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Thermal buckling of a nanoplate with small-scale effects

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Abstract In this paper, thermal buckling properties of a nanoplate with small-scale effects are studied. Based on the nonlocal continuum theory, critical temperatures for the nonlocal Kirchhoff and Mindlin plate theories are derived. The thermal buckling characteristics are presented with different models. The influences of the scale coefficients, half-wave numbers, width ratios, and the ratios of the width to the thickness are discussed. From this work, it can be observed that the small-scale effects are significant for the thermal buckling properties. Both the half-wave number and width ratio have influence. The nonlocal Kirchhoff plate theory is valid for the thin nanoplate, and the nonlocal Mindlin plate theory is more appropriate for simulating the mechanical behaviors of the thick nanoplate.

1 Introduction

Since the pioneer work on the carbon nanotube by Ijima [1], there is a growing interest on nanostructures [2–5]. With the excellent electronic, chemical, and mechanical characteristics, the carbon nanotube has become one of the most promising materials. As a result, many potential applications can be expected such as the atomic-force microscope, field emitters, and nanoscale electronic devices in nanoelectromechanical systems (NEMS). As a result, the mechanical behaviors of the carbon nanotube have drawn a lot of attention [6–11].

Because it is difficult to perform the experiment at the nanoscale and the molecular dynamics (MD) simulation, the continuum model has been applied for many investigations. Furthermore, among various numerical methods with the elastic continuum theory, the nonlocal continuum theory presented by Eringen [12, 13] is reliable and effective to present the mechanical behaviors of nanostructures. In the nonlocal continuum theory, the small-scale effects are considered [14–17] which are not included in the classical (i.e. local) elastic model. From the results, it can be observed that this theory can provide an appropriate simulation.

Besides the nanotube, the nanoplate is another typical structure of nanoscale systems, which can be deformed into the nanotube. In recent years, some works on the mechanical behaviors of the nanoplate have

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been reported, including the bending [18], buckling [19,20], vibration [21–23], and wave propagation characteristics [24,25]. However, compared with the nanotube, researches on the nanoplate seem to be rather limited and there are still a lot of problems to consider.

Moreover, it has been reported that the thermal effects are significant for the mechanical behavior of the nanotube [26–30]. In our recent work [31], the thermal effects on the vibration characteristics of the nanoplate are discussed. In the present paper, in order to illustrate the thermal effects on the mechanical behavior further, the thermal buckling properties of the nanoplate are studied. The results for the nonlocal Kirchhoff and Mindlin plate theories are shown, respectively. At last, the relation and difference between the two models are discussed.

2 Nonlocal continuum theory

The nanoplate is shown in Fig. 1 with the thickness h, and the widths l_a and l_b for the x and y directions. According to the nonlocal continuum theory [12,13], which accounts for the small-scale effects by assuming the stress at a reference point as a function of the strain at every point in the body, the constitutive relation can be represented as

$$\sigma_{kl,k} - \rho \ddot{u}_l = 0, \tag{1.1}$$

$$\sigma_{kl}(\mathbf{x}) = \int_V \alpha(\mathbf{x}, \mathbf{x}')\tau_{kl}(\mathbf{x}')\mathrm{d}V(\mathbf{x}'), \tag{1.2}$$

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \tag{1.3}$$

where σ_{kl} is the nonlocal stress tensor, ε_{kl} the strain tensor, ρ the mass density, u_l the displacement vector, $\tau_{kl}(\mathbf{x}')$ the classical (i.e. local) stress tensor, $\alpha(\mathbf{x}, \mathbf{x}')$ the kernel function which describes the influence of the strains at various locations $\mathbf{x}'$ on the stress at a given location $\mathbf{x}$, and V the entire body considered.

It can be observed that the spatial integrals are involved in the nonlocal constitutive relation, which results in the difficulty for the problem. However, these integral equations can be reduced to the partial differential forms, and the nonlocal constitutive relation can be employed conveniently. Then, the relation between the stress and strain is

$$\sigma_x - (e_0a)^2\nabla^2\sigma_x = \frac{E}{1-\upsilon^2}(\varepsilon_x + \upsilon\varepsilon_y), \tag{2.1}$$

$$\sigma_y - (e_0a)^2\nabla^2\sigma_y = \frac{E}{1-\upsilon^2}(\varepsilon_y + \upsilon\varepsilon_x), \tag{2.2}$$

$$\tau_{yz} - (e_0a)^2\nabla^2\tau_{yz} = \frac{E}{1+\upsilon}\varepsilon_{yz}, \tag{2.3}$$

$$\tau_{xz} - (e_0a)^2\nabla^2\tau_{xz} = \frac{E}{1+\upsilon}\varepsilon_{xz}, \tag{2.4}$$

$$\tau_{xy} - (e_0a)^2\nabla^2\tau_{xy} = \frac{E}{1+\upsilon}\varepsilon_{xy}, \tag{2.5}$$

where E is the Young's modulus, ν the Poisson's ratio, e_0 the constant appropriate to each material, and a the internal characteristic length (e.g. the length of C–C bond, the lattice spacing and the granular distance), and e_0a means the scale coefficient which denotes the small-scale effect on the mechanical characteristics. If $e_0a = 0$, this relation will be reduced to the classical elastic model.

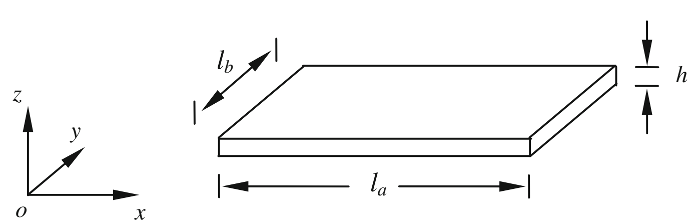

Fig. 1 Schematic diagram of the nanoplate

In order to study the thermal buckling properties of the nanoplate, nonlocal Kirchhoff and Mindlin plate models are proposed in the following.

3 Nonlocal Kirchhoff plate model

The stress bending moment for the nanoplate can be given in the following form:

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_x \mathrm{d}z,\ M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_y \mathrm{d}z,\ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xy} \mathrm{d}z. \tag{3}$$

According to the mechanical model for the nanoplate [32,33], the governing equation with the external load can be expressed as

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_T \left(\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) = 0, \tag{4}$$

where w is the transverse deflection, and N_T the thermal load caused by the temperature change which can be expressed as the following form:

$$N_T = -\frac{E\alpha T}{1-\upsilon} h, \tag{5}$$

where α is the thermal expansion coefficient and T the temperature change.

Based on Eqs. (2)–(4), the governing equations can be obtained as

$$D\nabla^2\nabla^2 w + N_T (e_0 a)^2 \nabla^2\nabla^2 w - N_T \nabla^2 w = 0, \tag{6}$$

where $D = Eh^3/12(1-\upsilon^2)$ is the bending stiffness, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ the Laplace operator.

For the simply supported condition, the solution can be assumed as

$$w = W \sin \alpha_m x \sin \beta_n y, \tag{7}$$

where W is the displacement amplitude, $\alpha_m = m\pi/l_a$, $\beta_n = n\pi/l_b$, m and n the integrals denoting the half-wave numbers for the x and y directions after the compression.

Based on Eqs. (5)–(7), the critical temperature for the nonlocal Kirchhoff plate theory is

$$T^K_{\text{cr\_Non}} = \frac{D(1-\upsilon)(\alpha_m^2 + \beta_n^2)}{E\alpha h[1 + (e_0 a)^2(\alpha_m^2 + \beta_n^2)]}. \tag{8}$$

As a result, the critical temperature ratio of the nonlocal to classical Kirchhoff plate models is defined in the following form:

$$R^K_{\text{cr}} = \frac{T^K_{\text{cr\_Non}}}{T^K_{\text{cr\_Cla}}}, \tag{9}$$

where $T^{\text{K}}_{\text{cr\_Cla}}$ is the critical temperature for the classical Kirchhoff plate model (i.e. $e_0 a = 0$).

Then, the thermal buckling properties for the nanoplate with the Kirchhoff plate theory are studied. The material constants are the Young's modulus $E = 1.06\,\text{TPa}$, the mass density $\rho = 2{,}250\,\text{kg/m}^3$, the Poisson's ratio $\nu = 0.25$, the plate widths $l_a = l_b = 10\,\text{nm}$, the thickness $h = 0.34\,\text{nm}$, the half-wave number for the y direction (i.e. n) is 1, and the temperature expansion coefficient $\alpha = 1.1 \times 10^{-6}$/K for the high temperature case. The relation between the critical temperature ratio (R^{K}_{cr}) and the scale coefficient ($e_0 a$) is shown in Fig. 2. Different half-wave numbers (m) for the nonlocal and classical Kirchhoff plate theories are considered.

It can be observed that the critical temperatures for the nonlocal and classical Kirchhoff plate models are almost the same when the scale coefficient is zero. However, the difference between the two models becomes obvious and the critical temperature ratio tends to be smaller with the scale coefficient increasing. Similar phenomena can be found for the vibration and buckling properties of nanotubes and nanoplates with the

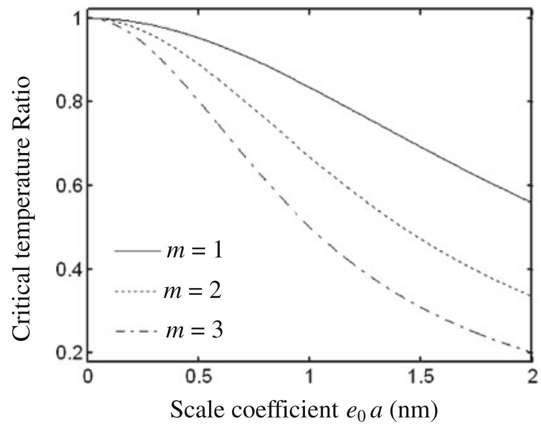

Fig. 2 Relation between the critical temperature ratio ($R_{\mathrm{cr}}^{\mathrm{K}}$) and the scale coefficient (e_0a) with different half-wave numbers (m) for the nonlocal and classical Kirchhoff plate theories

nonlocal continuum theory [19,20,34–37]. For a fixed scale coefficient, the larger the half-wave number is, the smaller the critical temperature ratio becomes. It means that the classical model will provide a higher estimation for thermal buckling properties of the nanoplate. The nonlocal continuum theory can predict a reliable result especially for larger half-wave numbers.

The relation between the critical temperature ratio ($R_{\mathrm{cr}}^{\mathrm{K}}$) and the width ratio is shown in Figs. 3a–c. It can be seen that for different half-wave numbers and scale coefficients, all of the critical temperature ratios tend to increase as the width ratio (i.e. l_a to l_b) becomes larger. Then, the results with the same scale coefficient tend to be an asymptotic value for different half-wave numbers. Moreover, we can observe that the asymptotic values for critical temperature ratios with $e_0a = 0.5\,\mathrm{nm}$, $e_0a = 1\,\mathrm{nm}$, and $e_0a = 2\,\mathrm{nm}$ are different.

4 Nonlocal Mindlin plate model

In this section, the Mindlin plate model is applied, the effects of transverse shear and rotary inertia are considered. The bending moments and shear forces are

$$M_x = \int_{-h/2}^{h/2} \sigma_x z\mathrm{d}z, \quad M_y = \int_{-h/2}^{h/2} \sigma_y z\mathrm{d}z, \quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\mathrm{d}z,$$
$$S_x = \int_{-h/2}^{h/2} \tau_{xz}\mathrm{d}z, \quad S_y = \int_{-h/2}^{h/2} \tau_{yz}\mathrm{d}z. \tag{10}$$

For the nonlocal Mindlin plate theory, the governing equations with the external load can be expressed in the following form [38–41]:

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + N_T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = I_0\frac{\partial^2 w}{\partial t^2}, \tag{11.1}$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - S_x = I_2\frac{\partial^2 \psi_x}{\partial t^2}, \tag{11.2}$$
$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - S_y = I_2\frac{\partial^2 \psi_y}{\partial t^2}, \tag{11.3}$$

where $I_0 = \int_{-h/2}^{h/2} \rho\mathrm{d}z = \rho h$ and $I_2 = \int_{-h/2}^{h/2} \rho h^2\mathrm{d}z = \rho h^3/12$ are the mass moments of inertia.

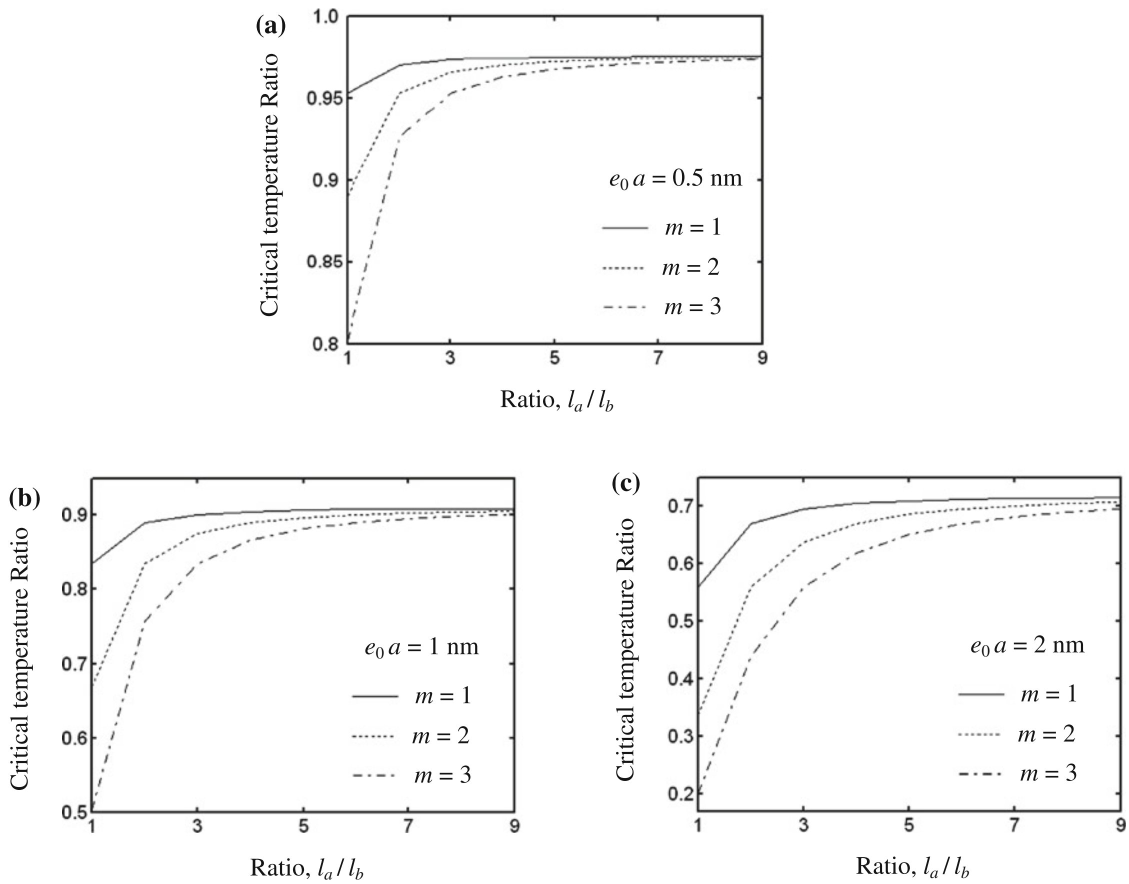

Fig. 3 Relation between the critical temperature ratio ($R_{\mathrm{cr}}^{\mathrm{K}}$) and the width ratio with different half-wave numbers (m) for the nonlocal and classical Kirchhoff plate theories. **a** $e_0 a = 0.5\,\mathrm{nm}$, **b** $e_0 a = 1\,\mathrm{nm}$, and **c** $e_0 a = 2\,\mathrm{nm}$

Based on Eqs. (2), (10), and (11), the following relations can be obtained:

$$\kappa G h\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial \psi_x}{\partial x}+\frac{\partial \psi_y}{\partial y}\right)+N_T[1-(e_0 a)^2\nabla^2]\nabla^2 w=0, \tag{12.1}$$

$$D\left[\frac{\partial^2 \psi_x}{\partial x^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2 \psi_x}{\partial y^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2 \psi_y}{\partial x \partial y}\right]-\kappa G h\left(\frac{\partial w}{\partial x}+\psi_x\right)=0, \tag{12.2}$$

$$D\left[\frac{\partial^2 \psi_y}{\partial y^2}+\frac{1}{2}(1-\upsilon)\frac{\partial^2 \psi_y}{\partial x^2}+\frac{1}{2}(1+\upsilon)\frac{\partial^2 \psi_x}{\partial x \partial y}\right]-\kappa G h\left(\frac{\partial w}{\partial y}+\psi_y\right)=0, \tag{12.3}$$

where G is the shear modulus, κ the shear correction factor, and ψ_x and ψ_y the local rotations for the x and y directions, respectively.

For the simply supported case, the boundary condition can be expressed as [39,40]

$$w=0, \quad \psi_y=0, \quad M_x=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a, \tag{13.1}$$
$$w=0, \quad \psi_x=0, \quad M_y=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b, \tag{13.2}$$

Then, the solution of Eqs. (12.1)–(12.3) can be expressed as

$$w = W \sin \alpha_m x \sin \beta_n y, \tag{14.1}$$
$$\psi_x = \Psi_x \cos \alpha_m x \sin \beta_n y, \tag{14.2}$$
$$\psi_y = \Psi_y \sin \alpha_m x \cos \beta_n y. \tag{14.3}$$

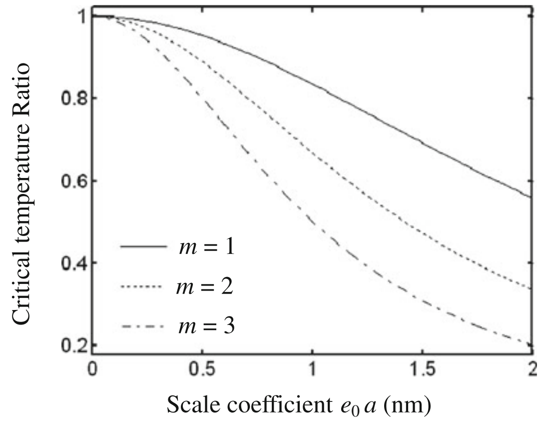

Fig. 4 Relation between the critical temperature ratio (R_{cr}^{M}) and the scale coefficient ($e_0 a$) with different half-wave numbers (m) for the nonlocal and classical Mindlin plate theories

Substituting Eqs. (13.1)–(13.3) into Eqs. (12.1)–(12.3), we can derive the following relations:

$$\{\kappa Gh + N_T[1 + (e_0 a)^2(\alpha_m^2 + \beta_n^2)]\}\,(\alpha_m^2 + \beta_n^2)W + \kappa Gh\alpha_m \Psi_x + \kappa Gh\beta_n \Psi_y = 0, \tag{15.1}$$

$$\kappa Gh\alpha_m W + \left[D\alpha_m^2 + \frac{1}{2}D(1-\upsilon)\beta_n^2 + \kappa Gh\right]\Psi_x + \frac{1}{2}D(1+\upsilon)\alpha_m\beta_n\Psi_y = 0, \tag{15.2}$$

$$\kappa Gh\beta_n W + \frac{1}{2}D(1+\upsilon)\alpha_m\beta_n\Psi_x + \left\{\kappa Gh + D\beta_n^2 + \frac{1}{2}D(1-\upsilon)\alpha_m^2\right\}\Psi_y = 0. \tag{15.3}$$

Then, for the existence of the nonzero solution, we can derive

$$\begin{vmatrix} (S_1 + N_T S_2)(\alpha_m^2 + \beta_n^2) & S_1\alpha_m & S_1\beta_n \\ S_1\alpha_m & S_1 + S_3 & S_5 \\ S_1\beta_n & S_5 & S_1 + S_4 \end{vmatrix} = 0, \tag{16}$$

where

$$S_1 = \kappa Gh, \tag{17.1}$$

$$S_2 = 1 + (e_0 a)^2(\alpha_m^2 + \beta_n^2), \tag{17.2}$$

$$S_3 = D\left[\alpha_m^2 + \frac{1}{2}(1-\upsilon)\beta_n^2\right], \tag{17.3}$$

$$S_4 = D\left[\beta_n^2 + \frac{1}{2}(1-\upsilon)\alpha_m^2\right], \tag{17.4}$$

$$S_5 = \frac{1}{2}D(1+\upsilon)\alpha_m\beta_n. \tag{17.5}$$

As a result, the critical temperature for the nonlocal Mindlin plate theory can be obtained from Eq. (17) and expressed as

$$T_{\text{cr\_Non}}^{M} = \frac{1-\upsilon}{E\alpha h}\left\{\frac{2S_1^2 S_5\alpha_m\beta_n - S_1^2[(S_1+S_3)\beta_n^2 + (S_1+S_4)\alpha_m^2]}{(\alpha_m^2+\beta_n^2)S_2[(S_1+S_3)(S_1+S_4) - S_5^2]} + \frac{S_1}{S_2}\right\}. \tag{18}$$

Similar to Sect. 3, the critical temperature ratio of the nonlocal to classical Mindlin plate models is defined in the following form:

$$R_{\text{cr}}^{\text{M}} = \frac{T_{\text{cr\_Non}}^{\text{M}}}{T_{\text{cr\_Cla}}^{\text{M}}}, \tag{19}$$

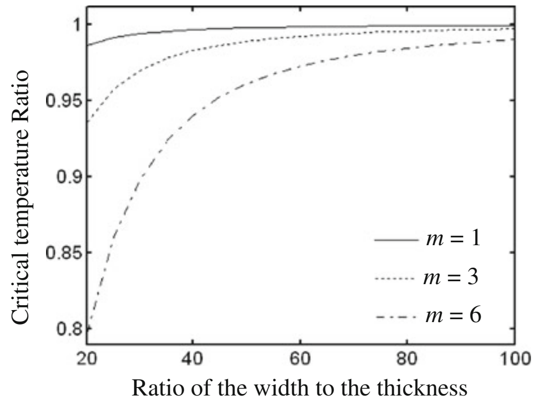

Fig. 5 Relation between the critical temperature ratio (T_{cr}^{M}/T_{cr}^{K}) and the ratio of the width to the thickness (l_b/h) with different half-wave numbers (m) for $l_a = l_b$ and $e_0a = 1\,\text{nm}$

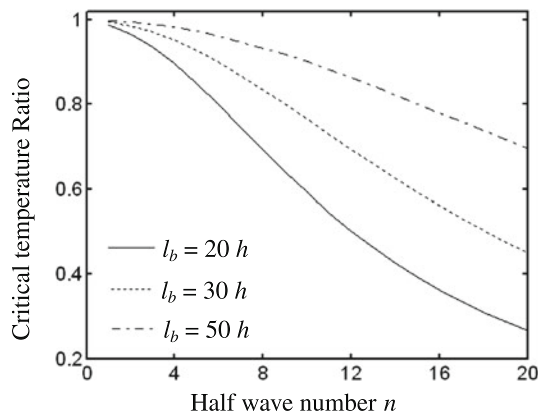

Fig. 6 Relation between the critical temperature ratio (T_{cr}^{M}/T_{cr}^{K}) and the half-wave number n with different ratios of the width to the thickness for $l_a = l_b$

where $T_{cr\_Cla}^{M}$ is the critical temperature for the classical Mindlin plate model.

The critical temperature ratio (R_{cr}^{M}) for the Mindlin plate theory is presented in Fig. 4. It can be seen that the thermal buckling behaviors with different scale coefficients and half-wave numbers are similar to those for the Kirchhoff plate theory which are illustrated in Fig. 2. Further consideration shows that the relation between the critical temperature ratio (R_{cr}^{M}) and the width ratio for the Kirchhoff plate theory in Figs. 3a–c can also be observed for this case. It means that the small-scale effects on the thermal buckling properties are similar for both the nonlocal Kirchhoff and Mindlin plate theories.

In order to discuss the difference between the nonlocal Mindlin and Kirchhoff plate models, the influence of the ratio of the width to the thickness on the critical temperature ratio is investigated. The results of T_{cr}^{M}/T_{cr}^{K} are presented in Fig. 5 with different half-wave numbers (m). The plate widths $l_a = l_b$ and the scale coefficient $e_0a = 1\,\text{nm}$. It can be seen that the difference between the two theories is not obvious for small half-wave numbers and large ratios of the width to the thickness. It can be concluded that the nonlocal Mindlin plate theory is more appropriate for the thick nanoplate and the nonlocal Kirchhoff plate theory is valid for the thin structure.

In this part, the effects of the half-wave numbers on the critical temperature ratio (T_{cr}^{M}/T_{cr}^{K}) by the nonlocal Mindlin and Kirchhoff plate models are discussed. The results are shown in Fig. 6 with different plate widths, $l_a = l_b = 20$, 30 and $50\,h$ are considered, respectively. We can see that all of the ratios with different widths become smaller with the half-wave number increasing. Moreover, such phenomenon is more obvious for a thick nanoplate.

5 Conclusions

In this paper, the thermal buckling properties of a nanoplate are investigated by the nonlocal continuum theory. The expressions of the critical temperature for the nonlocal Kirchhoff and Mindlin plate theories are derived, respectively. The critical temperature ratios for different nonlocal models are presented, and the small-scale effects are discussed. From the results, it can be concluded that the small-scale effects are obvious for larger half-wave numbers. The thermal buckling behaviors for the nonlocal Kirchhoff and Mindlin plate theories are similar. The nonlocal Mindlin plate theory is more appreciate for a thick nanoplate. Furthermore, the difference between the nonlocal Kirchhoff and Mindlin plate theories becomes more obvious for large half-wave numbers.

It is hoped that this work can present an effective model to analyze the mechanical characteristics of nanoscale devices.

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