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On fractional order generalized thermoelasticity with micromodeling

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Abstract This paper presents the theory of fractional order generalized thermoelasticity with microstructure modeling for porous elastic bodies and synthetic materials containing microscopic components and micro-cracks. Built upon the micromorphic theory, the theory of fractional order generalized micromorphic thermoelasticity ($FOGTE_{mm}$) is firstly established by introducing the fractional integral operator. To generalize the $FOGTE_{mm}$ theory, the general forms of the extended thermoelasticity, temperature rate dependent thermoelasticity, thermoelasticity without energy dissipation, thermoelasticity with energy dissipation, and dual-phase-lag thermoelasticity are introduced during the formulation. Secondly, the uniqueness theorem for $FOGTE_{mm}$ is established. Finally, a generalized variational principle of $FOGTE_{mm}$ is developed by using the semi-inverse method. For reference, the theories of fractional order generalized micropolar thermoelasticity ($FOGTE_{mp}$) and microstretch thermoelasticity ($FOGTE_{ms}$) and the corresponding generalized variational theorems are also presented.

1 Introduction

The generalized thermoelasticity theories are a series of theoretical models capable of predicting a finite speed of heat conduction in thermoelastic media. The first of such modeling is the extended thermoelasticity theory (ETE) of Lord and Shulman [1], who introduced the concept of thermal relaxation time into the classical Fourier law of heat conduction. Subsequently, modifying the stress versus strain relationship as well as the entropy relationship with relaxation time, Green and Lindsay [2] proposed the temperature rate dependent thermoelasticity (TRDTE) theory. Green and Naghdi [3–5] introduced further two such theories: thermoelasticity without energy dissipation (TEWOED) and thermoelasticity with energy dissipation (TEWED). There exist also other generalized thermoelasticity theories, such as the two-temperature generalized thermoelasticity [6], the low-temperature thermoelasticity [7,8], the dual-phase-lag thermoelasticity (DPLTE) [9], and the three-phase-lag thermoelasticity [10]. Recently, upon introducing the fractional integral operator into the generalized heat conduction law of ETE, Youssef [11,12] established the fractional order generalized thermoelasticity (FOGTE): both weak and strong heat conductivity in the context of generalized thermoelasticity were considered, and the corresponding variational theorem for FOGTE was developed. The theory was subsequently employed to solve two-dimensional thermal shock problems using Laplace and Fourier transforms [13] as well as half-space problems for elastic materials subjected to ramp-type heating by using Laplace

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transform and state-space methods [14]. Abouelregal [15] also established a model of fractional order generalized thermopiezoelectricity and used it to solve one-dimensional boundary value problems for semi-infinite piezoelectric media.

By implicitly averaging atomic scale dynamics over space and time domains, traditional continuum approaches for material deformation modeling are valid only for relatively large systems [16] and are further subjected to long acoustic wave limit [17]. As such, it has been established that continuum theories cannot explain experimental observations at microscale, especially in materials such as porous elastic bodies (e.g., bones and ceramics) and synthetic materials containing microscopic components and microcracks. This has motivated the rapid development of microscale modeling in recent years, leading to various microcontinuum field theories (or extensions of classical field theories to microscopic space and time scales), including the Cosserat theory [18], the couple stress theory [19], the micromorphic (mm) theory [20], the microstructure theory [21], the micropolar (mp) theory [22], the microstretch (ms) theory [23], and the nonlocal theory [24].

For materials containing microelements that can deform independently from their centroidal motions, Eringen [20] developed the micromorphic theory, which has been regarded as the most successful top-down formulation of a two-level continuum model [16]. While the deformation is expressed as a sum of macroscopic continuous deformation and internal microscopic deformation of the inner structure, the material body itself is envisioned as a continuous collection of deformable particles [25]. Each particle possesses finite size and directions representing its microstructure and has nine independent degrees of freedom describing both stretches and rotations, in addition to the three classical translational degrees of freedom of its center [26]. Subsequently, the concept of material forces was extended to micromorphic thermoelasticity, with detailed expressions of the Eshelby stress tensor, pseudo-momentum, and material forces derived [26]. Constitutive theories for generalized micromorphic solids and fluids were formulated by Lee and Wang [16]. Later, Eringen extended the theory to include thermal and memory effects [27], while Lee and Chen formulated a constitutive theory in Lagrangian form of micromorphic thermoplasticity [28]. Establishing a reciprocity relation for linear dynamic micromorphic thermoelasticity, Iesan [29] obtained uniqueness results with no definiteness assumption and derived the reciprocal theorem. For micromorphic thermoelastic solids of degree 1, Iesan [30] formulated a nonlinear theory of micromorphic elastic solids in Lagrangian description as well as a theory of prestressed micromorphic thermoelastic bodies with initial heat flux. Nappa [31] established a Gurtin-type variational theory for linear dynamic micromorphic thermoelasticity.

While Eringen and co-workers [32–34] established the micropolar theory, Kadowaki and Liu [35] introduced a multiscale approach for micropolar continuum modeling. The micropolar theory was subsequently generalized by Diebels [36] to porous media and fluid-saturated granular materials and further extended by Eringen [37], Nowacki [38], and Iesan [39] to include thermal effects. Passarella and Zampoli [40] established the reciprocal and variational theorem for micropolar thermoelasticity. The theory of micropolar generalized thermoelasticity was established by Boschi and Iesan [41] and Sherief et al. [42]. Ezzat and Awad [43] developed the linear theory of micropolar generalized thermoelasticity involving two temperatures, whereas Othman and Singh [44] studied the effect of rotation on generalized micropolar thermoelasticity.

To study the deformation and motions of such materials as animal bones, solids weakened by microcracks, cellular foams, and synthetic materials with pores or microreinforcements, Eringen [23] proposed the theory of microstretch elasticity. Typically, a microstretch elastic solid possesses seven degrees of freedom: three for translation, three for rotations, and a stretch required by the substructures [45]. To account for thermal effects in microstretch media, Eringen [23] introduced further the thermomicrostretch elasticity theory while Cicco and Nappa [46] derived the governing equations of the linear theory for thermomicrostretch elastic solids. Following these studies, Aouadi [47] established the linear theory for microstretch thermoelastic bodies with microtemperatures, and Othman et al. [48] formulated the equations of generalized thermomicrostretch elasticity with temperature dependent properties considered. Plane waves of a generalized thermomicrostretch elastic half-space were considered by Othman and Lotfy [49]. Interactions caused by thermal and mechanical sources in generalized thermomicrostretch elastic media were investigated by Aouadi [50].

As theoretical models in the fields of chemistry, physics, aerodynamics, etc. are increasingly expressed in terms of fractional order, investigations concerning fractional derivatives and fractional integrals have become increasingly important. As previously mentioned, to consider both weak and strong heat conductivity in the context of generalized thermoelasticity, Youssef [11] developed the FOGTE theory. However, on one hand, a fractional order generalized micromorphic thermoelasticity (FOGTE_{mm}) theory is yet to be established;

on the other hand, although studies in the context of generalized micropolar or microstretch thermoelasticity do exist, little work with respect to fractional order can be found in the open domain. Further, the generalized variational theorem of generalized thermoelasticity with micromodeling is not available. Consequently, the systematic framework of generalized thermoelasticity with micromodeling is at present incomplete. To address these deficiencies, this study firstly proposes the theory of fractional order generalized micromorphic thermoelasticity. To generalize the theory, the unified forms of several degenerated generalized thermoelasticity theories are introduced. Subsequently, the uniqueness theorem of the developed theory is established, so is a generalized variational theorem. For reference, the fractional order generalized micropolar and microstretch thermoelasticity theories and the corresponding generalized variational principles are listed in the Appendix.

2 Governing equations and general theory

Consider a body occupying region v and bounded by piecewise smooth surface s in Euclidean three-dimensional space. The classical summation and differentiation conventions are followed: Latin subscripts range over integers (1, 2, 3), summation over repeated subscripts is implied, suffix preceded by a comma denotes material derivative, while the superposed dot denotes derivative with respect to time. Following the linear theory for micromorphic solids, the equations governing motion may hence be written as [25]:

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i, \tag{1}$$

$$m_{kij,k} + \sigma_{ji} - s_{ji} + L_{ij} = \rho I_{jk} \ddot{\varphi}_{ik} \tag{2}$$

where σ_{ij} is the stress tensor, s_{ij} is the microstress tensor, m_{ijk} is the stress moment tensor, u_i is the displacement vector, φ_{ij} is the microdeformation tensor, f_i is the body force, L_{ij} is the body moment tensor, ρ indicates mass density, and I_{ij} is the micro-inertia. Correspondingly, the generalized strain versus displacement relations are given by [25]:

$$\varepsilon_{ij} = u_{j,i} - \varphi_{ji}, \tag{3}$$

$$2e_{ij} = \varphi_{ij} + \varphi_{ji}, \tag{4}$$

$$\gamma_{ijk} = \varphi_{ij,k} \tag{5}$$

where ε_{ij}, e_{ij} and γ_{ijk} are the linear strain tensors, respectively.

Upon introducing the fractional integral operator, the generalized heat conduction law may be expressed as:

$$w_1 q_i + (w_2 \tau_1 + w_3)\, \dot{q}_i = -w_4 k I^{\alpha-1} \theta_{,i} - w_5 k^* I^{\alpha-1} \theta_{,i} - w_6 k I^{\alpha-1} \dot{\theta}_{,i} - w_7 k \tau_2 I^{\alpha-1} \dot{\theta}_{,i} \tag{6}$$

where q_i, τ_1, k, k^*, θ and τ_2 are the heat flux vector, the thermal relaxation time in ETE or phase lags of the heat flux in DPL (dual-phase-lag), the coefficient of thermal conductivity, the material constant characteristic in TEWOED or TEWED, the conductive temperature, and the phase lags of conductive temperature gradient in DPL, respectively, with $w_i (i = 1, 2, \ldots, 7)$ introduced here to generalize the model proposed in this study. By sequentially setting $w_i (i = 1, 2, \ldots, 7)$ as zero or unity, the corresponding theories degenerated from the present generalized theory are summarized in Table 1. In addition, I indicates an integral operator defined as [51,52]:

$$I^{\alpha-1} f(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - \tau)^{\alpha-2} f(\tau)\, d\tau \tag{7}$$

Table 1 Illustration of $\mathrm{FOGTE_{mm}}$, $\mathrm{FOGTE_{mp}}$ and $\mathrm{FOGTE_{ms}}$ (see Appendices A and B)

Theories	Based on	$w_i = 1$	Theories ($\alpha = 1$)	
$\mathrm{FOGTE_{mm}}$	ETE	$i = 1, 2, 4$	$\mathrm{GTE_{mm}}$	
	TRDTE	$i = 1, 4, 8$		
	TEWOED	$i = 3, 5$		
	TEWED	$i = 3, 5, 6$		
	DPLTE	$i = 1, 2, 4, 7$		
$\mathrm{FOGTE_{mp}}$	Based on GTE (generalized thermoelasticity) as above		$\mathrm{GTE_{mp}}$	[41–43]
$\mathrm{FOGTE_{ms}}$			$\mathrm{GTE_{ms}}$	[47–49]

where $\Gamma(\alpha)$ is the gamma function, $0 < \alpha \leq 2$, and

$$I^0 f(t) = f(t),\ \ I^{-\alpha} f(t) = \frac{\partial^\alpha}{\partial t^\alpha} f(t)\,. \tag{8}$$

Youssef [11] addressed the physical meaning of fractional order: $0 < \alpha < 1$ indicates weak conductivity; $\alpha = 1$ normal conductivity; $1 < \alpha < 2$ strong conductivity. Ghazizadeh et al. [53] evaluated the fractional order basing on the experimental results of heat conduction implemented on processed meat by Mitra et al. [54] and found that $0 < \alpha < 1$ for meat, which may be suitable for the porous materials and synthetic materials containing microscopic components and microcracks. In the absence of any inner heat source, the equation for energy conservation is

$$q_{i,i} = -\rho T_0 \dot{\eta} \tag{9}$$

where T_0 and η are separately the reference temperature and entropy density.

The constitutive relations may be described as [25]:

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} + E_{ijkl}e_{kl} + F_{ijklm}\gamma_{klm} - \chi^\sigma_{ij}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{10}$$

$$s_{ij} = E_{klij}\varepsilon_{kl} + B_{ijkl}e_{kl} + G_{ijklm}\gamma_{klm} - \chi^s_{ij}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{11}$$

$$m_{kij} = F_{lmijk}\varepsilon_{lm} + G_{lmijk}e_{lm} + C_{ijklmn}\gamma_{lmn} - \chi^m_{ijk}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{12}$$

$$\rho\eta = \chi^\sigma_{ij}\varepsilon_{ij} + \chi^s_{ij}e_{ij} + \chi^m_{ijk}\gamma_{ijk} + \frac{\rho c_E}{T_0}\left(\theta + w_8\tau_4\dot{\theta}\right) \tag{13}$$

where $A_{ijkl},\ B_{ijkl},\ C_{ijklmn},\ E_{ijkl},\ F_{ijklm},\ G_{ijklm},\ \chi^\sigma_{ij},\ \chi^s_{ij}$ and χ^m_{ijk} are the constitutive coefficients, respectively; c_E indicates the specific heat at constant deformation; τ_3 and τ_4 are the relaxation times in TRDTE; and w_8 is introduced to include the TRDTE theory, serving the same purpose as $w_i (i = 1, 2, \ldots, 7)$.

Equations (1) to (13) are fundamental equations governing the force and temperature fields. Substituting Eq. (10) into Eq. (1) and considering Eqs. (3) to (5), one obtains the governing equations for the deformation vector, as:

$$\left[A_{jikl}\left(u_{k,l} - \varphi_{lk}\right) + 0.5E_{jikl}\left(\varphi_{kl} + \varphi_{lk}\right) + F_{jiklm}\varphi_{kl,m} - \chi^\sigma_{ji}\left(\theta + w_8\tau_3\dot{\theta}\right)\right]_{,j} + f_i = \rho\ddot{u}_i. \tag{14}$$

Similarly, introducing Eqs. (10)–(12) into Eq. (2) and considering Eqs. (3) to (5), one obtains the governing equations for the microdeformation tensor, as:

$$\begin{aligned}&\left[F_{lmijk}\left(u_{l,m} - \varphi_{ml}\right) + 0.5G_{lmijk}\left(\varphi_{lm} + \varphi_{ml}\right) + C_{ijklmn}\varphi_{lm,n} - \chi^m_{ijk}\left(\theta + w_8\tau_3\dot{\theta}\right)\right]_{,k}\\ &+ \left(A_{jikl} - E_{klji}\right)\left(u_{k,l} - \varphi_{lk}\right) + 0.5\left(E_{jikl} - B_{jikl}\right)\left(\varphi_{kl} + \varphi_{lk}\right) + \left(F_{jiklm} - G_{jiklm}\right)\varphi_{kl,m}\\ &- \left(\chi^\sigma_{ji} - \chi^s_{ji}\right)\left(\theta + w_8\tau_3\dot{\theta}\right) + L_{ij} = \rho I_{jk}\ddot{\varphi}_{ik}.\end{aligned} \tag{15}$$

Combining Eqs. (6), (8) and (9) and then introducing Eq. (13), one obtains the following governing equation for the temperature field:

$$\begin{aligned}&T_0\beta\left[\chi^\sigma_{ij}\left(u_{i,j} - \varphi_{ji}\right) + 0.5\chi^s_{ij}\left(\varphi_{ij} + \varphi_{ji}\right) + \chi^m_{ijk}\varphi_{ij,k} + \frac{\rho c_E}{T_0}\left(\theta + w_8\tau_4\dot{\theta}\right)\right]\\ &= w_4k\theta_{,ii} + w_5k^*\theta_{,ii} + w_6k\dot{\theta}_{,ii} + w_7k\tau_2\dot{\theta}_{,ii}\end{aligned} \tag{16}$$

where

$$\beta = w_1\frac{\partial^\alpha}{\partial t^\alpha} + \left(w_2\tau_1 + w_3\right)\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}.$$

In addition to the system of field equations, suitable boundary and initial conditions need to be introduced. On surfaces s_1 and s_2, the displacement and traction are prescribed as:

$$u_i = \bar{u}_i,\ \ \text{on } s_1 \times [0, \infty)\,;\ \sigma_{ij}n_j = F_i, \text{on } s_2 \times [0, \infty)\,. \tag{17}$$

On surfaces s_3 and s_4, the microdisplacement and stress moment satisfy:

$$\varphi_{ij}=\bar{\varphi}_{ij}, \quad \text{on } s_3 \times [0,\infty)\,;\, m_{kij}n_k = \bar{m}_{ij}, \text{on } s_4 \times [0,\infty)\,. \tag{18}$$

On surface s_5 and s_6, the temperature and heat flux satisfy:

$$\theta = \bar{\theta}, \quad \text{on } s_5 \times [0,\infty)\,;\, q_i n_i = \bar{q}, \text{on } s_6 \times [0,\infty)\,. \tag{19}$$

Note that, in the above boundary conditions, $s_1 + s_2 = s_3 + s_4 = s_5 + s_6 = s$ covers the total boundary surface, while $\bar{u}_i$, F_i, $\bar{\varphi}_{ij}$, $\bar{m}_{ij}$, $\bar{\theta}$ and $\bar{q}$ are the prescribed functions. Moreover, the initial conditions may be introduced as:

$$\begin{aligned} &u_i(\mathbf{x},0) = u_i^0(\mathbf{x}),\ \dot{u}_i(\mathbf{x},0) = \dot{u}_i^0(\mathbf{x})\,,\\ &\varphi_{ij}(\mathbf{x},0) = \varphi_{ij}^0(\mathbf{x}),\ \dot{\varphi}_{ij}(\mathbf{x},0) = \dot{\varphi}_{ij}^0(\mathbf{x})\,,\\ &\theta(\mathbf{x},0) = \theta^0(\mathbf{x}),\ \dot{\theta}(\mathbf{x},0) = \dot{\theta}^0(\mathbf{x}) \end{aligned} \tag{20}$$

where $u_i^0(\mathbf{x})$, $\dot{u}_i^0(\mathbf{x})$, $\varphi_{ij}^0(\mathbf{x})$, $\dot{\varphi}_{ij}^0(\mathbf{x})$, $\theta^0(\mathbf{x})$ and $\dot{\theta}^0(\mathbf{x})$ represent also the prescribed functions.

Thus far, the initial boundary value problem in the context of the present FOGTE$_{\mathrm{mm}}$ theory has been proposed. As summarized in Table 1, the FOGTE$_{\mathrm{mm}}$ theory can be simplified into several special theories of generalized micromorphic thermoelasticity by neglecting the fractional integral operator and assigning $w_i(i = 1, 2, \ldots, 8)$ as zero or unity. For clarity, it is necessary to illustrate the exact meaning of each degenerated theory in Table 1: ETE$_{\mathrm{mm}}$, taken as the example, indicates the micromorphic ETE theory, which will be described in detail in Sect. 3.

3 Uniqueness theorem

This section presents the uniqueness results obtained for the above-mentioned initial boundary value problem. Let M and N be nonnegative integers. If function h is continuous on $s \times [0,\infty)$ and the following functions:

$$\frac{\partial^m}{\partial x_i \partial x_j \ldots \partial x_r}\left(\frac{\partial^n h}{\partial t^n}\right)\ m \in \{0,1,\ldots,M\},\ n \in \{0,1,\ldots,N\},\ m+n \le \max\{M,N\}$$

exist and are continuous on $s \times [0,\infty)$, h is of class $C^{M,N}$ on $s \times [0,\infty)$. For conciseness, let $C^{M,M}$ be denoted as C^M.

We denote the external data system, using an ordered array, as:

$$X = \left\{f_i, L_{ij}, \bar{u}_i, F_i, \bar{\varphi}_{ij}, \bar{m}_{ij}, \bar{\theta}, \bar{q}, u_i^0, \dot{u}_i^0, \varphi_{ij}^0, \dot{\varphi}_{ij}^0, \theta^0, \dot{\theta}^0\right\}$$

which satisfies the following properties: (1) $u_i^0, \dot{u}_i^0, \varphi_{ij}^0, \dot{\varphi}_{ij}^0, \theta^0, \dot{\theta}^0$ are continuous on s; (2) $\bar{u}_i$ are of class C^0 on $s_1 \times [0,\infty)$; (3) F_i are of class C^0 on $s_2 \times [0,\infty)$; (4) $\bar{\varphi}_{ij}$ are of class C^0 on $s_3 \times [0,\infty)$; (5) $\bar{m}_{ij}$ are of class C^0 on $s_4 \times [0,\infty)$; (6) $\bar{\theta}$ are of class C^0 on $s_5 \times [0,\infty)$; (7) $\bar{q}$ are of class C^0 on $s_6 \times [0,\infty)$; and (8) f_i, L_{ij} are of class C^0 on $s \times [0,\infty)$.

Similarly, an admissible process may be expressed, as:

$$P = \{u_i, \varphi_{ij}, \theta, \varepsilon_{ij}, e_{ij}, \gamma_{ijk}, \sigma_{ij}, s_{ij}, m_{ijk}, q_i\}$$

in which the variables satisfy: (i) u_i, φ_{ij} are of class $C^{1,2}$ on $s \times [0,\infty)$; (ii) θ are of class $C^{2,1}$ on $s \times [0,\infty)$; (iii) $\varepsilon_{ij}, e_{ij}, \gamma_{ijk}, s_{ij}$ are of class C^0 on $s \times [0,\infty)$; (iv) σ_{ij}, m_{ijk} are of class $C^{1,0}$ on $s \times [0,\infty)$; and (v) q_i are of class C^1 on $s \times [0,\infty)$.

We note that a solution of the initial boundary problem with external data system X is an admissible process P that successfully satisfies Eqs. (14) to (20).

For illustration but without loss of generality, only the uniqueness results for the ETE$_{\mathrm{mm}}$ theory shown in Table 1 are established here. For clarity, we needs to restate the ETE$_{\mathrm{mm}}$ theory. Equations (1) to (5) and (9) remain unchanged, while Eq. (6) needs to be rewritten as the generalized Fourier law of heat conduction, as:

$$q_i + \tau_1 \dot{q}_i = -k\theta_{,i}\,. \tag{21}$$

Further, Eqs. (10) to (13) are expressed in the new form, as:

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} + E_{ijkl}e_{kl} + F_{ijklm}\gamma_{klm} - \chi_{ij}^{\sigma}\theta, \tag{22}$$

$$s_{ij} = E_{klij}\varepsilon_{kl} + B_{ijkl}e_{kl} + G_{ijklm}\gamma_{klm} - \chi_{ij}^{s}\theta, \tag{23}$$

$$m_{kij} = F_{lmijk}\varepsilon_{lm} + G_{lmijk}e_{lm} + C_{ijklmn}\gamma_{lmn} - \chi_{ijk}^{m}\theta, \tag{24}$$

$$\rho\eta = \chi_{ij}^{\sigma}\varepsilon_{ij} + \chi_{ij}^{s}e_{ij} + \chi_{ijk}^{m}\gamma_{ijk} + \frac{\rho c_E}{T_0}\theta. \tag{25}$$

Equations (1) to (5), (9) and (21) to (25) represent the fundamental equations of the $\mathrm{ETE_{mm}}$ theory. The governing equations of $\mathrm{ETE_{mm}}$, which can be straightforwardly formulated, read as:

$$\left[A_{jikl}\left(u_{k,l} - \varphi_{lk}\right) + \frac{1}{2}E_{jikl}\left(\varphi_{kl} + \varphi_{lk}\right) + F_{jiklm}\varphi_{kl,m} - \chi_{ji}^{\sigma}\theta\right]_{,j} + f_i = \rho\ddot{u}_i, \tag{26}$$

$$\begin{aligned}&\left[F_{lmijk}\left(u_{l,m} - \varphi_{ml}\right) + \frac{1}{2}G_{lmijk}\left(\varphi_{lm} + \varphi_{ml}\right) + C_{ijklmn}\varphi_{lm,n} - \chi_{ijk}^{m}\theta\right]_{,k}\\&+ \left(A_{jikl} - E_{klji}\right)\left(u_{k,l} - \varphi_{lk}\right) + \frac{1}{2}\left(E_{jikl} - B_{jikl}\right)\left(\varphi_{kl} + \varphi_{lk}\right)\\&+ \left(F_{jiklm} - G_{jiklm}\right)\varphi_{kl,m} - \left(\chi_{ji}^{\sigma} - \chi_{ji}^{s}\right)\theta + L_{ij} = \rho I_{jk}\ddot{\varphi}_{ik},\end{aligned} \tag{27}$$

$$\chi_{ij}^{\sigma}\left(\dot{u}_{i,j} - \dot{\varphi}_{ji}\right) + \frac{1}{2}\chi_{ij}^{s}\left(\dot{\varphi}_{ij} + \dot{\varphi}_{ji}\right) + \chi_{ijk}^{m}\dot{\varphi}_{ij,k} + \frac{\rho c_E}{T_0}\dot{\theta} + \frac{1}{T_0}q_{i,i} = 0, \tag{28}$$

$$q_i + \tau_1\dot{q}_i = -k\theta_{,i}. \tag{29}$$

Corresponding to the process P, the following energies and function W are considered [54,55]:

$$\Xi = \frac{1}{2}\left(A_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + B_{ijkl}e_{ij}e_{kl} + C_{ijklmn}\gamma_{ijk}\gamma_{lmn}\right) + E_{ijkl}\varepsilon_{ij}e_{kl} + F_{ijklm}\varepsilon_{ij}\gamma_{klm} + G_{ijklm}e_{ij}\gamma_{klm}, \tag{30}$$

$$\Upsilon = \frac{1}{2}\frac{\rho c_E}{T_0}\theta^2, \tag{31}$$

$$K = \frac{1}{2}\rho\left(\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_{mi}\dot{\varphi}_{mj}\right), \tag{32}$$

$$M = \frac{1}{2}Q_{ij}q_iq_j, \tag{33}$$

$$W = \int_v \left(f_i\dot{u}_i + I_{ij}\dot{\varphi}_{ij} - \frac{1}{\tau_1}Q_{ij}q_iq_j\right)\mathrm{d}v + \int_s \left(\sigma_{ij}\dot{u}_i + m_{ijk}\dot{\varphi}_{jk} - \frac{1}{T_0}q_i\theta\right)n_i\mathrm{d}s \tag{34}$$

and

$$\Pi = \int_v (\Xi + \Upsilon + K + M)\mathrm{d}v.$$

Lemma *Let* X *be an external data, and* P *the corresponding admissible process that satisfies the governing Eqs. (26) to (29) and the initial boundary conditions (17) to (20). The following energy conservation law then holds for* $t \in [0, \infty)$*:*

$$\Pi(t) = \Pi(0) + \int_0^t W(\tau)\,\mathrm{d}\tau. \tag{35}$$

Proof Multiplying Eqs. (26) to (28) sequentially by $\dot{u}_i$, $\dot{\varphi}_{ij}$ and θ and summing, then integrating over v and making use of the divergence theorem, one obtains:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_v (\Xi + \Upsilon + K)\mathrm{d}v = &\int_v \left(f_i\dot{u}_i + I_{ij}\dot{\varphi}_{ij} + \frac{1}{T_0}q_i\theta_{,i}\right)\mathrm{d}v\\&+ \int_s \left(\sigma_{ij}\dot{u}_i + m_{ijk}\dot{\varphi}_{jk} - \frac{1}{T_0}q_i\theta\right)n_i\mathrm{d}s.\end{aligned} \tag{36}$$

Following Ciarletta and Scarpetta [57], one derives from Eq. (29) that:

$$Q_{ij}q_iq_j + \tau_1 B_{ij}q_i\dot{q}_j = -\frac{\tau_1}{T_0}q_i\theta_{,i}$$

from which there follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v M\mathrm{d}v = -\frac{1}{\tau_1}\int_v Q_{ij}q_iq_j\mathrm{d}v - \frac{1}{T_0}\int_v q_i\theta_{,i}\mathrm{d}v. \tag{37}$$

Considering Eqs. (36) and (37), one gets:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi = \int_v \left(f_i\dot{u}_i + I_{ij}\dot{\varphi}_{ij} - \frac{1}{\tau_1}Q_{ij}q_iq_j\right)\mathrm{d}v + \int_s \left(\sigma_{ij}\dot{u}_i + m_{ijk}\dot{\varphi}_{jk} - \frac{1}{T_0}q_i\theta\right)n_i\mathrm{d}s = W, \tag{38}$$

Integrating Eq. (38) over $v\times[0,\infty)$, one obtains the identity equation of (35), and hence, the proof is complete.

Theorem *Assume that:*

(i) ρ is positive, namely:

$$\rho > 0; \tag{39}$$

(ii) I_{ij} are positive, namely:

$$I_{ij} > 0; \tag{40}$$

(iii) For any admissible process P the quadratic form $\Xi + \Upsilon$ is positive semi-definite, namely:

$$\Xi + \Upsilon \geq 0. \tag{41}$$

Then, the problem defined by the governing Eqs. (26) to (29) and the initial boundary conditions (17) to (20) has at most one solution.

Proof Due to linearity, one only needs to show that null data imply null solution, and hence, the initial boundary value problem with vanishing external data, $X = \{0, 0, \ldots, 0\}$, is considered. As a result, one has:

$$\Pi(0) = 0, \tag{42}$$

$$W = -\frac{1}{\tau_1}\int_v Q_{ij}q_iq_j\mathrm{d}v. \tag{43}$$

One obtains hence from Eqs. (35), (42) and (43) that:

$$\int_v (\Xi + \Upsilon + K + M)\mathrm{d}v = \int_0^t \left(-\frac{1}{\tau_1}\int_v Q_{ij}q_iq_j\mathrm{d}v\right)\mathrm{d}\tau. \tag{44}$$

Following Ciarletta and Scarpetta [57], one has:

$$\frac{1}{\tau_1}Q_{ij}q_iq_j \geq 0. \tag{45}$$

Combining (33) and (45) (or (44) and (45)), one obtains:

$$M \geq 0, \tag{46}$$

$$\int_v (\Xi + \Upsilon + K + M)\mathrm{d}v = \int_0^t \left(-\frac{1}{\tau_1}\int_v Q_{ij}q_iq_j\mathrm{d}v\right)\mathrm{d}\tau \leq 0. \tag{47}$$

Considering Eqs. (39) to (41), (46) and (47), one gets:

$$\Xi = \Upsilon = K = M = 0. \tag{48}$$

Finally, in view of Eqs. (30) to (33), one obtains:

$$P = \{0, 0, \ldots, 0\},$$

and the proof is complete.

4 Generalized variational theorem

In Sect. 3, a uniqueness theorem for the newly proposed $\mathrm{FOGTE_{mm}}$ theory is established. To provide a complete rationale for formulating numerical methods with $\mathrm{FOGTE_{mm}}$, a generalized variational principle is needed. In what follows, we aim to formulate such a principle using the semi-inverse method [58].

To make the problem self-adjoint and avoid the Gurtin-type variational theorem, the time-derivative term in the fundamental equations should be expressed as [55]:

$$\frac{\partial \psi}{\partial t} = \frac{\psi(t) - \psi(t_{n-1})}{\Delta t} \underline{\text{written as}} = \frac{\psi - \psi^{(n-1)}}{\Delta t} \tag{49}$$

where ψ indicates an arbitrary function and $\Delta t = t - t_{n-1}$ is the equal step length. In view of (49), the constitutive Eqs. (10) to (12) can be rewritten as:

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} + E_{ijkl}e_{kl} + F_{ijklm}\gamma_{klm} - Z_1\chi_{ij}^{\sigma}\theta + Z_2, \tag{50}$$

$$s_{ij} = E_{klij}\varepsilon_{kl} + B_{ijkl}e_{kl} + G_{ijklm}\gamma_{klm} - Z_1\chi_{ij}^{s}\theta + Z_3, \tag{51}$$

$$m_{kij} = F_{lmijk}\varepsilon_{lm} + G_{lmijk}e_{lm} + C_{ijklmn}\gamma_{lmn} - Z_1\chi_{ijk}^{m}\theta + Z_4 \tag{52}$$

where

$$Z_1 = 1 + \frac{1}{\Delta t}w_8\tau_3, \quad Z_2 = \frac{1}{\Delta t}w_8\tau_3\chi_{ij}^{\sigma}\theta^{(n-1)},$$
$$Z_3 = \frac{1}{\Delta t}w_8\tau_3\chi_{ij}^{s}\theta^{(n-1)}, \quad Z_4 = \frac{1}{\Delta t}w_8\tau_3\chi_{ijk}^{m}\theta^{(n-1)},$$

Upon considering (49), the energy conservation equation (9) has the new form, as:

$$\Delta t q_{i,i} = -T_0\left(\chi_{ij}^{\sigma}\varepsilon_{ij} + \chi_{ij}^{s}e_{ij} + \chi_{ijk}^{m}\gamma_{ijk}\right) - Z_5\theta + Z_6 \tag{53}$$

where

$$Z_5 = (\Delta t + w_8\tau_4)\rho c_E, \ Z_6 = T_0\left(\chi_{ij}^{\sigma}\varepsilon_{ij}^{(n-1)} + \chi_{ij}^{s}e_{ij}^{(n-1)} + \chi_{ijk}^{m}\gamma_{ijk}^{(n-1)}\right) + w_8\tau_4\rho c_E\theta^{(n-1)}.$$

When $0 < \beta \le 1$, Caputo's definition of time fractional derivative gives:

$$D^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial f}{\partial t}(t-\tau)^{-\tau}\mathrm{d}\tau. \tag{54}$$

Replacing the integral term in (54) with a summation, and approximating the first-order time derivative by first-order backward difference, one has:

$$\begin{aligned}
D^{\beta}f(t) &= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{\partial f}{\partial t}(t-\tau)^{-\beta}\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{f^{(i)} - f^{(i-1)}}{\Delta t}(t-\tau)^{-\beta}\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\beta)}\sum_{i=1}^{n}\frac{f^{(i)} - f^{(i-1)}}{\Delta t}\int_{t_{i-1}}^{t_i}(t-\tau)^{-\beta}\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\beta)}\frac{1}{1-\beta}\frac{1}{\Delta t^{\beta}}\sum_{i=1}^{n}\left(f^{(i)} - f^{(i-1)}\right)\left[(n-i+1)^{1-\beta} - (n-i)^{1-\beta}\right].
\end{aligned} \tag{55}$$

In view of Eq. (55), Eq. (6) ($0 < \alpha \leq 1$) has the alternative form:

$$w_1 q_i + (w_2\tau_1 + w_3)\,\dot{q}_i + \frac{Z_7}{\alpha\Gamma(\alpha)\Delta t^{1-\alpha}}\theta_{,i} + \frac{Z_8}{\alpha\Gamma(\alpha)\Delta t^{1-\alpha}}\dot{\theta}_{,i} = Z_{9(0<\alpha\leq 1)} \tag{56}$$

where

$$\begin{aligned}
&Z_7 = w_4 k + w_5 k^*, \quad Z_8 = k\,(w_6 + w_7\tau_2)\,,\\
Z_{9(0<\alpha\leq 1)} = &-Z_7\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\left[-\left(\theta_{,i}\right)^{(n-1)}\right] - Z_8\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\left[-\left(\dot{\theta}_{,i}\right)^{(n-1)}\right]\\
&-Z_7\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\sum_{i=1}^{n-1}\left[\left(\theta_{,i}\right)^{(i)} - \left(\theta_{,i}\right)^{(i-1)}\right]\times\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right]\\
&-Z_8\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\sum_{i=1}^{n-1}\left[\left(\theta_{,i}\right)^{(i)} - \left(\theta_{,i}\right)^{(i-1)}\right]\times\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right].
\end{aligned}$$

Given Eq. (49), Eq. (56) may be further expressed as:

$$Z_{10(0<\alpha\leq 1)} q_i + Z_{11(0<\alpha\leq 1)}\theta_{,i} = Z_{12(0<\alpha\leq 1)} \tag{57}$$

where

$$\begin{aligned}
Z_{10(0<\alpha\leq 1)} &= \Delta t w_1 + w_2\tau_1 + w_3, \quad Z_{11(0<\alpha\leq 1)} = \frac{\Delta t Z_7 + Z_8}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}},\\
Z_{12(0<\alpha\leq 1)} &= \Delta t Z_{9(0<\alpha\leq 1)} + (w_2\tau_1 + w_3)\,q_i^{(n-1)} + \frac{Z_8}{\Gamma(\alpha)}\frac{1}{\alpha}\frac{1}{\Delta t^{1-\alpha}}\left(\theta_{,i}\right)^{(n-1)}.
\end{aligned}$$

When $1 \leq \alpha \leq 2$, Eq. (6) may be written:

$$w_1 D^{\alpha-1} q_i + (w_2\tau_1 + w_3)\,D^{\alpha-1}\dot{q}_i = -w_4 k\nabla\theta - w_5 k^*\nabla\theta - w_6 k\nabla\dot{\theta} - w_7 k\tau_2\nabla\dot{\theta}. \tag{58}$$

Considering Eq. (55), one obtains:

$$\frac{w_1}{\Gamma(2-\alpha)\,(2-\alpha)\,\Delta t^{\alpha-1}}q_i + \frac{w_2\tau_1 + w_3}{\Gamma(2-\alpha)\,(2-\alpha)\,\Delta t^{\alpha-1}}\dot{q}_i + Z_7\theta_{,i} + Z_8\dot{\theta}_{,i} = Z_{9(1\leq\alpha\leq 2)} \tag{59}$$

where

$$\begin{aligned}
Z_{9(1\leq\alpha\leq 2)} = &\, w_1\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\left[-q^{(n-1)}\right] + (w_2\tau_1 + w_3)\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\left[-\dot{q}^{(n-1)}\right]\\
&-w_1\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\sum_{i=1}^{n-1}\left[q^{(i)} - q^{(i-1)}\right]\times\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right]\\
&-(w_2\tau_1 + w_3)\frac{1}{\Gamma(2-\alpha)}\frac{1}{2-\alpha}\frac{1}{\Delta t^{\alpha-1}}\sum_{i=1}^{n-1}\left[\dot{q}^{(i)} - \dot{q}^{(i-1)}\right]\times\left[(n-i+1)^{\alpha-1} - (n-i)^{\alpha-1}\right].
\end{aligned}$$

In view of (49), Eq. (59) has the form:

$$Z_{10(1\leq\alpha\leq 2)} q_i + Z_{11(1\leq\alpha\leq 2)}\theta_{,i} = Z_{12(1\leq\alpha\leq 2)} \tag{60}$$

where

$$\begin{aligned}
Z_{10(1\leq\alpha\leq 2)} &= \frac{\Delta t w_1 + w_2\tau_1 + w_3}{\Gamma(2-\alpha)\,(2-\alpha)\,\Delta t^{\alpha-1}}, \quad Z_{11(1\leq\alpha\leq 2)} = \Delta t Z_4 + Z_5,\\
Z_{12(1\leq\alpha\leq 2)} &= \Delta t Z_{9(1\leq\alpha\leq 2)} + \frac{w_2\tau_1 + w_3}{\Gamma(2-\alpha)\,(2-\alpha)\,\Delta t^{\alpha-1}}q^{(n-1)} + Z_8\left(\theta_{,i}\right)^{(n-1)}.
\end{aligned}$$

For convenience, the general form of (57) and (60) may be written, as:

$$Z_{10}q_i + Z_{11}\theta_{,i} = Z_{12} \tag{61}$$

which will be used in the following derivation.

An energy-like trial functional with independent variables $\left(\sigma_{ij}, u_i, \varepsilon_{ij}, s_{ij}, m_{ijk}, \varphi_{ij}, e_{ij}, \gamma_{ijk}, \theta, q_i\right)$ may be established as:

$$J\left(\sigma_{ij}, u_i, \varepsilon_{ij}, s_{ij}, m_{ijk}, \varphi_{ij}, e_{ij}, \gamma_{ijk}, \theta, q_i\right) = \int_{t^{(n-1)}}^{t^{(n)}} \int_{v} L\mathrm{d}v\mathrm{d}t + IB \tag{62}$$

where

$$L = u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + F,$$
$$\mathrm{IB} = \sum_{i=1}^{6} \int_{t^{(n-1)}}^{t^{(n)}} \int_{s_i} G_i\mathrm{d}s\mathrm{d}t \tag{63}$$

for which F and G_i $(i = 1, 2, \ldots, 8)$ are unknown functions to be determined below.

The stationary condition with respect to u_i in Eq. (62) has the form:

$$\sigma_{ji,j} + f_i - \rho\ddot{u}_i + \frac{\delta F}{\delta u_i} = 0. \tag{64}$$

To satisfy Eq. (1), one has:

$$\frac{\delta F}{\delta u_i} = 0$$

which indicates that F is not related to u_i and its derivatives.

The stationary condition for σ_{ij} appearing in (62) may be expressed as:

$$-u_{j,i} + \frac{\delta F}{\delta\sigma_{ij}} = 0. \tag{65}$$

In view of (3), one has:

$$F = \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} + F_1. \tag{66}$$

Introducing (66) into (63) leads to:

$$L = u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} + F_1. \tag{67}$$

The trial Euler equation for ε_{ij} in (62) may be written as:

$$\sigma_{ij} + \frac{\delta F_1}{\delta\varepsilon_{ij}} = 0. \tag{68}$$

Upon setting

$$F_1 = -\frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} + Z_1\chi_{ij}^{\sigma}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + F_2 \tag{69}$$

one finds that (68) satisfies (50). Substitution of (69) into (67) results in:

$$\begin{aligned} L = u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \\ -E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} + Z_1\chi_{ij}^{\sigma}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + F_2. \end{aligned} \tag{70}$$

The stationary condition with respect to φ_{ji} in (62) is:

$$\sigma_{ji} + \frac{\delta F_2}{\delta \varphi_{ij}} = 0.$$

Considering (2), one gets:

$$F_2 = m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + F_3. \tag{71}$$

Combination of (70) and (71) yields:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi_{ij}^{\sigma}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + F_3. \end{aligned} \tag{72}$$

The trial Euler equation for e_{ij} in (62) has the form:

$$-E_{ijkl}\varepsilon_{kl} + \frac{\delta F_3}{\delta e_{ij}} = 0.$$

Given (51), one has:

$$F_3 = s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi_{ij}^{s}e_{ij}\theta - Z_3e_{ij} + F_4. \tag{73}$$

Substitution of (73) into (72) leads to:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi_{ij}^{\sigma}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} \\ & - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi_{ij}^{s}e_{ij}\theta - Z_3e_{ij} + F_4. \end{aligned} \tag{74}$$

The stationary condition for m_{kij} may be written as:

$$-\varphi_{ij,k} + \frac{\delta F_4}{\delta m_{kij}} = 0. \tag{75}$$

Combination of (5) and (75) yields:

$$F_4 = \gamma_{ijk}m_{kij} + F_5. \tag{76}$$

Introducing (76) into (74), one obtains:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi_{ij}^{\sigma}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} \\ & - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi_{ij}^{s}e_{ij}\theta - Z_3e_{ij} + \gamma_{ijk}m_{kij} + F_5. \end{aligned} \tag{77}$$

The trial Euler equation with respect to s_{ij} in (62) is:

$$-\frac{1}{2}\left(\varphi_{ij} + \varphi_{ji}\right) + e_{ij} + \frac{\delta F_5}{\delta s_{ij}} = 0.$$

Together with Eq. (4), it follows that:

$$\frac{\delta F_5}{\delta s_{ij}} = 0 \tag{78}$$

which indicates that F is not related to s_{ij} and its derivatives.

The stationary condition with respect to γ_{klm} in Eq. (62) has the form:

$$-F_{ijklm}\varepsilon_{ij} - e_{ij}G_{ijklm} + m_{klm} + \frac{\delta F_5}{\delta \gamma_{klm}} = 0.$$

In view of (52), one gets:

$$F_5 = -\frac{1}{2}C_{ijklmn}\gamma_{lmn}\gamma_{ijk} + Z_1\chi^m_{ijk}\gamma_{ijk}\theta - Z_4\gamma_{ijk} + F_6$$

from which it follows that:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi^\sigma_{ij}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} \\ & - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi^s_{ij}e_{ij}\theta - Z_3e_{ij} + \gamma_{ijk}m_{kij} - \frac{1}{2}C_{ijklmn}\gamma_{lmn}\gamma_{ijk} + Z_1\chi^m_{ijk}\gamma_{ijk}\theta - Z_4\gamma_{ijk} + F_6. \end{aligned} \tag{79}$$

The trial Euler equation for θ in Eq. (62) reads:

$$Z_1\chi^\sigma_{ij}\varepsilon_{ij} + Z_1\chi^s_{ij}e_{ij} + Z_1\chi^m_{ijk}\gamma_{lmn} + \frac{\delta F_6}{\delta\theta} = 0. \tag{80}$$

Considering (53) and (80), one has:

$$F_6 = \frac{Z_1\theta}{T_0}\left(\Delta t q_{i,i} + \frac{1}{2}Z_5\theta - Z_6\right) + F_7. \tag{81}$$

Introducing (81) into (79), one obtains:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi^\sigma_{ij}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} \\ & - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi^s_{ij}e_{ij}\theta - Z_3e_{ij} + \gamma_{ijk}m_{kij} - \frac{1}{2}C_{ijklmn}\gamma_{lmn}\gamma_{ijk} + Z_1\chi^m_{ijk}\gamma_{ijk}\theta - Z_4\gamma_{ijk} \\ & + \frac{Z_1\theta}{T_0}\left(\Delta t q_{i,i} + \frac{1}{2}Z_5\theta - Z_6\right) + F_7. \end{aligned} \tag{82}$$

The stationary condition with respect to q_i may be written as:

$$-\frac{\Delta t Z_1}{T_0}\theta_{,i} + \frac{\delta F_7}{\delta q_i} = 0.$$

In view of (61), one gets:

$$F_7 = -\frac{\Delta t Z_1 q_i}{Z_{11}T_0}\left(\frac{1}{2}Z_{10}q_i - Z_{12}\right)$$

from which it follows that:

$$\begin{aligned} L = {} & u_i\left(\sigma_{ji,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + \varepsilon_{ij}\sigma_{ij} + \varphi_{ji}\sigma_{ij} - \frac{1}{2}A_{ijkl}\varepsilon_{kl}\varepsilon_{ij} - E_{ijkl}e_{kl}\varepsilon_{ij} - F_{ijklm}\gamma_{klm}\varepsilon_{ij} \\ & + Z_1\chi^\sigma_{ij}\theta\varepsilon_{ij} - Z_2\varepsilon_{ij} + m_{kji,k}\varphi_{ji} - s_{ij}\varphi_{ji} + L_{ji}\varphi_{ji} - \frac{1}{2}\rho I_{ik}\dot{\varphi}_{jk}\dot{\varphi}_{ji} + s_{ij}e_{ij} - \frac{1}{2}B_{ijkl}e_{kl}e_{ij} \\ & - G_{ijklm}\gamma_{klm}e_{ij} + Z_1\chi^s_{ij}e_{ij}\theta - Z_3e_{ij} + \gamma_{ijk}m_{kij} - \frac{1}{2}C_{ijklmn}\gamma_{lmn}\gamma_{ijk} + Z_1\chi^m_{ijk}\gamma_{ijk}\theta - Z_4\gamma_{ijk} \\ & + \frac{Z_1\theta}{T_0}\left(\Delta t q_{i,i} + \frac{1}{2}Z_5\theta - Z_6\right) - \frac{\Delta t Z_1 q_i}{Z_{11}T_0}\left(\frac{1}{2}Z_{10}q_i - Z_{12}\right). \end{aligned} \tag{83}$$

Applying Green's theory on the boundary results in:

$$\begin{aligned}
&\delta u_i :\ \frac{\partial G_i}{\partial u_i} = 0,\\
&\delta \sigma_{ij} :\ -u_i n_i + \frac{\partial G_i}{\partial \sigma_{ij}} = 0,\\
&\delta \varphi_{ij} :\ \frac{\partial G_i}{\partial \varphi_{ij}} = 0,\\
&\delta m_{kij} :\ -\varphi_{ij} n_j + \frac{\partial G_i}{\partial m_{kij}} = 0,\\
&\delta \theta :\ \frac{\partial G_i}{\partial \theta} = 0,\\
&\delta q_i :\ -\frac{Z_1 \Delta t}{T_0}\theta n_i + \frac{\partial G_i}{\partial q_i} = 0.
\end{aligned} \qquad (i = 1, 2, \ldots, 6) \tag{84}$$

Considering the boundary equations on s_i $(i = 1, 2, \ldots, 8)$, one obtains from (84) that:

$$\begin{aligned}
&G_1 = u_i\left(\sigma_{ij} n_j - F_i\right), \qquad G_2 = \sigma_{ij}\bar{u}_i n_j,\\
&G_3 = \varphi_{ij}\left(m_{ijk} n_k - \bar{m}_{ij}\right), \qquad G_4 = m_{ijk}\bar{\varphi}_{ij} n_k,\\
&G_5 = \frac{Z_1 \Delta t}{T_0}\theta\left(q_i n_i - \bar{q}\right), \qquad G_6 = \frac{Z_1 \Delta t}{T_0} q_i \bar{\theta} n_i.
\end{aligned} \tag{85}$$

Finally, substitution of (83) and (85) into (62) results in the generalized variational principle for the FOGTE$_{mm}$ theory. Note that several special variational theorems can be obtained by introducing suitable constraints.

5 Conclusion

Built upon the micromorphic theory, a theory of fractional order generalized micromorphic thermoelasticity (FOGTE$_{mm}$) is developed by employing the fractional integral operator. To generalize the new theory, the general forms of several generalized thermoelastic theories, such as the extended thermoelasticity (ETE), the temperature rate dependent thermoelasticity (TRDTE), the thermoelasticity without energy dissipation (TEWOED), the thermoelasticity with energy dissipation (TEWED), the two-temperature generalized thermoelasticity (TTGTE), and the dual-phase-lag thermoelasticity (DPLTE), are introduced. Corresponding uniqueness results are given and proven completely. For illustration, the generalized variational principle of the FOGTE$_{mm}$ theory is developed by using the semi-inverse method. In addition, the fractional order generalized micropolar (FOGTE$_{mp}$) and microstretch thermoelasticity (FOGTE$_{ms}$) as well as the corresponding generalized variational principles are also presented (see Appendices A and B).

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Appendix A: Theory of the fractional order generalized micropolar thermoelasticity (FOGTE$_{mp}$) and the corresponding variational principle

Here, with the linear theory of micropolar solids in mind, we introduce firstly the FOGTE$_{mp}$ theory and then propose the corresponding variational principle. The equations of motion may be written as [42]:

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i, \tag{A.1}$$

$$m_{ji,j} + \varepsilon_{ijk}\sigma_{jk} + G_i = \rho I_{ij}\ddot{\phi}_j \tag{A.2}$$

where σ_{ij} is the stress tensor, f_i is the body force, ρ is the mass density, u_i is the displacement vector, m_{ij} is the moment of couple stresses, ε_{ijk} is the alternating tensor, G_i is the body couple tensor, I_{ij} are the coefficients of inertia, and ϕ_i represents the micro-rotations field.

The generalized strain versus displacement relations are:

$$e_{ij} = u_{j,i} - \varepsilon_{ijk}\phi_k, \tag{A.3}$$

$$\kappa_{ij} = \phi_{j,i} \tag{A.4}$$

where e_{ij} and κ_{ij} are the linear strain tensors, respectively.

Upon introducing the fractional integral operator, the generalized heat conduction law may be expressed as:

$$w_1 q_i + (w_2\tau_1 + w_3)\,\dot{q}_i = -w_4 k I^{\alpha-1}\theta_{,i} - w_5 k^* I^{\alpha-1}\theta_{,i} - w_6 k I^{\alpha-1}\dot{\theta}_{,i} - w_7 k\tau_2 I^{\alpha-1}\dot{\theta}_{,i}. \tag{A.5}$$

In the absence of any inner heat source, the equation for energy conservation is:

$$q_{i,i} = -\rho T_0 \dot{\eta}. \tag{A.6}$$

The constitutive relations may be described as:

$$\sigma_{ij} = A_{ijkl}e_{kl} + E_{ijkl}\kappa_{kl} - \chi_{ij}^{\sigma}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{A.7}$$

$$m_{ij} = E_{ijkl}e_{kl} + C_{ijkl}\kappa_{kl} - \chi_{ij}^{m}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{A.8}$$

$$\rho\eta = \chi_{ij}^{\sigma}e_{ij} + \chi_{ij}^{m}\kappa_{ij} + \frac{\rho c_E}{T_0}\left(\theta + w_8\tau_4\dot{\theta}\right) \tag{A.9}$$

where $A_{ijkl},\ B_{ijkl},\ C_{ijkl},\ E_{ijkl},\ \chi_{ij}^{\sigma}$, and χ_{ij}^{m} are separately the constitutive coefficients; c_E indicates specific heat at constant deformation; τ_3 and τ_4 represent relaxation times in TRDTE.

A typical problem in the context of $\mathrm{FOGTE_{mp}}$ may be formulated, combining Eqs. (A.1)–(A.9) and corresponding initial and boundary conditions. Considering the conciseness of the micropolar theory among all the micromodeling theories, the reader is referred to [42] for the case of isotropic media, which is commonly existing in practical engineering. As shown in Sect. 4, an energy-like trial functional with independent variables $\left(\sigma_{ij}, u_i, e_{ij}, m_{ij}, \varphi_i, \kappa_{ij}, \theta, q_i\right)$ may be given as follows:

$$J\left(\sigma_{ij}, u_i, e_{ij}, m_{ij}, \varphi_i, \kappa_{ij}, \theta, q_i\right) = \int_{t^{(n-1)}}^{t^{(n)}} \int_{v} L\,\mathrm{d}v\,\mathrm{d}t + IB \tag{A.10}$$

where

$$\begin{aligned} L = {} & u_i\left(\sigma_{ij,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + e_{ij}\sigma_{ij} + \varepsilon_{ijk}\phi_k\sigma_{ij} - \frac{1}{2}A_{ijkl}e_{kl}e_{ij} - E_{ijkl}\kappa_{kl}e_{ij} + Z_1\chi_{ij}^{\sigma}\theta e_{ij} \\ & - Z_2 e_{ij} - m_{ji,j}\phi_i - G_i\phi_i + \frac{1}{2}\rho I_{ij}\dot{\phi}_j\dot{\phi}_i + m_{ij}\kappa_{ij} - \frac{1}{2}C_{ijkl}\kappa_{kl}\kappa_{ij} + Z_1\chi_{ij}^{m}\theta\kappa_{ij} - Z_{13}\kappa_{ij} \\ & + \frac{Z_1\theta}{T_0}\left(\Delta t q_{i,i} + \frac{1}{2}Z_5\theta - Z_{14}\right) - \frac{\Delta t Z_1 q_i}{Z_{11}T_0}\left(\frac{1}{2}Z_{10}q_i - Z_{12}\right), \end{aligned} \tag{A.11}$$

$$IB = \sum_{i=1}^{6}\int_{t^{(n-1)}}^{t^{(n)}}\int_{s_i} G_i\,\mathrm{d}s\,\mathrm{d}t \tag{A.12}$$

in which

$$Z_{13} = \frac{1}{\Delta t}w_8\tau_3\chi_{ij}^{m}\theta^{(n-1)},$$

$$Z_{14} = T_0\left(\chi_{ij}^{\sigma}e_{ij}^{(n-1)} + \chi_{ij}^{m}\kappa_{ij}^{(n-1)}\right) + w_8\tau_4\rho c_E\theta^{(n-1)}.$$

The variational operation of the energy-like trial functional J implies the generalized variational theorem of $\mathrm{FOGTE_{mp}}$.

Appendix B: Theory of the fractional order generalized microstretch thermoelasticity ($FOGTE_{ms}$) and the corresponding variational principle

Given that the microstretch theory is extended from the micropolar theory by considering microstretch, for conciseness, there is no need to restate the part shown in Appendix A. Consequently, to formulate the $FOGTE_{ms}$ theory, we just supplement or modify some contents of the $FOGTE_{mp}$ theory. The equation that needs to be added to the equations of motion, i.e., (A.1) and (A.2), is [46]:

$$\pi_{i,i} - g + L = \rho J \ddot{\psi} \tag{B.1}$$

where π_i is the microstress, g is the generalized internal body load, L is the generalized external body load, J is the coefficient of inertia, and ψ is the microstretch function.
The generalized strain versus displacement relations of (A.3) and (A.4) need to be supplemented by [59]:

$$\gamma_i = \psi_{,i}. \tag{B.2}$$

The constitutive relations (A.9) to (A.11) are modified and supplemented as:

$$\sigma_{ij} = A_{ijkl}e_{kl} + B_{ijkl}\kappa_{kl} + D_{ijk}\gamma_k + A_{ij}\psi - \chi^{\sigma}_{ij}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{B.3}$$

$$m_{ij} = B_{ijkl}e_{kl} + C_{ijkl}\kappa_{kl} + E_{ijk}\gamma_k + B_{ij}\psi - \chi^{m}_{ijk}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{B.4}$$

$$\pi_i = D_{ijk}e_{jk} + E_{ijk}\kappa_{jk} + D_{ij}\gamma_j + d_i\psi - \chi^{\pi}_{i}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{B.5}$$

$$g = A_{ij}e_{ij} + B_{ij}\kappa_{ij} + d_i\gamma_j + m\psi - \chi^{g}\left(\theta + w_8\tau_3\dot{\theta}\right), \tag{B.6}$$

$$\rho\eta = \chi^{\sigma}_{ij}e_{ij} + \chi^{m}_{ij}\kappa_{ij} + \chi^{\pi}_{i}\gamma_j + \chi^{g}\psi + \frac{\rho c_E}{T_0}\left(\theta + w_8\tau_4\dot{\theta}\right). \tag{B.7}$$

The equations (A.1) to (A.6) and (B.1) to (B.7) represent the fundamental equations of the $FOGTE_{ms}$ theory. Here, to establish the variational theorem, an energy-like trial functional with independent variables $\left(\sigma_{ij}, u_i, e_{ij}, m_{ij}, \varphi_i, \kappa_{ij}, \pi_i, \psi, \gamma_i, \theta, q_i\right)$ for initial boundary value problems of $FOGTE_{ms}$ may be expressed as:

$$J\left(\sigma_{ij}, u_i, e_{ij}, m_{ij}, \varphi_i, \kappa_{ij}, \pi_i, \psi, \gamma_i, \theta, q_i\right) = \int_{t^{(n-1)}}^{t^{(n)}} \int_v L \,\mathrm{d}v\mathrm{d}t + IB \tag{B.8}$$

where

$$\begin{aligned} L = {} & u_i\left(\sigma_{ij,j} + f_i\right) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i + e_{ij}\sigma_{ij} + \varepsilon_{ijk}\phi_k\sigma_{ij} - \frac{1}{2}A_{ijkl}e_{kl}e_{ij} - B_{ijkl}\kappa_{kl}e_{ij} \\ & - D_{ijk}\gamma_k e_{ij} - A_{ij}\psi e_{ij} + Z_1\chi^{\sigma}_{ij}\theta e_{ij} - Z_2 e_{ij} - m_{ji,j}\phi_i - G_i\phi_i + \frac{1}{2}\rho I_{ij}\dot{\phi}_j\dot{\phi}_i + m_{ij}\kappa_{ij} \\ & - \frac{1}{2}C_{ijkl}\kappa_{kl}\kappa_{ij} - E_{ijk}\gamma_k\kappa_{ij} - B_{ij}\psi\kappa_{ij} + Z_1\chi^{m}_{ij}\theta\kappa_{ij} - Z_{13}\kappa_{ij} + \pi_{i,i}\psi - d_i\gamma_i\psi \\ & - \frac{1}{2}m\psi^2 + Z_1\chi^{g}\theta\psi - Z_{15}\psi + L\psi - \frac{1}{2}\rho J\dot{\psi}^2 + \pi_i\gamma_i - \frac{1}{2}\gamma_i D_{ij}\gamma_j + Z_1\chi^{\pi}_{i}\theta\gamma_i - Z_{16}\gamma_i \\ & + \frac{Z_1\theta}{T_0}\left(\Delta t q_{i,i} + \frac{1}{2}Z_5\theta - Z_{17}\right) - \frac{\Delta t Z_1 q_i}{Z_{11}T_0}\left(\frac{1}{2}Z_{10}q_i - Z_{12}\right), \end{aligned} \tag{B.9}$$

$$IB = \sum_{i=1}^{8} \int_{t^{(n-1)}}^{t^{(n)}} \int_{s_i} G_i \,\mathrm{d}s\mathrm{d}t \tag{B.10}$$

in which

$$Z_{15} = \frac{1}{\Delta t}w_8\tau_3\chi^{g}\theta^{(n-1)}, \quad Z_{16} = \frac{1}{\Delta t}w_8\tau_3\chi^{\pi}_{i}\theta^{(n-1)},$$
$$Z_{17} = T_0\left(\chi^{\sigma}_{ij}e^{(n-1)}_{ij} + \chi^{m}_{ij}\kappa^{(n-1)}_{ij} + \chi^{\pi}_{i}\gamma^{(n-1)}_{i} + \chi^{g}\psi^{(n-1)}\right) + w_8\tau_4\rho c_E\theta^{(n-1)}.$$

The generalized variational theorem of $FOGTE_{ms}$ is obtained via variational operation of the energy-like trial functional J.

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