Acta Mech 225, 825–834 (2014)
DOI 10.1007/s00707-013-0997-9

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Non-smooth bending and buckling of a strain gradient elastic beam with non-convex stored energy function

Received: 11 April 2013 / Revised: 24 July 2013 / Published online: 29 September 2013

Abstract Non-smooth strain gradient fields are studied in bending beams, in the context of strain gradient elasticity. It is found that strain fields with continuous curvature, but discontinuous curvature derivatives (evolutes) are possible. The pure bending and buckling problems of a simply supported beam are investigated.

1 Introduction

Ericksen [1], trying to interpret nonelastic phenomena in the context of elasticity such as elastic-plastic states, suggested the existence of globally stable equilibrium states. On the contrary, the conventional stable equilibrium states locally minimize the potential energy functional. In fact, Ericksen [2] allowed the existence of deformations with smooth displacement fields but non-differentiable. This phenomenon was called co-existence of phases phenomenon. In fact, there exist internal boundaries, phase boundaries, separating the low strain (elastic) states from the high strain (nonelastic) ones. James [3,4], an Ericksen's student, studied the co-existence of phases phenomena in the context of Finite Elasticity and the propagation of phase boundaries in elastic bars. A flow of research papers appeared then, connecting the co-existence of phases phenomenon in elastic deformations with crystal deformations and phenomena of twinning, Bhattacharya [5] and James and Hane [6]. In fact, all these theories were based on minimization procedures using Maxwell's rule. Another way to look for minimization procedures with smooth but non-differentiable fields was the application of the Weierstrass-Erdmann corner conditions, Gelfand and Fomin [7]. Fosdick and James [8] and James [9] studied non-convex beam bending elastica problems with smooth elastic curves but non-smooth curvature fields. Introducing a version of the existence of phases phenomena, just to study the elastoplastic buckling, Lazopoulos [10] presented a model with the strain energy density depending upon non-smooth variable parameters. According to that model, the displacement and strain fields were smooth but the internal variable material parameters were non-smooth. Hence, a phase boundary emerges with discontinuity of the material parameters. That idea was transferred into Finite Elasticity by Lazopoulos and Ogden [11], where co-existence of phases phenomena with smooth displacement and strain fields was studied. Furthermore, Ogden [12] discussed Mullin's effect, based on that theory. In addition, Aifantis and Serrin [13,14] discussed the fluid interfaces using Maxwell's rule. Furthermore, it is known that if the problem is non-convex, then it has multiple (local) solutions. To find global and local optimal solutions is fundamentally difficult by traditional direct methods. Based on the canonical duality and triality theory, Gao [15], a complete set of analytical solutions for 1-D non-convex problems has been presented in [16,17], and a complete set of numerical solutions for post-buckling analysis of a nonlinear beam was recently obtained in [18].

However, regularizing the non-smooth strain solutions, smooth strain diagrams connecting the low and high values of the strain may be established, invoking strain gradient elasticity, Ru and Aifantis [19].

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An extensive literature exists regarding gradient strain linear elasticity, concerning materials with the influence of the microstructure which the conventional elasticity fails to introduce, Altan and Aifantis [20]. Considering that kind of elasticity, many singularities, active in problems of the conventional elasticity, have been lifted, Aifantis [21]. The question whether non-smooth strain gradient deformations exist with smooth strain and smooth displacement in the context of gradient elasticity has been posed by Lazopoulos et al. [22]. One-dimensional bar models have been studied for the uni-axial extension problem of a bar with nonlinear geometric deformations and non-convex strain energy density function due to a strain gradient. The mathematical variational problem is similar to the definition of the Weierstrass-Erdmann corner conditions, Fomin and Gelfand [7]. This problem has been solved, and the conditions for a non-smooth strain gradient have been clarified. Those conditions require the continuity not only of the balance stress but also the smoothness of the double stress with convexification of the strain energy density function. Discontinuity of the strain gradient has been proposed in fiber-reinforced materials, interpreting kinking phenomena, Soldatos [23].

The present problem discusses the bending and buckling problems of a strain gradient elastic beam with non-convex strain energy densities due to the curvature gradient. Allowing non-smooth curvature gradients, the relaxation of the strain energy density yields the coexistence of phases phenomena with non-smooth curvature gradients. Conditions for the strain (curvature) gradient discontinuity are defined. An implementation of the proposed theory is presented with applications.

2 Revised model of a strain gradient elastic beam under pure bending

A geometrically nonlinear strain gradient Bernoulli beam model will be proposed, adapting linear constitutive stress–strain relations, nonlinear hyperstrains, but nonlinear geometry. The theory follows the approximations adopted in the conventional Bernoulli bending theory. Following a simple version of Mindlin's linear theory of elasticity with microstructure, a widely used micro-elasticity theory equipped with one additional constitutive coefficient, apart from the Lamé constants, is adopted. The intrinsic bulk length g is the additional constitutive parameter that has length dimensions and is called intrinsic length. Indeed, the strain energy density function, for the present case, is expressed by,

$$W = \frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + g^2\left(\frac{1}{2}\lambda\varepsilon_{kmm}\varepsilon_{knn} + G\varepsilon_{kmn}\varepsilon_{knm}\right) \tag{1}$$

where ε_{ij} denotes the nonlinear strain and ε_{ijk} the strain gradient, respectively, with

$$\varepsilon_{ijk} = \varepsilon_{ikj} = \partial_i\varepsilon_{kj} \tag{2}$$

and $u_i = u_i\,(x_k)$, the displacement field.

The constitutive stresses are defined by the relations,

$$\tau_{ij} = \frac{\partial W}{\partial\varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \tag{3}$$

and the hyper-stresses by

$$\mu_{ijk} = \frac{\partial W}{\partial\varepsilon_{ijk}} = g^2\left(\lambda\varepsilon_{inn}\delta_{jk} + 2G\varepsilon_{ijk}\right). \tag{4}$$

For the present study, we consider a beam shown in Fig. 1. The x axis is the axis of the beam, whereas the y axis is directed along the deflection axis. The elastic line lies on the x–y plane.

Considering Bernoulli–Euler principle, the infinitesimal strain of the beam is defined by

$$\varepsilon_{xx} = -yk \tag{5}$$

with k denoting the curvature of the inextensible elastic line.

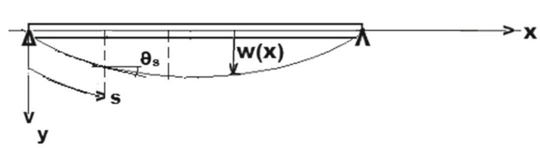

Fig. 1 Geometry of the beam

For the formulation of the present problem, we need the constitutive stress

$$\tau_{xx} = \frac{\partial W}{\partial \varepsilon_{xx}} = E\varepsilon_{xx} \tag{6}$$

where E is the elastic Young's modulus and the hyperstresses

$$\mu_{xxx} = g^2 E\varepsilon_{xxx}, \tag{7}$$
$$\mu_{yxx} = g^2 E\varepsilon_{yxx} \tag{8}$$

with the hyperstrains

$$\varepsilon_{xxx} = -y\frac{\partial^2 \vartheta}{\partial s^2} \quad \text{and} \quad \varepsilon_{yxx} = -\frac{\partial \vartheta}{\partial s}. \tag{9}$$

The total potential requires,

$$V = W_b - U \tag{10}$$

where ϑ is the angle of the elastic line with the axis of the beam and s denotes the arc of the inextensible elastic line, with U denoting the work of the external forces and W_b the strain energy of the beam. Since non-smooth strain gradient elastic fields will be discussed, the strain energy density should be a non-convex function of the strain gradient field. Consequently, non-convex terms of the strain gradient should be added. Recalling the symmetry of the problem, the present beam strain energy is defined by

$$W_b = 2\int_0^L \int_{-h/2}^{h/2} \left(\frac{E}{2}\varepsilon_{xx}^2 + \frac{g^2E}{2}\left(\varepsilon_{xxx}^2 + \varepsilon_{yxx}^2\right) - \frac{g^2\beta E}{4}\varepsilon_{xxx}^4\right)dxdy. \tag{11}$$

Performing the algebra with the help of Eqs. (6–9), the strain energy of the beam becomes,

$$W_b = 2\int_0^L \left(\frac{EI}{2}\left(1+cg^2\right)\vartheta,_s^2 + \frac{g^2EI}{2}\vartheta,_{ss}^2 - g^2\frac{3\beta EI}{16}h^2\vartheta,_{ss}^4\right)dx \tag{12}$$

where E is Young's modulus, I inertia moment, and A area of the cross-section, h the height, $2L$ the length of the beam and β the non-convex parameter for the strain energy density, and s is the length of the inextensible elastic line. Further, $c = \frac{A}{I}$ is the squared inverse of the inertia radius. In addition, the work of the external forces is expressed in the present case of the pure bending by

$$U = -\left[M\vartheta\right]_{-L}^{L} - \left[m\vartheta,_s\right]_{-L}^{L} \tag{13}$$

where M and m denote the bending moments and hypermoments of the beam. Hence, the total potential V yields

$$V = 2\int_0^L \left(\frac{EI}{2}\left(1+cg^2\right)\vartheta,_s^2 + \frac{g^2EI}{2}\vartheta,_{ss}^2 - g^2\frac{3\beta EI}{16}h^2\vartheta,_{ss}^4 - M\vartheta,_s - m\vartheta,_{ss}\right)dx. \tag{14}$$

Considering the minimum of the potential V with smooth curvature field $\vartheta,_s$ but non-smooth derivative of the curvature $\vartheta,_{ss}$, the governing equilibrium equation is defined by,

$$\left(1+cg^2\right)\vartheta,_s - g^2\vartheta,_{ss} + \frac{9}{4}\beta g^2h^2\vartheta,_{ss}^2 \cdot \vartheta,_{sss} = \frac{M}{EI}. \tag{15}$$

Further, the boundary conditions are defined by

$$\frac{m}{EI} = -g^2\vartheta,_{ss} + \frac{3}{4}\beta g^2h^2\vartheta,_{ss}^3 \quad \text{at x} = -L, L \tag{16}$$

where m denotes the hypermoment.

In addition, the corner conditions at the point x_o corresponding to smooth curvature field but to non-smooth curvature derivative field are expressed by

$$\left[\frac{\partial W}{\partial \vartheta,_{ss}}\right] = 0, \tag{17}$$

$$\left[W - \vartheta,_{ss}\frac{\partial W}{\vartheta,_{ss}}\right] = 0 \tag{18}$$

where the discontinuity $[(\bullet)]_{x_o} = (\bullet)_{x_{o+}} - (\bullet)_{x_{o-}}$.

In the present case, the discontinuity conditions, Eqs. (17, 18) yield

$$\vartheta,_{ss}(x_{o+}) - \vartheta,_{ss}(x_{o-}) = \frac{3}{4}\beta h^2\left(\vartheta,_{ss}(x_{o+})^3 - \vartheta,_{ss}(x_{o-})^3\right), \tag{19}$$

$$\vartheta,_{ss}(x_{o+})^2 - \vartheta,_{ss}(x_{o-})^2 = \frac{9}{8}\beta h^2\left(\vartheta,_{ss}(x_{o+})^4 - \vartheta,_{ss}(x_{o-})^4\right). \tag{20}$$

The Eqs. (15, 16, 19, 20) yield the equilibrium of the pure bending problem under a non-smooth curvature field, caused by non-convex stored energy density function.

A simple symmetric solution for the governing Eq. (15) and the boundary conditions, Eqs. (16), and the corner conditions corresponding to non-smooth strain gradient field, Eqs. (19, 20), is expressed by

$$\vartheta,_s = \frac{M}{EI\left(1+cg^2\right)} + B_1\cosh(\omega x) \quad \text{for} \quad 0 < x < x_o, \tag{21}$$

$$\vartheta,_s = \frac{M}{EI\left(1+cg^2\right)} + A_2\sinh(\omega x) + B_2\cosh(\omega x) \quad \text{for} \quad x_o < x < L \tag{22}$$

with $\omega = \sqrt{\frac{1+cg^2}{g^2}}$ and x denoting the x_1 coordinate. Further, algebra on the corner system, Eqs. (19, 20), reveals that

$$\vartheta,_{ss}(x_o) = \pm\frac{2}{\sqrt{3\beta}\,h}. \tag{23}$$

Considering that $\vartheta,_{ss}(x_{o-}) = \frac{2}{\sqrt{3\beta}\,h}$ and $\vartheta,_{ss}(x_{o+}) = -\frac{2}{\sqrt{3\beta}\,h}$ that is proved to be the unique solution of the corner system, the simplified solution, Eqs. (21, 22), yields the following equations:

$$B_1\cosh(\omega x_o) = A_2\sinh(\omega x_o) + B_2\cosh(\omega x_o). \tag{24}$$

Corresponding to the continuity of the curvature, Eqs. (21, 23) yield

$$B_1\,\omega\sinh(\omega x_o) = \frac{2}{\sqrt{3\beta}\,h}. \tag{25}$$

Further, the relation $\vartheta,_{ss}(x_{o-})/\vartheta,_{ss}(x_{o+}) = -1$ yields

$$(B_1 + B_2)\tanh(\omega x_o) + A_2 = 0. \tag{26}$$

In addition, the boundary condition, Eq. (16), yields,

$$m/EI = -g^2\omega\left(A_2\cosh(\omega L) + B_2\sinh(\omega L)\right). \tag{27}$$

The system of the four Eqs. (24–27) defines the four unknowns $B_1, A_2, B_2, and\ x_o$.

Implementation of the theory is found in the application that follows. Let us consider a beam with $h = 0.02,\ g = 0.03,\ 2L = 0.20$, and then, with $m = -0.030$, we get the curvature function defined by

$$\vartheta_s = 0.55\frac{M}{EI} + B_1\cosh(\omega x) \qquad 0 < x < x_o = 0.03, \tag{28}$$

$$\vartheta_s = 0.55\frac{M}{EI} + A_2\sinh(\omega x) + B_2\cosh(\omega x) \qquad x_o = 0.03 < x < 0.1. \tag{29}$$

In this case, $\omega = 176, 4$, and the curvature corresponding to Eq. (28, 29) is shown in Fig. 2.

It is simply evident that at the non-smooth point there is a maximum (peak) of the curvature.

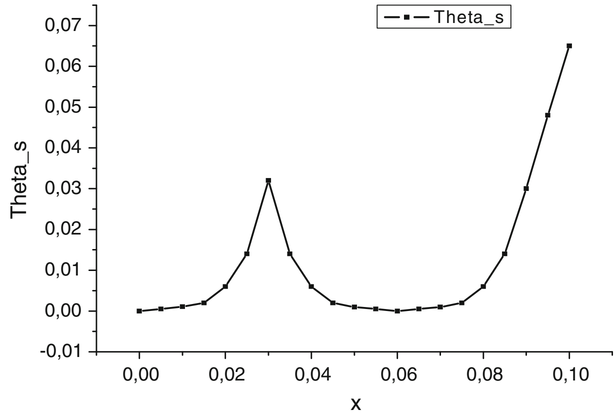

Fig. 2 Diagram of the continuous curvature θ_s versus x

3 The buckling of a beam with non-convex strain energy due to non-smooth strain gradient fields

The buckling problem of a strain gradient elastic beam will be studied, with non-convex strain energy density function. Allowing for non-smooth fields of the strain gradient, the strain fields remain smooth. Let us consider a beam of inextensible elastic line of length L applying the compressible force P at one end, see Fig. 3. The strain energy density is similar to the one proposed in the preceding section, and the total potential is expressed by

$$\tilde{V} = \int_0^L \left(\frac{EI}{2} \left(1 + cg^2\right) \vartheta,_s^2 + \frac{g^2 EI}{2} \vartheta,_{ss}^2 - g^2 \frac{3\beta EI}{16} h^2 \vartheta,_{ss}^4 - P\left(1 - \cos\vartheta\right) \right) dx \tag{30}$$

where ϑ is the angle of the elastic line with the axis of the beam and w is the beam displacement, while s denotes the arc of the elastic line. Due to the inextensibility condition, the curvature $\vartheta,_s$ is expressed by

$$\vartheta,_s \approx \frac{w''}{\sqrt{1 + w'^2}} \tag{31}$$

and $\cos\vartheta \approx \sqrt{1 - w'^2}$ while w denotes the beam displacement.

If $V = \tilde{V}/EI$, and $p = P/EI$, then the total potential energy may be approximated by

$$V = \frac{1}{2}\left(-pw'^2 + \left(1 + cg^2\right) w''^2 + gw'''^2\right) + \frac{1}{2}\left(-\left(1 + cg^2\right) w'^2 w''^2 - 2g^2 w' w''^2 w''' - g^2 w'^2 w'''^2\right) - \frac{3}{16} \beta g^2 h^2 w'''^4. \tag{32}$$

A simple variational procedure for the functional V yields the nonlinear governing equation

$$\begin{aligned} -g^2 w^{VI} + \left(1 + cg^2\right) w^{IV} + pw'' &= \left(1 + cg^2\right) w''^3 - 10g^2 w''^2 w^{IV} - 10g^2 w' w''' w^{IV} \\ &+ w''\left(-15g^2 w'''^2 + w'\left(1 + cg^2\right) w''' - 6g^2 w^V\right) + w'^2\left(\left(1 + cg^2\right) w^{IV} - g^2 w^{VI}\right) \\ &- \frac{9}{4} \beta g^2 h^2 \left(2 w^{IV^3} + 6 w''' w^{IV} w^V + w'''^2 w^{VI}\right) \end{aligned} \tag{33}$$

with the boundary conditions

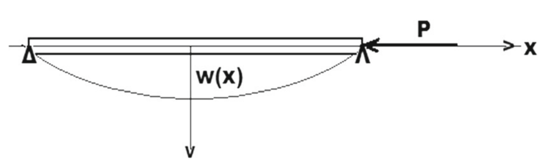

Fig. 3 The buckling beam

$$Q/EI = -pw' - 0.5\left(1+cg^2\right)w''' + g^2 w^V + \left(1+cg^2\right)\left(w'w''^2 + w'^2 w'''\right)$$
$$-g^2\left(\left(-3w'''\left(2w''^2 + w'w'''\right)\right)\right) - g^2\left(4w'w''w^{IV} + \frac{9}{4}\beta h^2 w'''\left(2w^{IV^2} + w'''w^V\right) + w'^2 w^V\right)$$
$$or \quad \delta w = 0 \quad at \quad x = 0, L, \tag{34}$$
$$M/EI = \left(1+cg^2\right)w'' - g^2 w^{IV} - \left(1+cg^2\right)w'^2 w'' + \frac{g^2}{2}\left(4w'w''w''' + \frac{9}{2}\beta h^2 w^{IV} w^{(3)^2} + 2w'^2 w^{IV}\right)$$
$$\text{or} \quad \delta w' = 0 \quad \text{at} \quad x = 0, L, \tag{35}$$
$$m/EI = g^2 w''' - g^2 w' w''^2 - g^2 w''' w'^2 - \frac{3}{4}\beta h^2 w'''^3 \quad \text{or} \quad \delta w'' = 0 \quad \text{at } x = 0, L. \tag{36}$$

Further, the corner type conditions for the non-smooth strain gradient are expressed by Eqs. (17, 18), corresponding to

$$3\beta h^2\left(w\left(x_{o-}\right),_{xxx}^2 + w\left(x_{o-}\right),_{xxx} w\left(x_{o+}\right),_{xxx} + w\left(x_{o+}\right),_{xxx}^2\right) + 4w\left(x_o\right),_{xx}^2 - 4 = 0 \tag{37}$$

and

$$\left(w\left(x_{o-}\right),_{xxx} + w\left(x_{o+}\right),_{xxx}\right)\left(-8 + 8w\left(x_{o+}\right),_{xx}^2 + 9\beta h^2\left(w\left(x_{o-}\right),_{xxx}^2 + w\left(x_{o+}\right),_{xxx}^2\right)\right) = 0 \tag{38}$$

where x_o is the point of non-smoothness of the strain gradient field. The solution of the system of Eqs. (37, 38) yields

$$w\left(x_{o\mp}\right),_{xxx} = \pm\frac{2\sqrt{1 - w\left(x_o\right),_{xx}^2}}{\sqrt{3\beta}h}. \tag{39}$$

Since the buckling problem depends on the solution of the linear problem composed by the governing equation,

$$-g^2 w^{VI} + \left(1+cg^2\right)w^{IV} + pw'' = 0, \tag{40}$$

and the boundary conditions,

$$Q/EI = -pw' - 0.5\left(1+cg^2\right)w''' + g^2 w^V \quad \text{or} \quad \delta w = 0 \text{ at } \quad x = 0, L, \tag{41}$$
$$M/EI = \left(1+cg^2\right)w'' - g^2 w^{IV} \quad \text{or} \quad \delta w' = 0 \quad \text{at} \quad x = 0, L, \tag{42}$$
$$m/EI = g^2 w''' \quad \text{or} \quad \delta w'' = 0 \quad \text{at } x = 0, L, \tag{43}$$

the solution to the buckling problem Eq. (40) is defined by the function

$$w(x) = \xi\left(c_2 + c_1 x + \sin\left(r_1 x\right) + c_3\cos\left(r_1 x\right) + c_4\sinh\left(r_2 x\right) + c_5\cosh\left(r_2 x\right)\right) \tag{44}$$

with $r_1 = \left(\frac{-a+\sqrt{a^2+4g^2p}}{2g^2}\right)^{1/2}$, $r_2 = \left(\frac{a+\sqrt{a^2+4g^2p}}{2g^2}\right)^{1/2}$ where $a = \frac{I+g^2A}{I}$, satisfying the boundary conditions, Eqs.(42–44). Details for the buckling solution of the linear problem may be found in Lazopoulos and Lazopoulos [24]. The critical load p_o, along with the coefficients c_i $i = 1, \ldots, 5$, may be defined by the system of six equations derived by the substitution of the $w(x)$ function in the six boundary conditions, Eqs. (42, 43).

Increasing the critical loading,

$$p = p_o(1+\lambda), \tag{45}$$

where $\lambda \approx O\left(\xi^2\right)$, the equilibrium system may compactly be denoted by

$$L(w) = R(\lambda, w) \tag{46}$$

Applying the well-known Fredholm Alternative theorem, Troger and Steindl [25] for the buckling of the beam in the present case, the relation between the applied thrust and deflection of the beam in the post-critical area is defined. The proposed methodology will be implemented in the simply supported beam.

4 Buckling of a simply supported beam

The present section clarifies the procedure proposed in the preceding section. The buckling and post-critical behavior of a simply supported beam, Fig. 3, strain gradient elastic beam with non-smooth strain gradient, will be studied.

Taking into consideration the symmetry of the problem, we change the boundaries from $x = 0, L$ to $x = -L/2, L/2$. The symmetric solution, see Eq. (44), yields, in the present case,

$$w(x) = c_1 + c_2 \cos(r_1 x) + c_3 \cosh(r_2 x) \quad \text{for} \quad 0 < x < x_o, \tag{47.1}$$

$$w(x) = c_4 + c_5 x + c_6 \cos(r_1 x) + c_7 \cosh(r_2 x) + c_8 \sin(r_1 x) + c_9 \sinh(r_2 x) \quad \text{for } x_o < x < L/2, \tag{47.2}$$

The boundary conditions at $x = L/2$ are corresponding to

$$Q = M = m = 0. \tag{48}$$

That is,

$$Q = -pw' - 0.5\left(1 + cg^2\right) w''' + g^2 w^V = 0 \quad \text{at } x = L/2, \tag{49}$$

$$M = \left(1 + cg^2\right) w'' - g^2 w^{IV} = 0 \quad \text{at } x = L/2, \tag{50}$$

$$m = g^2 w''' = 0 \quad \text{at } x = L/2. \tag{51}$$

Furthermore, we impose the condition:

$$w(L/2) = 0. \tag{52}$$

And at point $x = x_0$, the boundary conditions are expressed by:

$$w(x_{o\mp})_{,xxx} = \pm \frac{2}{\sqrt{3\beta h}}, \tag{53}$$

$$w(x_{o+}) = w(x_{o-}), \tag{54}$$

$$w'(x_{o+}) = w'(x_{o-}), \tag{55}$$

$$w''(x_{o+}) = w''(x_{o-}). \tag{56}$$

From these 9 equations, the 9 unknowns $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$, will be defined and various diagrams will be plotted.

Let us first consider a beam with length L, with h the height of the cross-section, and b the breadth of the cross-section. This beam is loaded by a compressive force at the right end of the beam Fig. (3). The beam is simply supported at the ends. The displacement equation is given by the Eqs. (47.1) and (47.2). It is then required to calculate the constant variables $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$. To be more specific, a detailed numerical example is presented, implementing the theory.

The input for this calculation is as follows:

The length of the beam is $L = 1$ mm, while $b = 0.1L$ is the breadth of the cross-section of the beam, and $h = 0.09$L is the height of the cross-section of the beam (Figs. 4, 5, 6).

The area of the cross-section of the beam is $A = bh$, $g = 0.05L$ is the intrinsic length, $x_0 = L/10$ is the point where we have the discontinuity of $w'''(x)$, while $I = 1/12bh^3$ is the moment of inertia of the cross-section.

$P = 0.8(3.14^2 EI)/L^2$ is the compressive force applied on the beam (where E is Young's modulus), $p = P/((EI))$ is the compressive force of the beam, $\alpha = (I + g^2 bh)/I$ is the α shape coefficient, $r_1 = \left(\frac{-a+\sqrt{a^2+4g^2p}}{2g^2}\right)^{1/2}$ is the r_1 coefficient, $r_2 = \left(\frac{a+\sqrt{a^2+4g^2p}}{2g^2}\right)^{1/2}$ is the r_2 coefficient of the beam, $\beta = 0.2$ is the non-convex parameter in the strain energy density function, and $c = \frac{A}{I}$ is the c coefficient denoting the inverse of the square of radius of gyration of the beam cross-section.

This input is mainly inserted in Eqs. (47.1), (47.2) so that we have the corresponding equations for $w(x_{o+})$ and $w(x_{o-})$. Considering the boundary conditions (49)–(56), a linear system consisting of 9 equations with

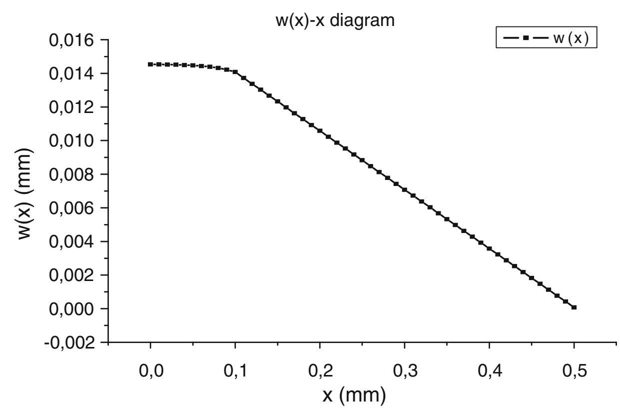

Fig. 4 Displacement of the beam versus x diagram

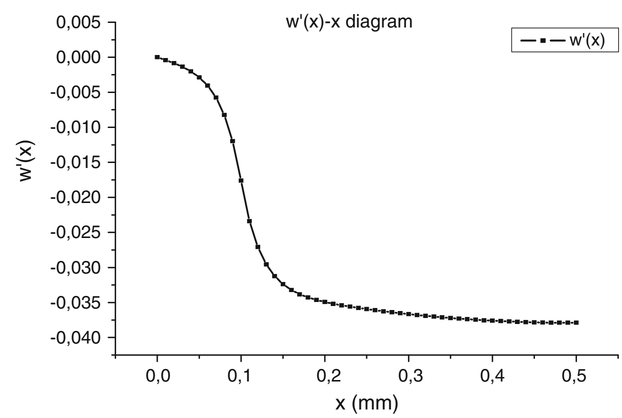

Fig. 5 First derivative of the displacement versus x diagram

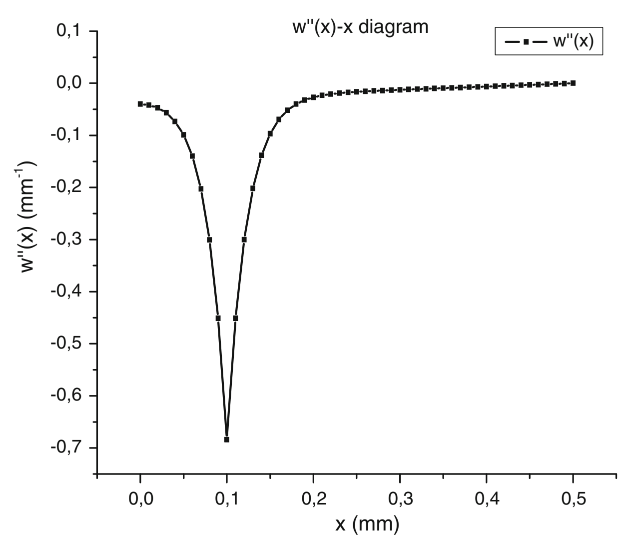

Fig. 6 Second derivative of the displacement versus x diagram

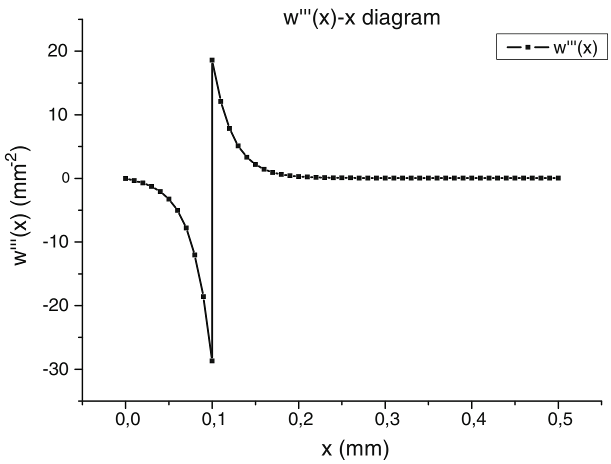

Fig. 7 Third derivative of displacement versus x diagram

nine unknowns ($c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$) is formulated. With the help of the computerized algebra *Mathematica package* [26], Eqs. (49)–(56) are solved and the parameters are defined,

$$c_1 = 0.00098 \text{ mm}, c_2 = 0.01356 \text{ mm}, c_3 = -9.160210^{-6} \text{ mm}, c_4 = 0.00005 \text{ mm},$$
$$c_5 = -0.0001 \text{ mm}, c_6 = 0.01753 \text{ mm}, c_7 = -0.02687 \text{ mm}, c_8 = -0.02319 \text{ mm}, c_9 = 0.02687 \text{ mm}.$$

Then, the displacement of the beam $w(x)$ is given by:

$$w(x) = \mathrm{c}_1 + c_2 \cos(r_1 x) - c_3 \cosh(r_2 x) \quad \text{for } 0 < x < x_0, \tag{57}$$

$$w(x) = c_4 + c_5\, x + c_6 \cos(r_1\, x) + c_7 \cosh(r_2\, x) + c_8 \sin(r_1\, x) + c_9 \sinh(r_2\, x)$$
$$\text{for } x_0 < x < L/2. \tag{58}$$

After replacing c_i from (57), (58), it is obvious that the displacement $w(x)$ is smooth.

That is shown in the $w(x)$ diagram, Fig. 4.

The first derivative of the displacement of the beam $w'(x)$ is:

$$w'(x) = -c_2\, r_1 \sin(r_1\, x) + c_3 r_2 \sinh(r_2\, x) \quad \text{for } 0 < x < x_0, \tag{59}$$

$$w'(x) = c_5 - c_6\, r_1 \sin(r_1 x) + c_7\, r_2 \sinh(r_2\, x) + c_8\, r_1 \cos(r_1\, x) + c_9\, r_2 \cosh(r_2\, x)$$
$$\text{for } x_0 < x < L/2. \tag{60}$$

From (59) and (60) (after replacing c_i), it is easy to conclude that the first derivative $w'(x)$ is also continuous.

And the $w'(x)$ is shown in Fig. 5.

The second derivative (the curvature or the strain) $w''(x)$ of the displacement of the beam is:

$$w''(x) = -c_2\, r_1^2 \cos(r_1\, x) + c_3\, r_2^2 \cosh(r_2\, x) \quad \text{for } 0 < x < x_0, \tag{61}$$

$$w''(x) = -c_6\, r_1^2 \cos(r_1\, x) + c_7\, r_2^2 \cosh(r_2\, x) - c_8\, r_1^2 \sin(r_1\, x) + c_9\, r_2^2 \sinh(r_2\, x)$$
$$\text{for } x_0 < x < L/2. \tag{62}$$

From Fig. 6, it is obvious that the second derivative of the displacement (the strain) of the beam $w''(x)$ is continuous, with discontinuous third derivative at $x = x_o = L/10$.

Finally, the third derivative of the displacement of the beam $w'''(x)$ is found:

$$w'''(x) = c_2 r_1^3 \sin(r_1 x) + c_3 r_2^3 \sinh(r_2 x) \quad \text{for } 0 < x < x_0, \tag{63}$$

$$w'''(x) = c_6 r_1^3 \sin(r1\, x) + c_7\, r_2^3 \sinh(r_2\, x) - c_8\, r_1^3 \cos(r_1\, x) + c_9\, r_2^3 \cosh(r_2\, x)$$
$$\text{for } x_0 < x < L/2, \tag{64}$$

and the discontinuous diagram of $w'''(x)$ is Fig. 7.

In Fig. 7, the discontinuity of the values $w(x_{o+}), w(x_{o-})$ is obvious $(at\ x_o = L/10)$.

5 Conclusions

A new beam bending theory is proposed in the context of strain gradient elasticity. Continuous displacement with smooth elastic lines with continuous tangent and continuous curvature fields is considered but with discontinuous evolutes. The proposed method extends the strain gradient elastic theory in one-dimensional bars with smooth displacement and strain fields, but with non-smooth strain gradient fields proposed by Lazopoulos [11]. The proposed theory describes the coexistence of phases phenomena, before the appearance of discontinuous strain fields.

Acknowledgments The author thanks the reviewers for their thoughtful comments that helped in improving the present work.

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