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Topology optimization for reinforcement of no-tension structures

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Abstract In the paper, the reinforcement of no-tension structures by the application of superposed high-strength sheets, or by the insertion of tensile bars, has been considered with the purpose to set up a design path aiming at the positioning of the new material according to some optimal criterion. In detail, no-tension models are adopted which are recognized as an effective tool for analyzing a wide class of structures (e.g., masonry and reinforced concrete members), and the equilibrium and the failure analysis of the reinforced body are developed with particular reference to its ultimate limit state of collapse. Finally, an approach through the "topologic optimization" is proposed for the identification of the optimal distribution of the reinforcement, and some of the obtained results are shown.

1 Introduction

At the present state of the art, modern material technology offers many opportunities to strengthen structural organisms by integration of the original body with another material yielding a kind of prosthesis suitably applied to the preexistent structures.

In recent years, a big spread has been recorded as regards the adoption of new materials such as the fiber-reinforced polymers (or FRP) (see, e.g., [1–3]). In particular, an increasing interest about the composites from the factory and research fields has established their exploitation both in predamage and post-damage strategies, especially for ancient constructions since they are able to provide effective and low invasive reinforcement interventions. Under such a perspective, the exploitation of the composites seems particularly suitable for applications related to the restoration and/or reinforcement of ancient constructions; on the other hand, the adoption of innovative materials and technologies, including smart materials, is strongly time limited since a very deep knowledge of their behavior and interaction with the structure is required in order to avoid risks of loss or irreversible damage of the heritage. Anyway, new materials offer the opportunity to distribute reinforcement over an existing structure almost in an arbitrary way, since they can conform to any predefined shape and are easy to apply. Therefore, in this field, the most felt problem becomes the design of the reinforcement, and where to apply the reinforcement or not for the optimal effectiveness of the refurbishment. Moreover, the application of the reinforcement should be carefully considered in a way to make the intervention as light as possible, and one way to approach the problem is via the "topological optimization".

In literature, the bibliography is usually concerned with different mathematical approaches to the optimization problem, but there are no applications referred to the behavior of masonry structures aiming to search for

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the minimum optimal reinforcement to be applied. However, neglecting the generic bibliography on masonry, many references by the authors can be founded for further backup and for a complete approach to the problem. Targeted knowledge on the subject can be acquired by means of works about the theoretical models of the material (see, e.g., [4–7]) constituting the structure, referring to no-tension models (see, e.g., [8–12]) also validated by means of numerical applications and/or experimental laboratory tests on masonry arches and vaults (see, e.g., [13–16]), walls (see, e.g., [17,18]), and staircases (see, e.g., [19,20]), depending on the masonry typology under examination, also in the presence of FRP provisions (see, e.g., [21,22]).

In synthesis in the present paper, the reference domain for identifying where to place the reinforcement via topology optimization coincides with the existing structure, and optimization traduces in a black and white (or a grayscale) picture showing the additional material to be applied for reinforcement. To this purpose, a 0–1 multiplicative function is introduced dependent on the reinforcement's presence or not, at each point of the single element of the structure. This leads to very cumbersome integer programming problems, and it is desirable to regularize the problem by approximating the factor by a function (the density of the reinforcement) continuously variable in (0–1). Even in this case, the final solution is often a 0–1 result, or, anyway, some skill in the interpretation of the density distribution can produce a well-distinguished solution.

In particular, the problem is approached with reference to a two-dimensional no-tension (or NT) body (for details see, e.g., [8–10]), that is intended to be assimilated to a masonry or concrete beam and panel. It is clear that in such cases it would be helpful to have a procedure to optimize the distribution of the reinforcement, for at least two reasons: (i) most times, the reinforcement introduces a disturbance in the authenticity of the original fabric; (ii) the reinforcement interferes with finishes and with technological plants.

Here, the reinforcement is conceived as the application of an additional resistant sheet that is superposed to the existent structure, thus yielding additional strength; in reinforced concrete members, it can also be made by steel bars incorporated in the concrete casting. In particular, the reinforcement may be intended to help the material to improve its load-carrying capacity versus some stresses it is not suited to resist. Examples may be compressive stresses in thin membranes, where reinforcement can be viewed at as a stiffening provision, or tensile stresses in masonry, where reinforcement is intended as the capacity to resist tensile stresses. In the paper, the structural element is modeled through the assumption that the basic material (e.g., masonry or concrete or anything else) is an elastic not-resisting-tension material (or NT material) and the reinforcement is an indefinitely elastic sheet, possibly with variable thickness, to be glued over the panel.

The problem can be approached through *topological optimization*, that is, nowadays receiving much attention (see, e.g., [23–27]). Structural topology optimization is concerned with the optimal distribution of resisting material in the interior of a given domain, able to resist given loads, subject to some constraints (e.g., the quantity of material involved, and/or the maximum stress/strain in the material) and aiming at optimizing some performance index or some design objectives. Essentially, two basic problems can be set in topology optimization. The first one is concerned with structures that are discrete in nature and are conceived as the assemblage of a number of connected components, and in this case, the problem is set with the objective to identify the number, the dimensions and the arrangement of the members. For continuum systems, essentially the shape of the body able to resist the loads is optimized, and the solution consists in deciding whether any point in the domain Ω where the structure must be included is filled with material or not (Fig. 1).

This approach leads to a 0–1 optimization that may be not always tractable from the point of view of mathematical programming; the problem can be regularized by making recourse to a *density function* $\rho(\mathbf{x})$ that distributes the material with continuity over the domain, despite the fact that in the solution the optimal layout may nevertheless be of a well-defined 0–1 type.

In the optimization process, some additional difficulties may arise from the fact that at some steps of the process the absence of material at some key points in the domain may produce singularities in the constitutive equations of the system, thus blocking the solution process. For the case of the reinforcement that is the object of the present paper, this problem should not be encountered, in that the body to be reinforced is basically existent, and optimization only regards the addition of material.

For the case of no-tension panels, anyway, the applied load pattern might be out of the load-carrying capacity of the basic body, or even at some intermediate stage of the procedure, and the optimization should be carried out on following admissible paths. Similar problems are met with other instances of topology optimization when the structural response is governed by nonlinear relationships.

In the following, the fundamentals for reinforced NT panels will be outlined, and the extension to reinforcement will be treated. The optimum problem for the distribution of the reinforcement will be set up, and some preliminary solution strategies will be outlined.

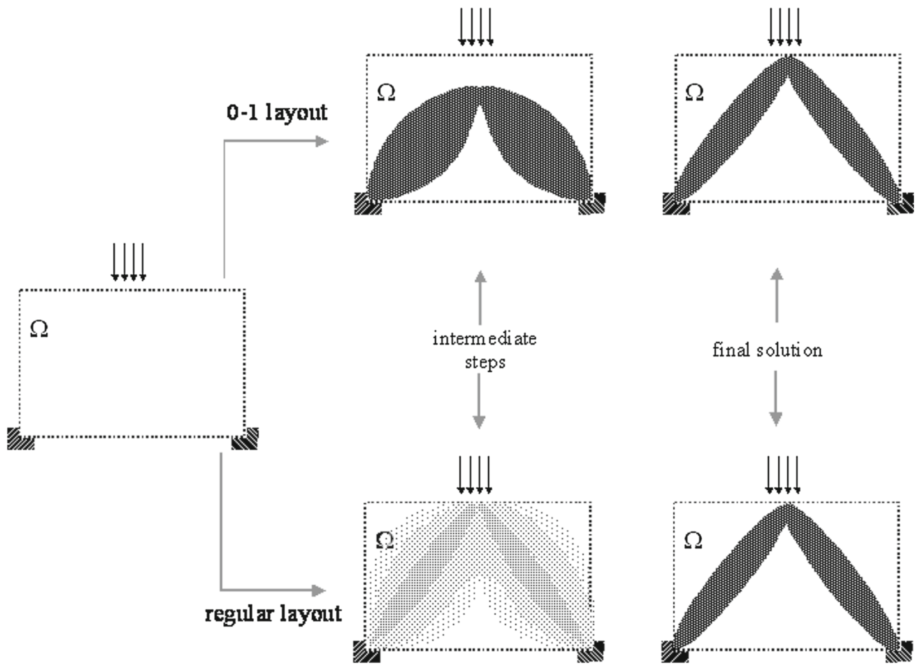

Fig. 1 Illustration of 0–1 and regularized topology optimization

2 Basic relationships for no-tension body with reinforcement

Let us consider the domain Ω, that is occupied by the considered NT material (Fig. 2a), subject to surface tractions **p** and body forces **f** (Fig. 2b), and the contour Γ of the domain Ω, that is split in the constrained part Γ_u, where displacements $\mathbf{u}_t$ are imposed and forces correspond to the reactions, and the part Γ_p, where displacements are free and forces are data.

Moreover, let $\mathbf{u}(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ be, respectively, the displacement field and the total strain field relevant to the current point **x** in, Ω and $\mathbf{u}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x})$ on Γ_u. Denote by the suffix "b" the basic structural body and the relevant material, and by the suffix "r" the reinforcing material, the FRP reinforcement is usually placed on the basic body and it solidly moves with the same displacements and strains.

After the reinforcement has been gradually added (see, e.g., Fig. 2c–d), and the forces are applied again, the response of the system is ruled by the following usual compatibility and equilibrium equations:

(a) The basic relations are

$$\begin{aligned} \varepsilon(\mathbf{x}) &= \varepsilon_b(\mathbf{x}) = \varepsilon_r(\mathbf{x}), \\ \varepsilon_b(\mathbf{x}) &= \varepsilon_{be}(\mathbf{x}) + \varepsilon_f(\mathbf{x}), \\ \boldsymbol{\sigma}(\mathbf{x}) &= \boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x}), \end{aligned} \quad (1)$$

where $\varepsilon_b(\mathbf{x})$ and $\varepsilon_r(\mathbf{x})$ are the total strain field in the basic body and in the reinforcement, $\varepsilon_{be}(\mathbf{x})$ and $\varepsilon_{bf}(\mathbf{x})$ are the elastic strain field in the basic body and the fracture strain field in the basic body, $\boldsymbol{\sigma}(\mathbf{x})$ is the total stress in the reinforced body, $\boldsymbol{\sigma}_b(\mathbf{x})$ and $\boldsymbol{\sigma}_r(\mathbf{x})$ are the stress in the basic body and the stress in the reinforcement, respectively and $\rho(\mathbf{x})$ is the density function of the reinforcement. Trivially the symbols $\boldsymbol{\sigma}_b(\mathbf{x}) \leq \mathbf{0}$ or $\varepsilon_{bf}(\mathbf{x}) \geq \mathbf{0}$ simply mean that the relevant tensors possess non-positive or, respectively, non-negative eigenvalues.

(b) The compatibility conditions are

$$\begin{aligned} \varepsilon(\mathbf{x}) &= \nabla \mathbf{u}(\mathbf{x}) \quad \forall \mathbf{x} \in \boldsymbol{\Omega}, \\ \mathbf{u}(\mathbf{x}) &= \mathbf{u}_t(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u, \end{aligned} \quad (2)$$

with stress and fracture admissibility being expressed as

$$\left.\begin{aligned} \varepsilon_{bf}(\mathbf{x}) &\geq \mathbf{0} \\ \boldsymbol{\sigma}_b(\mathbf{x}) &\leq \mathbf{0} \end{aligned}\right\} \quad \forall \mathbf{x} \in \boldsymbol{\Omega} \quad (3)$$

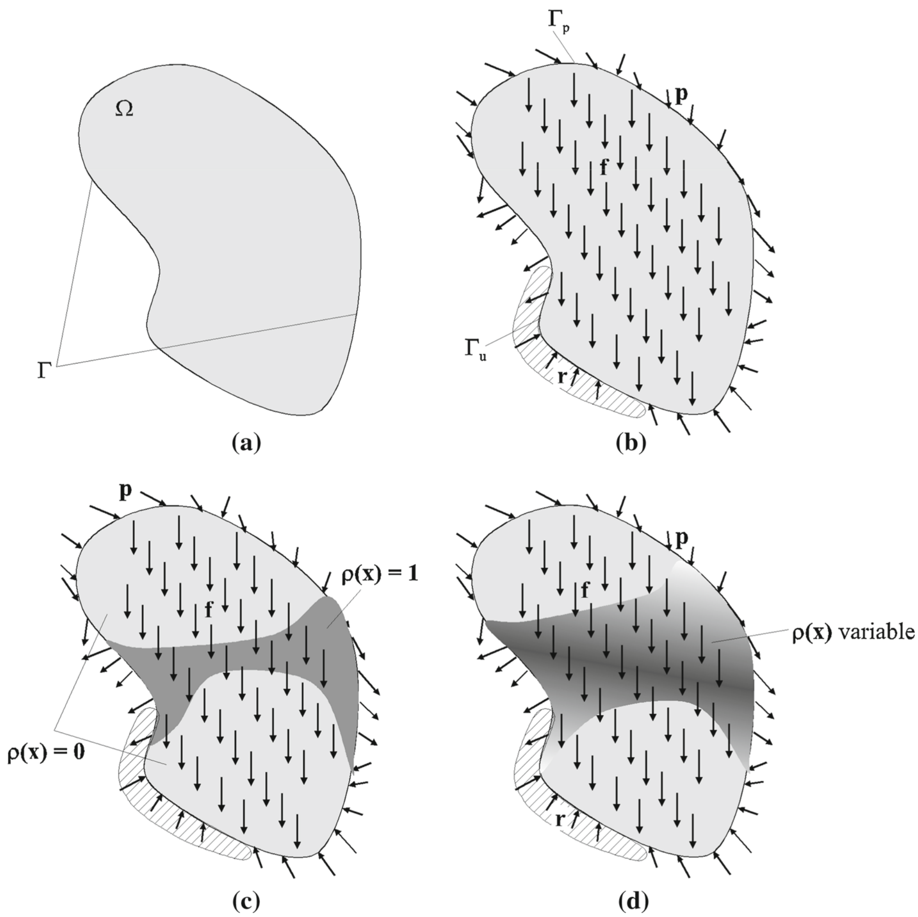

Fig. 2 **a** Basic NT body, **b** body with some constraints and applied forces, **c** reinforced body and **d** body with a variable density reinforcement

(c) The constitutive relations are

$$\left.\begin{array}{l}\boldsymbol{\sigma}_b(\mathbf{x}) = \mathbf{D}_b\,\boldsymbol{\varepsilon}_{be}(\mathbf{x}) \le \mathbf{0}\\ \boldsymbol{\sigma}_r(\mathbf{x}) = \mathbf{D}_r\,\boldsymbol{\varepsilon}_r(\mathbf{x}) = \mathbf{D}_r\,\boldsymbol{\varepsilon}(\mathbf{x})\\ \boldsymbol{\sigma}_b(\mathbf{x})\cdot\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \mathbf{0}\end{array}\right\}\quad \forall\mathbf{x}\in\boldsymbol{\Omega} \tag{4}$$

where $\mathbf{D}_b$, $\mathbf{D}_r$ are the elastic tensors of the basic material and of the reinforcement.

(d) For the equilibrium condition, one gets

$$\begin{array}{c}\mathrm{Div}\,\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}\quad \forall\mathbf{x}\in\boldsymbol{\Omega},\\ \boldsymbol{\sigma}(\mathbf{x})\,\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x})\quad \forall\mathbf{x}\in\Gamma_p\end{array} \tag{5}$$

which, after some substitution, yield

$$\begin{array}{c}\mathrm{Div}\,\boldsymbol{\sigma}_b(\mathbf{x}) + \mathrm{Div}\left[\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x}) = \mathbf{0}\quad \forall\mathbf{x}\in\boldsymbol{\Omega}\\ \left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x})\quad \forall\mathbf{x}\in\Gamma_p,\end{array} \tag{6}$$

with the strength constraint

$$\begin{array}{l}\boldsymbol{\sigma}_b(\mathbf{x}) \le \mathbf{0},\\ \boldsymbol{\sigma}_r(\mathbf{x}) \in \mathbf{D},\end{array} \tag{7}$$

D being the admissible domain for the reinforcement material. Since now on, it is assumed that the reinforcement has an indefinitely elastic behavior and no matter the intensity of stress it is subject to, so the second constraint in Eq. (7) is ineffective.

Some statements can be generically enunciated: i) *compatible displacement fields* are the fields $\mathbf{u}(\mathbf{x})$ obeying the constraint boundary conditions, i.e., $\mathbf{u}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u$; ii) *statically admissible* stress fields are defined such that equilibrium and admissibility stress conditions are fully verified; iii) *kinematically compatible* strain fields are defined such that compatibility (i.e., $\varepsilon(\mathbf{x}) = \varepsilon_b(\mathbf{x}) = \varepsilon_{be}(\mathbf{x}) + \varepsilon_{bf}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$) and admissibility (i.e., $\varepsilon_{bf}(\mathbf{x}) \geq \mathbf{0}$) strain conditions are fully verified.

For any couple of admissible stress and fracture fields $\boldsymbol{\sigma}_b(\mathbf{x})$ and $\varepsilon_{bf}(\mathbf{x})$, the following inequality holds:

$$\boldsymbol{\sigma}_b(\mathbf{x}) \cdot \varepsilon_{bf}(\mathbf{x}) \leq \mathbf{0}. \tag{8}$$

3 Equilibrium analysis: the principle of minimum complementary energy

Let the complementary energy functional be as follows:

$$U(\boldsymbol{\sigma}) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \mathbf{D}_b(\mathbf{x})\, \boldsymbol{\sigma}_b(\mathbf{x})\, \mathrm{d}A + \frac{1}{2}\int_\Omega \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x}) \cdot \mathbf{D}_r(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x})\, \mathrm{d}A - \int_{\Gamma_u} [\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s. \tag{9}$$

If $\boldsymbol{\sigma}_o$ is the solution stress field, it is possible to prove that the condition $U(\boldsymbol{\sigma}) \geq U(\boldsymbol{\sigma}_o)$ holds for any statically admissible stress field $\boldsymbol{\sigma}$. So by considering the following difference:

$$\begin{aligned}\Delta U(\boldsymbol{\sigma}, \boldsymbol{\sigma}_o) = U(\boldsymbol{\sigma}) - U(\boldsymbol{\sigma}_o) &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \mathbf{D}_b(\mathbf{x})\, \boldsymbol{\sigma}_b(\mathbf{x})\, \mathrm{d}A + \frac{1}{2}\int_\Omega \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x}) \cdot \mathbf{D}_r(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x})\, \mathrm{d}A \\ &- \int_{\Gamma_u} [\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s - \frac{1}{2}\int_\Omega \boldsymbol{\sigma}_{bo}(\mathbf{x}) \cdot \mathbf{D}_b(\mathbf{x})\, \boldsymbol{\sigma}_{bo}(\mathbf{x})\, \mathrm{d}A \\ &- \frac{1}{2}\int_\Omega \rho(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x}) \cdot \mathbf{D}_r(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x})\, \mathrm{d}A + \int_{\Gamma_u} [\boldsymbol{\sigma}_{bo}(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s,\end{aligned} \tag{10}$$

after some algebra, one gets

$$\begin{aligned}\Delta U(\boldsymbol{\sigma}, \boldsymbol{\sigma}_o) &= U(\boldsymbol{\sigma}) - U(\boldsymbol{\sigma}_o) = L(\Delta\boldsymbol{\sigma}_b) + L_\rho(\Delta\boldsymbol{\sigma}_r) \\ &+ \int_\Omega [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})] \cdot \mathbf{D}_b(\mathbf{x})\, \boldsymbol{\sigma}_{bo}(\mathbf{x})\, \mathrm{d}A + \int_\Omega [\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})]\, \rho(\mathbf{x}) \cdot \mathbf{D}_r(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x})\, \mathrm{d}A \\ &- \int_{\Gamma_u} [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s - \int_{\Gamma_u} \{[\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})]\, \mathbf{n}(\mathbf{x})\}\, \rho(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s,\end{aligned} \tag{11}$$

with

$$\begin{aligned}L(\Delta\boldsymbol{\sigma}_b) &= \frac{1}{2}\int_\Omega [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})] \cdot \mathbf{D}_b(\mathbf{x})\, [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})]\, \mathrm{d}A \geq 0 \\ L_\rho(\Delta\boldsymbol{\sigma}_r) &= \frac{1}{2}\int_\Omega \rho(\mathbf{x})\, [\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})] \cdot \mathbf{D}_r(\mathbf{x})\, [\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})]\, \mathrm{d}A \geq 0.\end{aligned} \tag{12}$$

The application of the principle of virtual works yields

$$\int_\Omega [\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\sigma}_o(\mathbf{x})] \cdot \varepsilon_o(\mathbf{x})\, \mathrm{d}A - \int_{\Gamma_u} [\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\sigma}_o(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s = 0, \tag{13}$$

whence, after some algebra,

$$\begin{aligned}
&\int_{\Omega} [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})] \cdot \mathbf{D}_b(\mathbf{x})\, \boldsymbol{\sigma}_{bo}(\mathbf{x})\, \mathrm{d}A + \int_{\Omega} \rho(\mathbf{x}) [\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})] \cdot \mathbf{D}_r(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x})\, \mathrm{d}A \\
&\quad - \int_{\Gamma_u} [\boldsymbol{\sigma}_b(\mathbf{x}) - \boldsymbol{\sigma}_{bo}(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s - \int_{\Gamma_u} \rho(\mathbf{x}) [\boldsymbol{\sigma}_r(\mathbf{x}) - \boldsymbol{\sigma}_{ro}(\mathbf{x})]\, \mathbf{n}(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})\, \mathrm{d}s \\
&= \int_{\Omega} [\boldsymbol{\sigma}_{bo}(\mathbf{x}) - \boldsymbol{\sigma}_b(\mathbf{x})] \cdot \boldsymbol{\varepsilon}_{bfo}(\mathbf{x})\, \mathrm{d}A \geq 0,
\end{aligned} \tag{14}$$

and in combination with Eq. (11), it results in

$$\Delta U(\boldsymbol{\sigma}, \boldsymbol{\sigma}_o) = U(\boldsymbol{\sigma}) - U(\boldsymbol{\sigma}_o) \geq 0, \tag{15}$$

for any statically admissible stress field $\boldsymbol{\sigma}(\mathbf{x})$, which also means

$$U(\boldsymbol{\sigma}_o) = \min_{\boldsymbol{\sigma} \in \Sigma} U(\boldsymbol{\sigma}), \tag{16}$$

being Σ the set of all statically admissible stress fields for the reinforced body, the condition $\boldsymbol{\sigma} \in \Sigma$ means

$$\begin{aligned}
&\text{(i)} \quad \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x}), \\
&\text{(ii)} \quad \begin{cases} \mathrm{Div}\boldsymbol{\sigma}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{0}, \\ \boldsymbol{\sigma}(\mathbf{x})\, \mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}), \end{cases} \\
&\text{(iii)} \quad \boldsymbol{\sigma}_b(\mathbf{x}) \leq \mathbf{0}.
\end{aligned} \tag{17}$$

4 Equilibrium analysis: the principle of minimum potential energy

Let the potential energy functional be as follows:

$$\begin{aligned}
E(\mathbf{u}, \boldsymbol{\varepsilon}_{bf}) &= \frac{1}{2} \int_{\Omega} \left[\nabla \mathbf{u}(\mathbf{x}) - \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x}) \left[\nabla \mathbf{u}(\mathbf{x}) - \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\right] \mathrm{d}A \\
&\quad + \frac{1}{2} \int_{\Omega} \rho(\mathbf{x})\, \nabla \mathbf{u}(\mathbf{x}) \cdot \mathbf{C}_r(\mathbf{x})\, \nabla \mathbf{u}(\mathbf{x})\, \mathrm{d}A - \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}s - \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}A,
\end{aligned} \tag{18}$$

with $\mathbf{C}_b$ and $\mathbf{C}_r$ the inverse tensors of $\mathbf{D}_b$ and $\mathbf{D}_r$. If $\mathbf{u}_o$, $\boldsymbol{\varepsilon}_{bfo}$ are the displacements and fractures in the solution, it is possible to verify that $E(\mathbf{u}, \boldsymbol{\varepsilon}_{bf}) \geq E(\mathbf{u}_o, \boldsymbol{\varepsilon}_{bfo})$ for any compatible displacement field $\mathbf{u}$ and any admissible fracture field $\boldsymbol{\varepsilon}_{bf}$.

So by considering the difference

$$\begin{aligned}
&\Delta E(\mathbf{u}, \boldsymbol{\varepsilon}_{bf}; \mathbf{u}_o, \boldsymbol{\varepsilon}_{bfo}) = E(\mathbf{u}, \boldsymbol{\varepsilon}_{bf}) - E(\mathbf{u}_o, \boldsymbol{\varepsilon}_{bfo}) \\
&= \frac{1}{2} \int_{\Omega} \left[\nabla \mathbf{u}(\mathbf{x}) - \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x}) \left[\nabla \mathbf{u}(\mathbf{x}) - \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\right] \mathrm{d}A + \frac{1}{2} \int_{\Omega} \rho(\mathbf{x})\, \nabla \mathbf{u}(\mathbf{x}) \cdot \mathbf{C}_r(\mathbf{x})\, \nabla \mathbf{u}(\mathbf{x})\, \mathrm{d}A \\
&\quad - \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}s - \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}A - \frac{1}{2} \int_{\Omega} \left[\nabla \mathbf{u}_o(\mathbf{x}) - \boldsymbol{\varepsilon}_{bfo}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x}) \left[\nabla \mathbf{u}_o(\mathbf{x}) - \boldsymbol{\varepsilon}_{bfo}(\mathbf{x})\right] \mathrm{d}A \\
&\quad - \frac{1}{2} \int_{\Omega} \rho(\mathbf{x})\, \nabla \mathbf{u}_o(\mathbf{x}) \cdot \mathbf{C}_r(\mathbf{x})\, \nabla \mathbf{u}_o(\mathbf{x})\, \mathrm{d}A + \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}_o(\mathbf{x})\, \mathrm{d}s + \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}_o(\mathbf{x})\, \mathrm{d}A,
\end{aligned} \tag{19}$$

with the condition $\varepsilon_{be}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x}) - \varepsilon_{bf}(\mathbf{x})$, $\varepsilon(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$ and after some algebra, one gets

$$\begin{aligned}
&\Delta E\left(\mathbf{u}, \varepsilon_{bf}; \mathbf{u}_o, \varepsilon_{bfo}\right) = E\left(\mathbf{u}, \varepsilon_{bf}\right) - E\left(\mathbf{u}_o, \varepsilon_{bfo}\right) \\
&= \frac{1}{2}\int_\Omega \left[\varepsilon_{be}(\mathbf{x}) - \varepsilon_{beo}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x}) \left[\varepsilon_{be}(\mathbf{x}) - \varepsilon_{beo}(\mathbf{x})\right] \mathrm{d}A \\
&+ \frac{1}{2}\int_\Omega \rho(\mathbf{x}) \left[\varepsilon(\mathbf{x}) - \varepsilon_o(\mathbf{x})\right] \cdot \mathbf{C}_r(\mathbf{x}) \left[\varepsilon(\mathbf{x}) - \varepsilon_o(\mathbf{x})\right] \mathrm{d}A + \int_\Omega \left[\varepsilon_{be}(\mathbf{x}) - \varepsilon_{beo}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x})\, \varepsilon_{beo}(\mathbf{x})\, \mathrm{d}A \\
&+ \int_\Omega \rho(\mathbf{x}) \left[\varepsilon(\mathbf{x}) - \varepsilon_o(\mathbf{x})\right] \cdot \mathbf{C}_r(\mathbf{x})\, \varepsilon_o(\mathbf{x})\, \mathrm{d}A - \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \left[\mathbf{u}(\mathbf{x}) - \mathbf{u}_o(\mathbf{x})\right] \mathrm{d}s \\
&- \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \left[\mathbf{u}(\mathbf{x}) - \mathbf{u}_o(\mathbf{x})\right] \mathrm{d}A.
\end{aligned} \tag{20}$$

By default

$$\begin{aligned}
&\varepsilon(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x}); \quad \varepsilon_o(\mathbf{x}) = \nabla \mathbf{u}_o(\mathbf{x}), \\
&\varepsilon_{bf}(\mathbf{x}) \geq \mathbf{0}; \quad \varepsilon_{bfo}(\mathbf{x}) \geq \mathbf{0}.
\end{aligned} \tag{21}$$

Moreover, in the solution, the following conditions shall hold:

$$\begin{aligned}
&\text{(i)} \quad \boldsymbol{\sigma}_o(\mathbf{x}) = \boldsymbol{\sigma}_{bo}(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_{ro}(\mathbf{x}) = \mathbf{C}_b(\mathbf{x})\, \varepsilon_{beo}(\mathbf{x}) + \rho(\mathbf{x})\, \mathbf{C}_r(\mathbf{x})\, \varepsilon_{ro}(\mathbf{x}), \\
&\text{(ii)} \quad \boldsymbol{\sigma}_{bo}(\mathbf{x}) = \mathbf{C}_b(\mathbf{x})\, \varepsilon_{beo}(\mathbf{x}), \\
&\text{(iii)} \quad \boldsymbol{\sigma}_{bo}(\mathbf{x}) \cdot \varepsilon_{bfo}(\mathbf{x}) = 0; \; \boldsymbol{\sigma}_{bo}(\mathbf{x}) \cdot \varepsilon_{bf}(\mathbf{x}) \leq 0,
\end{aligned} \tag{22}$$

where $\boldsymbol{\sigma}_{bo}$ must be admissible (i.e., $\boldsymbol{\sigma}_{bo}(\mathbf{x}) \leq 0$). So let apply the principle of virtual works,

$$\begin{aligned}
\Delta L_i = \int_\Omega \boldsymbol{\sigma}_o(\mathbf{x}) \cdot \left[\varepsilon(\mathbf{x}) - \varepsilon_o(\mathbf{x})\right] \mathrm{d}V &= \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \left[\mathbf{u}(\mathbf{x}) - \mathbf{u}_o(\mathbf{x})\right] \mathrm{d}s \\
&+ \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \left[\mathbf{u}(\mathbf{x}) - \mathbf{u}_o(\mathbf{x})\right] \mathrm{d}A = \Delta L_e,
\end{aligned} \tag{23}$$

and after some algebra, remembering the (iii) conditions in Eq. (22), one gets

$$\begin{aligned}
&\int_\Omega \left[\varepsilon_{be}(\mathbf{x}) - \varepsilon_{beo}(\mathbf{x})\right] \cdot \mathbf{C}_b(\mathbf{x})\, \varepsilon_{beo}(\mathbf{x})\, \mathrm{d}A + \int_\Omega \rho(\mathbf{x}) \left[\varepsilon(\mathbf{x}) - \varepsilon_o(\mathbf{x})\right] \cdot \mathbf{C}_r(\mathbf{x})\, \varepsilon_o(\mathbf{x})\, \mathrm{d}A - \Delta L_e \\
&= \int_\Omega \boldsymbol{\sigma}_{bo}(\mathbf{x}) \cdot \left[\varepsilon_{bfo}(\mathbf{x}) - \varepsilon_{bf}(\mathbf{x})\right] \mathrm{d}A \geq 0,
\end{aligned} \tag{24}$$

which combined with Eq. (20) yields

$$\Delta E\left(\mathbf{u}, \varepsilon_{bf}; \mathbf{u}_o, \varepsilon_{bfo}\right) \geq 0 \Leftrightarrow E\left(\mathbf{u}, \varepsilon_{bf}\right) \geq E\left(\mathbf{u}_o, \varepsilon_{bfo}\right) \tag{25}$$

for any compatible displacement field $\mathbf{u}$ and any admissible fracture field ε_{bf}.

In other word, this condition means that

$$E\left(\mathbf{u}_o, \varepsilon_{bfo}\right) = \min_{\substack{\mathbf{u} \in U_t \\ \varepsilon_{bf} \in \mathbb{E}}} E\left(\mathbf{u}, \varepsilon_{bf}\right), \tag{26}$$

with $\mathbf{U}_t$ the set of all compatible displacement fields and $\mathbb{E}$ the set of all admissible fracture fields for the basic body, i.e., the set of fields $\mathbf{u}(\mathbf{x})$ and $\varepsilon_{fb}(\mathbf{x})$ such that

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u, \\
&\varepsilon_{bf}(\mathbf{x}) \geq \mathbf{0} \quad \forall \mathbf{x} \in \boldsymbol{\Omega}.
\end{aligned} \tag{27}$$

5 Limit analysis basics for reinforced NT solids

On the basis of the relations governing the problem introduced in the previous sections, one can establish conditions for equilibrium to be possible or for collapse to be unavoidable under given forces **p**, **f**. So in this section, the basic theorems for a limit analysis of reinforced NT panels will be extended.

Consider basically that any displacement field $\mathbf{u}(\mathbf{x})$, associated with a compatible strain field $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0}$ is a potential collapse mechanism. It is a *kinematically possible* mechanism if the external work is positive,

$$L_e(\mathbf{u}) = \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}s + \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}A > 0. \tag{28}$$

Moreover, consider that the reinforcement is assumed to enjoy very large strength, infinite in the limit. Therefore, fractures cannot occur where the reinforcement is applied, and for a potential collapse mechanism the equality $\boldsymbol{\rho}(\mathbf{x})\, \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \mathbf{0}$ holds everywhere in the solid.

Two criteria are stated as basis for deciding if collapse occurs or not:

(i) if no statically admissible stress field exists under the given forces, collapse occurs;
(ii) if no kinematically possible displacement field exists, collapse cannot occur.

In other words, the existence of a statically admissible stress field is an a priori necessary condition for stability; the existence of a kinematically possible mechanism is an a priori necessary condition for collapse.

5.1 The static theorem approach

By assuming that a statically admissible field $\boldsymbol{\sigma}$ exists, it is expressed as a linear combination of the stress field relevant to the basic body $\boldsymbol{\sigma}_b$ and the stress field relevant to the reinforcement $\boldsymbol{\sigma}_r$, so that $\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x})$. Equilibrium requires that Eq. (6) is verified for admissibility.

Considering any compatible mechanism $(\mathbf{u}, \boldsymbol{\varepsilon}_{bf})$ with $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$ and $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0} \forall \mathbf{x} \in \boldsymbol{\Omega}$, the principle of virtual work yields

$$\int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A = L_e(\mathbf{u}). \tag{29}$$

After some algebra, remembering that $\rho(\mathbf{x})\, \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = 0$, the first term yields

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A &= \int_{\Omega} \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A + \int_{\Omega} \rho(\mathbf{x})\, \boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A \\ &= \int_{\Omega} \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A + \int_{\Omega} \boldsymbol{\sigma}_r(\mathbf{x}) \cdot \left[\rho(\mathbf{x})\, \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\right] \mathrm{d}A \\ &= \int_{\Omega} \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A. \end{aligned} \tag{30}$$

Hence, for any compatible mechanism

$$L_e(\mathbf{u}) = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})\, \mathrm{d}A \leq 0, \tag{31}$$

that means that no kinematically possible mechanism exists.

In conclusion, by the above criterion (ii), "if any admissible stress field exists, no kinematically possible mechanism exists and the structure cannot collapse" (Static Theorem).

5.2 The kinematical theorem approach

By assuming that a kinematically possible collapse mechanism $(\mathbf{u}, \varepsilon_f)$ exists, the relevant external work is positive,

$$L_e(\mathbf{u}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{x}) \cdot \varepsilon_{bf}(\mathbf{x})\, \mathrm{d}A > 0. \tag{32}$$

After Eq. (13), that is in conflict with Eq. (12), in this case no statically admissible stress field would exist.

In conclusion, by the above criterion (i), "if any kinematically possible mechanism exists, no statically admissible stress field exists and the structure must collapse" (Kinematical Theorem).

In synthesis, the static theorem, combined with criterion (i), is a necessary and sufficient condition for stability, and the kinematical theorem, combined with criterion (ii), is a necessary and sufficient condition for collapse.

Note that if a kinematically possible mechanism exists, the structure will collapse, not necessarily according to the considered mechanism. In this sense, any kinematically possible mechanism can be considered equivalent to the *collapse mechanism*.

6 Optimal design of reinforcement by limit analysis approach

The objective is to find the most economical distribution of $\rho(\mathbf{x})$ such that all above conditions are satisfied. For simplicity's sake it is assumed that the reinforcement has a much larger quality than the basic material (e.g., masonry walls reinforced by C-FRP tissues), so that strength conditions for the reinforcement are largely verified and do not need be included in the optimization process.

6.1 The problem layout

The problem can be approached in very different ways; one of these is to consider the objective function as the quantity of reinforcement that is applied to the basic material void volume produced by fractures in the basic body,

$$F(\rho) = \int_\Omega \rho(\mathbf{x})\, \mathrm{d}A = \min, \tag{33}$$

with $\rho(\mathbf{x})$ the local density of the reinforcement, a function with values in (0, 1).

The constraints are:

(a) Equilibrium is to be supplied by a stress field

$$\begin{aligned} &\mathrm{Div}\boldsymbol{\sigma}_b(\mathbf{x}) + \mathrm{Div}\left[\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \boldsymbol{\Omega}, \\ &\left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] \mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_p. \end{aligned} \tag{34}$$

(b) The reaction on the constrained contour Γ_u is equal to

$$\mathbf{r}(\mathbf{x}) = \left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] \mathbf{n}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u. \tag{35}$$

(c) The stress field must be admissible (i.e., $\boldsymbol{\sigma}_b(\mathbf{x}) \le 0$ in $\boldsymbol{\Omega}$)

$$\begin{cases} I_{1b}(\mathbf{x}) \le 0 \\ I_{2b}(\mathbf{x}) \ge 0 \end{cases} \quad \forall \mathbf{x} \in \boldsymbol{\Omega}, \tag{36}$$

with I_{1b} and I_{2b} the first and second invariant of the stresses $\boldsymbol{\sigma}_b(\mathbf{x})$ in the basic material,

$$\begin{aligned} &\boldsymbol{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\ &\boldsymbol{\sigma}_b(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\sigma}_{bx} & \tau_b \\ \tau_b & \boldsymbol{\sigma}_{by} \end{bmatrix}; \quad \boldsymbol{\sigma}_t(\mathbf{x}) = \mathbf{R}^T \boldsymbol{\sigma}_b(\mathbf{x})\, \mathbf{R} = \begin{bmatrix} \boldsymbol{\sigma}_{by} & -\tau_b \\ -\tau_b & \boldsymbol{\sigma}_{bx} \end{bmatrix}, \\ &I_{1b}(\mathbf{x}) = \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \boldsymbol{\Delta}; \quad I_{2b}(\mathbf{x}) = \frac{1}{2}\left[\mathbf{R}^T \boldsymbol{\sigma}_b(\mathbf{x})\, \mathbf{R}\right] \cdot \boldsymbol{\sigma}_b(\mathbf{x}) = \frac{1}{2}\boldsymbol{\sigma}_t(\mathbf{x}) \cdot \boldsymbol{\sigma}_b(\mathbf{x}). \end{aligned} \tag{37}$$

(d) The reinforcement density $\rho(\mathbf{x})$ is everywhere not smaller than 0 and not larger than 1,

$$\begin{cases} \rho(\mathbf{x}) \geq 0 \\ \rho(\mathbf{x}) - 1 \leq 0 \end{cases} \quad \forall \mathbf{x} \in \boldsymbol{\Omega}. \tag{38}$$

6.2 Necessary conditions in the solution

The Lagrangian functional of the problem set up in the previous section, with the introduction of suitable multipliers, can be written down as follows:

$$\begin{aligned}
&\mathcal{L}\left[\rho(\mathbf{x}), \boldsymbol{\sigma}_b(\mathbf{x}), \boldsymbol{\sigma}_r(\mathbf{x}); \boldsymbol{\theta}(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}), \lambda(\mathbf{x}), \mu(\mathbf{x}), \omega(\mathbf{x}), \beta(\mathbf{x})\right] \\
&= \int_\Omega \rho(\mathbf{x})\,\mathrm{d}A + \int_\Omega \boldsymbol{\theta}^T(\mathbf{x})\left\{\mathbf{Div}\left[\boldsymbol{\sigma}_b(\mathbf{x})\right] + \mathbf{Div}\left[\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x})\right\}\mathrm{d}A \\
&+ \int_{\Gamma_p} \boldsymbol{\varphi}^T(\mathbf{x})\left\{\left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) - \mathbf{p}(\mathbf{x})\right\}\mathrm{d}s \\
&+ \int_{\Gamma_u} \boldsymbol{\psi}^T(\mathbf{x})\left\{\left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) - \mathbf{r}(\mathbf{x})\right\}\mathrm{d}s \\
&+ \int_\Omega \lambda(\mathbf{x})\,I_{1b}(\mathbf{x})\,\mathrm{d}A - \int_\Omega \mu(\mathbf{x})\,I_{2b}(\mathbf{x})\,\mathrm{d}A + \int_\Omega \omega(\mathbf{x})\left[\rho(\mathbf{x}) - 1\right]\mathrm{d}A - \int_\Omega \beta(\mathbf{x})\,\rho(\mathbf{x})\,\mathrm{d}A,
\end{aligned} \tag{39}$$

where it is intended that $\boldsymbol{\theta}(\mathbf{x})$, $\boldsymbol{\varphi}(\mathbf{x})$ and $\boldsymbol{\psi}(\mathbf{x})$ are vector fields, $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$, $\omega(\mathbf{x})$ and $\beta(\mathbf{x})$ are non-negative scalar fields, and $\mathbf{n}(\mathbf{x})$ the normal to the boundary, and all the following conditions are satisfied:

$$\begin{aligned}
&\left.\begin{array}{lll}
\lambda(\mathbf{x})\,I_{1b}(\mathbf{x}) = 0 & ;\ I_{1b}(\mathbf{x}) \leq 0 & ;\ \lambda(\mathbf{x}) \geq 0 \\
\mu(\mathbf{x})\,I_{2b}(\mathbf{x}) = 0 & ;\ -I_{2b}(\mathbf{x}) \leq 0 & ;\ \mu(\mathbf{x}) \geq 0 \\
\omega(\mathbf{x})\left[\rho(\mathbf{x}) - 1\right] = 0 & ;\ \rho(\mathbf{x}) - 1 \leq 0 & ;\ \omega(\mathbf{x}) \geq 0 \\
\beta(\mathbf{x})\,\rho(\mathbf{x}) = 0 & ;\ -\rho(\mathbf{x}) \leq 0 & ;\ \beta(\mathbf{x}) \geq 0
\end{array}\right\} \text{ in } \Omega, \\
&\boldsymbol{\theta}^T(\mathbf{x})\left\{\mathbf{Div}\left[\boldsymbol{\sigma}_b(\mathbf{x})\right] + \mathbf{Div}\left[\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x})\right\} = 0;\ \mathbf{Div}\left[\boldsymbol{\sigma}_b(\mathbf{x})\right] + \mathbf{Div}\left[\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right] + \mathbf{f}(\mathbf{x}) = \mathbf{0} \quad \text{in} \quad \Omega, \\
&\phi^T(\mathbf{x})\left\{\left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) - \mathbf{p}(\mathbf{x})\right\} = 0;\ \left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \quad \text{on} \quad \Gamma_p, \\
&\boldsymbol{\psi}^T(\mathbf{x})\left\{\left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) - \mathbf{r}(\mathbf{x})\right\} = 0;\ \mathbf{r}(\mathbf{x}) = \left[\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) \quad \text{on} \quad \Gamma_u.
\end{aligned} \tag{40}$$

The following variational conditions must be fulfilled in the solution, taking into account that basic variables are the tensor fields $\boldsymbol{\sigma}_b(\mathbf{x})$ and $\boldsymbol{\sigma}_r(\mathbf{x})$, the reaction field $\mathbf{r}(\mathbf{x})$, and the scalar field $\rho(\mathbf{x})$.

1. *For independent variation of the stress field $\boldsymbol{\sigma}_r(\mathbf{x})$ in the reinforcement*

$$\begin{aligned}
&\delta_{\boldsymbol{\sigma}_r} L\left[\rho(\mathbf{x}), \boldsymbol{\sigma}_b(\mathbf{x}), \boldsymbol{\sigma}_r(\mathbf{x}); \boldsymbol{\theta}(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}), \lambda(\mathbf{x}), \mu(\mathbf{x}), \omega(\mathbf{x}), \beta(\mathbf{x})\right] = 0 \quad \forall \delta\boldsymbol{\sigma}_r(\mathbf{x}) \\
&\Downarrow \\
&\int_\Omega \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A + \int_{\Gamma_p} \rho(\mathbf{x})\,\boldsymbol{\varphi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s \\
&\quad + \int_{\Gamma_u} \rho(\mathbf{x})\,\boldsymbol{\psi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s = 0 \quad \forall \delta\boldsymbol{\sigma}_r(\mathbf{x}).
\end{aligned} \tag{41}$$

From the identity (valid for symmetric tensors), one gets

$$\begin{aligned}
&\mathrm{div}\left[\rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right] = \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\right] + \rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \nabla\boldsymbol{\theta}(\mathbf{x}), \\
&\Downarrow \\
&\boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\right] = \mathrm{div}\left[\rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right] - \rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \nabla\boldsymbol{\theta}(\mathbf{x}),
\end{aligned} \tag{42}$$

and by the Gauss theorem applied to the first term of Eq. (41)

$$\begin{aligned}\int_{\Omega} \theta^T(\mathbf{x})\mathrm{Div}\left[\rho(\mathbf{x})\delta\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A &= \int_{\Omega} \mathrm{div}\left[\rho(\mathbf{x})\delta\boldsymbol{\sigma}_r\theta(\mathbf{x})\right]\mathrm{d}A - \int_{\Omega} \rho(\mathbf{x})\delta\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\theta(\mathbf{x})\mathrm{d}A \\ &= \int_{\Gamma} \left[\rho(\mathbf{x})\delta\boldsymbol{\sigma}_r(\mathbf{x})\theta(\mathbf{x})\right]\cdot\mathbf{n}(\mathbf{x})\mathrm{d}s - \int_{\Omega} \rho(\mathbf{x})\delta\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\theta(\mathbf{x})\mathrm{d}A \\ &= \int_{\Gamma} \left[\rho(\mathbf{x})\theta(\mathbf{x})^T\delta\boldsymbol{\sigma}_r(\mathbf{x})\right]\cdot\mathbf{n}(\mathbf{x})\mathrm{d}s - \int_{\Omega} \rho(\mathbf{x})\delta\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\theta(\mathbf{x})\mathrm{d}A. \end{aligned} \tag{43}$$

The variation in Eq. (41) becomes

$$\begin{aligned} &\left.\begin{aligned} &\int_{\Gamma} \rho(\mathbf{x})\,\theta^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s - \int_{\Omega} \rho(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\theta(\mathbf{x})\,\mathrm{d}A, \\ &+\int_{\Gamma_p} \rho(\mathbf{x})\,\boldsymbol{\phi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s + \int_{\Gamma_u} \rho(\mathbf{x})\,\boldsymbol{\psi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s = 0 \end{aligned}\right\} \quad \forall\delta\boldsymbol{\sigma}_r(\mathbf{x}), \\ &\qquad\Downarrow \\ &\left.\begin{aligned} &-\int_{\Omega} \rho(\mathbf{x})\,\nabla\theta(\mathbf{x})\cdot\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathrm{d}A + \int_{\Gamma_p} \rho(\mathbf{x})\left[\boldsymbol{\phi}(\mathbf{x}) + \theta(\mathbf{x})\right]^T\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s \\ &+\int_{\Gamma_u} \rho(\mathbf{x})\left[\boldsymbol{\psi}(\mathbf{x}) + \theta(\mathbf{x})\right]^T\delta\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s = 0 \end{aligned}\right\} \quad \forall\delta\boldsymbol{\sigma}_r(\mathbf{x}). \end{aligned} \tag{44}$$

So that Eq. (44) is verified if and only if it happens that all the terms are null,

$$\rho(\mathbf{x})\,\nabla\theta(\mathbf{x}) = \mathbf{0} \quad \forall\mathbf{x}\in\boldsymbol{\Omega}, \tag{45}$$

$$\begin{aligned} \boldsymbol{\phi}(\mathbf{x}) &= -\theta(\mathbf{x}) \quad \text{on} \quad \Gamma_p, \\ \boldsymbol{\psi}(\mathbf{x}) &= -\theta(\mathbf{x}) \quad \text{on} \quad \Gamma_u. \end{aligned} \tag{46}$$

2. *For independent variation of the stress field $\boldsymbol{\sigma}_b(\mathbf{x})$ in the basic material*

$$\begin{aligned} &\delta_{\boldsymbol{\sigma}_b}\mathcal{L}\left[\rho(\mathbf{x}),\boldsymbol{\sigma}_b(\mathbf{x}),\boldsymbol{\sigma}_r(\mathbf{x});\theta(\mathbf{x}),\boldsymbol{\varphi}(\mathbf{x}),\boldsymbol{\psi}(\mathbf{x}),\lambda(\mathbf{x}),\mu(\mathbf{x}),\omega(\mathbf{x}),\beta(\mathbf{x})\right] = 0 \quad \forall\delta\boldsymbol{\sigma}_b(\mathbf{x}), \\ &\qquad\Downarrow \\ &\left.\begin{aligned} &\int_{\Omega} \theta^T(\mathbf{x})\,\mathbf{Div}\left[\delta\boldsymbol{\sigma}_b(\mathbf{x})\right]\mathrm{d}A + \int_{\Gamma_p} \boldsymbol{\varphi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s + \int_{\Gamma_u} \boldsymbol{\psi}^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s \\ &\qquad + \int_{\Omega} \lambda(\mathbf{x})\,\delta I_{1b}(\mathbf{x})\,\mathrm{d}A - \int_{\Omega} \mu(\mathbf{x})\,\delta I_{2b}(\mathbf{x})\,\mathrm{d}A = 0 \end{aligned}\right\} \quad \forall\delta\boldsymbol{\sigma}_b(\mathbf{x}), \\ &\qquad\Downarrow \\ &\left.\begin{aligned} &\int_{\Omega} \theta^T(\mathbf{x})\,\mathbf{Div}\left[\delta\boldsymbol{\sigma}_b(\mathbf{x})\right]\mathrm{d}A - \int_{\Gamma} \theta^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s \\ &+\int_{\Omega} \lambda(\mathbf{x})\,\delta I_{1b}(\mathbf{x})\,\mathrm{d}A - \int_{\Omega} \mu(\mathbf{x})\,\delta I_{2b}(\mathbf{x})\,\mathrm{d}A = 0 \end{aligned}\right\} \quad \forall\delta\boldsymbol{\sigma}_b(\mathbf{x}), \end{aligned} \tag{47}$$

where the result in Eq. (46) was considered, too. Since the following identity holds:

$$\int_{\Omega} \theta^T(\mathbf{x})\,\mathbf{Div}\left[\delta\boldsymbol{\sigma}_b(\mathbf{x})\right]\mathrm{d}A = \int_{\Gamma} \theta^T(\mathbf{x})\,\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathbf{n}(\mathbf{x})\,\mathrm{d}s - \int_{\Omega} \nabla\theta(\mathbf{x})\cdot\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathrm{d}A, \tag{48}$$

one gets

$$-\int_{\Omega} \nabla\theta(\mathbf{x})\cdot\delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathrm{d}A + \int_{\Omega} \lambda(\mathbf{x})\,\delta I_{1b}(\mathbf{x})\,\mathrm{d}A - \int_{\Omega} \mu(\mathbf{x})\,\delta I_{2b}(\mathbf{x})\,\mathrm{d}A = 0 \quad \forall\delta\boldsymbol{\sigma}_b(\mathbf{x}), \tag{49}$$

and remembering the conditions in Eq. (37)

$$\delta I_{1b} = \boldsymbol{\Delta}\cdot\delta\boldsymbol{\sigma}_b; \quad \delta I_{2b} = \frac{1}{2}\left(\boldsymbol{\sigma}_b\cdot\delta\boldsymbol{\sigma}_t + \boldsymbol{\sigma}_t\cdot\delta\boldsymbol{\sigma}_b\right) = \boldsymbol{\sigma}_t\cdot\delta\boldsymbol{\sigma}_b, \tag{50}$$

the variation becomes

$$\int_{\Omega} [\lambda(\mathbf{x})\,\boldsymbol{\Delta} - \mu(\mathbf{x})\,\boldsymbol{\sigma}_t(\mathbf{x}) - \nabla\boldsymbol{\theta}(\mathbf{x})] \cdot \delta\boldsymbol{\sigma}_b(\mathbf{x})\,\mathrm{d}A = 0 \quad \forall\delta\boldsymbol{\sigma}_b(\mathbf{x}), \tag{51}$$

whence

$$\nabla\boldsymbol{\theta}(\mathbf{x}) = \lambda(\mathbf{x})\,\boldsymbol{\Delta} - \mu(\mathbf{x})\,\boldsymbol{\sigma}_t(\mathbf{x}) \quad \forall\mathbf{x} \in \boldsymbol{\Omega}, \tag{52}$$

where $\boldsymbol{\sigma}_t(\mathbf{x})$ is coaxial with $\boldsymbol{\sigma}_b(\mathbf{x})$, and the two tensors possess the same invariants and non-positive eigenvalues. Hence, considering that $\lambda(\mathbf{x}) \geq 0$ and $\mu(\mathbf{x}) \geq 0$

$$\nabla\boldsymbol{\theta}(\mathbf{x}) \geq \mathbf{0} \quad \forall\mathbf{x} \in \boldsymbol{\Omega}. \tag{53}$$

Moreover, by considering the product

$$\nabla\boldsymbol{\theta}(\mathbf{x}) \cdot \boldsymbol{\sigma}_b(\mathbf{x}) = \lambda(\mathbf{x})\,\boldsymbol{\Delta} \cdot \boldsymbol{\sigma}_b(\mathbf{x}) - \mu(\mathbf{x})\,\boldsymbol{\sigma}_t(\mathbf{x}) \cdot \boldsymbol{\sigma}_b(\mathbf{x}) = \lambda(\mathbf{x})\,I_{1b}(\mathbf{x}) - 2\mu(\mathbf{x})\,I_{2b}(\mathbf{x}) = 0 \quad \forall\mathbf{x} \in \boldsymbol{\Omega}, \tag{54}$$

it can be intended that the multiplier function $\boldsymbol{\theta}(\mathbf{x})$ is proportional to some admissible displacement field $\mathbf{u}(\mathbf{x})$ by means of a positive factor $k > 0$,

$$\boldsymbol{\theta}(\mathbf{x}) = k\mathbf{u}(\mathbf{x}), \tag{55}$$

whence

$$\begin{aligned} \nabla\boldsymbol{\theta}(\mathbf{x}) &= k\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq \mathbf{0}, \\ \rho(\mathbf{x})\,\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) &= 0 \quad \forall\mathbf{x} \in \boldsymbol{\Omega}. \end{aligned} \tag{56}$$

3. *For independent variation of the reaction $r(x)$ on Γ_u*

$$\begin{gathered} \delta_{\mathbf{r}}\mathcal{L}[\rho(\mathbf{x}), \boldsymbol{\sigma}_b(\mathbf{x}), \boldsymbol{\sigma}_r(\mathbf{x}); \boldsymbol{\theta}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}), \lambda(\mathbf{x}), \mu(\mathbf{x}), \omega(\mathbf{x}), \beta(\mathbf{x})] = 0 \quad \forall\delta\mathbf{r}(\mathbf{x}) \\ \Downarrow \\ -\int_{\Gamma_u} \boldsymbol{\psi}^T(\mathbf{x})\,\delta\mathbf{r}(\mathbf{x})\,\mathrm{d}s = 0 \quad \forall\delta\mathbf{r}(\mathbf{x}) \\ \Downarrow \\ \boldsymbol{\psi}(\mathbf{x}) = -\boldsymbol{\theta}(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma_u. \end{gathered} \tag{57}$$

In general, so if one considers the variations of the stress field in the reinforcement, in the basic body and of the reaction [developed into the points 1, 2 and 3], one concludes that the field $\boldsymbol{\theta}(\mathbf{x})$ yields a potential collapse mechanism with unlimited k.

If the work produced by the applied loads is considered, it results non-positive as

$$\begin{aligned} L_e &= \int_{\Omega} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A + \int_{\Gamma_p} \mathbf{p}(\mathbf{x}) \cdot \boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}s = \int_{\Omega} [\boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})] \cdot \nabla\boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A \\ &= \int_{\Omega} \boldsymbol{\sigma}_b(\mathbf{x}) \cdot \nabla\boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A \leq 0, \end{aligned} \tag{58}$$

being $\rho(\mathbf{x}) \cdot \nabla\boldsymbol{\theta}(\mathbf{x}) = \mathbf{0}$ [by point (1)] and $\nabla\boldsymbol{\theta}(\mathbf{x}) \geq \mathbf{0}$ [by point (2)] with $\boldsymbol{\sigma}_b(\mathbf{x}) \leq \mathbf{0}$, and collapse cannot occur according to the mechanism $\boldsymbol{\theta}(\mathbf{x})$.

4. *For independent variation of the reinforcement distribution $\boldsymbol{\rho}(\mathbf{x})$*

$$\delta_\rho \mathcal{L}\left[\rho(\mathbf{x}), \boldsymbol{\sigma}_b(\mathbf{x}), \boldsymbol{\sigma}_r(\mathbf{x}); \boldsymbol{\theta}(\mathbf{x}), \boldsymbol{\varphi}(\mathbf{x}), \boldsymbol{\psi}(\mathbf{x}), \lambda(\mathbf{x}), \mu(\mathbf{x}), \omega(\mathbf{x}), \beta(\mathbf{x})\right] = 0 \quad \forall \delta\rho(\mathbf{x}),$$
$$\Downarrow$$
$$\left.\begin{array}{l} \int\limits_\Omega \delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \omega(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A - \int\limits_\Omega \beta(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A \\ \quad + \int\limits_{\Gamma_p} \boldsymbol{\varphi}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s + \int\limits_{\Gamma_u} \boldsymbol{\psi}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s = 0 \end{array}\right\} \forall \delta\rho(\mathbf{x}). \tag{59}$$

Remembering that

$$\left\{\begin{array}{l} \boldsymbol{\phi}(\mathbf{x}) = -\boldsymbol{\theta}(\mathbf{x}) = -k\mathbf{u}(\mathbf{x}) \text{ on } \Gamma_p, \\ \boldsymbol{\psi}(\mathbf{x}) = -\boldsymbol{\theta}(\mathbf{x}) = -k\mathbf{u}(\mathbf{x}) \text{ on } \Gamma_u, \end{array}\right. \tag{60}$$

$$\left.\begin{array}{l} \int\limits_\Omega \delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A - \int\limits_{\Gamma_p} \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s + \\ \quad - \int\limits_{\Gamma_u} \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s + \int\limits_\Omega \omega(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A - \int\limits_\Omega \beta(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A = 0 \end{array}\right\} \forall \delta\rho(\mathbf{x}),$$
$$\Downarrow \tag{61}$$
$$\left.\begin{array}{l} \int\limits_\Omega \delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \left[\omega(\mathbf{x}) - \beta(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A + \\ \quad - \int\limits_\Gamma \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s = 0 \end{array}\right\} \forall \delta\rho(\mathbf{x}),$$

and by considering that for symmetric tensors as $\boldsymbol{\sigma}_r(\mathbf{x})$

$$\begin{array}{c} \operatorname{div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right] = \left[\mathbf{Div}\,\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\cdot\boldsymbol{\theta}(\mathbf{x}) + \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x}) \\ = \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\,\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x}) + \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x}) \\ \Downarrow \\ \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\,\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x}) = \operatorname{div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right] - \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x}), \end{array} \tag{62}$$

the third term in Eq. (61) becomes

$$\begin{aligned} \int\limits_\Omega \boldsymbol{\theta}^T(\mathbf{x})\,\mathbf{Div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\right]\mathrm{d}A &= \int\limits_\Omega \operatorname{div}\left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right]\mathrm{d}A - \int\limits_\Omega \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A \\ &= \int\limits_\Gamma \left[\delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\,\boldsymbol{\theta}(\mathbf{x})\right]\cdot\mathbf{n}(\mathbf{x})\,\mathrm{d}s - \int\limits_\Omega \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A \\ &= \int\limits_\Gamma \delta\rho(\mathbf{x})\,\boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\mathrm{d}s - \int\limits_\Omega \delta\rho(\mathbf{x})\,\boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\,\mathrm{d}A \\ &= \int\limits_\Gamma \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s - \int\limits_\Omega \boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A, \end{aligned} \tag{63}$$

and the variation in Eq. (61) becomes

$$\left.\begin{array}{l} \int\limits_\Omega \delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Omega \omega(\mathbf{x})\left[2\rho(\mathbf{x}) - 1\right]\delta\rho(\mathbf{x})\,\mathrm{d}A + \int\limits_\Gamma \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s \\ - \int\limits_\Gamma \boldsymbol{\theta}^T(\mathbf{x})\left[\boldsymbol{\sigma}_r(\mathbf{x})\,\mathbf{n}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}s - \int\limits_\Omega \boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\,\delta\rho(\mathbf{x})\,\mathrm{d}A = 0 \end{array}\right\} \forall \delta\rho(\mathbf{x}),$$
$$\Downarrow \tag{64}$$
$$\int\limits_\Omega \left[1 + \omega(\mathbf{x}) - \beta(\mathbf{x}) - \boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x})\right]\delta\rho(\mathbf{x})\,\mathrm{d}A = 0 \quad \forall \delta\rho(\mathbf{x})$$

which is verified if and only if the following condition is verified:

$$1 + \omega(\mathbf{x}) - \beta(\mathbf{x}) - \boldsymbol{\sigma}_r(\mathbf{x})\cdot\nabla\boldsymbol{\theta}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \boldsymbol{\Omega}. \tag{65}$$

7 Some results for a combined limit analysis and optimization approach

From the conditions that have been set for the optimal reinforcement distribution, one can conclude that the reinforcement should be distributed according to the following rules.

(a) If $\nabla\theta(\mathbf{x}) = k\varepsilon_{bf} > \mathbf{0}$, and remembering that $\boldsymbol{\sigma}_r(\mathbf{x}) \geq \mathbf{0}$, Eq. (65) yields

$$\beta(\mathbf{x}) - \omega(\mathbf{x}) = 1 - k\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}). \tag{66}$$

Since the coefficient k is arbitrarily large, one can assume

$$k > \sup_{\mathbf{x}\in\boldsymbol{\Omega}:\boldsymbol{\sigma}_r(\mathbf{x})\cdot\boldsymbol{\varepsilon}_{bf}(\mathbf{x})>0} \left[\frac{1}{\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x})}\right] > 0, \tag{67}$$

leading to

$$1 - k\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) < 0 \rightarrow \beta(\mathbf{x}) - \omega(\mathbf{x}) < 0, \tag{68}$$

and by Eq. (40)

$$\omega(\mathbf{x}) > 0 \rightarrow \rho(\mathbf{x}) = 1; \quad \beta(\mathbf{x}) = 0. \tag{69}$$

(b) By contrast, if $\nabla\theta(\mathbf{x}) = k\varepsilon_{bf}(\mathbf{x}) = \mathbf{0}$, by Eq. (65) one gets

$$k\boldsymbol{\sigma}_r(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = 0 \rightarrow 1 + \omega(\mathbf{x}) - \beta(\mathbf{x}) = 0, \tag{70}$$

whence

$$\beta(\mathbf{x}) > 0 \rightarrow \rho(\mathbf{x}) = 0; \quad \omega(\mathbf{x}) = 0, \tag{71}$$

still agreeing into Eq. (45). One corollary is that $\boldsymbol{\rho}(\mathbf{x})$ in the solution is equal to 0 without any reinforcement or 1 with a full reinforcement.

In order to accomplish for a first numerical application, a beam made by basic no-tension material has been considered with the possibility to incorporate a tensile-resistant material (e.g., FRP sheets or steel beams, or others) (Fig. 3).

Through the search of an NT equilibrium displacement field, according to the algorithms recalled in Sect. 4, fractures $\boldsymbol{\varepsilon}_{bf}(\mathbf{x})$ are subsequently introduced in the basic material, while displacements $\mathbf{u}(\mathbf{x})$ reproduce a collapse mechanism, i.e., converge toward the displacements $\theta(\mathbf{x})$.

In a self-made basic program during the iterations, the reinforcement is progressively applied where the condition (a) in the above holds and it is removed when the condition (b) is encountered. Some results are presented in Fig. 4, where the program automatically reproduces the usual reinforcement with steel bars.

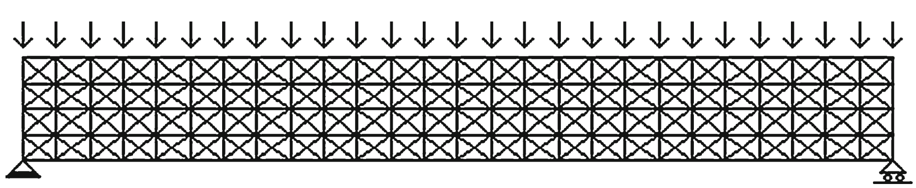

Fig. 3 No-tension beam with constraints and applied loads used in numerical tests

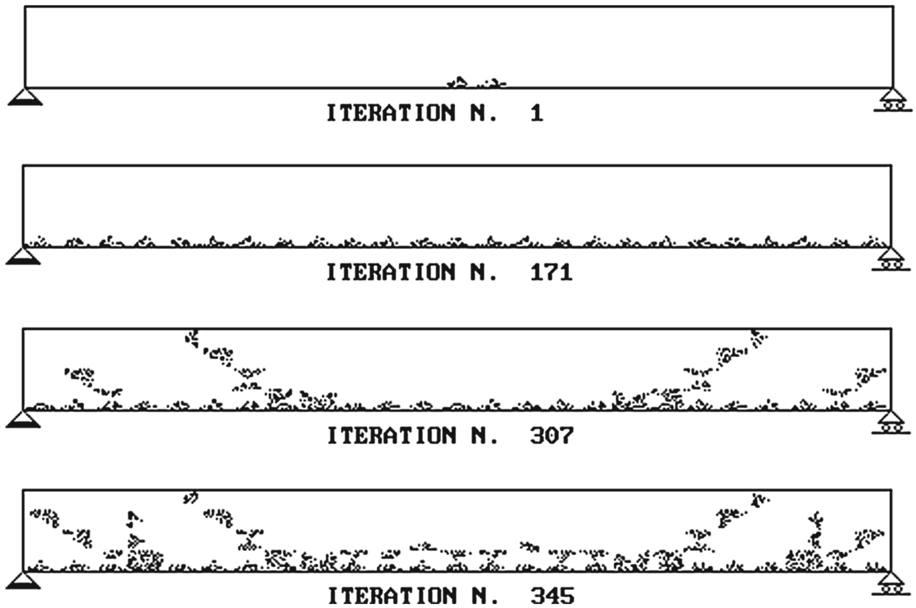

Fig. 4 Subsequent phases of optimal reinforcement distribution after different iterations

8 Conclusions

In the paper, the reinforcement of no-tension structures by the application of superposed high-strength sheets or insertion of tensile bars has been considered, with the purpose to set up a design path aiming to distribute the new material according to some optimal criterion. An approach through "topologic optimization" is proposed, a topic that is receiving increasing attention in the area of Structural Design. No-tension models are recognized as an effective tool for analyzing a large class of civil engineering structures (say, e.g., masonry and reinforced concrete members), so that the problem is strongly felt also from the point of view of practical technique.

Moreover, for simplicity's sake, the reinforcement is subject to a stress condition very far from the limit state, so that failure of reinforcement is not included in the treatment, provided that a posteriori tests are implied.

Under such simple assumptions (that could anyway be easily relaxed), the equilibrium and the failure analysis of the reinforced body are dealt, for static assessment of the reinforced system, with particular reference to the ultimate limit state of collapse. Thereafter, optimal reinforcement is approached by stating one possible criterion that would allow to design the best distribution of reinforcement over the existing body, and the relevant objective function and constraints have been formulated. A first implementation of a search procedure has been performed, with reference to the location and shaping of steel bars or, equivalently, of FRP stripes.

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