\lambda=0}$ and $\partial_{\tau=t}$ with the tangent functor T, and to identify T with ∇. Thus, with $\mathbf{v}_t = \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,t}$ and $\delta\mathbf{v}_t = \partial_{\lambda=0}\,\boldsymbol{\varphi}_{\lambda,t}$, from (113), we get

$$\dot{\mathbf{F}}_t := \partial_{\tau=t}\,T\boldsymbol{\varphi}_{\tau,t} = \nabla\mathbf{v}_t\,, \qquad \delta\mathbf{F}_t := \partial_{\lambda=0}\,T\boldsymbol{\varphi}_{\lambda,t} = \nabla\delta\mathbf{v}_t\,,$$

and hence, (114) may be written as

$$\partial_{\tau=t}\,\delta\tfrac{1}{2}(\mathbf{F}^A\mathbf{F})_{\tau,t} = \mathrm{sym}_{\mathbf{g}}((\nabla\mathbf{v}_t)^A \cdot \nabla\delta\mathbf{v}_t). \tag{115}$$

The result in (115), which refers to the special context of 3D bodies in motion in the EUCLID space with the standard connection by translation, is the same as the one obtained in (110) by specialization of (108).

[16] In [18, 6.3], this assumption is stated as *…because the virtual velocities are not a function of the configuration* …. Dependence on the choice of a parallel transport is not evidenced, in line with the common way of doing in the literature.

14 Elasticity

The elastic stretching is defined by the new rate-elastic constitutive law contributed in [12][17]

$$\mathbf{el} := \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}. \tag{116}$$

A purely rate-elastic behavior is characterized by the equality

$$\mathbf{el} = \boldsymbol{\varepsilon}(\mathbf{v}), \tag{117}$$

where $\boldsymbol{\varepsilon}(\mathbf{v}) := \frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}})$ by (89). Substituting (116) and (117) into the RVPP, the variational principle governing rate elastostatics writes

$$\langle \dot{\mathbf{f}}(\mathbf{v}), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \mathbf{H}(\boldsymbol{\sigma})^{-1} \cdot \boldsymbol{\varepsilon}(\mathbf{v}), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}(\mathbf{v}, \delta\mathbf{v}) \rangle \, \mathbf{m}. \tag{118}$$

Let us denote by d_F the fiber-derivative, viz. the derivative performed by taking the base point of the tensor field fixed.

Existence of a stretching-valued elastic potential $\boldsymbol{\Psi}$ such that $d_F\boldsymbol{\Psi} = \mathbf{H}$ is called CAUCHY integrability and is equivalent to the former of the following symmetry conditions:

$$(d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma})^A = d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}, \tag{119}$$

$$\mathbf{H}(\boldsymbol{\sigma})^A = \mathbf{H}(\boldsymbol{\sigma}). \tag{120}$$

The adjoint $\mathbf{H}(\boldsymbol{\sigma})^A$ of the linear operator $\mathbf{H}(\boldsymbol{\sigma})$ is defined, for all $\delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma}$, by the identity (here the metric plays no role)

$$\langle \mathbf{H}(\boldsymbol{\sigma})^A \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle = \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle. \tag{121}$$

Both conditions are equivalent to GREEN integrability which assures the existence of a scalar-valued elastic stress potential Ξ such that $d_F\Xi = \boldsymbol{\Psi}$. [18]

Elasticity (or resp. hyper-elasticity) is recovered by assuming that the rate-elastic operator $\mathbf{H}$ is a time invariant and also CAUCHY (resp. GREEN) integrable, so that

$$\mathbf{el} = d_F\boldsymbol{\Psi}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, \qquad (\mathbf{el} = d_F^2\Xi(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}),$$

with time-invariance expressed by the pullback conditions

$$\boldsymbol{\Psi} = \boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\Psi}, \qquad (\Xi = \boldsymbol{\varphi}_\alpha\!\downarrow\!\Xi).$$

Conservation of mass together with time-invariance and GREEN integrability of the rate-elastic constitutive behavior, expressed in terms of the KIRCHHOFF stress tensor field, ensures conservation of mechanical energy [13,14].

15 Elastic tangent stiffness and nonlinear geometric stiffness

It terms of mixed spatial tensors over the immersed trajectory, the RVPP is expressed by the variational condition

$$\langle \dot{\mathbf{f}}(\mathbf{v}), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}_t} \langle \dot{\mathbf{K}}, \mathbf{D}_{\mathcal{T}}(\delta\mathbf{v}) \rangle_{\mathbf{g}_{\mathcal{T}}} \, \mathbf{m} + \int_{\boldsymbol{\Omega}_t} \langle \mathbf{K}, \dot{\mathbf{D}}_{\mathcal{T}}(\mathbf{v}, \delta\mathbf{v}) \rangle_{\mathbf{g}_{\mathcal{T}}} \, \mathbf{m}. \tag{122}$$

Here, $\mathbf{K} = \boldsymbol{\sigma} \circ \mathbf{g}_{\mathcal{T}}$ is the KIRCHHOFF mixed stress tensor and $\dot{\mathbf{K}} := \dot{\boldsymbol{\sigma}} \circ \mathbf{g}_{\mathcal{T}}$ is the mixed KIRCHHOFF stressing. We underline that $\dot{\mathbf{K}}$ is *not* the LIE derivative of $\mathbf{K}$. The bracket $\langle \bullet, \bullet \rangle_{\mathbf{g}_{\mathcal{T}}}$ is the inner product between mixed tensors induced by the material metric.

[17] Hypo-elasticity was introduced by TRUESDELL in [34] with a different definition.

[18] Integrability was analyzed in [35] with improper arguments. The results in (119), (120), (121) are taken from [12,14].

The rate-elastic law (116) written in the form

$$\mathbf{D}_{\mathcal{T}}(\mathbf{v}) = \mathbf{H}(\mathbf{K}) \cdot \dot{\mathbf{K}} \tag{123}$$

may then be substituted into the RVPP variational condition (122), to get the expression of the rate elastostatic variational principle:

$$\langle \dot{\mathbf{f}}(\mathbf{v}), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \mathbf{H}(\mathbf{K})^{-1} \cdot \mathbf{D}_{\mathcal{T}}(\mathbf{v}), \mathbf{D}_{\mathcal{T}}(\delta\mathbf{v}) \rangle_{\mathbf{g}_{\mathcal{T}}} \, \mathbf{m} + \int_{\boldsymbol{\Omega}} \langle \mathbf{K}, \dot{\mathbf{D}}_{\mathcal{T}}(\mathbf{v}\,, \delta\mathbf{v}) \rangle_{\mathbf{g}_{\mathcal{T}}} \, \mathbf{m}. \tag{124}$$

The last integral provides the variational expression of the nonlinear geometric stiffness which is independent of the constitutive law.

In general, the nonlinear dependence of the terms $\dot{\mathbf{D}}(\mathbf{v}\,, \delta\mathbf{v})$ and $\dot{\mathbf{f}}(\mathbf{v})$ on the spatial velocity field $\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; V\mathcal{E})$ requires an iterative algorithm for the evaluation of the nonlinear equilibrium problem, even for purely elastic structural models, as will be outlined in Sect. 17.

16 Coordinate-induced connections

Two reference frames are associated with a coordinate system on a Riemann manifold. The former is the natural one whose basis vectors at any point are the velocities of coordinate lines. The latter is obtained from the former by normalizing the basis vectors.

These two reference frames coincide in an Euclid affine manifold endowed with a Descartes coordinate system. In the general case, two linear connections may be introduced by defining path-independent parallel transports in which the components of tangent vectors are left invariant. In the two reference systems, these connections are, respectively, called the *natural* and the *normalized* (or *engineering*) connection.

By definition, in both connections, the parallel derivatives of basis vector fields vanish identically by tensoriality; in computing the curvature and the torsion at a given point, the tangent vector fields may be chosen to be generated by the path-independent parallel transport defined by the connection. Hence, parallel derivatives vanish, the curvature operator vanishes, and according to the definition (27), the torsion form is equal to the opposite of the Lie bracket:

$$\mathrm{TORS}(\mathbf{u}\,, \mathbf{w}) = -[\mathbf{u}\,, \mathbf{w}]. \tag{125}$$

In dynamics, this last property leads to formulate the Poincaré equations of motion as a special case of the equations expressed in terms of an arbitrary linear connection [31,32].

In the natural connection, the Lie bracket of basis vector fields vanishes identically, due to the commutation of the corresponding flows [23]. Hence, the torsion vanishes, but the parallel derivative of the metric tensor does not vanish.

Assuming time-invariance of virtual velocity fields $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}_{\mathcal{E}}\,; V\mathcal{E})$ and adopting the connection induced by a Descartes coordinate system, being $\mathrm{TORS} = 0$ and $\mathbf{G} = 0$, the time rate of the virtual stretching along the motion may be computed by the formula (108).

In a curvilinear coordinate system, the connection induced by the natural frame is not metric, in general, but is torsion-free. Then, $\mathbf{Y} = \nabla$, and the general expression in formula (90) of Proposition 1 writes

$$\mathbf{D}(\delta\mathbf{v}) = \tfrac{1}{2}\mathbf{G}(\delta\mathbf{v}) + \mathrm{sym}_{\mathbf{g}}\nabla(\delta\mathbf{v}), \tag{126}$$

and hence, (99) becomes

$$\begin{aligned}\dot{\mathbf{D}}(\mathbf{v}\,, \delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}(&\tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{E}}}\mathbf{G}(\delta\mathbf{v}) + \mathbf{G}(\mathbf{v}) \cdot \mathbf{G}(\delta\mathbf{v}) + \mathbf{G}(\delta\mathbf{v}) \cdot \nabla(\mathbf{v}) \\ &+\nabla_{\mathbf{v}_{\mathcal{E}}}\mathrm{sym}_{\mathbf{g}}\nabla(\delta\mathbf{v}) + 2\,\mathbf{G}(\mathbf{v}) \cdot \mathrm{sym}_{\mathbf{g}}\nabla(\delta\mathbf{v}) + 2\,\mathrm{sym}_{\mathbf{g}}\nabla(\delta\mathbf{v}) \cdot \nabla(\mathbf{v})).\end{aligned} \tag{127}$$

On the other hand, the connection induced by the normalized mobile frame is metric but with torsion equal to the negative of Lie bracket, as stated by (125). Then, $\mathbf{G} = 0$, $\mathbf{Y} = \nabla + \mathrm{TORS}$, with $\mathrm{TORS}(\mathbf{u}) \cdot \mathbf{w} = -[\mathbf{u}\,, \mathbf{w}]$.

Then, the expression (98) writes

$$\mathbf{D}(\delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\delta\mathbf{v})\,, \tag{128}$$

and hence, (99) becomes

$$\dot{\mathbf{D}}(\mathbf{v}\,, \delta\mathbf{v}) = \mathrm{sym}_{\mathbf{g}}(\nabla_{\mathbf{v}_{\mathcal{E}}}\mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\delta\mathbf{v}) + 2\,\mathbf{G}(\mathbf{v}) \cdot \mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\delta\mathbf{v}) + 2\,\mathrm{sym}_{\mathbf{g}}\mathbf{Y}(\delta\mathbf{v}) \cdot \mathbf{Y}(\mathbf{v})). \tag{129}$$

17 Computational issues

Let us denote by $p \in \mathcal{R}^n$ and $\delta p \in \mathcal{R}^n$ the active parameters describing the finite dimensional linear subspaces of discretized velocities and virtual velocities. Setting $\mathbf{v} = \mathbf{N} \cdot p$ and $\delta\mathbf{v} = \mathbf{N} \cdot \delta p$, we have that

$$\nabla\mathbf{v} = \nabla\mathbf{N} \cdot p\,, \quad (\nabla\mathbf{v})^A = (\nabla\mathbf{N} \cdot p)^A = (\nabla\mathbf{N})^A \cdot p\,, \tag{130}$$

$$\nabla\delta\mathbf{v} = \nabla\mathbf{N} \cdot \delta p\,, \quad (\nabla\delta\mathbf{v})^A = (\nabla\mathbf{N} \cdot \delta p)^A = (\nabla\mathbf{N})^A \cdot \delta p. \tag{131}$$

The parametric forcing $\dot{F}(p)$ is defined by the virtual power identity

$$\langle \dot{F}(p), \delta p \rangle = \langle \dot{\mathbf{f}}(\mathbf{N} \cdot p), \mathbf{N} \cdot \delta p \rangle\,, \tag{132}$$

and the discretized rate-elastic equilibrium problem writes

$$\langle \dot{F}(p), \delta p \rangle \;= \langle \text{ELASTIFF}(\mathbf{K}) \cdot p, \delta p \rangle + \langle \mathrm{G}(\mathbf{K}, p), \delta p \rangle\,, \tag{133}$$

so that the nonlinear algebraic problem to solve is

$$\text{ELASTIFF}(\mathbf{K}) \cdot p + \mathrm{G}(\mathbf{K}, p) = \dot{F}(p)\,, \tag{134}$$

where $\text{ELASTIFF}(\mathbf{K})$ is the elastic stiffness matrix, symmetric and positive definite, and $\mathrm{G}(\mathbf{K}, \bullet)$ is the nonlinear geometric stiffness with $\mathrm{G}(\mathbf{K}, 0) = 0$, evaluated according to the formulae of Sect. 15. In well-behaved elastostatic problems, an iterative algorithm for the solution of this nonlinear problem could consist in starting with the evaluation of p_0 by the linear rate-elastic problem

$$\text{ELASTIFF}(\mathbf{K}) \cdot p_0 = \dot{F}(p)\,, \tag{135}$$

assuming a time invariant velocity field and looping according to the iterative scheme

$$\text{ELASTIFF}(\mathbf{K}) \cdot p_i = \dot{F}(p_{i-1}) - \mathrm{G}(\mathbf{K}, p_{i-1})\,, \tag{136}$$

for $i \geq 1$, until a fixed point is attained with a prescribed accuracy. In more general situations, a dynamical computational approach ought to be adopted.

18 Conclusions

Rate equilibrium problems are formulated in computational mechanics in the context of iterative solution algorithms for geometrically nonlinear problems. The issue has been analyzed here in a suitable geometric setting with the aid of basic notions and results of differential geometry.

The starting point is a direct geometric interpretation of the basic issues in nonlinear continuum mechanics concerning the motion of a body along a space-time trajectory. A peculiar characteristic of the treatment is that no reference placement or manifold is adopted to introduce the fundamentals and that bodies of any dimensionality are dealt with on the same ground.

These are innovative aspects of the theory in comparison with usual treatments of continuum mechanics.

A further qualifying point is the careful and innovative definition of material and spatial fields involved in the analysis and the detection of the relevant tools for comparing their variation along actual or virtual motions.

The theory leads a natural way to consider LIE derivatives and parallel derivatives in performing time rate, respectively, of material and spatial fields. The outcome is a general formulation of equilibrium and of rate equilibrium, with virtual velocity fields defined on the trajectory manifold by parallel transport along the motion.

Rate formulations are thus dependent on the choice of a spatial connection, but equivalence to equilibrium along the trajectory is not affected and virtual velocities can be assumed, without loss of generality, to be time-independent along the motion, as explicated in Sect. 11. The geometric stiffness and the forcing operators depend in general in a nonlinear way on the motion velocity, so that iterative procedures are required for the solution of rate equilibrium problems.

By adopting a LEVI- CIVITA spatial connection, a linear geometric stiffness is obtained which, for path-dependent parallel transports, involves a curvature term. This term vanishes for a connection with path-independent parallel transport such as the standard EUCLID connection by translation.

The general formula for the time rate of the virtual stretching extends the one derived in the literature for 3D bodies in motion in the flat ambient EUCLID space with parallel transport by translation.

Standard treatments [15–20] are in fact limited to the analysis of rate equilibrium of 3D structural models, a class of problems in which the need for a nonlinear geometric analysis is less compelling than for slender structures modeled as one- or two-dimensional continua.

The geometric treatment developed in this paper provides an expression of rate equilibrium applicable to continuous bodies of any dimensionality. Also, linear connections other than the standard EUCLID translation may be considered in space. Instances are provided by analysis of continua in curvilinear coordinates, with connections induced by natural or mobile frames, as outlined in Sect. 16.

The geometric approach leads moreover to a natural rate formulation of elasticity [12], with the stress rate univocally defined as time rate of a material tensor field. This formulation is suited to provide the variational expression (124) of rate elastostatic problems in the full nonlinear range and for continuous bodies of any dimensionality.

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