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Conformal invariance of Mei symmetry for discrete Lagrangian systems

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Abstract The conformal invariance of the Mei symmetry and the conserved quantities are investigated for discrete Lagrangian systems under the infinitesimal transformation of the Lie group. The difference Euler–Lagrange equations on regular lattices of the discrete Lagrangian systems are presented via the transformation operators in the space of the discrete variables. The conformal invariance of the Mei symmetry is defined for the discrete Lagrangian systems. The criterion equations and the determining equations are proposed. The conserved quantities of the systems are derived from the structure equation governing the gauge function. Two examples are given to illustrate the application of the results.

1 Introduction

All variables are measured in a certain time interval, and all simulations can be implemented in meshes. Therefore, discrete systems defined by difference equations provide with a potentially powerful way to model dynamics in real-world problems [1]. Lagrangian systems are physically significant. There have been many attempts at the development of discrete Lagrangian systems. Cadzow [2] developed the concept of discrete calculus of variations and obtained the discrete Euler–Lagrange equation in 1970. Logan [3] researched the invariance of the discrete Lagrangian and constructed the first integrals of the discrete Euler–Lagrange equation. He clearly connected the discrete Euler–Lagrange equations and the discrete Noether's theorem. Marsden [4] dealt with the symmetry reduction in discrete Lagrangian mechanics and proposed the conclusion that the Poisson structure coincides with the reduction under the symmetry group of the canonical discrete Lagrange 2-form. It is a general and useful method of locating the first integrals of difference equations when the difference equations can be cast into Lagrangian systems [5–7].

The symmetry in mechanics is a theoretically significant topic [8]. It has been discovered that the conserved quantities can be derived from the symmetries. Noether [9] and Lutzky [10] revealed the relations about symmetries and conserved quantities. The Mei symmetry (the form invariance) [11] is a new kind of symmetry, which means that the dynamical functions are replaced by the transformed functions under the infinitesimal

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transformations of the group, the forms of the differential equations of motion keep invariant. The Mei symmetry of continuous systems has been paid much attention [8]. Due to the importance of the symmetries of discrete mechanical systems, the Mei symmetry of difference systems has been recently investigated. Shi [12] presented the Mei symmetry of general discrete holonomic systems and obtained the definition of the Mei symmetry of discrete motion equations. Zhang [13] proposed the Mei symmetry of discrete mechanico-electrical systems and obtained the discrete Mei conserved quantity induced directly by the Mei symmetry as well as its form. Mei symmetries and conserved quantities for difference systems with non-conservative constraints are also studied [14]. The Mei symmetry of discrete mechanical systems presented methods of getting clear about the local physical state of difference ones.

Conformal invariance is a new method for finding conserved quantities for dynamical systems. It generalizes the scale invariance, the translation invariance, rotational invariance and a variety of interactions. In 1997, Galiulin et al. [15] proposed conformal invariance (or conformal symmetry) for Birkhoff systems under special infinitesimal transformations and got the Noether conserved quantity. There are two types of conformal invariance. One is the conformal invariance of Lie symmetry, and the other is conformal invariance of the Mei symmetry. Conformal invariance of Lagrange systems under Lie point transformation is studied in [16]. And it is treated in holonomic systems [17] and non-holonomic systems [18]. In this aspect, the relationship between the system's conformal invariance and Lie symmetry is discussed by introducing the conformal factor. The conformal factor is the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system. The Mei symmetry can also lead to conformal invariance under certain conditions. This type of conformal invariance is the conformal invariance of the Mei symmetry [19,20], in which the operators both satisfy the determining equation of the Mei symmetry and the conformal invariance. So far, these two types of conformal invariances have only been proposed for continuous systems. There have been no investigations on conformal invariance of discrete mechanical systems. To address the lacks of research in this aspect, the present work focuses on the conformal invariance of the Mei symmetry for discrete Lagrangian systems.

The manuscript is organized as follows: Sect. 2 proves some general propositions regarding the conformal invariance of the Mei symmetry for discrete Lagrangian systems. Section 3 proposes a condition under which a conformal invariance of the Mei symmetry can lead to a conserved quantity. Section 4 presents 2 examples to demonstrate the application of the results. Section 5 ends the paper by the conclusions.

2 Conformal invariance of discrete Lagrangian systems

Consider the space $\tilde{Z}$ of sequence $(t, \boldsymbol{q})$, a mechanical system whose configurations are determined by one independent variable t and dependent variable $\boldsymbol{q} = (q_1,\ q_2, \ldots, q_n)$. A finite-difference equation for a Lagrangian system is [21]

$$\frac{\partial L_d}{\partial q_s} - \underset{-h}{D}\left(\frac{\partial L_d}{\partial q_{s,t}}\right) = 0, \tag{1}$$

which is a generalized difference Euler equation on a uniform mesh, the function $L_d = L_d(t, q_s,\ q_{s,t})$ is called a mesh (or discrete, or finite-difference) Lagrangian function, $q_{s,t} = \underset{+h}{D}(q_s) = \frac{q_s^+ - q_s}{h^+}$ is the right discrete first derivative, $\underset{-h}{D}$ is the left difference derivative in the lattices space $Z = (t, t^+, t^-, \boldsymbol{q}, \boldsymbol{q}^+, \boldsymbol{q}^-)$, and any solution of Eq. (1) is called an extremal.

For dynamical systems, Mei proposed a novel method of searching conserved quantities. It is the form invariance (the Mei symmetry) [11] which has practical significance. The Mei symmetry is an invariance of the differential equation satisfied by physical quantities such as Lagrangian, non-potential generalized forces and generalized constrained forces under the infinitesimal transformation with respect to the time and generalized coordinates. In the following, the case of one independent variable t and dependent variable $\boldsymbol{q}$ will be considered. Introduce the infinitesimal transformations

$$t^* = t + \varepsilon\xi(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\ q_s^*(t^*) = q_s + \varepsilon\eta_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{2}$$

where ε is an infinitesimal parameter and ξ, η_s are infinitesimal generators. References [12,13] gave the form of determining equations of the Mei symmetry of discrete holonomic systems and mechanico-electrical systems. For discrete Lagrangian systems, under the infinitesimal transformations Eq. (2), the function L_d

becomes L_d^*. Expanding function $L_d^* = L_d + \mathrm{Pr}X^{(1)}(L_d) + o(\varepsilon^2)$, substituting it into $L_d^* = 0$, and ignoring high-order small quantities, we can get the determining equations of the Mei symmetry of discrete Lagrangian systems

$$E_s^d \left\{ \mathrm{Pr}X(L_d) \right\} \Big|_{E_s^d(L_d)=0} = 0, \tag{3}$$

where $E_s^d = \frac{\partial}{\partial q_s} - \underset{-h}{D}\left(\frac{\partial}{\partial q_{s,t}}\right)$ are difference Euler's operators,

$$\mathrm{Pr}\, X = \xi \frac{\partial}{\partial t} + \eta_s \frac{\partial}{\partial q_s} + \xi^- \frac{\partial}{\partial t^-} + \eta_s^- \frac{\partial}{\partial q_s^-} + \xi^+ \frac{\partial}{\partial t^+} + \eta_s^+ \frac{\partial}{\partial q_s^+} \tag{4}$$

are the corresponding coefficients of the operator being obtained by shifting the arguments to the difference stencil points adjacent to $(t, \boldsymbol{q})$, and $\xi^- = \xi(t^-, \boldsymbol{q}^-)$, $\eta_s^- = \eta_s(t^-, \boldsymbol{q}^-)$, $\xi^+ = \xi(t^+, \boldsymbol{q}^+)$ and $\eta_s^+ = \eta_s(t^+, \boldsymbol{q}^+)$.

Definition For discrete Lagrangian equations (1), if there exists a nonsingular matrix $M_{s,d}^k$ satisfying

$$E_s^d \left\{ \mathrm{Pr}X(L_d) \right\} = M_{s,d}^k \left\{ E_s^d(L_d) \right\}, \tag{5}$$

then the discrete Lagrangian equations (1) maintain conformal invariance of the Mei symmetry under the single parameter infinitesimal transformations (2). Equation (5) is called the determining equation of conformal invariance of the Mei symmetry for the discrete Lagrangian systems in Eq. (1), and $M_{s,d}^k$ is called the conformal factor of discrete Lagrangian systems.

Theorem 1 *For the discrete Lagrangian $L_d = L_d(t, q_s, q_{s,t})$, if the generators ξ and η_s of the infinitesimal transformations (2) satisfy*

$$\mathrm{Pr}X(L_d) = ML_d + C_1\varphi_d + C_2 q_{s,t}, \tag{6}$$

then the discrete Lagrangian equations (1) are the conformal invariance of the Mei symmetry, where C_1, C_2 and M are constants, $\varphi_d = \varphi_d(t, t^+)$.

Proof Since

$$E_s^d(\varphi_d) = \frac{\partial \varphi_d}{\partial q_s} - \underset{-h}{D}\left(\frac{\partial \varphi_d}{\partial q_{s,t}}\right) = 0,\ E_s^d(q_{s,t}) = \frac{\partial q_{s,t}}{\partial q_s} - \underset{-h}{D}\left(\frac{\partial q_{s,t}}{\partial q_{s,t}}\right) = 0, \tag{7}$$

from Eqs. (5)–(7), we can get

$$\begin{aligned} E_s^d \left\{ \mathrm{Pr}X(L_d) \right\} &= ME_s^d(L_d) + C_1 E_s^d(\varphi_d) + C_2 E_s^d(q_{s,t}) \\ &= ME_s^d(L_d) = M_{s,d}^k \left\{ E_s^d(L_d) \right\},\ s, k = 1, \ldots, n, \end{aligned} \tag{8}$$

where $M_{s,d}^k = \delta_s^k M \left(\delta_s^k = \begin{cases} 1, s = k \\ 0, s \neq k \end{cases} \right)$. So the discrete Lagrangian equations (1) are the conformal invariance of the Mei symmetry.

Theorem 2 *If a Lagrangian system maintains the conformal invariance of the Mei symmetry, then it maintains the Mei symmetry, too.*

Proof Substituting Eq. (1) into Eq. (8), one can obtain the determining equations of the Mei symmetry for discrete Lagrangian equations as below,

$$E_s^d \left\{ \mathrm{Pr}X(L_d) \right\} \Big|_{E_s^d(L_d)=0} = M_{s,d}^k \left\{ E_s^d(L_d) \right\} \Big|_{E_s^d(L_d)=0} = 0. \tag{9}$$

3 Discrete structure equation and conserved quantity

A large number of references have contributed to the discovery of the relationships between conservation laws and symmetries. The conformal invariance of discrete equations can be used to find integrals of a discrete dynamical system. For conformal invariance of the Mei symmetry, conserved quantities can be proposed by the famous Noether's theorem. According to Noether's theorem, if the integral of a variational problem is invariant under the finite or continuous transformation group G_α, then of Lagrangian functions there exist α linear connections, each of which is a divergence. The point of the theorem is that every single symmetry (transformation) is corresponding to a conserved quantity. For discrete systems, the discrete analogs of Noether's theorem for difference equations were reported in references [21–23]. The operator identity

$$\begin{aligned}\mathrm{Pr}X(L_d) + L_d \underset{+h}{D}(\xi) + \underset{+h}{D}(G_s^d) &\equiv \xi\left(\frac{\partial L_d}{\partial t} + \frac{h^-}{h^+}\frac{\partial L_d^-}{\partial t} - \underset{+h}{D}(L_d)\right)\\ &+\eta_s\left(\frac{\partial L_d}{\partial q_s} + \frac{h^-}{h^+}\frac{\partial L_d^-}{\partial q^s}\right) + \underset{+h}{D}\left(h^-\eta_s\frac{\partial L_d^-}{\partial q_s} + h^-\xi\frac{\partial L_d^-}{\partial t} + \xi L_d^- + G_s^d\right) \qquad (10)\end{aligned}$$

holds for a Lagrangian which has the form $L_d = L_d(t, q_s, \frac{q_s^+ - q_s}{t^+ - t}) = L_d(t, t^+, q_s, q_s^+)$ and vector field $\mathrm{Pr}X$ of Eq. (4). $\underset{-h}{D}$ are the right difference derivatives. This identity can be established by a straightforward computation. If the left-hand side of Eq. (10) equals zero, i.e.,

$$\mathrm{Pr}X(L_d) + L_d \underset{+h}{D}(\xi) + \underset{+h}{D}(G_s^d) = 0, \qquad (11)$$

this is called the discrete version of generalized Noether-type identity for the systems. Here, the right-hand side of equation (10) equals zero, when there exists

$$\xi\left(\frac{\partial L_d}{\partial t} + \frac{h^-}{h^+}\frac{\partial L_d^-}{\partial t} - \underset{+h}{D}(L_d)\right) + \eta_s\left(\frac{\partial L_d}{\partial q_s} + \frac{h^-}{h^+}\frac{\partial L_d^-}{\partial q^s}\right) = 0, \qquad (12)$$

which are called the generalized quasi-extremal equations for this discrete Lagrangian systems, and then the systems possess the discrete version of conservation law,

$$\underset{+h}{D}\left(h^-\eta_s\frac{\partial L_d^-}{\partial q_s} + h^-\xi\frac{\partial L_d^-}{\partial t} + \xi L_d^- + G_s^d\right) = 0. \qquad (13)$$

Namely,

$$I = h^-\eta_s\frac{\partial L_d^-}{\partial q_s} + h^-\xi\frac{\partial L_d^-}{\partial t} + \xi L_d^- + G_s^d = \text{const}. \qquad (14)$$

The discrete Eqs. (14) are called the difference version of Noether's conservation laws associated with discrete Lagrangian systems.

Theorem 3 *If a discrete gauge function $G_s^d = G_s^d(t, t^+, q_s, q_s^+)$ exists such that the infinitesimal transformation generators ξ, η_s or the symmetry operators $pr\,X$ satisfy the discrete Noether's identity* (11)*, then the discrete Lagrangian systems* (1) *possess the discrete conserved quantities* (14).

The discrete Lagrangian version of Noether's theorem, as formulated here, leads to the conditions of finding first integrals of discrete Lagrangian equations once their symmetries are obtained. The conditions also account for the reasons that not all symmetries of discrete Lagrangian equations yield a first integral.

4 Examples

In the following, two examples are presented to illustrate the above results.

Example 1 Suppose that the Lagrangian equation of a discrete Lagrangian system is

$$L_d = \frac{1}{2}\left(\frac{q_1^+ - q_1}{h}\right)^2 + \frac{1}{2}\left(\frac{q_2^+ - q_2}{h}\right)^2 + \left(\frac{q_1^+ + q_1}{2}\right)^2 + t^+. \tag{15}$$

Firstly, study the Mei symmetry and conserved quantity of this system:

$$\begin{aligned}\mathrm{Pr}X\,(L_d) = &\left(\frac{q_1^+ - q_1}{h}\right)\cdot\left(\frac{\eta_1^+ - \eta_1}{h}\right) + \left(\frac{q_2^+ - q_2}{h}\right)\cdot\left(\frac{\eta_2^+ - \eta_2}{h}\right)\\ &+2\left(\frac{q_1^+ + q_1}{2}\right)\cdot\left(\frac{\eta_1^+ + \eta_1}{2}\right) + \xi^+.\end{aligned} \tag{16}$$

Let

$$X_1 = -2t\frac{\partial}{\partial t} + q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}, \tag{17}$$

$$X_2 = \frac{\partial}{\partial t} + \frac{\partial}{\partial q_2}, \tag{18}$$

$$X_3 = \frac{\partial}{\partial q_2}, \tag{19}$$

$$X_4 = \frac{\partial}{\partial t}. \tag{20}$$

The operators (17)–(20) satisfy the determining equations (3) of the Mei symmetry of discrete Lagrangian systems. So the operators (17)–(20) are Mei symmetrical. If a discrete gauge function $G_s^d = G_s^d(t, t^+, q_s, q_s^+)$ exists such that the infinitesimal transformation generators ξ, η_s or the symmetry operators PrX of the Mei symmetry satisfy the discrete Noether's identity (1), then the discrete Lagrangian systems (1) possess the discrete conserved quantities

$$\begin{aligned}I_1 = &\frac{q_1(q_1 - q_1^-)}{h} + \frac{q_2(q_2 - q_2^-)}{h} - t\left(\frac{q_1 - q_1^-}{h}\right)^2\\ &-t\left(\frac{q_2 - q_2^-}{h}\right)^2 - 2t\left(\frac{q_1 + q_1^-}{2}\right)^2 + 2ht = \mathrm{const},\end{aligned} \tag{21}$$

$$I_2 = \frac{q_2 - q_2^-}{h} + L_d^- + t = \mathrm{const}, \tag{22}$$

$$I_3 = \frac{q_2 - q_2^-}{h} = \mathrm{const}, \tag{23}$$

$$I_4 = L_d^- + t = \mathrm{const}. \tag{24}$$

Note that

$$I_2 = I_3 + I_4, \tag{25}$$

$$I_1 - q_2 I_3 + 2t I_4 = 2(t^2 + t^{+2} + ht^+) + \frac{q_1(q_1 - q_1^-)}{h}. \tag{26}$$

For operator (17), we have

$$\mathrm{Pr}X\,(L_d) = 2L_d - 4t^+. \tag{27}$$

Therefore,

$$E_s^d\{\Pr X(L_d)\} = E_s^d(2L_d - 4t^{+2}) = 2E_s^d(L_d) = M_{s,d}^k E_s^d(L_d) = 0 \tag{28}$$

which implies that the system maintains the Mei symmetry as well as conformal invariance of the Mei symmetry. The conformal factor is $M_{s,d}^k = 2\delta_s^k M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

From Eq. (11), one can obtain

$$-4t^+ + \underset{+h}{D}(G) = 0,\ G = 2t^{+2} - 2ht^+. \tag{29}$$

From the conserved quantity in Eq. (14), we can conclude the first integral I_1 of Eq. (21).

For operators (18)–(20), they do not satisfy the determining Eq. (3) of conformal invariance of the Mei symmetry.

Example 2 Consider a discretization of the ordinary differential equation $q_{tt} = q$ [25],

$$\frac{q^+ - 2q + q^-}{(t^+ - t)^2} = q, \tag{30}$$

$$t^+ - 2t + t^- = 0. \tag{31}$$

Its discrete Lagrangian equation can be written as

$$L_d = \frac{1}{2}\left(\frac{q^+ - q}{h}\right)^2 - \frac{1}{2}q^2, \tag{32}$$

$$\Pr X\,(L_d) = \left(\frac{q^+ - q}{h}\right)\cdot\left(\frac{\eta^+ - \eta}{h}\right) - q\eta. \tag{33}$$

Let

$$X = -2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}. \tag{34}$$

One has

$$\Pr X\,(L_d) = 2L_d. \tag{35}$$

Therefore,

$$E_s^d\{\Pr X(L_d)\} = E_s^d(2L_d) = 2E_s^d(L_d) = M_{s,d}^k E_s^d(L_d) = 0. \tag{36}$$

From the conserved quantity in Eq. (14), one can conclude that

$$I = \frac{q(q - q^-)}{h} - t\left(\frac{q - q^-}{h}\right)^2 + tq^{-2} = \text{const}. \tag{37}$$

The symmetry operator is different from the results in [25], and the conserved quantity (37) is also not the conservation of energy in [24].

The conserved quantities (21)–(24) and (37) are discrete schemes. It can guide how to construct conservative finite-difference schemes in the Lagrangian framework, which is significant in numerical implementations. By use of the Noether symmetry, it may be difficult to derive first integrals through the conformal invariance for equations (especially nonlinear equation) of motion of the discrete mechanical systems. For example, the Noether symmetry and conserved quantities about the nonlinear equation $q_{tt} = q^2$ were treated on the page 187 of [1]. The discrete Lagrangian equation is $L_d = 3\left(\frac{q^+ - q}{h}\right)^2 + 2q^3$. Taking the transformation operator $X = \partial/\partial t$, the author presented the first integral $I = 2q^3 - 3\left(\frac{q^+ - q}{h}\right)^2 = \text{const}$. But for the conformal invariance of the Mei symmetry, the transformation operators are not found to satisfy the determining Eqs. (5). So the corresponding conserved quantities cannot be obtained. It is difficult to get first integrals through the conformal invariance because the operators must satisfy both the Noether identity and the determining equations of the conformal invariance.

5 Conclusions

The difference equations and the conformal invariance of the Mei symmetry are proposed for discrete Lagrangian systems. If the infinitesimal generators satisfy the determining equations of conformal invariance of the Mei symmetry, these generators are both the Mei symmetry and the conformal invariance. This may be a novel approach to find conserved quantities of discrete systems. A conformal invariance of the Mei symmetry does not always imply a conserved quantity for discrete Lagrangian systems. A condition is presented under which a conformal invariance of the Mei symmetry can lead to a conserved quantity. The condition is the satisfaction of both the determining equation and the Noether identity.

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