Acta Mech 225, 413–430 (2014)
DOI 10.1007/s00707-013-0969-0

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Hybrid reliability analysis of structures with multi-source uncertainties

Received: 6 June 2013 / Published online: 18 August 2013

Abstract A new hybrid reliability analysis technique based on the convex modeling theory is developed for structures with multi-source uncertainties, which may contain randomness, fuzziness, and non-probabilistic boundedness. By solving the convex modeling reliability problem and further analyzing the correlation within uncertainties, the structural hybrid reliability is obtained. Considering various cases of uncertainties of the structure, four hybrid models including the convex with random, convex with fuzzy random, convex with interval, and convex with other three are built, respectively. The present hybrid models are compared with the conventional probabilistic and the non-probabilistic models by two typical numerical examples. The results demonstrate the accuracy and effectiveness of the proposed hybrid reliability analysis method.

1 Introduction

With the growing complexity of practical engineering problems, the uncertainty relating to material properties, loads, boundary conditions, etc., has become more and more profound [1–5]. Traditional analytic approaches derived from probability models and fuzzy models have been widely applied to varieties of industrial communities in past decades [6–10]. Traditional structural reliability analyses require precise probability distributions or membership functions of the uncertain parameters based on a large amount of experimental samples. However, in many engineering applications, the experimental data are often limited, and thus, the requirement of the available data to justify either the probabilistic reliability model or the fuzzy reliability model is not satisfied. The given subjective assumptions on description of the uncertainty characteristics are likely to bring about a serious error of the reliability analysis [11–14].

Some non-probabilistic methods for analyzing reliability via limited parametric data have been developed and been paid more and more attention during the past two decades. Ben-Haim [15] first proposed the concept of structural non-probabilistic safety based on the convex model. Elishakoff [16] first proposed a quantitative measure of the non-probabilistic safety based on interval analysis. Guo et al. [17–19] extended the traditional first-order reliability method (FORM) into the interval convex model, and thereby quantified the uncertain structural parameters as interval variables and proposed another measure of the 'non-probabilistic reliability,' which was taken as the shortest distance from the origin to the failure surface. Qiu et al. [14,20,21] suggested a non-probabilistic model of convex reliability using the partial order relation of the superscribed hyper-rectangle or hyper-ellipsoid. Jiang et al. [22,23] carried out a correlation analysis for the non-probabilistic

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convex models, and further developed an effective method of construction of the multi-dimensional ellipsoids on the uncertainty in order to overcome the drawback of the non-probabilistic convex reliability in complex structural engineering. Several reliability-based optimization design methods were also developed by treating the non-probabilistic reliability indexes as constraints [24–26].

However, most of the existing reliability analyses generally employ the single-source uncertainty models, which consider randomness, fuzziness, or non-probabilistic (interval/convex) uncertainty separately rather than their combination. In view of the complexity in practical applications, there is considerable interest in developing efficient methods for dealing with problems comprising of mixed uncertain variables [27].

In recent years, researchers have studied the hybrid reliability analysis structures. When the probabilistic and interval variables appear in the same problem, numerical methods have been proposed. These include the function approximation technique [28], the iterative rescaling method [29], the probability bounds approach [30], the mixed perturbation Monte Carlo method [31], and the complex nesting optimization algorithm [32], among others [6,33–37]. Randomness and fuzziness/convexity have also been combined for hybrid reliability analysis [8,38,39].

Nevertheless, the hybrid reliability analysis is still in its preliminary stage, and some important issues still remain unsolved. One difficulty is the construction and solution of the mixed models containing multiple types of uncertainties, such as randomness, fuzziness, and non-probabilistic uncertainty. Moreover, the interval variables and the convex variables have been rarely investigated simultaneously. Therefore, it is necessary to develop effective hybrid reliability analytical techniques and propose a series of safety assessments of the practical complicated structures based on multi-source uncertainties.

This paper aims at developing a new reliability analysis method for uncertain structures with the mixture of randomness, fuzziness, and non-probabilistic uncertainty. The remainder of this paper is organized as follows. First, the traditional reliability analysis deduced by single-source uncertainty is introduced. Second, four hybrid reliability analysis models including the convex with random, convex with fuzzy random, convex with interval, and convex with other three are proposed, respectively. Two numerical examples are then provided to demonstrate the effectiveness of the present method, followed by some conclusions.

2 Probabilistic reliability and fuzzy random reliability

2.1 Structural probability-based reliability model

Traditional probabilistic reliability can typically be measured by the probability of structural functions that satisfy certain requirements. The structural function is expressed by the limit state function, which is determined by the failure criteria. Consider a limit state function of the structure in the following form:

$$M = g(\boldsymbol{X}) = g(X_1, X_2, \ldots, X_n) \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ is the n-dimensional random variable vector. $M = g(\boldsymbol{X}) = 0$ represents the failure surface, which divides the variable space into two parts, namely the failure region and the safety region. Hence, the reliability of the structure can be expressed as

$$R_s = 1 - P_f = 1 - \int\int_{\Omega_f} \cdots \int f_X(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \tag{2}$$

where P_f is the failure probability, Ω_f is the failure region, and $f_X(x_1, x_2, \ldots, x_n)$ is the joint probability density function of the basic random variables $X_1, X_2, \ldots, X_n$. The random reliability index β is defined as the minimum distance between the origin and the failure surface of the standard normal variable space, i.e.,

$$\beta = \min\left\{\|\boldsymbol{u}\|_2^2\right\} = \min\left\{\sum_{i=1}^{n} u_i^2\right\} \tag{3}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T \in (-\infty, \infty)$ are standard normal variables. Consider a linear performance function

$$M = g(\boldsymbol{X}) = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \tag{4}$$

where a_i $(i = 1, 2, \ldots, n)$ are constants. The reliability index β can be obtained by

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2}} \tag{5}$$

where μ and σ represent the mean value and the standard deviation, respectively. Consequently, the structural reliability based on the probabilistic model can be rewritten as follows:

$$R_s = 1 - \Phi(-\beta) \tag{6}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

If the normal random variables are correlated with each other, their correlation coefficients are necessary to derive the reliability. For problems with non-Gaussian random variables, some techniques, such as Rosenblatt's transformation [40] and Rackwitz–Fiessler transformation [41], can be adopted to transform the distribution into approximately equivalent normal distribution. Subsequently, FORM [42] can be implemented for solving the multi-fold integration in Eq. (2).

2.2 Structural fuzzy random reliability model

Fuzziness is usually involved in the basic random variables. For instance, structural stress is determined by various factors, such as external loads, geometry size, and supporting conditions. The fuzziness of the stress is entirely determined by the fuzziness of these factors. Similar to Eq. (1), the fuzzy failure surface can be written as

$$M = g\left(\tilde{\boldsymbol{X}}\right) = g\left(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n\right) = 0 \tag{7}$$

where $\tilde{\boldsymbol{X}}$ denotes the n-dimensional fuzzy random vector. Let $\mu_{\tilde{X}}(\boldsymbol{X})$ be the membership function of $\tilde{\boldsymbol{X}}$, the failure probability of the fuzzy random structure is expressed as [43]

$$P_f = E\left[\mu_{\tilde{X}}(\boldsymbol{X})\right] = \int\int_{\Omega_f}\cdots\int \mu_{\tilde{X}}(x_1, x_2, \ldots, x_n)\, f_X(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \tag{8}$$

where $E[\cdot]$ is the mathematical expectation. The fuzzy random reliability can be obtained as $R_s = 1 - P_f$.

3 Structural safety estimation based on non-probabilistic set theory

The above two methods based on probability approach and fuzzy theory need to have sufficient information to determine the probability distributions and the membership functions, respectively. However, experimental data are often limited, which causes the requirement of the available data to justify the probabilistic reliability model, or the fuzzy reliability model may not be satisfied. Under this circumstance, the convex method based on non-probabilistic set theory is attracting more attention. Two typical models for structural safety measure are described in this section.

3.1 Reliability analysis based on interval model

Assuming that $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^T$ represents the basic interval variable vector, Y_i can be expressed as

$$Y_i \in Y_i^I = \left[\underline{Y_i}, \overline{Y_i}\right] \quad i = 1, 2, \ldots, n \tag{9}$$

where $\underline{Y_i}$ and $\overline{Y_i}$ represent the lower and upper bounds of Y_i, respectively.

Similar to the probabilistic model, the limit state function of the uncertain structure is given by

$$M = g(\boldsymbol{Y}) = g(Y_1, Y_2, \ldots, Y_n) = 0. \tag{10}$$

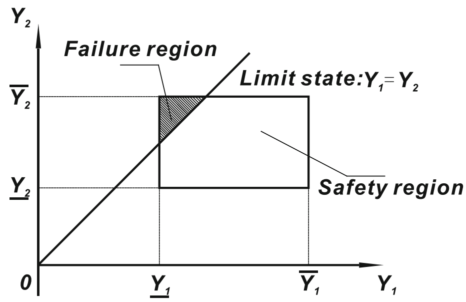

Fig. 1 The safety region and failure region for the two-dimensional interval model

In Eq. (10), the hyper-rectangular domain enclosed by the interval variables Y_i is divided into the failure region ($M < 0$) and the safety region ($M > 0$). The measure of the structural failure can be defined as the ratio of the hyper-volume of the failure region to the whole region, which is

$$P_f = \eta (M < 0) = \eta (g (Y_1, Y_2, \ldots, Y_n) < 0) = \frac{V_{\text{failure}}}{V_{\text{total}}} \tag{11}$$

where η represents the possibility. Consequently, the non-probabilistic measure of structural safety is

$$R_s = 1 - P_f = \eta (M > 0) = \eta (g (Y_1, Y_2, \ldots, Y_n) > 0) = \frac{V_{\text{safety}}}{V_{\text{total}}} \tag{12}$$

As an example, Fig. 1 illustrates the case of a two-dimensional interval reliability model, in which the structural safety is defined when $Y_1 > Y_2$.

3.2 Reliability analysis based on convex model

Supposing an n-dimensional uncertain variable vector $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^T$, the boundary of each variable is determined by the following hyper-ellipsoid:

$$\begin{aligned} \mathbf{Z} \in \Omega &= \left\{ \mathbf{Z} : (\mathbf{Z} - \mathbf{Z}^c)^T \mathbf{W} (\mathbf{Z} - \mathbf{Z}^c) \leq 1 \right\} \\ &\rightarrow \left(\frac{Z_1 - Z_1^c}{Z_1^r}, \ldots, \frac{Z_n - Z_n^c}{Z_n^r} \right)^T \left(\frac{Z_1 - Z_1^c}{Z_1^r}, \ldots, \frac{Z_n - Z_n^c}{Z_n^r} \right) \leq 1 \end{aligned} \tag{13}$$

where Ω is the hyper-ellipsoid convex set, W is a characteristic matrix, $\boldsymbol{Z^c} = \left(Z_1^c, Z_2^c, \ldots, Z_n^c\right)$ and $\boldsymbol{Z^r} = \left(Z_1^r, Z_2^r, \ldots, Z_n^r\right)$, respectively, denote the median value and the radius of $\mathbf{Z}$. By normalizing the variables Z_i, Eq. (13) can be rewritten as

$$\boldsymbol{V} \in \Omega_{\text{standard}} = \left\{ \boldsymbol{V} : \boldsymbol{V}^T \boldsymbol{V} \leq 1 \right\} \rightarrow V_1^2 + V_2^2 + \cdots + V_n^2 \leq 1 \tag{14}$$

where $\boldsymbol{V} = (V_1, V_2, \ldots, V_n)^T$ and $V_i = \frac{Z_i - Z_i^c}{Z_i^r}$. Thus, the uncertain variables are redefined into a unit hyper-sphere. Relating with the limit state function $M = g(\boldsymbol{V}) = g(V_1, V_2, \ldots, V_n) = 0$, the failure/safety measure of the structure is given mathematically as

$$P_f = \eta (g (V_1, V_2, \ldots, V_n) < 0) \quad \text{and} \quad R_s = 1 - P_f = \eta (g (V_1, V_2, \ldots, V_n) > 0) . \tag{15}$$

Similarly, for a bi-variable problem with uncertain parameters Z_1 and Z_2 as shown in Fig. 2, the ellipsoidal convex model will degenerate into an ellipse and further into a circle with normalization, as shown in Fig. 3.

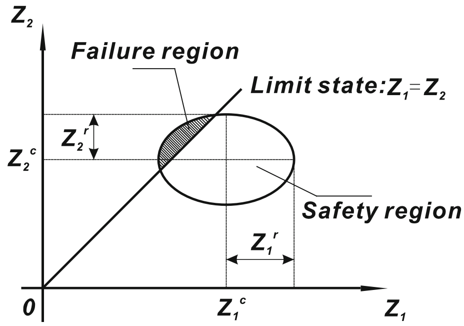

Fig. 2 The safety region and failure region for the case of the convex model

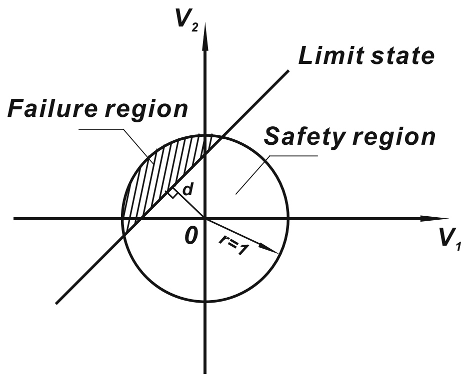

Fig. 3 The structural non-probabilistic reliability based on the convex model

Assume that the structure is safe if $Z_1 > Z_2$. In this case, the failure/safety measure of the structure can be deduced from the following expressions:

$$\begin{aligned} P_f &= \frac{S_{\text{failure}}}{S_{\text{total}}} = \frac{\cos^{-1} d - d\sqrt{1-d^2}}{\pi} \\ &= \frac{1}{\pi}\left[\cos^{-1}\left(\frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\right) - \frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\sqrt{1-\left(\frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\right)^2}\right] \end{aligned} \tag{16}$$

and

$$\begin{aligned} R_s &= \frac{S_{\text{safe}}}{S_{\text{total}}} = 1 - \frac{\cos^{-1} d - d\sqrt{1-d^2}}{\pi} \\ &= 1 - \frac{1}{\pi}\left[\cos^{-1}\left(\frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\right) - \frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\sqrt{1-\left(\frac{Z_1^c - Z_2^c}{\sqrt{\left(Z_1^r\right)^2 + \left(Z_2^r\right)^2}}\right)^2}\right] \end{aligned} \tag{17}$$

where d is the distance from the origin to the limit state function (shown in Fig. 3).

As above mentioned, the non-probabilistic reliability analysis based on the convex model may show superiority to some extent when available information of uncertainties is insufficient. Moreover, the convex model has some advantages over the interval model. On the one hand, the uncertain parameters enclosed by the convex model no longer satisfy the assumption of independence. On the other hand, the uncertain variables in the convex model can be explicitly expressed as continuously differentiable equations, whereas those in the interval model not.

Due to the increasing complexity of engineering structures, the study on multi-source uncertainties, especially the hybrid reliability analysis, is of profound significance. In the following section, several cases of the convex model combined with different types of uncertain factors will be proposed.

4 Hybrid reliability analysis based on convex modeling theory

In this section, four typical combined models based on the convex method are proposed for estimation of the structural safety under different cases of multi-source uncertainties. These models or algorithms are alternatives to the current hybrid uncertainty analysis.

4.1 Reliability analysis of the convex and random mixed model

If both random variables and non-probabilistic convex variables are contained in the basic variables relating to the limit state function, the failure surface can be expressed as

$$M = g(\boldsymbol{X}, \boldsymbol{Z}) = g(X_1, \ldots, X_m, Z_{m+1}, \ldots, Z_n) = 0 \tag{18}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)^T$ denotes the m-dimensional random variable vector, and $\boldsymbol{Z} = (Z_{m+1}, Z_{m+2}, \ldots, Z_n)^T$, represents the $(n-m)$-dimensional convex modeling variables.

Assuming that the random vector $\boldsymbol{X}$ is taken as a constant one, and hence, the hybrid model can be transformed into a non-probabilistic convex model. Similarly, it will be transformed into a random model when the convex vector $\boldsymbol{Z}$ is confirmed. Therefore, the reliability analytical model based on a single uncertainty source is generally the special case of the mixed one.

Let one implementation $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^T$ be the initial random vector $\boldsymbol{X}$. According to the convex theory, the non-probabilistic reliability of $\boldsymbol{x}$ can be derived as [14]

$$\eta(M(\boldsymbol{x}, Z_{m+1}, Z_{m+2}, \ldots, Z_n) > 0) = \eta(\boldsymbol{x}). \tag{19}$$

By virtue of the distributional density function of $\boldsymbol{X}$, the structural hybrid reliability can be defined as

$$R_s = E[\eta(\boldsymbol{x})]. \tag{20}$$

As $\boldsymbol{x}$ ultimately decides the expression of $\eta(\boldsymbol{x})$, the subsection solution method should be applied for realization of Eq. (20).

A linear limit state function is considered as

$$M = aX + b_1 Z_1 - b_2 Z_2 \tag{21}$$

where X is a random variable and its probability density function is $f(x)$. Z_1 and Z_2 are convex modeling variables and are limited in the following ellipse $\left(\frac{Z_1 - Z_1^c}{Z_1^r}\right)^2 + \left(\frac{Z_2 - Z_2^c}{Z_2^r}\right)^2 \leq 1$. It is assumed that coefficients a, b_1 and b_2 are all positive.

Introducing normalized variables V_1 and V_2

$$V_1 = \frac{Z_1 - Z_1^c}{Z_1^r} \quad \text{and} \quad V_2 = \frac{Z_2 - Z_2^c}{Z_2^r}, \tag{22}$$

the original ellipse becomes $V_1^2 + V_2^2 \leq 1$. The limit state function can then be rewritten as

$$M = aX + b_1 Z_1^c - b_2 Z_2^c + b_1 Z_1^r V_1 - b_2 Z_2^r V_2. \tag{23}$$

Different values of X will directly affect the position of the failure surface $M = 0$, and further change the interference condition between the limit state function and the feasible region of the normalized variables. $\eta(x)$ derived from the convex theory is a piecewise function of X. In view of this, four cases are shown in Fig. 4.

(i) If the failure surface is located in region ①, X ranges from $-\infty$ to $\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$. In this case, $\eta^{①}(x) = 0$. According to Eq. (20), the hybrid reliability is also zero, i.e., $R_s^{①} = 0$.

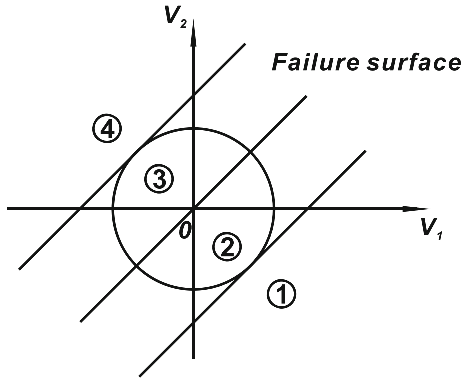

Fig. 4 Position of failure surface given different values of X

(ii) When $x \in \left[\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c}{a}\right]$, the failure surface is in region ②. Utilizing Eqs. (16) and (17), $\eta^{②}(x)$ can be obtained by

$$\eta^{②}(x) = \frac{1}{\pi}\left[\cos^{-1}\left(d^{②}(x)\right) - d^{②}(x)\sqrt{1 - \left(d^{②}(x)\right)^2}\right] \tag{24}$$

where $d^{②}(x) = \frac{b_2 Z_2^c - b_1 Z_1^c - ax}{\sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}$. The hybrid reliability is

$$R_s^{②} = \int_{\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} \eta^{②}(x)\, f(x)\, dx. \tag{25}$$

(iii) In region ③, the span of X is

$$\left[\frac{b_2 Z_2^c - b_1 Z_1^c}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]. \tag{26}$$

In consideration of the geometric symmetry of this case and case 2, $\eta^{③}(x)$ and $R_s^{③}$ are both easily given as

$$\eta^{③}(x) = 1 - \frac{1}{\pi}\left[\cos^{-1}\left(d^{③}(x)\right) - d^{③}(x)\sqrt{1 - \left(d^{③}(x)\right)^2}\right] \tag{27}$$

and

$$R_s^{③} = \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}} \eta^{③}(x)\, f(x)\, dx \tag{28}$$

where $d^{③}(x) = -d^{②}(x) = \frac{ax + b_1 Z_1^c - b_2 Z_2^c}{\sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}$.

(iv) If $x \in \left[\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1)^2}}{a}, +\infty\right)$, the failure surface will no longer intersect the feasible region of convex modeling variables. $\eta^{\textcircled{4}}(x)$ is always equal to unity, and the hybrid reliability is

$$R_s^{\textcircled{4}} = \int_{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{+\infty} f(x)\, \mathrm{d}x. \tag{29}$$

The final hybrid reliability based on the convex and the random mixed model is summation of the four regions, that is,

$$R_s = E[\eta(x)] = R_s^{\textcircled{1}} + R_s^{\textcircled{2}} + R_s^{\textcircled{3}} + R_s^{\textcircled{4}}. \tag{30}$$

4.2 Reliability analysis of the convex and fuzzy random mixed model

In this model, the random variables will be replaced by the fuzzy random variables. Thus, the failure surface can be rewritten as

$$M = g\left(\tilde{\boldsymbol{X}}, \boldsymbol{Z}\right) = g\left(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_m, Z_{m+1}, \ldots, Z_n\right) = 0. \tag{31}$$

For a given $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)^T$, the non-probabilistic reliability $\eta_{\tilde{X}}(\boldsymbol{x})$ can be known by the convex method. Then, the structural hybrid reliability is

$$R_s = E\left[\eta_{\tilde{X}}(\boldsymbol{x})\right] = E\left[\eta_{\tilde{X}}(M(\boldsymbol{x}, Z_{m+1}, Z_{m+2}, \ldots, Z_n) > 0)\right]. \tag{32}$$

Taking into account the influence of $f_X(\boldsymbol{X})$ and $\mu_{\tilde{X}}(\boldsymbol{X})$ on $\eta_{\tilde{X}}(\boldsymbol{x})$, a subregional treatment should be carried out for the computation of Eq. (32). It is convenient to consider a linear limit state function as

$$M = a\tilde{X} + b_1 Z_1 - b_2 Z_2. \tag{33}$$

The approximate analytical approach as in Sect. 4.1 is used again to obtain the final hybrid reliability as

$$\begin{aligned} R_s &= E\left[\eta_{\tilde{X}}(x)\right] = \overset{\frown}{R_s^{\textcircled{1}}} + \overset{\frown}{R_s^{\textcircled{2}}} + \overset{\frown}{R_s^{\textcircled{3}}} + \overset{\frown}{R_s^{\textcircled{4}}} = \int_{\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} \eta^{\textcircled{2}}(x) f(x) \mu_{\tilde{X}}(x)\, \mathrm{d}x \\ &+ \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}} \eta^{\textcircled{3}}(x) f(x) \mu_{\tilde{X}}(x)\, \mathrm{d}x + \int_{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{+\infty} f(x) \mu_{\tilde{X}}(x)\, \mathrm{d}x. \end{aligned} \tag{34}$$

4.3 Reliability analysis of the convex and interval mixed model

The limit state function contains both the convex and interval non-probabilistic uncertainties, i.e.,

$$M = g(\boldsymbol{Y}, \boldsymbol{Z}) = g(Y_1, \ldots, Y_m, Z_{m+1}, \ldots, Z_n). \tag{35}$$

The feasible region of the uncertain parameters would be formed into a hyper-volume, which lies between the hyper-rectangle and the hyper-ellipsoid. Figure 5 illustrates a three-dimensional case. In this circumstance, the

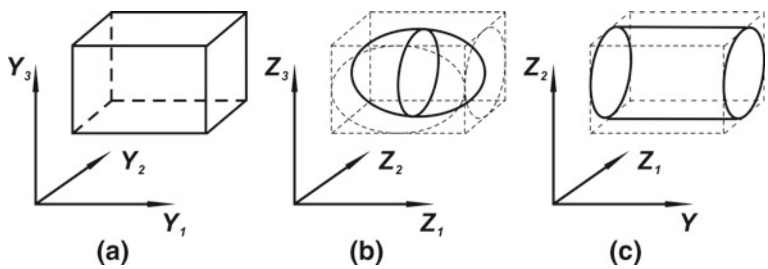

Fig. 5 Three-dimensional models for non-probabilistic uncertainties. **a** Three-dimensional interval model. **b** Three-dimensional convex model. **c** Three-dimensional convex and interval mixed model

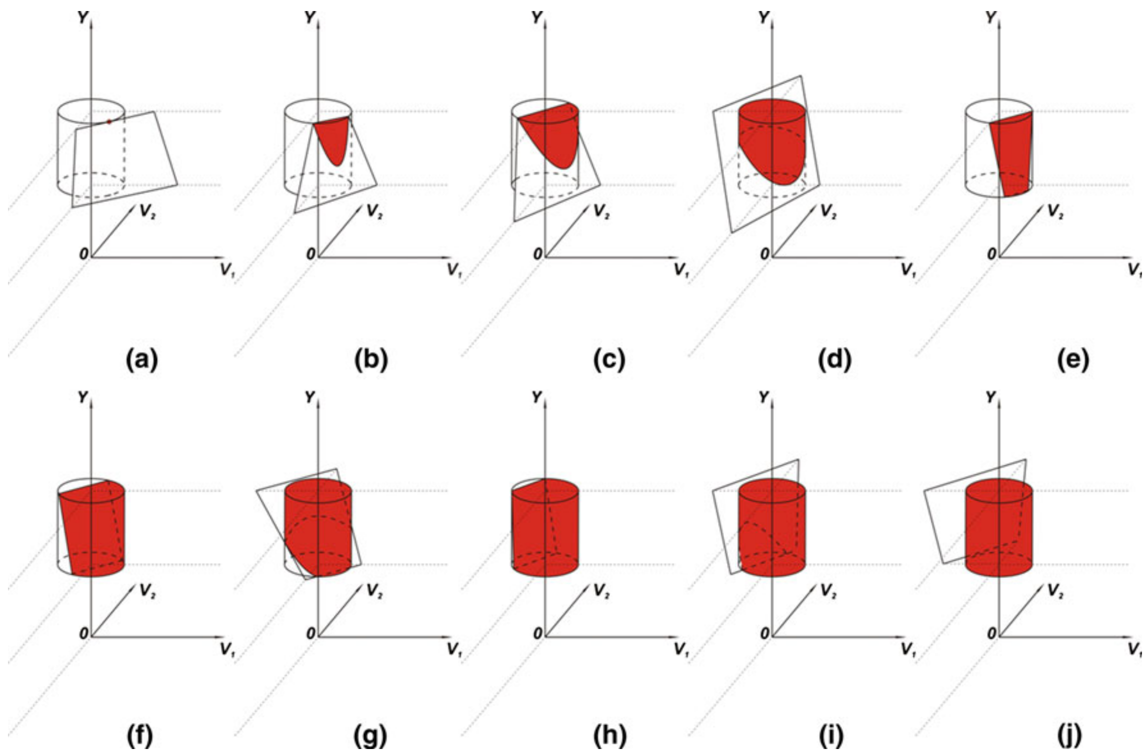

Fig. 6 Different cases of the convex and interval mixed model given different interval variable Y

failure region and the safety region are divided by Eq. (35), and the structural failure/safety measure based on the non-probabilistic set theory is still applicable to the hybrid model with minor modifications.

For ease of presentation, introducing a linear limit state function as

$$M = aY + b_1 Z_1 - b_2 Z_2 \tag{36}$$

where $Y \in [\underline{Y}, \overline{Y}]$ is an interval variable, Z_1 and Z_2 are normalized as $\left(\frac{Z_1 - Z_1^c}{Z_1^r}\right)^2 + \left(\frac{Z_2 - Z_2^c}{Z_2^r}\right)^2 \leq 1$, and a, b_1 and b_2 are positive constants.

With normalized variables V_1 and V_2 defined in Eq. (22), the limit state function becomes

$$M = aY + b_1 Z_1^c - b_2 Z_2^c + b_1 Z_1^r V_1 - b_2 Z_2^r V_2. \tag{37}$$

$\underline{Y}$ and $\overline{Y}$ directly change the intersection between the hyper-volume domain and the failure surface. Through comprehensive analysis, the following cases should be considered (see Fig. 6 for details).

Case I: (Fig. 6a): When $\overline{Y} \in \left(-\infty, \frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$, the hybrid reliability is zero, i.e., $R_s^I = 0$.

Case II: When $\underline{Y} \in \left(-\infty, \frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$, and the upper bound $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ to $\frac{b_2 Z_2^c - b_1 Z_1^c}{a}$ (Fig. 6b), the hybrid reliability R_s^{II} is

$$R_s^{\mathrm{II}} = \frac{\int_{\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{\overline{Y}} s^{\mathrm{II}}(y)\,\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} \tag{38}$$

where $s^{\mathrm{II}}(y) = \cos^{-1}\left(d^{\mathrm{II}}(y)\right) - d^{\mathrm{II}}(y)\sqrt{1 - \left(d^{\mathrm{II}}(y)\right)^2}$ and $d^{\mathrm{II}}(y) = \frac{b_2 Z_2^c - b_1 Z_1^c - ay}{\sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}$.

Case III: When $\underline{Y} \in \left(-\infty, \frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$, and $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c}{a}$ to $\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ (Fig. 6c), the hybrid reliability R_s^{III} is

$$R_s^{\mathrm{III}} = \frac{\int_{\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} s^{\mathrm{II}}(y)\,\mathrm{d}y + \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\overline{Y}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} \tag{39}$$

where $s^{\mathrm{III}}(y) = \cos^{-1}\left(d^{\mathrm{III}}(y)\right) - d^{\mathrm{III}}(y)\sqrt{1 - \left(d^{\mathrm{III}}(y)\right)^2}$ and $d^{\mathrm{III}}(y) = -d^{\mathrm{II}}(y)$.

Case IV: When $\underline{Y} \in \left(-\infty, \frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$, and $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ to $+\infty$ (Fig. 6d), the hybrid reliability R_s^{IV} is

$$R_s^{\mathrm{IV}} = \frac{\int_{\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} s^{\mathrm{II}}(y)\,\mathrm{d}y + \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} + \left(\frac{a\overline{Y} - b_2 Z_2^c + b_1 Z_1^c - \sqrt{\left(b_2 Z_2^r\right)^2 + \left(b_1 Z_1^r\right)^2}}{\left(\overline{Y} - \underline{Y}\right)a}\right). \tag{40}$$

Case V: When the lower bound $\underline{Y}$ and the upper bound $\overline{Y}$ are both $\left[\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c}{a}\right]$ (Fig. 6e), the hybrid reliability R_s^{V} is

$$R_s^{V} = \frac{\int_{\underline{Y}}^{\overline{Y}} s^{\mathrm{II}}(y)\,\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi}. \tag{41}$$

Case VI: When $\underline{Y} \in \left[\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c}{a}\right]$ and $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c}{a}$ to $\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ (Fig. 6f), the hybrid reliability R_s^{VI} is

$$R_s^{\mathrm{VI}} = \frac{\int_{\underline{Y}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} s^{\mathrm{II}}(y)\,\mathrm{d}y + \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\overline{Y}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi}. \tag{42}$$

Case VII: When $\underline{Y} \in \left[\frac{b_2 Z_2^c - b_1 Z_1^c - \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c}{a}\right]$ and $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ to $+\infty$ (Fig. 6g), the hybrid reliability R_s^{VII} is

$$R_s^{\mathrm{VII}} = \frac{\int_{\underline{Y}}^{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}} s^{\mathrm{II}}(y)\,\mathrm{d}y + \int_{\frac{b_2 Z_2^c - b_1 Z_1^c}{a}}^{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} + \left(\frac{a\overline{Y} - b_2 Z_2^c + b_1 Z_1^c - \sqrt{\left(b_2 Z_2^r\right)^2 + \left(b_1 Z_1^r\right)^2}}{\left(\overline{Y} - \underline{Y}\right)a}\right). \tag{43}$$

Case VIII: When the lower bound $\underline{Y}$ and the upper bound $\overline{Y}$ are both $\left[\frac{b_2 Z_2^c - b_1 Z_1^c}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$ (Fig. 6h), the hybrid reliability R_s^{VIII} is

$$R_s^{\mathrm{VIII}} = \frac{\int_{\underline{Y}}^{\overline{Y}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} = 1 - \frac{\int_{\underline{Y}}^{\overline{Y}} s^{\mathrm{III}}(y)\,\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi}. \tag{44}$$

Case IX: When $\underline{Y} \in \left[\frac{b_2 Z_2^c - b_1 Z_1^c}{a}, \frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}\right]$ and $\overline{Y}$ ranges from $\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}$ to $+\infty$ (Fig. 6i), the hybrid reliability R_s^{IX} is

$$R_s^{\mathrm{IX}} = \frac{\int_{\underline{Y}}^{\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}} \left(\pi - s^{\mathrm{III}}(y)\right)\mathrm{d}y}{\left(\overline{Y} - \underline{Y}\right)\pi} + \left(\frac{a\overline{Y} - b_2 Z_2^c + b_1 Z_1^c - \sqrt{\left(b_2 Z_2^r\right)^2 + \left(b_1 Z_1^r\right)^2}}{\left(\overline{Y} - \underline{Y}\right)a}\right). \tag{45}$$

Case X: When the lower bound $\underline{Y}$ and the upper bound $\overline{Y}$ are both $\left[\frac{b_2 Z_2^c - b_1 Z_1^c + \sqrt{(b_2 Z_2^r)^2 + (b_1 Z_1^r)^2}}{a}, +\infty\right)$ (Fig. 6j), the hybrid reliability is unity, i.e., $R_s^X = 1$.

As mentioned above, the convex method based on non-probabilistic set theory can be effectively utilized to deal with the reliability analysis under the interval and the convex mixed model. Particularly for the problem stated by Eq. (36), once the lower and the upper bounds of the interval variable Y are assured, one of the ten cases can be selected and its formula for hybrid reliability will be further applicable to estimate the structural safety.

4.4 Hybrid reliability model containing randomness, fuzziness, and non-probabilistic uncertainty based on convex theory

In this section, a more complex model containing four types of uncertainties (random, fuzzy random, interval, and convex) is discussed. The failure surface is taken as

$$M = g\left(\boldsymbol{X}, \widetilde{\boldsymbol{X}}, \boldsymbol{Y}, \boldsymbol{Z}\right) = g\left(X_1, \ldots, X_{m_1}, \tilde{X}_{m_1+1}, \ldots, \tilde{X}_{m_2}, Y_{m_2+1}, \ldots, Y_{m_3}, Z_{m_3+1}, \ldots, Z_n\right) = 0. \tag{46}$$

Given the specified values of X_i and $\widetilde{X}_j$ ($i = 1, 2, \ldots, m_1$ and $j = m_1 + 1, m_1 + 2, \ldots, m_2$), the structural state of safety or failure can be determined from the hybrid reliability analysis of the interval and convex mixed model, namely $\eta(M > 0) = \eta(\boldsymbol{x}, \widetilde{\boldsymbol{x}})$. Furthermore, by means of the patterns of the probability density function and the membership function, the hybrid reliability can be obtained by the following equation:

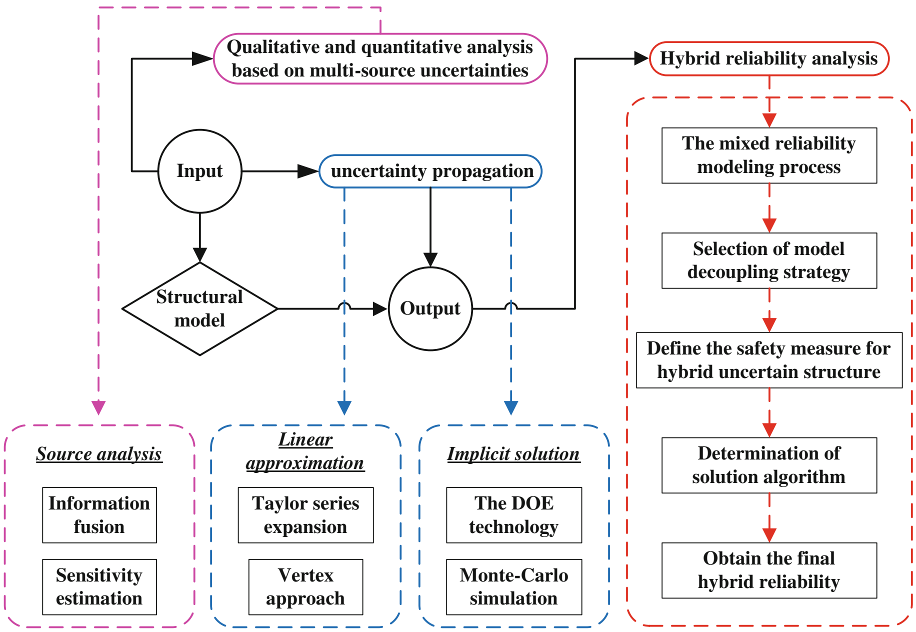

Fig. 7 Combination between numerical simplified technologies and the hybrid reliability analytical methods

$$
\begin{aligned}
R_s &= E\left[\eta\left(\boldsymbol{x}, \widetilde{\boldsymbol{x}}\right)\right] = \int \mu_{\widetilde{X}}\left(\widetilde{\boldsymbol{x}}\right) \int f_X\left(\boldsymbol{x}\right) \eta\left(\boldsymbol{x}, \widetilde{\boldsymbol{x}}\right) \mathrm{d}\boldsymbol{x} \mathrm{d}\widetilde{\boldsymbol{x}} \\
&= \int\int_{\Omega_{\widetilde{X}}} \mu_{\widetilde{X}}\left(\widetilde{x}_{m_1+1}, \widetilde{x}_{m_2+2}, \ldots, \widetilde{x}_{m_2}\right) \cdot \left(\int\int_{\Omega_X} \cdots \int f_X\left(x_1, x_2, \ldots, x_{m_1}\right)\right. \\
&\quad \left. \cdot \eta\left(x_1, x_2, \ldots, x_{m_1}, x_{m_1+1}, x_{m_1+2}, \ldots, x_{m_2}\right) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{m_1}\right) \mathrm{d}\widetilde{x}_{m_1+1} \mathrm{d}\widetilde{x}_{m_1+2} \ldots \mathrm{d}\widetilde{x}_{m_2}
\end{aligned} \tag{47}
$$

where Ω_X and $\Omega_{\widetilde{X}}$ are, respectively, the feasible regions of $\boldsymbol{X}$ and $\widetilde{\boldsymbol{X}}$.

Nevertheless, uncertainties of practical structures are complicated. For example, they may embody a multi-variable, nonlinear limit state function, implicit solution, and so forth. Obtaining the exact solutions of Eq. (47) will be difficult, and some approximate techniques may be employed. For example, when dealing with the multi-source uncertainties, the information fusion theory or the sensitivity analysis based on the uncertain parameters can be adopted. If the limit state function is nonlinear, the linear approximation techniques, such as the Taylor series expansion or the vertex approach can be used. With regard to the implicit expression of structural responses, such as stress or displacement, the design of experiment (DOE) method as well as the Monte Carlo simulations may be considered.

It is noted that each model or algorithm has its own feasibility and limitation. The amount of uncertain information, the complexity of the structures, and the requirements of accuracy and efficiency are the core factor in selecting appropriate models. Figure 7 illustrates more details.

5 Numerical examples

5.1 A cantilever beam

As the first example, we consider a cantilever beam as shown in Fig. 8. The cantilever beam is subjected to two concentrated forces applied at distances $b_1 = 2.0\,\text{m}$ and $b_2 = 5.0\,\text{m}$ from the fixed end. The structure is identified as failure if $|m_{\max}| \geq m_{cr}$, where $m_{\max}$ is the maximum actual moment and m_{cr} is the moment capacity of the beam. Two cases with different uncertain parameter settings are studied as follows:

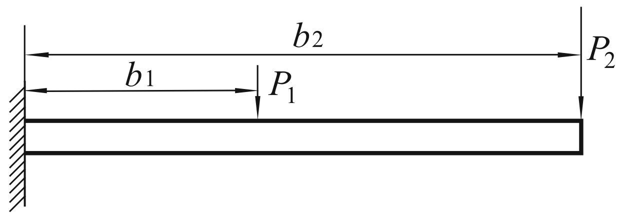

Fig. 8 A cantilever beam

Table 1 Uncertainty characteristics of the cantilever beam with mixed uncertainties

P_1 P_2	m_{cr}		
Convex modeling	Random	Fuzzy random	Interval
$\left(\frac{16(P_1-5)}{15}\right)^2 + (4(P_2-2))^2 \le 1$	$m_{cr} \sim N(23, a^2)$ $a \in [1,2]$	$m_{cr} \sim N(23, a^2)$ $\mu_{\widetilde{M_{cr}}}(m_{cr}) = \begin{cases} \frac{m_{cr}-19}{2} & 19 \le m_{cr} < 21 \\ 1 & 21 \le m_{cr} < 25 \\ \frac{31-m_{cr}}{6} & 19 \le m_{cr} < 21 \\ 0 & \text{otherwise} \end{cases}$	$\underline{m_{cr}} = 23 - ka$ $\overline{m_{cr}} = 23 + ka$ $a \in [1,2]$ $k = 1, 2, 3$

Table 2 Uncertainty characteristics of the cantilever beam with single-source uncertainty

	P_1	P_2	m_{cr}
Probabilistic model	$P_1 \sim N\left(5, \left(\frac{5}{16}\right)^2\right)$	$P_2 \sim N\left(2, \left(\frac{1}{12}\right)^2\right)$	$m_{cr} \sim N(23, a^2)$ $a \in [1,2]$
Convex model	$\left(\frac{16(P_1-5)}{15}\right)^2 + (4(P_2-2))^2 + \left(\frac{(m_{cr}-23)}{3a}\right)^2 \le 1$		$a \in [1,2]$
Interval model	$P_1 \in [4.0625, 5.9375]$	$P_2 \in [1.75, 2.25]$	$m_{cr} \in [23-ka, 23+ka]$ $a \in [1,2]$ $k = 1, 2, 3$

Case 1: P_1, P_2, and m_{cr} are of different uncertainty types. Assuming that P_1 and P_2 are expressed as the convex modeling variables, and m_{cr} is defined as random variable, fuzzy random variable, and interval variable, respectively. The uncertainty characteristics are listed in Table 1, where a denotes the change factor of interval and ranges between 1 and 2, and coefficient $k (k = 1, 2, 3)$ represents the interval ranges.

Case 2: Consider that P_1, P_2 and m_{cr} are of the same type of single-source uncertainty. The uncertainty characteristics are listed in Table 2.

The limit state function of this example can be expressed as

$$M = m_{cr} - b_1 P_1 - b_2 P_2. \tag{48}$$

Based on the proposed hybrid reliability models in Sect. 4, the structural reliability of case (i) is obtained and shown in Fig. 9. From the reliability analysis of single-source uncertainty in Sect. 3, the structural reliability of Case 2 is also obtained and shown in Fig. 10. The numerical results of Case 1 and 2 are compared in Table 3 for $a = 1, 1.5, 2$.

From the results in Figs. 9, 10, and Table 3, the following points can be summarized:

(i) The reliability results given by either the hybrid models or the single-source models with different combinations of uncertain parameters decrease as the change factor a increases, as expected. This indicates that a higher uncertainty leads to a lower structural safety.

(ii) The hybrid reliability obtained by the convex and random mixed model is coincident with that derived from the convex and fuzzy random mixed model when the smaller value a. With the increase of a, however, due to the existence of fuzziness, the results based on the latter are more conservative.

(iii) The results obtained by the convex and interval mixed model are very sensitive to the interval parameters. The reliability decreases as the coefficient increases. In particular, when $k = 3$, the reliability is much lower than those deduced by the convex and random models as well as the convex and fuzzy random models.

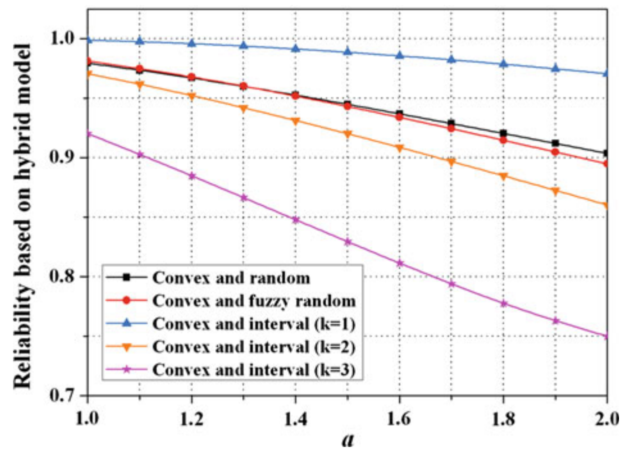

Fig. 9 The hybrid reliability for various mixed models

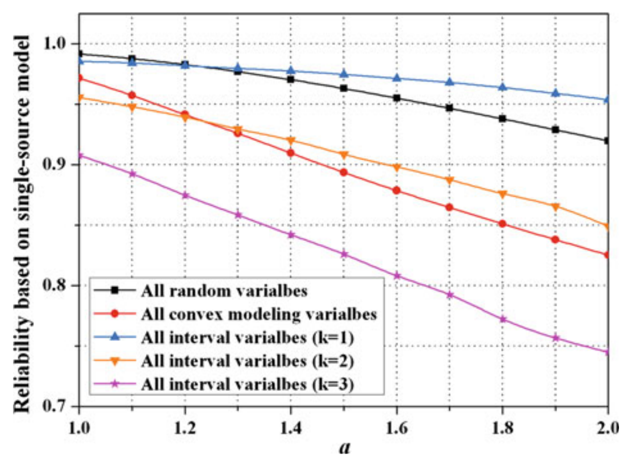

Fig. 10 The reliability of various single-source uncertainties

Table 3 Reliability analysis results of the cantilever beam structure

	Reliability based on hybrid model				
	Convex and random	Convex and fuzzy random	Convex and interval ($k = 1$)	Convex and interval ($k = 2$)	Convex and interval ($k = 3$)
$a = 1$	0.9796	0.9813	0.9989	0.9707	0.9203
$a = 1.5$	0.9450	0.9432	0.9888	0.9203	0.8296
$a = 2$	0.9037	0.8949	0.9707	0.8604	0.7500
	Reliability based on single-source model				
	Random	Convex modeling	Interval ($k = 1$)	Interval ($k = 2$)	Interval ($k = 3$)
$a = 1$	0.9918	0.9718	0.9856	0.9557	0.9078
$a = 1.5$	0.9631	0.8938	0.9748	0.9087	0.8261
$a = 2$	0.9199	0.8253	0.9539	0.8491	0.7450

(iv) By comparisons of the results obtained by the single-source reliability models and the hybrid reliability models, we also can obtain some meaningful conclusions: on the one hand, the assumption of precise probabilistic distributions for all of the uncertain variables may be dangerous; on the other hand, the interval analytic methods, in which all uncertainties are quantified by interval variables, may lead to excessively conservative results so that higher economic costs have to be paid on safety consideration for structural design. It should be emphasized that the structural reliability is closely related to the uncertain parameters, and hence, subjective assumptions may yield unreliable results.

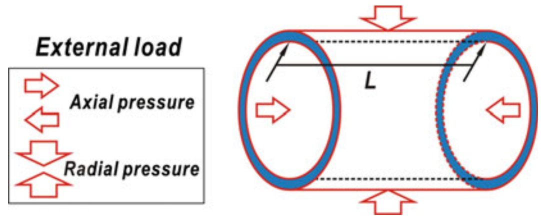

Fig. 11 Configuration of a composite cylindrical shell under external pressure load

Table 4 Experimental data of the elastic moduli for composite cylindrical shell [44]

No.	E_1 (GPa)	E_2 (GPa)	υ_{21}	G_{12} (GPa)	No.	E_1 (GPa)	E_2 (GPa)	υ_{21}	G_{12} (GPa)
1	129.20	9.34	0.28	5.23	9	132.19	9.07	0.30	4.85
2	131.59	9.53	0.33	4.97	10	132.00	9.73	0.35	5.00
3	130.63	9.08	0.33	5.16	11	130.39	9.21	0.34	5.34
4	132.01	9.34	0.33	5.15	12	128.28	8.67	0.33	4.98
5	131.04	8.94	0.34	5.15	13	135.30	9.18	0.32	5.13
6	120.61	9.04	0.33	4.81	14	137.33	9.28	0.33	5.25
7	127.69	8.99	0.32	5.11	15	141.69	10.73	0.31	5.47
8	133.65	9.36	0.35	5.08	16	126.91	9.39	0.33	5.65

Table 5 Dimensionless uncertainty characteristics of the composite cylindrical shell

	$e_1 = \frac{E_1}{131\times10^9}$ $\quad e_2 = \frac{E_2}{9.4\times10^9}$ $\quad \mu_{21} = \frac{\upsilon_{21}}{0.3}$ $\quad g_{12} = \frac{G_{12}}{5.3\times10^9}$ $\quad p^* = \frac{p}{2.0254\times10^6}$
Hybrid model	$e_1 \in [0.9207, 1.0816]$; $e_2 \sim N\left(1.0319, 0.0365^2\right)$; $\left(\frac{\mu_{21}-1.05}{0.1167}\right)^2 + \left(\frac{g_{12}-0.9868}{0.0793}\right)^2 \le 1$; $p^* \sim N\left(1, 0.01^2\right)$; $\mu_{\widetilde{P^*}}(p^*) = \begin{cases} 20p^*-19 & 0.95 \le p^* < 1 \\ 21-20p^* & 1 \le p^* < 1.05 \\ 0 & \text{otherwise} \end{cases}$
Probabilistic model	$e_1 \sim N\left(1.0012, 0.0268^2\right)$ $\quad e_2 \sim N\left(1.0319, 0.0365^2\right)$ $\quad \mu_{21} \sim N\left(1.05, 0.0389^2\right)$ $\quad g_{12} \sim N\left(0.9868, 0.0264^2\right)$ $\quad p^* \sim N\left(1, 0.01^2\right)$
Convex model	$\left(\frac{e_1-1.0012}{0.0804}\right)^2 + \left(\frac{e_2-1.0319}{0.1096}\right)^2 + \left(\frac{\mu_{21}-1.05}{0.1167}\right)^2 + \left(\frac{g_{12}-0.9868}{0.0793}\right)^2 + \left(\frac{p^*-1}{0.03}\right)^2 \le 1$
Interval model	$e_1 \in [0.9207, 1.0816]$ $\quad e_2 \in [0.9223, 1.1415]$ $\quad \mu_{21} \in [0.9333, 1.1667]$ $\quad g_{12} \in [0.9075, 1.066]$ $\quad p^* \in [0.97, 1.03]$

5.2 Buckling problem of laminated composite shell

In order to illustrate the validity and feasibility of the presented hybrid reliability method, the buckling problem of a composite shell will be used to investigate the influence of multi-source uncertainties in material properties and external loads on the structural reliability.

Consider a 10-layer symmetric laminated composite cylindrical shell with cross-ply $[\theta/(90^\circ+\theta)/\theta/(90^\circ+\theta)/\theta]_{symmetric}$, where the thickness of each laminate is $t = 0.5$ mm, and the ply angle θ may range from 0° to 90°. The radius of cylindrical shell is $R = 125.0$ mm, and the length is $L = 2{,}000.0$ mm. The density of the composite material equals 1,380.0 kg/m^3. Both ends of the cylindrical shell are simply supported, and the external loads include the axial pressure and radial pressure as shown in Fig. 11.

The laminated composite shell will be identified as buckling failure if $|p| \ge p_{cr}$, where p is the external pressure and p_{cr} is the limit criterion. Additionally, due to the dispersion of composites, the elastic moduli $\boldsymbol{E} = (E_1, E_2, \upsilon_{21}, G_{12})^T$ are also regarded as the uncertain parameters. The experimental data of elastic moduli from Ref. [44] are listed in Table 4. Several cases including one hybrid uncertainty problem and three single-source uncertainty problems are considered, and the dimensionless uncertainty characteristics are summarized in Table 5.

According to the basic equations of the buckling problem for a compressed composite shell, the closed-form of the buckling load obtained from Ref. [45] can be used, and hence, the limit state function is expressed

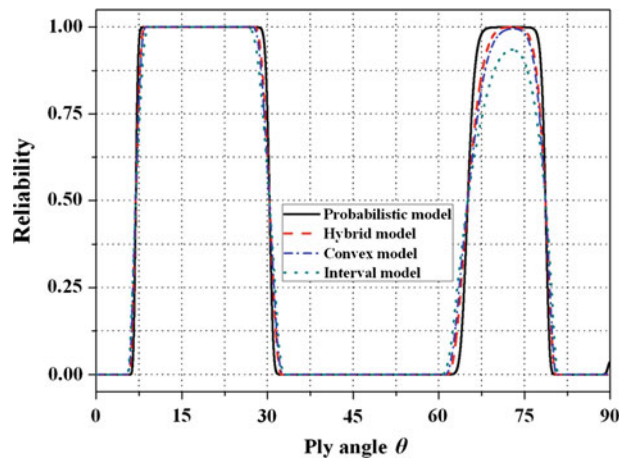

Fig. 12 Structural buckling reliability for the composite cylindrical shell obtained by four different uncertainty analytical models

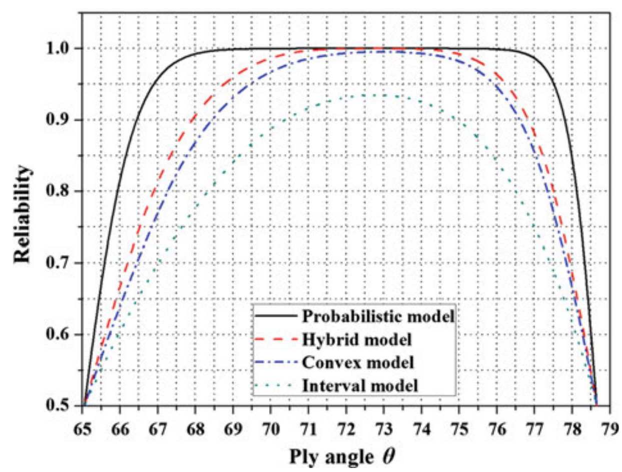

Fig. 13 Structural buckling reliability for the composite cylindrical shell in typical domain of θ obtained by four different uncertainty analytic models

as

$$\begin{aligned} M &= g\left(\boldsymbol{E}, p\right) = g\left(e_1, e_2, \mu_{21}, g_{12}, p\right) = p_{cr}\left(e_1, e_2, \mu_{21}, g_{12}\right) - p \\ &= \frac{1}{\lambda_m^2 + \lambda_n^2 R}\left[T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right] - p \end{aligned} \tag{49}$$

where T_{ij} $(i, j = 1, 2, 3)$ is the element of flexural stiffness matrix $\boldsymbol{T}$, m, and n denote the buckling wave numbers. Based on the proposed hybrid reliability method, the reliability results of the structural buckling are shown in Fig. 12, and the partially enlarged region for a typical domain of θ is shown in Fig. 13. In addition, those for given specific values of ply angle θ are summarized in Table 6.

The reliability results given by either the hybrid model or the other three types of single-source uncertainty models reflect the same increasing or decreasing trend along with the change of the ply angle θ. This implies that the proposed hybrid analytic method can be properly applied to complex structural problems. Furthermore, the mechanical properties of the composite cylindrical shell may vary significantly with the laminate configuration. For example, when θ equals 20°, the laminated structure is definitely safe with a unity reliability; when θ equals 45°, however, the composite cylindrical shell will be under the state of complete failure with a null reliability.

The single-source uncertainty models including the probabilistic model, convex model, and interval model have been, respectively, analyzed for comparison. The numerical results show that the probabilistic model

Table 6 Reliability analysis results of the composite cylindrical shell

Ply angle θ	0°	8°	20°	29°	45°	68°	76°	84°
Probabilistic model	0	0.9992	1	0.9915	0	0.9925	0.9987	0
Hybrid model	0	0.9681	1	0.8946	0	0.9072	0.9619	0
Convex model	0	0.9422	1	0.8550	0	0.8698	0.9441	0
Interval model	0	0.8683	1	0.7838	0	0.7780	0.8394	0

gives the largest buckling reliability, the hybrid model the second, then the convex, and the interval model gives the smallest, for a certain θ.

6 Conclusions

In engineering analysis and design, it is necessary to properly deal with the uncertainties that affect the structural performance. As the uncertainties may consist of multi-source and multi-dimensional parameters in practical structural problems, the current reliability analytical techniques based on single-source uncertainty models are infeasible anymore. In order to fill the gap, four new hybrid reliability models including convex with random, convex with random fuzzy, convex with interval, and convex with other three types are, respectively, investigated in this paper. Numerical examples show the feasibility and effectiveness of the presented methodology. The results derived from different reliability models indicate that the uncertainty plays an important role in the mechanical behavior and structural safety.

The presented hybrid reliability technique has broad applications. It can deal with a variety of different situations such as both linear and nonlinear state functions, explicit or implicit solution, and multi-source and multi-dimensional mixed uncertainties. In contrast with the existing mixed models based on probabilistic reliability theory, the models proposed are less dependent on the distribution characteristics of the uncertain parameters. It will lead to a more reliable result under the circumstances of insufficient sample data. In addition, as compared with the hybrid reliability analysis obtained by interval models, the convex modeling variables are taken into account to reflect the correlation between the uncertain-but-bounded parameters.

The present paper presents hybrid uncertainty models as alternatives to dealing with the structural reliability analysis for multi-source uncertainties. The type and the amount of the uncertain information determine which model can be applied more effectively. The results from numerical examples indicate that the nature of the uncertain parameters should be the key points to determine the choice of the reliability analytic models, and thus, the developed hybrid reliability method may have a wider application space in complex engineering. In summary, the fewer assumptions we make, the more reliable the results we get.

Acknowledgments The authors would like to thank the Construction Industry Institute (Hong Kong)/PolyU Innovation Fund (No. 5-ZJD3), the National Nature Science Foundation of China (No. 11002013), Defense Industrial Technology Development Program (No. A2120110001, No. B2120110011), and the Aeronautical Science Foundation of China (No. 2012ZA51010) for the financial supports.

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