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A two-dimensional generalized thermoelastic diffusion problem for a half-space subjected to harmonically varying heating

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Abstract A two-dimensional problem for a thermoelastic half-space is considered within the context of the theory of generalized thermoelastic diffusion with one relaxation time. The upper surface of the half-space is taken to be traction free and subjected to harmonically varying heating with constant angular frequency of thermal vibration. Laplace and Fourier transform techniques are used. The solution in the transformed domain is obtained by a direct approach. Numerical inversion techniques are used to obtain the inverse double transforms. Numerical results are discussed and represented graphically.

1 Introduction

Recently, the studying of diffusion became increasingly important. This is due mainly to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, and form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors.

In most of these applications, the concentration is calculated using what is known as Fick's law. The Fick's law is analogous to the relationships discovered in the same era by other eminent scientists: Darcy's law (hydraulic flow), Ohm's law (charge transport), and Fourier's Law (heat transport). Equations based on Fick's law have been commonly used to model transport processes in foods, neurons, biopolymers, pharmaceuticals, porous soils, population dynamics, nuclear materials, semiconductor doping processes, etc.

The coupled theory of thermoelasticity was developed by Biot [1] to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories, however, are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations.

Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. Dhaliwal and Sherief [3] extended this theory to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity.

Uniqueness of solution for this theory was proved under different conditions by Ignaczak [4]. Sherief and El-Maghraby [5,6] solved two crack problems. Sherief et al. [7] have solved a dynamical problem for an infinitely long hollow cylinder for short time. A one-dimensional problem for a half-space under the action of a body force has been solved by Saleh in [8]. Sherief et al. [9] have solved a stochastic thermal shock problem in

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generalized thermoelasticity. Sherief and Hamza [10,11] and Elhagary [12] have solved some two-dimensional problems. Sherief et al. extended this theory to include micropolar effects [13] and viscoelastic materials [14]. This theory was generalized using fractional derivatives by Sherief et al. [15].

Nowacki [16–19] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Recently, Sherief et al. [20] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. In this theory, Sherief et al. [20] studied the uniqueness and introduced a variational and reciprocity theorem. Sherief and Saleh [21] solved a one-dimensional thermoelastic diffusion problem for a half-space, Sherief and El-Maghraby [22] solved a thick plate problem in the theory of generalized thermoelastic diffusion. Elhagary [23,24] solved some one-dimensional problems in the theory of generalized thermoelastic diffusion in cylindrical and spherical coordinates.

In the present work, the author considers a two-dimensional problem for a half-space. The bounding surface is taken to be traction free and subjected to harmonically varying heating with constant angular frequency of thermal vibration. Laplace and exponential Fourier transform techniques are used.

2 Formulation of the problem

We consider a homogeneous isotropic thermoelastic solid occupying the half-space $y \geq 0$. The y-axis is taken perpendicular to the bounding plane pointing inward. We also assume that the initial state of the medium is quiescent. The surface of this medium is taken to be traction free and subjected to harmonic heating. The chemical potential is also assumed to be a known function of time on the surface of the half-space.

The displacement vector thus has the form $\mathbf{u} = (u, v, 0)$, and the cubical dilatation e is given by

$$e = \operatorname{div}\underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{1}$$

The equation of motion in the absence of body forces can be written as [20]:

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial e}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \beta_1 \frac{\partial T}{\partial x} - \beta_2 \frac{\partial C}{\partial x}, \tag{2}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial e}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \beta_1 \frac{\partial T}{\partial y} - \beta_2 \frac{\partial C}{\partial y}, \tag{3}$$

where T is the absolute temperature, C is the concentration of the diffusion material in the elastic body, λ, μ are Lamé constant, ρ is the density, β_1 and β_2 are material constants given by $\beta_1 = (3\lambda + 2\mu)\alpha_t$ and $\beta_2 = (3\lambda + 2\mu)\alpha_c$, α_t is the coefficient of linear thermal expansion, and α_c is the coefficient of linear diffusion expansion

The energy equation has the form [20]

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) [\rho c_E T + \beta_1 T_0 e + a T_0 C], \tag{4}$$

where k is the thermal conductivity, c_E is the specific heat at constant strain, τ_0 is the thermal relaxation time, a is the measure of thermo-diffusion effect, and T_0 is a reference temperature assumed to obey the inequality $|(T - T_0)/T_0| << 1$.

The diffusion equation has the form [20]

$$D\beta_2 \nabla^2 e + Da\nabla^2 T + \left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2} \right) C = Db\nabla^2 C, \tag{5}$$

where D is the diffusion coefficient, b is a measure of diffusion effect, and τ is the diffusion relaxation time.

The constitutive equations are given by [20] as follows:

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij} [\lambda e_{kk} - \beta_1 (T - T_0) - \beta_2 C], \tag{6a}$$

$$P = -\beta_2 e_{kk} + bC - a (T - T_0), \tag{6b}$$

where P is the chemical potential, and σ_{ij} are the components of the stress tensor.

In our case, by using Eq. (1), Eq. (6a) become

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda e - \beta_1 (T - T_0) - \beta_2 C$$
$$= (\lambda + 2\mu) e - 2\mu\frac{\partial v}{\partial y} - \beta_1 (T - T_0) - \beta_2 C. \tag{7a}$$

Similarly,

$$\sigma_{yy} = (\lambda + 2\mu) e - 2\mu\frac{\partial u}{\partial x} - \beta_1 (T - T_0) - \beta_2 C, \tag{7b}$$
$$\sigma_{zz} = \lambda e - \beta_1 (T - T_0) - \beta_2 C, \tag{7c}$$
$$\sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{7d}$$
$$\sigma_{xz} = \sigma_{yz} = 0. \tag{7e}$$

We shall use the following nondimensional variables:

$$x' = c\eta x, \quad y' = c\eta y, \quad u' = c\eta u, \quad v' = c\eta v, \quad t' = c^2\eta t, \quad \tau'_0 = c^2\eta\tau_0,$$
$$\tau' = c^2\eta\tau, \quad \theta = \frac{\beta_1 (T - T_0)}{(\lambda + 2\mu)}, \quad C' = \frac{\beta_2 C}{(\lambda + 2\mu)}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\lambda + 2\mu}, \quad P' = \frac{P}{\beta_2}$$

where $\eta = \frac{\rho c_E}{k}$, and $c = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the speed of propagation of isothermal elastic waves.

Using these nondimensional variables, the governing equations (1–7) take the following form (dropping the primes for convenience):

$$\beta^2\frac{\partial^2 u}{\partial t^2} = \left(\beta^2 + 1\right)\frac{\partial e}{\partial x} + \nabla^2 u - \beta^2\frac{\partial\theta}{\partial x} - \beta^2\frac{\partial C}{\partial x}, \tag{8}$$

$$\beta^2\frac{\partial^2 v}{\partial t^2} = \left(\beta^2 + 1\right)\frac{\partial e}{\partial y} + \nabla^2 v - \beta^2\frac{\partial\theta}{\partial y} - \beta^2\frac{\partial C}{\partial y}, \tag{9}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)[\theta + \varepsilon e + \varepsilon\alpha_1 C], \tag{10}$$

$$\nabla^2 e + \nabla^2\theta + \alpha_2\left(\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right)C = \alpha_3\nabla^2 C, \tag{11}$$

$$\sigma_{xx} = \beta^2 e - 2\frac{\partial v}{\partial y} - \beta^2\theta - \beta^2 C, \tag{12a}$$

$$\sigma_{yy} = \beta^2 e - 2\frac{\partial u}{\partial x} - \beta^2\theta - \beta^2 C, \tag{12b}$$

$$\sigma_{zz} = \left(\beta^2 - 2\right)e - \beta^2\theta - \beta^2 C, \tag{12c}$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{12d}$$

$$P = \alpha_3 C - e - \alpha_1\theta, \tag{12e}$$

where $\varepsilon = \frac{\beta_1^2 T_0}{\rho c_E(\lambda+2\mu)}$, $\alpha_1 = \frac{a(\lambda+2\mu)}{\beta_1\beta_2}$, $\alpha_2 = \frac{\lambda+2\mu}{\beta_2^2 D\eta}$, and $\alpha_3 = \frac{b(\lambda+2\mu)}{\beta_2^2}$.

Combining Eqs. (8) and (9), upon using Eq. (1) we get

$$\frac{\partial^2 e}{\partial t^2} = \nabla^2 e - \nabla^2\theta - \nabla^2 C, \tag{13}$$

The boundary conditions, at $y = 0$, are taken as

$$\text{(i) } \sigma_{yy}(x,0,t) = \sigma_{xy}(x,0,t) = 0, \tag{14a}$$

$$\text{(ii) } \theta(x,0,t) = \theta_0 \text{Cos}(\omega t)\, H(d - |x|), \tag{14b}$$

$$\text{(iii) } P(x,0,t) = P_0 H(t), \tag{14c}$$

where θ_0, P_0 and d are constants while ω is the angular frequency of thermal vibration ($\omega = 0$ for a thermal shock). $H(.)$ is the Heaviside unit step function. Thus, the surface $y = 0$ is traction free and it is heated on a band of width $2d$ around the x-axis.

3 Solution in the Laplace transformed domain

Applying the Laplace transform with parameter s (denoted by a bar) of both sides of Eqs. (8)–(13), we obtain the following set of equations:

$$\left(\nabla^2 - \beta^2 s^2\right)\bar{u} = \frac{\partial}{\partial x}\left(\beta^2\left(\bar{\theta} + \bar{C}\right) - \left(\beta^2 - 1\right)\bar{e}\right), \tag{15}$$

$$\left(\nabla^2 - \beta^2 s^2\right)\bar{v} = \frac{\partial}{\partial y}\left(\beta^2\left(\bar{\theta} + \bar{C}\right) - \left(\beta^2 - 1\right)\bar{e}\right), \tag{16}$$

$$\left(\nabla^2 - s^2\right)\bar{e} = \nabla^2\bar{\theta} + \nabla^2\bar{C}, \tag{17}$$

$$\left(\nabla^2 - s(1 + \tau_0 s)\right)\bar{\theta} = \varepsilon s(1 + \tau_0 s)\left[\bar{e} + \alpha_1\bar{C}\right], \tag{18}$$

$$\left(\alpha_3\nabla^2 - \alpha_2 s(1 + \tau_0 s)\right)\bar{C} = \nabla^2\bar{e} + \alpha_1\nabla^2\bar{\theta}, \tag{19}$$

$$\bar{\sigma}_{xx} = \beta^2\left(\bar{e} - \bar{\theta} - \bar{C}\right) - 2\frac{\partial\bar{v}}{\partial y}, \tag{20a}$$

$$\bar{\sigma}_{yy} = \beta^2\left(\bar{e} - \bar{\theta} - \bar{C}\right) - 2\frac{\partial\bar{u}}{\partial x}, \tag{20b}$$

$$\bar{\sigma}_{zz} = \left(\beta^2 - 2\right)\bar{e} - \beta^2\left(\bar{\theta} - \bar{C}\right), \tag{20c}$$

$$\bar{\sigma}_{xy} = \left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right), \tag{20d}$$

$$\bar{P} = \alpha_3\bar{C} - \bar{e} - \alpha_1\bar{\theta}. \tag{20e}$$

Eliminating $\bar{C}$ and $\bar{e}$ between Eqs. (17)–(19), we get

$$\left(\nabla^6 - a_1\nabla^4 + a_2\nabla^2 - a_3\right)\bar{\theta} = 0, \tag{21}$$

where

$$a_1 = \frac{s}{\alpha_3 - 1}\left[(1 + \tau_0 s)\left(\alpha_1\varepsilon\left((\alpha_1 + 2) + \alpha_3(\varepsilon + 1) - 1\right)\right) + \alpha_2(1 + \tau s) + \alpha_3 s\right],$$

$$a_1 = \frac{s^2}{\alpha_3 - 1}\left[(1 + \tau_0 s)\left(\alpha_1\varepsilon s^2 + \alpha_3 s + \alpha_2(1 + \tau s)(\varepsilon + 1)\right) + +\alpha_2(1 + \tau s)s\right],$$

$$a_3 = \frac{s^4\alpha_2}{\alpha_3 - 1}(1 + \tau s)(1 + \tau_0 s).$$

In a similar manner, we can show that $\bar{e}$ and $\bar{C}$ satisfy the equations

$$\left(\nabla^6 - a_1\nabla^4 + a_2\nabla^2 - a_3\right)\bar{e} = 0, \tag{22}$$

$$\left(\nabla^6 - a_1\nabla^4 + a_2\nabla^2 - a_3\right)\bar{C} = 0. \tag{23}$$

Equation (21) can be factorized as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\left(\nabla^2 - k_3^2\right)\bar{\theta} = 0, \tag{24}$$

where k_1, k_2 and k_3 are the roots with positive real parts of the characteristic equation

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0. \tag{25}$$

The roots k_1, k_2 and k_3 are given by

$$k_1 = \sqrt{\frac{1}{3}\,[2p\,\sin(q) + a_1]}, \tag{26}$$

$$k_2 = \sqrt{\frac{1}{3}\left[a_1 - p\left(\sqrt{3}\cos(q) + \sin(q)\right)\right]}, \tag{27}$$

$$k_3 = \sqrt{\frac{1}{3}\left[a_1 + p\left(\sqrt{3}\cos(q) - \sin(q)\right)\right]}, \tag{28}$$

where $p = \sqrt{a_1^2 - 3a_2}$, $q = \frac{\sin^{-1}(\gamma)}{3}$ and $\gamma = -\left(\frac{2a_1^3 - 9a_1 a_2 + 27a_3}{2p^3}\right)$.

The general solution of Eq. (24) can be written as the sum

$$\bar{\theta} = \sum_{i=1}^{3} \bar{\theta}_i,$$

where $\bar{\theta}_i$ is the solution of the partial differential equation

$$\left(\nabla^2 - k_i^2\right)\bar{\theta}_i = 0 \quad i = 1, 2, 3.$$

We consider, in general, the solution of equation of the form

$$\left(\nabla^2 - k^2\right) f(x, y, s) = 0. \tag{29}$$

We use the Fourier exponential transform defined by the relation [25]

$$f^*(q, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} f(x, y, s)\mathrm{d}x$$

with its corresponding inversion formula

$$f(x, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} f^*(q, y, s)\,\mathrm{d}q \quad \text{where } i = \sqrt{-1}.$$

We assume that all the relevant functions (such as temperature and stress) are sufficiently smooth on the real line such that the Fourier transforms of these functions exist.

Taking the Fourier transform of both sides of Eq. (29), we obtain

$$\left(D^2 - q^2 - k^2\right) f^*(q, y, s) = 0,$$

where $D = \partial/\partial y$. The solution of this equation bounded for $y > 0$ has the form

$$f^*(q, y, s) = A_1 \mathrm{e}^{-hy} + A_2 \mathrm{e}^{hy} \quad \text{with } h = \sqrt{q^2 + k^2}.$$

We thus obtain the Fourier transform of the bounded solution of Eq. (24) in the form

$$\bar{\theta}^* = \sum_{i=1}^{3} A_i e^{-h_i y}, \tag{30}$$

where $h_i = \sqrt{q^2 + k_i^2}$ and $A_i (i = 1, 2, 3)$ are parameters depending on s.

Similarly, the solutions of Eqs. (22) and (23) can be written as

$$\bar{e}^* = \sum_{i=1}^{3} A'_i \mathrm{e}^{-h_i y}, \tag{31}$$

$$\bar{C}^* = \sum_{i=1}^{3} A''_i \mathrm{e}^{-h_i y}, \tag{32}$$

where A'_i and $A"_i (i = 1, 2, 3)$ are parameters depending only on s.

Substituting from Eqs. (30)–(32) into Eqs. (17)–(19), we get

$$A'_i = \frac{k_i^2 \left[k_i^2 - (1 - \varepsilon\alpha_1)\left(s + \tau_0 s^2\right)\right]}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i,$$

$$A''_i = \frac{k_i^4 - k_i^2 \left[s^2 + (\varepsilon + 1)\left(s + \tau_0 s^2\right)\right] + s^3 (1 + \tau_0 s)}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i.$$

we thus have

$$\bar{e}^* = \sum_{i=1}^{3} \frac{k_i^2 \left[k_i^2 - (1 - \varepsilon\alpha_1)\left(s + \tau_0 s^2\right)\right]}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{33}$$

$$\bar{C}^* = \sum_{i=1}^{3} \frac{k_i^4 - k_i^2 \left[s^2 + (\varepsilon + 1)\left(s + \tau_0 s^2\right)\right] + s^3 (1 + \tau_0 s)}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}. \tag{34}$$

Taking the Fourier transform of both sides of Eqs. (15)–(20), we obtain the following set of equations

$$\left(D^2 - q^2 - \beta^2 s^2\right) \bar{u}^* = iq\left(\beta^2\left(\bar{\theta}^* + \bar{C}^*\right) - \left(\beta^2 - 1\right)\bar{e}^*\right), \tag{35}$$

$$\left(D^2 - q^2 - \beta^2 s^2\right) \bar{v}^* = D\left(\beta^2\left(\bar{\theta}^* + \bar{C}^*\right) - \left(\beta^2 - 1\right)\bar{e}^*\right), \tag{36}$$

$$\bar{\sigma}^*_{xx} = \beta^2\left(\bar{e}^* - \bar{\theta}^* - \bar{C}^*\right) - 2D\bar{v}^*, \tag{37a}$$

$$\bar{\sigma}^*_{yy} = \beta^2\left(\bar{e}^* - \bar{\theta}^* - \bar{C}^*\right) - 2iq\bar{u}^*, \tag{37b}$$

$$\bar{\sigma}^*_{zz} = \left(\beta^2 - 2\right)\bar{e}^* - \beta^2\left(\bar{\theta}^* + \bar{C}^*\right), \tag{37c}$$

$$\bar{\sigma}^*_{xy} = \left(D\bar{u}^* + iq\bar{v}^*\right), \tag{37d}$$

$$\bar{P}^* = \alpha_3\bar{C}^* - \bar{e}^* - \alpha_1\bar{\theta}^*, \tag{37e}$$

$$\bar{e}^* = iq\bar{u}^* + D\bar{v}^*. \tag{38}$$

Substituting from Eqs. (30), (33), and (34) into the right hand side of Eqs. (35) and (36), we get

$$\left(D^2 - q^2 - \beta^2 s^2\right) \bar{u}^* = iq \sum_{i=1}^{3} \frac{\left(k_i^2 - \beta^2 s^2\right)\left[k_i^2 - (1 - \varepsilon\alpha_1)\left(s + \tau_0 s^2\right)\right]}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{39}$$

$$\left(D^2 - q^2 - \beta^2 s^2\right) \bar{v}^* = D \sum_{i=1}^{3} \frac{\left(k_i^2 - \beta^2 s^2\right)\left[k_i^2 - (1 - \varepsilon\alpha_1)\left(s + \tau_0 s^2\right)\right]}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}. \tag{40}$$

The general solution of Eqs. (39) and (40)

$$\bar{u}^* = B\mathrm{e}^{-hy} + iq \sum_{i=1}^{3} \frac{\left[k_i^2 - (1 - \varepsilon\alpha_1)\left(s + \tau_0 s^2\right)\right]}{\varepsilon\left(s + \tau_0 s^2\right)\left[(1 + \alpha_1) k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{41}$$

$$\bar{v}^* = \frac{iq}{h} B\mathrm{e}^{-hy} - \sum_{i=1}^{3} h_i \frac{\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{42}$$

where $h = \sqrt{q^2 + \beta^2 s^2}$ and B is a parameter depending on s and q.

Substituting from Eqs. (30), (33), (34), (41) and (42) into (37), we get

$$\bar{\sigma}_{xx}^* = 2iqB\mathrm{e}^{-hy} + \sum_{i=1}^{3} \frac{\left(\beta^2 s^2 - 2h_i^2\right)\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{43a}$$

$$\bar{\sigma}_{yy}^* = -2iqB\mathrm{e}^{-hy} + \sum_{i=1}^{3} \frac{\left(\beta^2 s^2 + 2q^2\right)\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{43b}$$

$$\bar{\sigma}_{zz}^* = \sum_{i=1}^{3} \frac{\left(\beta^2 s^2 - 2k_i^2\right)\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{43c}$$

$$\bar{\sigma}_{xy}^* = -\left(\frac{h^2+q^2}{h}\right) B\mathrm{e}^{-hy} - 2iq\sum_{i=1}^{3} \frac{h_i\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i \mathrm{e}^{-h_i y}, \tag{43d}$$

$$\lfloor\bar{P}^* = \frac{\alpha_2(1+\tau s)}{\varepsilon(1+\tau_0 s)} \sum_{i=1}^{3} \frac{k_i^4 - k_i^2\lfloor s^2 + (\varepsilon+1)\left(s+\tau_0 s^2\right)\rfloor + s^3(1+\tau_0 s)}{k_i^2\lfloor(1+\alpha_1)k_i^2 - \alpha_1 s^2\rfloor} A_i \mathrm{e}^{-h_i y}. \tag{43e}$$

The boundary conditions (14) in the Laplace and Fourier transforms take the form

$$\text{(i)}\quad \bar{\sigma}_{yy}^*(q,0,s) = \bar{\sigma}_{xy}^*(q,0,s) = 0, \tag{44a}$$

$$\text{(ii)}\quad \bar{\theta}^*(q,0,s) = \sqrt{\frac{2}{\pi}}\theta_0\left(\frac{s}{s^2+\omega^2}\right)\frac{\sin(dq)}{q}\left(1 - i\pi q\delta(q)\right), \tag{44b}$$

$$\text{(iii)}\quad \bar{P}^*(q,0,s) = \frac{P_0}{s}\delta(q). \tag{44c}$$

Equation (44) immediately give the following system of four linear equations in the unknown parameters A_1, A_2, A_3 and B:

$$-2iqB + \sum_{i=1}^{3} \frac{\left(\beta^2 s^2 + 2q^2\right)\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i = 0, \tag{45a}$$

$$\left(\frac{h^2+q^2}{h}\right) B + 2iq\sum_{i=1}^{3} \frac{h_i\left[k_i^2 - (1-\varepsilon\alpha_1)\left(s+\tau_0 s^2\right)\right]}{\varepsilon\left(s+\tau_0 s^2\right)\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i = 0, \tag{45b}$$

$$A_1 + A_2 + A_2 = \sqrt{\frac{2}{\pi}}\theta_0 \frac{s}{s^2+\omega^2}\frac{\sin(dq)}{q}\left(1 - i\pi q\delta(q)\right), \tag{45c}$$

$$\sum_{i=1}^{3} \frac{k_i^4 - k_i^2\lfloor s^2 + (\varepsilon+1)\left(s+\tau_0 s^2\right)\rfloor + s^3(1+\tau_0 s)}{k_i^2\left[(1+\alpha_1)k_i^2 - \alpha_1 s^2\right]} A_i = \frac{\delta(q)P_0}{s}\frac{\varepsilon(1+\tau_0 s)}{\alpha_2(1+\tau s)}. \tag{45d}$$

Solving the linear system of equations (45), we can obtain the parameters $A_i (i = 1, 2, 3)$ and B. This completes the solution of the problem in the Laplace transform domain.

4 Inversion of the double transforms

We shall now outline the numerical inversion method used to fine the solution in the physical domain. Let $\bar{f}^*(x, q, s)$ be the double Fourier-Laplace transform of a function $f(x, y, t)$. First, we use the inversion formula of the Fourier transform mentioned earlier to obtain a Laplace transform expression $\bar{f}(x, y, s)$ of the form

$$\bar{f}(x, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} \bar{f}^{*}(x, q, s)\, dq,$$
$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left(\cos(qy)\, \bar{f}_e^{*}(x, q, s) + \sin(qy)\, \bar{f}_o^{*}(x, q, s)\right) dq,$$

where $\bar{f}_e^{*}$ and $\bar{f}_o^{*}$ denote to the even and odd parts of $\bar{f}^{*}(x, q, s)$, respectively.

The inversion formula for Laplace transforms can be written as

$$f(x, y, t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{st} \bar{f}(x, y, s)\, ds,$$

where d is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(x, y, s)$. Taking $s = d + iy$, the preceding integral takes the form

$$f(x, y, t) = \frac{e^{dt}}{2\pi i} \int_{-\infty}^{\infty} e^{iyt} \bar{f}(x, y, d + iy)\, dy.$$

Expanding the function $g(x, y, t) = \exp(-dt) f(x, y, t)$ into a Fourier series in the interval $[0, 2T]$, we obtain the approximate formula [26]

$$f(x, y, t) = f_{\infty}(x, y, t) + E_D,$$

where

$$f_{\infty}(x, y, t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \quad \text{for} \quad 0 \le t \le 2T \tag{46}$$

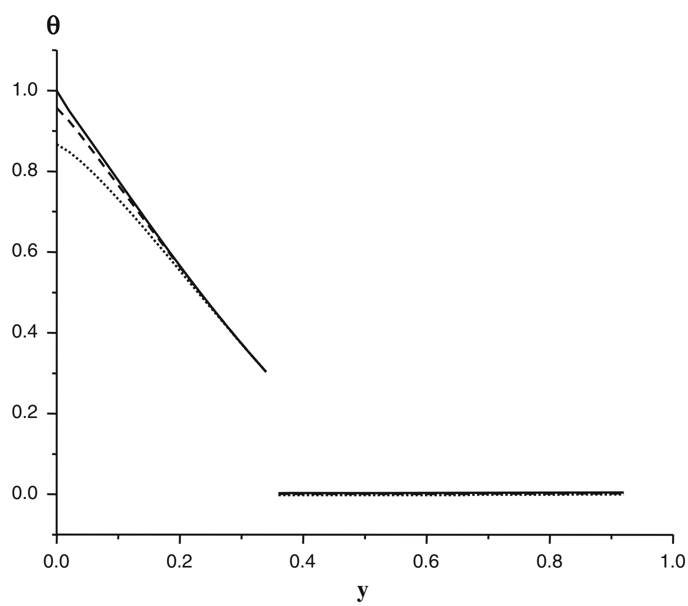

Fig. 1 Temperature distribution

and

$$c_k = \frac{e^{\mathrm{d}t}}{T} Re\left[\mathrm{e}^{ik\pi t/T}\bar{f}\left(x, y, d + ik\pi/T\right)\right]. \tag{47}$$

The discretization error, E_D, can be made arbitrarily small by choosing d large enough [26]. Since the infinity series in Eq. (46) only be summed up to a finite number N of terms, the approximate value of $f(x, y, t)$ becomes

$$f_N\left(x, y, t\right) = \frac{c_0}{2} + \sum_{k=1}^{N} c_k \quad \text{for} \quad 0 \le t \le 2T. \tag{48}$$

Using this formula to evaluate $f(x, y, t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total error.

Tow methods are used to reduce the total error. First, the "Korrektur" method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence:

The Korrektur method uses the following formula to evaluate the function $f(x, y, t)$:

$$f\left(x, y, t\right) = f_\infty\left(x, y, t\right) - \mathrm{e}^{-2\mathrm{d}T} f_\infty\left(x, y, 2T + t\right) + E_D^{'},$$

where the discretization errors $\left|E_D^{'}\right| << |E_D|$. Thus, the approximate value of $f(x, y, t)$ becomes

$$f_{Nk}\left(x, y, t\right) = f_N\left(x, y, t\right) - \mathrm{e}^{-2\mathrm{d}T} f_{N'}\left(x, y, 2T + t\right), \tag{49}$$

where N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in (48). Let N be an odd natural number and let

$$s_n = \sum_{k=1}^{n} c_k$$

be the sequence of partial sums of (48). We define the ε-sequence by

$$\varepsilon_{0,n} = 0, \varepsilon_{1,n} = s_n \qquad n = 1, 2, 3, \ldots\ldots.$$

and

$$\varepsilon_{m+1,n} = \varepsilon_{m-1,n+1} + \frac{1}{\varepsilon_{m,n+1} - \varepsilon_{m,n}} \qquad m, n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots.$$

It can be shown [26] that the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots\ldots\ldots, \varepsilon_{N,1}$$

converges to $f(x, y, t) + E_D - c_0/2$ faster than the sequence of partial sums

$$s_n \quad n = 1, 2, 3, \ldots\ldots\ldots$$

The actual procedure used to invert the Laplace transforms consists of using Eq. (49) together with the ε-algorithm. The values of d and T are chosen according to the criteria outline in [26].

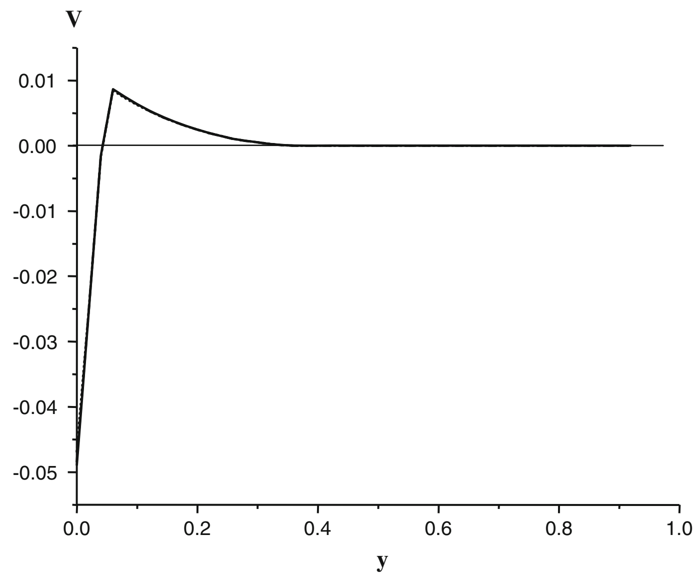

Fig. 2 Displacement distribution

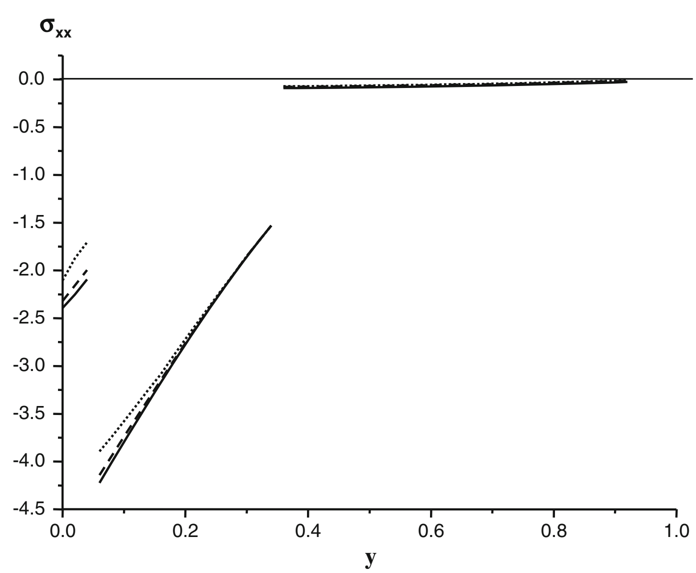

Fig. 3 Stress component σ_{xx} distribution

5 Numerical results

The copper material was chosen for purposes of numerical evaluations. The parameters of the problem are thus given in SI units by [27]

$$T_0 = 293K,\ \rho = 8954\,\text{kg/m}^3,\ \tau_0 = 0.02\,\text{s},\ \tau = 0.2\,\text{s},$$
$$c_E = 383.1\,\text{J/(kg K)},\ \alpha_t = 1.78(10)^{-5}\,\text{K}^{-1},\ \alpha_c = 1.98(10)^{-4}\,\text{m}^3/\text{kg},$$
$$\mu = 3.86(10)^{10}\,\text{kg/(m s}^2),\ \lambda = 7.76(10)^{10}\,\text{kg/(m s}^2),\ k = 386\,\text{W/(m K)},$$
$$D = 0.85(10)^{-8}\,\text{kg s/m}^3,\ a = 1.2(10)^4\,\text{m}^2/(\text{s}^2\text{K}),\ b = 0.9(10)^6\,\text{m}^5/(\text{kg s}^2).$$

From the above values, it was found that

$$\eta = 8886.73,\ \varepsilon = 0.0168,\ \beta^2 = 4,\ \alpha_1 = 5.43,\ \alpha_2 = 0.533 \quad \text{and} \quad \alpha_3 = 36.24.$$

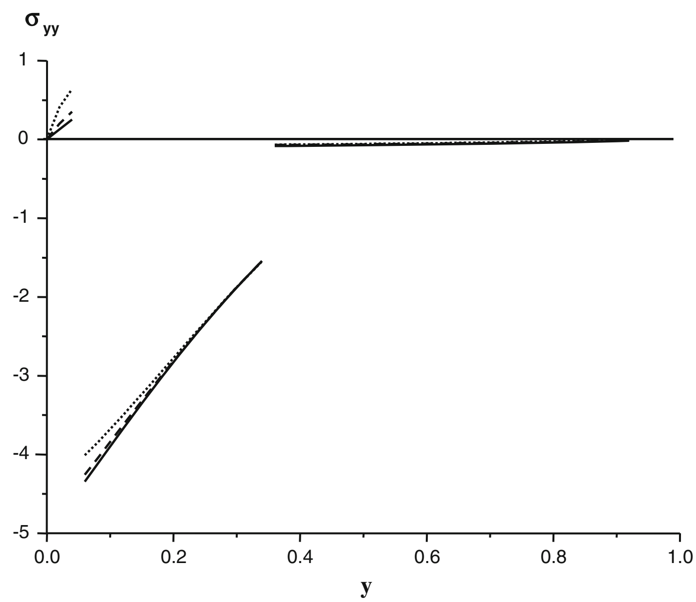

Fig. 4 Stress component σ_{yy} distribution

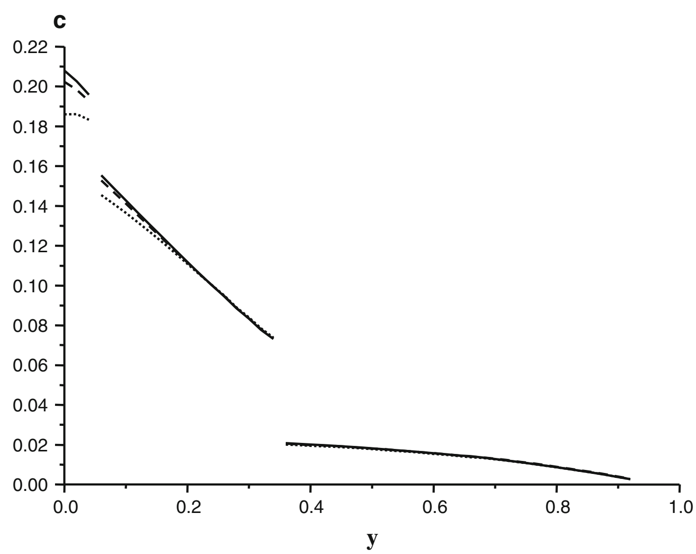

Fig. 5 Concentration distribution

The computations were carried out for one value of time, namely $t = 0.05$ with three different values of the angular frequency, namely for $\omega = 0$, 5 and 10. The results are illustrated graphically in Figs. 1, 2, 3, 4, 5, and 6 for the temperature increment θ, the displacement component v, the stress component σ_{xx}, the stress component σ_{yy}, concentration C, and chemical potential P distributions, respectively. All the functions were evaluated inside the medium on the y-axis ($x = 0$) as functions of y. In all figures, the solid lines represent the case when $\omega = 0$ (the case of thermal shock), the dashed lines represent the case when $\omega = 5$, while the dotted lines represent the case when $\omega = 10$. Due to the symmetry, the displacement component u is identically zero on the y-axis.

All the figures show that the heat, elastic, and diffusion waves propagate with finite speeds. We can see that all the functions considered vanish identically for $y > 0.92$. The fronts of these waves are depicted in the figures as discontinuities in the functions in Figs. 1 and 3, 4, 5, and 6 or in the first derivative in Fig. 2 because the displacement is a continuous function. Of course, some of these discontinuities are very small to

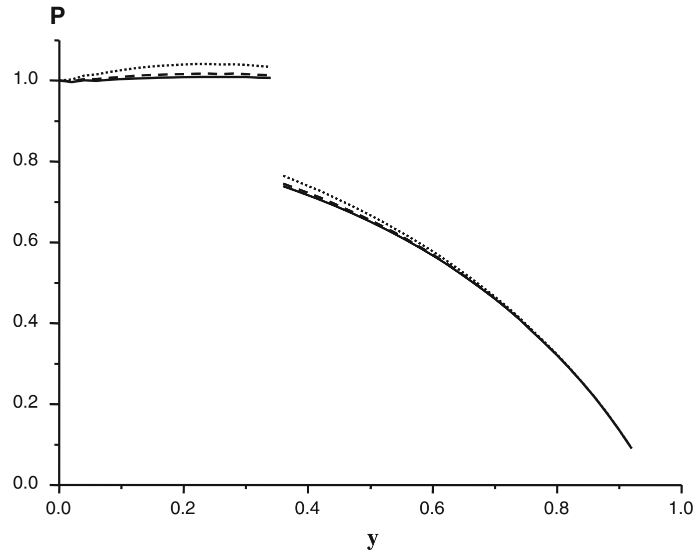

Fig. 6 Chemical potential distribution

show in the figures. It was found that the three wave fronts are located at the positions $y = 0.04$, $y = 0.34$, and $y = 0.92$.

From the graphs, we can see that the effect of diffusion on the temperature and displacement is very weak but has a noticeable effect on the stress. Also, these graphs show that the change of the angular frequency of thermal vibration ω has a significant effect on all the studied fields.

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