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The J-integral for single-lap joint using the stress field from the mixed variational principle

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Abstract The closed-form solution for the J-integral of a single-lap joint is presented based on the stress field derived from Reissner's mixed variational principle. In an adhesive-bonded joint, loads are carried by the surface of the adherends in shear through an adhesive layer, and thus, the shear effect is important. To improve the accuracy of shear response in fracture analysis, all transverse effects of the shear and peel stresses are considered, and then the constitutive equations and the equilibrium equations are derived from the variational principle. The obtained J-integral gives additional terms on the transverse shear part of the total integral compared with the results from previous conventional analysis, and illustrative examples are provided to show the effects of the current approach. Also, the formulation proposed in this paper can deal with non-identical adherends and laminates easily.

1 Introduction

Adhesively bonded joints are widely used because of high strength/weight ratio, no strength degradation by cutouts, and less corrosion problems associated with mechanical fasteners. However, the geometric discontinuity at the ends of a bonded joint causes peak shear and peel stresses in the adhesive layer, which can result in local failure and crack initiation. The adhesive bonding can transfer the load smoothly from one adherend to the other, and also the peak shear and peel stresses in the adhesive layer are to be minimized as much as possible. Thus, extensive research efforts have been devoted to the analysis and design of adhesively bonded joints.

In their pioneering work, Goland and Reissner [1] presented a closed-form solution of stress distribution in lap joints using a 2-D elasticity theory. Ojalvo and Eidinoff [2] extended the results of Goland and Reissner by introducing a more complete relation between shear strain and displacement corresponding to linearly varying displacements through the adhesive thickness. Roberts [3] proposed an analytical procedure based on beam theory and the assumption of relatively flexible adhesive layers. To handle the complexity in geometry, boundary conditions, and material properties, finite element methods have been used for static, and fatigue loading analysis of adhesively bonded joints and the extensive research results are reviewed in [4].

Also, the strength of bonded joints has been researched by many authors. In early researches, the linear elastic fracture mechanics (LEFM) approaches have been used to investigate the fracture of bonded laminates [5,6]. Adams [7] used finite element method to consider the nonlinear mechanics and material behavior for a Volkerson type joint. Hamoush and Ahmad [8] used energy release rate as a criterion to predict the Mode I and Mode II failure loads of adhesive joints. An engineering approach to fracture load predictions for an adhesive

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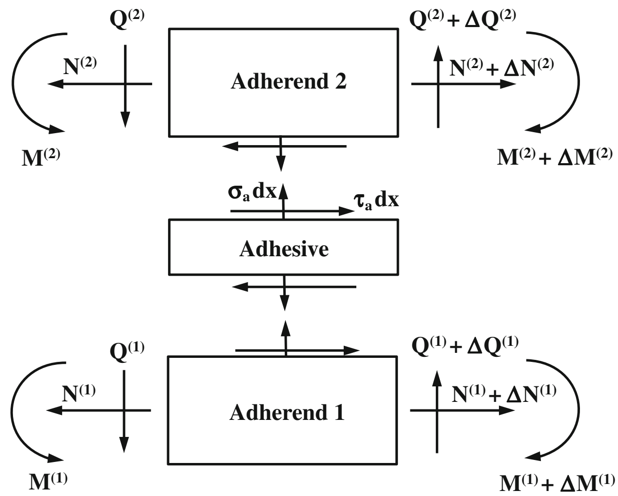

Fig. 1 Equilibrium of adherends and adhesive layer

joint was proposed by Fernlund et al. [9–11], in which the premise that in-situ strength of the bondline can be characterized by the fracture envelope for a specific adhesive system was used. Tong [12] used the arbitrary nonlinear stress-strain curves in both shear and peel for the adhesive and formulated two coupled governing equations for single-lap joints.

Some papers dealt with the J-integral as a failure criterion for specific geometries, for example, the double cantilever beam by Yamada [13,14] and for generic adhesive sandwiches by Fernlund and Spelt [15]. Fraisse and Schmit [16] calculated the closed-form of the J-integral for single-lap joints based on Goland and Reissner's stress analysis. The J-integral approach has several advantages [16]; thus, it can be an efficient tool for fracture analysis in bonded joints. It can be integrated on a path located far from the crack front, and therefore, the singular stress zone can be avoided in the calculation. Also, it is not restricted to the case of linear elastic materials.

Most papers in bonded joints fracture analysis have used a beam theory or classical Goland and Reissner's theory. In adhesive bonding, loads are carried by the surface of the adherends in shear through an adhesive layer. Thus, the shear effect is important in the adhesive-bonded joints, and the results from the above-mentioned references can have a certain amount of error in some cases. In this paper, the stress fields of a single-lap joint is derived by considering all transverse effects of the shear and peel stresses in the adherend. For this, Reissner's mixed variational principle [17] is applied to the single-lap joint, and the constitutive equations and the equilibrium equations are derived from the functional. Then, the closed-from solution of the J-integral is calculated for illustrative problems, and the results are compared with previous ones from conventional stress analysis. The formulation in this study can deal with non-identical adherends or laminates easily; for the sake of simplicity and comparison, the adherends are assumed to be identical and isotropic.

2 Mixed formulation for single-lap joint

A single-lap adhesive joint as in Fig. 1, which shows a typical infinitesimal element, is considered. To determine the equilibrium equations and constitutive equations using Reissner's mixed variational principle [17], the following basic assumptions are applied:

(A) Plane stress in each adherend
(B) Longitudinal stress in the adherend is neglected
(C) Longitudinal and transverse deflection in the adhesive vary linearly through the thickness
(D) Shear stress in the adhesive is constant through the thickness

The constitutive equations for the adherend and adhesive are obtained by the variational formulation. First, the functional for the adhrends is

$$\pi_R = \int_V \left[\tau_{ij}\varepsilon_{ij} - W^*(\tau_{ij})\right]\mathrm{d}V = \int_V \left[\sigma_{xx}\varepsilon_{xx} + 2\tau_{xz}\varepsilon_{xz} + \sigma_{zz}\varepsilon_{zz} - \frac{1}{2}\tau_{ij}\varepsilon_{ij}\right]\mathrm{d}V. \tag{1}$$

In Eq. (1), $W^*(\tau_{ij})$ is a complimentary energy and the stress-strain relation used for that term is

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu\sigma_{zz}}{E}, \quad \varepsilon_{xz} = \frac{\tau_{xz}}{2G}, \quad \varepsilon_{zz} = \frac{\sigma_{zz} - \nu\sigma_{xx}}{E}. \tag{2}$$

Thus, Eq. (1) can be written as

$$\pi_R = \int_0^L \left\{ \sum_{k=1}^{2} \int_{-h_k/2}^{h_k/2} \left[\sigma_{xx}^{(k)}\varepsilon_{xx}^{(k)} + 2\tau_{xz}^{(k)}\varepsilon_{xz}^{(k)} - \frac{1}{2}\sigma_{xx}^{(k)}\left(\frac{\sigma_{xx} - \nu\sigma_{zz}}{E}\right)^{(k)} - \tau_{xz}^{(k)}\left(\frac{\tau_{xz}}{2G}\right)^{(k)} - \frac{1}{2}\sigma_{zz}^{(k)}\left(\frac{\sigma_{zz} - \nu\sigma_{xx}}{E}\right)^{(k)} \right] \mathrm{d}z^{(k)} \right\} \mathrm{d}x. \tag{3}$$

Then, the variational form of Eq. (3) is

$$\delta\pi_R = \int_0^L \left\{ \sum_{k=1}^{2} \int_{-h_k/2}^{h_k/2} \left[\delta\varepsilon_{xx}^{(k)}\sigma_{xx}^{(k)} + 2\delta\varepsilon_{xz}^{(k)}\tau_{xz}^{(k)} + \delta\tau_{xz}^{(k)}\left(2\varepsilon_{xz}^{(k)} - \frac{\tau_{xz}^{(k)}}{G^{(k)}}\right) + \delta\sigma_{zz}^{(k)}\left(\nu^{(k)}\varepsilon_{xx}^{(k)} - \left(\frac{1-\nu^2}{E}\right)^{(k)}\sigma_{zz}^{(k)}\right) \right] \mathrm{d}z^{(k)} \right\} \mathrm{d}x. \tag{4}$$

The functional for the adhesive can be defined in a similar way:

$$\pi'_R = \int_V \left[2\tau_{xz}\varepsilon_{xz} + \sigma_{zz}\varepsilon_{zz} - \frac{1}{2}\left(\sigma_{zz}\varepsilon_{zz} + 2\tau_{xz}\varepsilon_{xz}\right)\right]\mathrm{d}V = \int_V \left[\tau_a\gamma_a + \sigma_a\varepsilon_a - \frac{1}{2}\frac{\sigma_a^2}{E_a} - \frac{1}{2}\frac{\tau_a^2}{G_a}\right]\mathrm{d}V. \tag{5}$$

Also, the variational form is

$$\delta\pi'_R = \int_V \left[\delta\gamma_a\tau_a + \delta\varepsilon_a\sigma_a + \delta\tau_a\left(\gamma_a - \frac{\tau_a}{G_a}\right) + \delta\sigma_a\left(\varepsilon_a - \frac{\sigma_a}{E_a}\right)\right]\mathrm{d}V. \tag{6}$$

Hence, Reissner's mixed variational principle applied to the single-lap joint becomes

$$\int_0^L \left\{ \sum_{k=1}^{2} \int_{-h_k/2}^{h_k/2} \left[\delta\varepsilon_{xx}^{(k)}\sigma_{xx}^{(k)} + 2\delta\varepsilon_{xz}^{(k)}\tau_{xz}^{(k)} + \delta\tau_{xz}^{(k)}\left(\frac{\partial u^{(k)}}{\partial z} + \frac{\partial w^{(k)}}{\partial x} - \frac{\tau_{xz}^{(k)}}{G^{(k)}}\right) \right.\right.$$
$$\left. + \delta\sigma_{zz}^{(k)}\left(\nu^{(k)}\frac{\partial u^{(k)}}{\partial x} - \left(\frac{1-\nu^2}{E}\right)^{(k)}\sigma_{zz}^{(k)}\right)\right] \mathrm{d}z^{(k)}$$
$$\left. + \int_{-h_a/2}^{h_a/2} \left[\delta\gamma_a\tau_a + \delta\varepsilon_a\sigma_a + \delta\tau_a\left(\gamma_a - \frac{\tau_a}{G_a}\right) + \delta\sigma_a\left(\varepsilon_a - \frac{\sigma_a}{E_a}\right)\right]\mathrm{d}z \right\} \mathrm{d}x = 0, \tag{7}$$

where the strain-displacement relations for small displacement were used.

For further derivation, the displacement field of the adherends and the strain-displacement relations in the adhesive can be defined as follows:

$$\begin{aligned} u^{(k)}(x,z) &= U^{(k)}(x) + z^{(k)}\theta^{(k)}(x), \\ w^{(k)}(x,z) &= W^{(k)}(x), \end{aligned} \tag{8}$$

where $U^{(k)}(x), \theta^{(k)}(x), W^{(k)}(x)$ are mid-plane displacements in the kth adherend. The strain-displacement relations in the adhesive are

$$\begin{aligned} \gamma_a &= \left[u^{(1)}\left(x, -\frac{h_1}{2}\right) - u^{(2)}\left(x, \frac{h_2}{2}\right)\right] \Big/ h_a, \\ \varepsilon_a &= \left[w^{(1)}\left(x, -\frac{h_1}{2}\right) - w^{(2)}\left(x, \frac{h_2}{2}\right)\right] \Big/ h_a. \end{aligned} \tag{9}$$

The τ_{xz} and σ_{xz} distribution through the thickness can be obtained by integrating the equilibrium equations, and the results are

$$\begin{aligned} \tau_{xz}^{(1)} &= \frac{3Q^{(1)}}{2h_1}\left[1 - 4\left(\frac{z^{(1)}}{h_1}\right)^2\right] + \tau_a\left[3\left(\frac{z^{(1)}}{h_1}\right)^2 - \frac{z^{(1)}}{h_1} - \frac{1}{4}\right], \\ \sigma_{zz}^{(1)} &= \frac{\sigma_a}{2}\left[1 - 3\left(\frac{z^{(1)}}{h_1}\right) + 4\left(\frac{z^{(1)}}{h_1}\right)^3\right], \end{aligned} \tag{10}$$

$$\begin{aligned} \tau_{xz}^{(2)} &= \frac{3Q^{(2)}}{2h_2}\left[1 - 4\left(\frac{z^{(2)}}{h_2}\right)^2\right] + \tau_a\left[3\left(\frac{z^{(2)}}{h_2}\right)^2 + \frac{z^{(2)}}{h_2} - \frac{1}{4}\right], \\ \sigma_{zz}^{(2)} &= \frac{\sigma_a}{2}\left[1 + 3\left(\frac{z^{(2)}}{h_2}\right) - 4\left(\frac{z^{(2)}}{h_2}\right)^3\right], \end{aligned} \tag{11}$$

where

$$Q^{(k)} = \int_{-h_k/2}^{h_k/2} \tau_{xz}^{(k)} \mathrm{d}z^{(k)}. \tag{12}$$

Substituting Eqs. (8), (9), (10), and (11) into Eq. (7) and then integrating each term gives the equilibrium equation and the constitutive equations including the shear correction factor:

Equilibrium equations

$$\frac{\mathrm{d}N^{(1)}}{\mathrm{d}x} - \tau_a = 0, \quad \frac{\mathrm{d}N^{(2)}}{\mathrm{d}x} + \tau_a = 0, \tag{13}$$

$$\frac{\mathrm{d}M^{(1)}}{\mathrm{d}x} - Q^{(1)} + \frac{h_1}{2}\tau_a = 0, \quad \frac{\mathrm{d}M^{(2)}}{\mathrm{d}x} - Q^{(2)} + \frac{h_2}{2}\tau_a = 0, \tag{14}$$

$$\frac{\mathrm{d}Q^{(1)}}{\mathrm{d}x} - \sigma_a = 0, \quad \frac{\mathrm{d}Q^{(2)}}{\mathrm{d}x} + \sigma_a = 0, \tag{15}$$

where the resultants are defined as

$$N^{(k)} = \int_{-h_k/2}^{h_k/2} \sigma_{xx}^{(k)} \mathrm{d}z^{(k)}, \quad M^{(k)} = \int_{-h_k/2}^{h_k/2} \sigma_{xx}^{(k)} z^{(k)} \mathrm{d}z^{(k)}. \tag{16}$$

Constitutive equations

$$\theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x} - \frac{6Q^{(1)}}{5G^{(1)}h_1} + \frac{\tau_a}{10G^{(1)}} = 0, \tag{17}$$

$$\theta^{(2)} + \frac{\mathrm{d}W^{(2)}}{\mathrm{d}x} - \frac{6Q^{(2)}}{5G^{(2)}h_2} + \frac{\tau_a}{10G^{(2)}} = 0, \tag{18}$$

$$\frac{Q^{(1)}}{10G^{(1)}} + \frac{Q^{(1)}}{10G^{(1)}} - \frac{2}{15}\left(\frac{h_1}{G^{(1)}} + \frac{h_1}{G^{(1)}}\right)\tau_a + U^{(1)} - \frac{h_1}{2}\theta^{(1)} - U^{(2)} - \frac{h_2}{2}\theta^{(2)} - \frac{h_a}{G_a}\tau_a = 0, \tag{19}$$

$$\sigma_a = \frac{E_a}{h_a}\left(W^{(1)} - W^{(2)}\right). \tag{20}$$

The explicit expressions for $Q^{(1)}$, $Q^{(2)}$, and τ_a can be obtained by manipulating Eqs. (17)–(19) and written as

$$\begin{aligned} Q^{(1)} &= \left(\frac{5G^{(1)}h_1}{6} + \frac{h_1^2 R}{144}\right)\left(\theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x}\right) + \frac{h_1 R}{12}\left(U^{(1)} - \frac{h_1}{2}\theta^{(1)} - U^{(2)} - \frac{h_2}{2}\theta^{(2)}\right) \\ &\quad + \frac{h_1 h_2 R}{144}\left(\theta^{(2)} + \frac{\mathrm{d}W^{(2)}}{\mathrm{d}x}\right), \end{aligned} \tag{21}$$

$$\begin{aligned} Q^{(2)} &= \frac{h_1 h_2 R}{144}\left(\theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x}\right) + \frac{h_2 R}{12}\left(U^{(1)} - \frac{h_1}{2}\theta^{(1)} - U^{(2)} - \frac{h_2}{2}\theta^{(2)}\right) \\ &\quad + \left(\frac{5G^{(2)}h_2}{6} + \frac{h_2^2 R}{144}\right)\left(\theta^{(2)} + \frac{\mathrm{d}W^{(2)}}{\mathrm{d}x}\right), \end{aligned} \tag{22}$$

$$\tau_a = \frac{h_1 R}{12}\left(\theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x}\right) + R\left(U^{(1)} - \frac{h_1}{2}\theta^{(1)} - U^{(2)} - \frac{h_2}{2}\theta^{(2)}\right) + \frac{h_2 R}{12}\left(\theta^{(2)} + \frac{\mathrm{d}W^{(2)}}{\mathrm{d}x}\right), \tag{23}$$

where

$$R \equiv 1 \Bigg/ \left(\frac{h_1}{8G^{(1)}} + \frac{h_2}{8G^{(2)}} + \frac{h_a}{G_a} \right).$$

The remaining constitutive equations for $N^{(k)}$, $M^{(k)}$ are determined as

$$N^{(1)} = E^{(1)} h_1 \frac{\mathrm{d}U^{(1)}}{\mathrm{d}x}, \tag{24}$$

$$M^{(1)} = \frac{E^{(1)} h_1^3}{12} \frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x}, \tag{25}$$

$$N^{(2)} = E^{(2)} h_2 \frac{\mathrm{d}U^{(2)}}{\mathrm{d}x}, \tag{26}$$

$$M^{(2)} = \frac{E^{(2)} h_2^3}{12} \frac{\mathrm{d}\theta^{(2)}}{\mathrm{d}x}. \tag{27}$$

If we define the generalized stress and strain vector as follows:

$$\{\sigma\}^T = \left\{ N^{(1)} \quad M^{(1)} \quad Q^{(1)} \quad \tau_a \quad \sigma_a \quad N^{(2)} \quad M^{(2)} \quad Q^{(2)} \right\}, \tag{28}$$

$$\{\varepsilon\}^T = \left\{ \frac{\mathrm{d}U^{(1)}}{\mathrm{d}x} \quad \frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x} \quad \theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x} \quad U^{(1)} - \frac{h_1}{2}\theta^{(1)} - U^{(2)} - \frac{h_2}{2}\theta^{(2)} \quad W^{(1)} - W^{(2)} \right.$$
$$\left. \frac{\mathrm{d}U^{(1)}}{\mathrm{d}x} \quad \frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x} \quad \theta^{(1)} + \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x} \right\}, \tag{29}$$

then the constitutive equation can be expressed in matrix form as in Eq. (30):

$$\{\sigma\} = [D]\{\varepsilon\}. \tag{30}$$

$[D]$ is a symmetric matrix and its nonzero components are

$$\begin{aligned}
&D_{11} = E^{(1)} h_1; \quad D_{22} = \frac{E^{(1)} h_1^3}{12}, \\
&D_{33} = \frac{5G^{(1)} h_1}{6} + \frac{h_1^2 R}{144}; \quad D_{34} = \frac{h_1 R}{12}; \quad D_{38} = \frac{h_1 h_2 R}{144}, \\
&D_{44} = R; \quad D_{48} = \frac{h_2 R}{12}, \\
&D_{55} = \frac{E_a}{h_a}; \quad D_{66} = E^{(2)} h_2; \quad D_{77} = \frac{E^{(2)} h_2^3}{12}; \quad D_{88} = \frac{5G^{(2)} h_2}{6} + \frac{h_2^2 R}{144}.
\end{aligned} \tag{31}$$

For calculating the J-integral in a later section, the inverse of $[D]$ has to be calculated. It can be obtained by symbolic linear algebra or simple numerical calculation. Thus, it is defined as

$$\{\varepsilon\} = [C]\{\sigma\}. \tag{32}$$

3 J-integral for a cracked joint

For the J-integral with the stress field derived in the previous section, we consider a bonded structure with a debond or a crack in the adhesive layer, which was also studied by Fraisse and Schmit [16]. The schematic and integral direction is depicted in Fig. 2.

The integral path is

$$\begin{aligned}
J &= J_{O'A} + J_{AB} + J_{BC} + J_{CD} + J_{DE} + J_{EF} + J_{FG} + J_{GH} + J_{HO} \\
&= J_{AB} + J_{CD} + J_{DE} + J_{EF} + J_{GH},
\end{aligned} \tag{33}$$

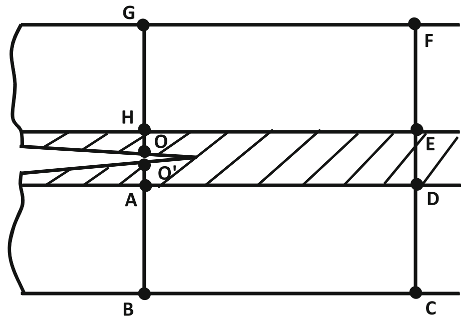

Fig. 2 Interface crack in a single-lap joint

where $J_{O'A} = J_{HO} = J_{BC} = J_{FG} = 0$ because those sections are unloaded.

First, J_{GH} is

$$J_{GH} = -\int_{-h_1/2}^{h_1/2} \left[w - \sigma_{xx}\frac{\partial u}{\partial x} - \tau\frac{\mathrm{d}W}{\mathrm{d}x} \right] \mathrm{d}z. \tag{34}$$

The first term is from the strain energy density and its result is

$$\begin{aligned}(J_{GH})_1 &= \int_{-h_1/2}^{h_1/2} w\mathrm{d}z = \int_{-h_1/2}^{h_1/2} \int_0^{\varepsilon} \left[\sigma_x \mathrm{d}\varepsilon_x + \tau\mathrm{d}\gamma_{xy}\right]\mathrm{d}z \\ &= \int_{-h_1/2}^{h_1/2} \left[h_1 C_{11} \int_0^{\sigma} \sigma_x \mathrm{d}\sigma_x + \frac{h_1}{D_{33}} \int_0^{\tau} \tau\mathrm{d}\tau \right] \mathrm{d}z \\ &= \int_{-h_1/2}^{h_1/2} h_1 C_{11} \int \left(\frac{N^{(1)}}{h_1} + z\frac{C_{22}M^{(1)}}{h_1 C_{11}} \right) \left(\frac{1}{h_1}\mathrm{d}N^{(1)} + z\frac{C_{22}}{h_1 C_{11}}\mathrm{d}M^{(1)} \right) \mathrm{d}z \\ &\quad + \int_{-h_1/2}^{h_1/2} \frac{h_1}{D_{33}} \int \frac{Q^{(1)}}{h_1} \left(\frac{\mathrm{d}Q^{(1)}}{h_1} \right) \mathrm{d}z. \end{aligned} \tag{35}$$

This yields

$$(J_{GH})_1 = \frac{1}{2}\left[C_{11}\left(N_0^I\right)^2 + C_{22}\left(M_0^I\right)^2 + \frac{\left(Q_0^I\right)^2}{D_{33}} \right], \tag{36}$$

where the subscript "0" means the value at the crack tip ($x = 0$) and the superscript "I" is for the adherend 1.

The second term in Eq. (34) can be integrated in a similar way:

$$\begin{aligned}(J_{GH})_2 &= \int_{-h_1/2}^{h_1/2} \sigma_x \frac{\partial}{\partial x}\left[U^{(1)} + z\theta^{(1)}\right]\mathrm{d}z \\ &= \int_{-h_1/2}^{h_1/2} \left(\frac{N^{(1)}}{h_1} + z\frac{C_{22}M^{(1)}}{h_1 C_{11}} \right) \left(C_{11}N^{(1)} + zC_{22}M^{(1)} \right) \mathrm{d}z \\ &= C_{11}\left(N_0^I\right)^2 + C_{22}\left(M_0^I\right)^2. \end{aligned} \tag{37}$$

The third integrand in Eq. (34), which is the additional term obtained from the mixed formulation, is

$$(J_{GH})_3 = \int_{-h_1/2}^{h_1/2} \frac{Q^{(1)}}{h_1} \left(C_{33} Q^{(1)} + C_{34} \tau_a - \theta^{(1)} \right) \mathrm{d}z$$
$$= C_{33} \left(Q_0^I \right)^2 - Q_0^I \theta_0^I + C_{34} Q_0^I \tau_{a0}. \quad (38)$$

Thus,

$$J_{GH} = \frac{1}{2} \left[C_{11} \left(N_0^I \right)^2 + C_{22} \left(M_0^I \right)^2 \right] + \left(C_{33} - \frac{1}{2} \frac{1}{D_{33}} \right) \left(Q_0^I \right)^2 - Q_0^I \theta_0^I + C_{34} Q_0^I \tau_{a0}, \quad (39)$$

where τ_{a0} is defined from the constitutive equations.

J_{EF} can be calculated in the same way as J_{GH}:

$$J_{EF} = \int_{-h_1/2}^{h_1/2} \left[w - \sigma_x \frac{\partial}{\partial x} \left(U^{(1)} + z\theta^{(1)} \right) - \tau \frac{\mathrm{d}W^{(1)}}{\mathrm{d}x} \right] \mathrm{d}z$$
$$= -\frac{1}{2} \left[C_{11} \left(N_l^I \right)^2 + C_{22} \left(M_l^I \right)^2 \right] + \left(\frac{1}{2} \frac{1}{D_{33}} - C_{33} \right) \left(Q_l^I \right)^2 + Q_l^I \theta_l^I - C_{34} Q_l^I \tau_{al}, \quad (40)$$

where the subscript "l" means the value at $x = l$. And the integration for the lower adherend can be determined in a similar way as the upper one. Thus, the J-integral in Eq. (33) becomes

$$\begin{aligned} J = & \frac{1}{2} \left[C_{11} \left(N_0^I \right)^2 + C_{22} \left(M_0^I \right)^2 \right] + \left(C_{33} - \frac{1}{2} \frac{1}{D_{33}} \right) \left(Q_0^I \right)^2 - Q_0^I \theta_0^I + C_{34} Q_0^I \tau_{a0} \\ & + \frac{1}{2} \left[C_{66} \left(N_0^{II} \right)^2 + C_{77} \left(M_0^{II} \right)^2 \right] + \left(C_{88} - \frac{1}{2} \frac{1}{D_{88}} \right) \left(Q_0^{II} \right)^2 - Q_0^{II} \theta_0^{II} + C_{84} Q_0^{II} \tau_{a0} \\ & - \frac{1}{2} \left[C_{66} \left(N_l^{II} \right)^2 + C_{77} \left(M_l^{II} \right)^2 \right] - \left(C_{88} - \frac{1}{2} \frac{1}{D_{88}} \right) \left(Q_l^{II} \right)^2 + Q_l^{II} \theta_l^{II} - C_{84} Q_l^{II} \tau_{al} \\ & - \frac{1}{2} \left[C_{11} \left(N_l^I \right)^2 + C_{22} \left(M_l^I \right)^2 \right] - \left(C_{33} - \frac{1}{2} \frac{1}{D_{33}} \right) \left(Q_l^I \right)^2 + Q_l^I \theta_l^I - C_{34} Q_l^I \tau_{al} + J_{DE}. \end{aligned} \quad (41)$$

Rearranging Eq. (41),

$$\begin{aligned} J = & \frac{1}{2} \left[C_{11} \left\{ \left(N_0^I \right)^2 - \left(N_l^I \right)^2 \right\} + C_{66} \left\{ \left(N_0^{II} \right)^2 - \left(N_l^{II} \right)^2 \right\} \right] \\ & + \frac{1}{2} \left[C_{22} \left\{ \left(M_0^I \right)^2 - \left(M_l^I \right)^2 \right\} + C_{77} \left\{ \left(M_0^{II} \right)^2 - \left(M_l^{II} \right)^2 \right\} \right] \\ & + \left(C_{33} - \frac{1}{2} \frac{1}{D_{33}} \right) \left[\left(Q_0^I \right)^2 - \left(Q_l^I \right)^2 \right] + \left(C_{88} - \frac{1}{2} \frac{1}{D_{88}} \right) \left[\left(Q_0^{II} \right)^2 - \left(Q_l^{II} \right)^2 \right] \\ & - Q_0^I \theta_0^I + Q_l^I \theta_l^I - Q_0^{II} \theta_0^{II} + Q_l^{II} \theta_l^{II} + \tau_{a0} \left(C_{34} Q_0^I + C_{84} Q_0^{II} \right) - \tau_{al} \left(C_{34} Q_l^I + C_{84} Q_l^{II} \right), \end{aligned} \quad (42)$$

where J_{DE} is assumed to be zero because the stresses in the adhesive decrease exponentially from the ends of the bond.

Based on the assumption of nearly zero stress state on the section DE, Eq. (42) can be further simplified. It can be thought that the displacements of the upper and lower surfaces of the adhesive and their derivatives are equal. Thus,

$$\frac{\mathrm{d}W^{(1)}}{\mathrm{d}x}=\frac{\mathrm{d}W^{(2)}}{\mathrm{d}x}\ ;\quad C_{33}Q_l^I-\theta_l^I=C_{88}Q_l^{II}-\theta_l^{II}, \tag{43}$$

$$\frac{\mathrm{d}\theta^{(1)}}{\mathrm{d}x}=\frac{\mathrm{d}\theta^{(2)}}{\mathrm{d}x}\ ;\quad C_{22}M_l^I=C_{77}M_l^{II}, \tag{44}$$

$$u^{(1)}=u^{(2)}\ ;\quad U_0^{(1)}-\frac{h_1}{2}\theta_l^I=U_o^{(2)}+\frac{h_2}{2}\theta_l^{II}, \tag{45}$$

$$\frac{\mathrm{d}u^{(1)}}{\mathrm{d}x}=\frac{\mathrm{d}u^{(1)}}{\mathrm{d}x}\ ;\quad C_{11}N_l^I-\frac{h_1}{2}C_{22}M_l^I=C_{66}N_l^{II}-\frac{h_2}{2}C_{77}M_l^{II}. \tag{46}$$

Also, the force and moment can be represented by equivalent ones which act on the neutral axis on the section *DE*. Hence,

$$\begin{aligned}
N_0&=N_0^I+N_0^{II},\\
Q_0&=Q_0^I+Q_0^{II},\\
M_0&=M_l^I+\frac{C_{11}}{C_{11}+C_{66}}\left[\frac{h_1+h_2}{2}\right]N_l^I+M_l^{II}-\frac{C_{66}}{C_{11}+C_{66}}\left[\frac{h_1+h_2}{2}\right]N_l^{II}.
\end{aligned} \tag{47}$$

First, let us consider the in-plane load terms in the J-integral:

$$J_N=\frac{1}{2}\left[C_{11}\left\{\left(N_0^I\right)^2-\left(N_l^I\right)^2\right\}+C_{66}\left\{\left(N_0^{II}\right)^2-\left(N_l^{II}\right)^2\right\}\right]. \tag{48}$$

Using Eqs. (44), (46), and (47), eliminating N_0^{II} and N_l^{II} in Eq. (48) gives

$$J_N=\frac{1}{2}\left[C_{11}\left(N_0^I\right)^2+C_{66}\left(N_0^{II}\right)^2-\frac{C_{11}C_{66}}{C_{11}+C_{66}}N_0^2\right]-\frac{1}{8}\frac{(h_1+h_2)^2}{C_{11}+C_{66}}\left(C_{77}M_l^{II}\right)^2. \tag{49}$$

The bending moment terms can be represented in a similar way:

$$\begin{aligned}
J_M&=\frac{1}{2}\left[C_{22}\left\{\left(M_0^I\right)^2-\left(M_l^I\right)^2\right\}+C_{77}\left\{\left(M_0^{II}\right)^2-\left(M_l^{II}\right)^2\right\}\right]\\
&=\frac{1}{2}\left[C_{22}\left(M_0^I\right)^2+C_{77}\left(M_0^{II}\right)^2\right]-\frac{1}{2}\left[\frac{1}{C_{22}}+\frac{1}{C_{77}}+\frac{1}{4}\frac{(h_1+h_2)^2}{C_{11}+C_{66}}\right]^{-1}\\
&\quad M_0^2+\frac{1}{8}\frac{(h_1+h_2)^2}{C_{11}+C_{66}}\left(C_{77}M_l^{II}\right)^2.
\end{aligned} \tag{50}$$

Lastly, the shear terms are considered:

$$\begin{aligned}
J_Q=&\left(C_{33}-\frac{1}{2}\frac{1}{D_{33}}\right)\left[\left(Q_0^I\right)^2-\left(Q_l^I\right)^2\right]+\left(C_{88}-\frac{1}{2}\frac{1}{D_{88}}\right)\left[\left(Q_0^{II}\right)^2-\left(Q_l^{II}\right)^2\right]\\
&-Q_0^I\theta_0^I+Q_l^I\theta_l^I-Q_0^{II}\theta_0^{II}+Q_l^{II}\theta_l^{II}+\tau_{a0}\left(C_{34}Q_0^I+C_{84}Q_0^{II}\right).
\end{aligned} \tag{51}$$

Rearrangement of Eq. (51) can be written as

$$\begin{aligned}
J_Q=&\left(C_{33}-\frac{1}{2}\frac{1}{D_{33}}\right)\left(Q_0^I\right)^2+\left(C_{88}-\frac{1}{2}\frac{1}{D_{88}}\right)\left(Q_0^{II}\right)^2-\left[\left(C_{33}-\frac{1}{2}\frac{1}{D_{33}}\right)C_{88}^2+\left(C_{88}-\frac{1}{2}\frac{1}{D_{88}}\right)C_{33}^2\right]\\
&\left(\frac{Q_0}{C_{33}+C_{88}}\right)^2-Q_0^I\theta_0^I-Q_0^{II}\theta_0^{II}+Q_0\theta_0+\tau_{a0}\left(C_{34}Q_0^I+C_{84}Q_0^{II}\right).
\end{aligned} \tag{52}$$

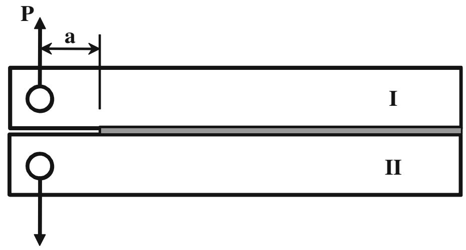

Fig. 3 Double cantilever beam specimen

Thus, the J-integral can be obtained by adding Eqs. (49), (50), and (52):

$$
\begin{aligned}
J &= J_N + J_M + J_Q \\
&= \frac{1}{2}\left[C_{11}\left(N_0^I\right)^2 + C_{66}\left(N_0^{II}\right)^2 - \frac{C_{11}C_{66}}{C_{11}+C_{66}}N_0^2\right] \\
&+\frac{1}{2}\left[C_{22}\left(M_0^I\right)^2 + C_{77}\left(M_0^{II}\right)^2 - \left\{\frac{1}{C_{22}}+\frac{1}{C_{77}}+\frac{1}{4}\frac{(h_1+h_2)^2}{C_{11}+C_{66}}\right\}^{-1} M_0^2\right] \\
&+\left(C_{33}-\frac{1}{2}\frac{1}{D_{33}}\right)\left(Q_0^I\right)^2 + \left(C_{88}-\frac{1}{2}\frac{1}{D_{88}}\right)\left(Q_0^{II}\right)^2 - \left[\left(C_{33}-\frac{1}{2}\frac{1}{D_{33}}\right)C_{88}^2\right. \\
&\left.+\left(C_{88}-\frac{1}{2}\frac{1}{D_{88}}\right)C_{33}^2\right]\left(\frac{Q_0}{C_{33}+C_{88}}\right)^2 - Q_0^I\theta_0^I - Q_0^{II}\theta_0^{II} + Q_0\theta_0 + \tau_{a0}\left(C_{34}Q_0^I + C_{84}Q_0^{II}\right).
\end{aligned} \tag{53}
$$

The final term containing τ_{a0} in Eq. (54) is given by the mixed formulation, which is distinct from previous conventional formulations [15,16].

4 J-integral examples

Two examples are considered, which have been solved by Fraisse and Schmit [16], to compare the J-integrals from the current formulation and the conventional approach. The first example is a double cantilever beam as shown in Fig. 3. In this case, the boundary loading conditions are defined as

$$
\begin{aligned}
&N_0^I = N_0^{II} = 0\,; \quad M_0^I = M_0^{II} = 0\,; \quad Q_0^I = -Q_0^{II} = -P, \\
&N_0 = 0\,; \quad M_0 = 0\,; \quad Q_0 = 0, \\
&\theta_0^I = \frac{1}{2}C_{22}Pa^2\,; \quad \theta_0^{II} = -\frac{1}{2}C_{77}Pa^2\,; \quad \theta_0 = 0.
\end{aligned} \tag{54}
$$

If two adherends have identical material properties, then the J-integral from Eq. (54) gives no difference compared with the conventional formulation. (The τ_{a0} term in Eq. (54) automatically becomes zero, since $Q_0^I = -Q_0^{II}$.) Thus, the dissimilar adherends are considered to investigate the effect of $E^{(1)}/E^{(2)}$ on the J-integral. Figure 4 shows the J-integral results with varying the material property ratio. In the plot, J_0 is the value when $E^{(1)} = E^{(2)}$ and it is shown that the difference increases as $E^{(1)}/E^{(2)}$ becomes larger. Also, it is compared with the results by Wang and Qiao [18], who proposed the shear deformable bi-layer beam theory for interface fracture analysis and compared their solutions with FEA results.

Considering that the J-integral is equal to the energy release rate, it can be thought that the crack growth is retarded by the refined shear model in this paper. This means that the crack propagation predicted by the conventional analysis might not happen; thus, the results can be too conservative in some cases. Also, it indicates that the proposed methodology is more appropriate to dissimilar adherends or laminated joints.

As next example, a mixed mode flexure specimen depicted in Fig. 5 is considered. The boundary loading and displacement conditions are

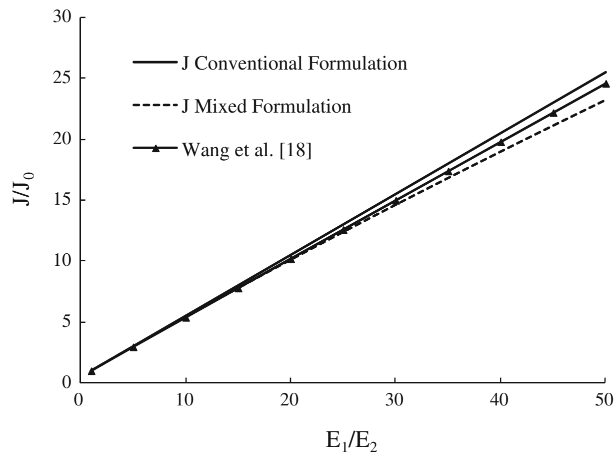

Fig. 4 The effect of $E^{(1)}/E^{(2)}$ on J-integral

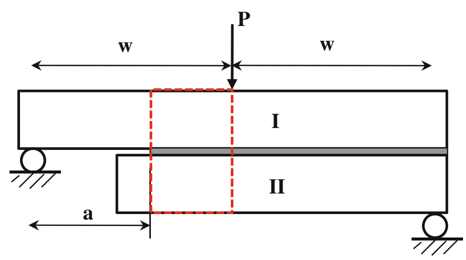

Fig. 5 Mixed mode flexure specimen

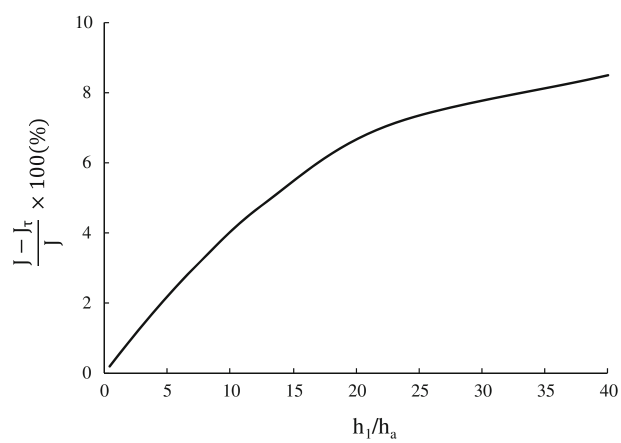

Fig. 6 The effect of adhesive layer thickness on J-integral

$$
\begin{aligned}
&N_0^I = N_0^{II} = 0\,; \quad M_0^I = M_0^{II} = 0\,; \quad Q_0^I = -\frac{1}{2}P, \quad Q_0^{II} = 0, \\
&N_0 = 0\,; \quad M_0 = -\frac{1}{2}Pw\,; \quad Q_0 = \frac{1}{2}P, \\
&\theta_0^I = \frac{1}{4}C_{22}Pa^2\,; \quad \theta_0^{II} = 0\,; \quad \theta_0 = \frac{1}{4}\left\{\frac{1}{C_{22}} + \frac{1}{C_{77}} + \frac{1}{4}\frac{(h_1+h_2)^2}{C_{11}+C_{66}}\right\}^{-1} P\left(w^2 - a^2\right).
\end{aligned}
\tag{55}
$$

In this case, the contribution of τ_a to J_Q is distinct from the conventional analysis; thus, the effect of adhesive thickness on the J-integral is investigated. In Fig. 6, the result is shown, where J and J_τ are from the conventional analysis and the mixed formulation, respectively. As the thickness h_a decreases, the J-integral from the mixed formulation gives lower values. This difference is because τ_a from the conventional analysis [1,2] does not give accurate results when the adhesive layer thickness becomes small compared with that of the adherend. The conventional stress-displacement relation in the adhesive layer gives the singular behavior in τ_a when the adhesive thickness h_a approaches zero.

5 Conclusions

The J-integral of a single-lap adhesive joint was derived using the stress fields from the mixed variational principle. Compared with the conventional analysis, the proposed formulation gave the more accurate adhesive shear stress and the appropriate shear correction factor automatically in the constitutive equations. Two types of specimen were considered, and the results showed the effectiveness of the current approach in case of dissimilar adherends and relatively thin adhesive layer. For many adhesively bonded joints, the laminates are used as adherend. The present method can easily consider those cases, and also, the simple analytic type of solutions will be helpful in designing the joint.

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