Acta Mech 224, 479–498 (2013)
DOI 10.1007/s00707-012-0768-z

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An explicit, direct approach to obtain multi-axial elastic potentials which accurately match data of four benchmark tests for rubbery materials—Part 2: general deformations

Received: 21 July 2012 / Revised: 12 October 2012 / Published online: 17 November 2012

Abstract In a previous study (Xiao in Acta Mechanica 223:2039–2063, 2012), an explicit, straightforward approach has been proposed to obtain multi-axial elastic potentials for incompressible rubberlike materials. With a new idea of treating compressibility behavior, we extend this explicit, straightforward approach for incompressible deformations to a general case of finite compressible deformations. From data of a uniaxial test and a simple shear test, we obtain unified forms of multi-axial compressible elastic potentials which accurately match data of four benchmark tests. Reduced results are presented for the slight compressibility case. In particular, we apply the new approach with a simple form of rational interpolating function with two poles and from the uniaxial case derive a simple form of multi-axial compressible potential with a strain limit. It is found that this strain limit is a counterpart of the well-known von Mises limit for stress in elastoplasticity. For highly elastic materials with strain stiffening effects, this simple compressible potential is shown to be in good accord with data of four tests from small to large deformations.

1 Introduction

In Part 1 of this serial study (cf. [70]), an explicit, direct approach has been proposed to obtain multi-axial elastic potentials which exactly match data of four benchmark tests for highly elastic solids such as rubberlike materials. The results therein are for incompressible deformations. Accordingly, a hyper-elastic stress–strain relation subjected to the constraint of incompressibility should be introduced, which incorporates an indeterminate hydrostatic stress part. In numerical implementations, particular procedures should be introduced to deal with the incompressibility condition for the purpose of bypassing possible related issues. However, such issues will not be involved without assuming the incompressibility condition and then with a stress–strain relation based on a compressible elastic potential. This consideration in a broad sense applies to realistic material behavior. Indeed, the incompressibility is not the reality but merely an idealization, albeit volumetric deformations of rubberlike materials are actually very small, as shown in experimental studies earlier in, for example, [15,22,31,34,37,61,67,68] and recently in, for example, [3,18], and many others. Extensive data may be found in the monograph by Treloar [66]. Constitutive models for rubber elasticity with compressibility behavior were investigated earlier in, for example, [1,13,19–21,27,50,54–56] and recently in [5,6,11,12,23,25,29,30,42–46,49,51–53,62,64,65] and many others. Summaries/reviews of results may be found in [7,28] and [57–60].

The statistical and phenomenological approaches for obtaining elastic potentials (strain-energy or stored-energy functions) have been discussed in Part 1, and related representative references may be found therein. Here we relax the constraint of incompressibility and treat compressible deformations in a broad sense. Toward

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this goal, we shall extend and develop the main ideas and procedures suggested in Part 1, in conjunction with new ideas of treating compressibility behavior. The main content is as follows. In Sect. 2, following Beatty and Stalnaker [10] we introduce the Poisson function in the case of uniaxial extension/compression, which will play an essential role in characterizing the compressibility behavior in the subsequent development. In Sect. 3, with Hencky strain we introduce bridging and matching invariants for a general compressible case and study their properties for future use. In Sect. 4, we first introduce a multi-axial bridging procedure based on the bridging invariants given in Sect. 3 and then extend two one-dimensional potentials derived separately from the uniaxial case and the shear case to obtain two multi-axial potentials; after that we further introduce a multi-axial matching procedure based on the matching invariants given in Sect. 3 to eventually obtain a unified form of multi-axial potential and then derive a unified form of multi-axial stress–strain relation. In Sect. 5, we apply the explicit approach proposed with a simple form of rational interpolating function with two poles for the uniaxial case and obtain a simple form of multi-axial potential with strain limit. We then show that this limit is a counterpart of the well-known von Mises limit for stress in elastoplasticity. Finally, we derive the predictions of the obtained model for the uniaxial, biaxial, plane-strain compression and simple shear cases and compare the model predictions with test data. In Sect. 6 we present discussions and remarks concerning a few related respects.

At the end of this introduction, we explain some relevant notations and facts concerning finite deformation kinematics and hyper-elastic formulation. An account in a broad sense may be found in Haupt [35]. Let $(\lambda_1,\ \lambda_2,\ \lambda_3)$ and $(\boldsymbol{n}_1,\ \boldsymbol{n}_2, \boldsymbol{n}_3)$ be the three principal stretches and three corresponding orthonormal principal axis vectors (Eulerian triad). Hencky strain of Eulerian type is of the form

$$\boldsymbol{h} = \sum_{r=1}^{3} (\ln \lambda_r) \boldsymbol{n}_r \otimes \boldsymbol{n}_r .$$

The first, second, and third Hencky invariants are given by

$$i_r = \mathrm{tr}\boldsymbol{h}^r = \lambda_1^r + \lambda_2^r + \lambda_3^r, \quad r = 1, 2, 3,$$

and the second and third invariants of the deviatoric Hencky strain $\tilde{\boldsymbol{h}}$ by

$$j_2 = \mathrm{tr}\tilde{\boldsymbol{h}}^2 = i_2 - \frac{1}{3} i_1^2, \quad j_3 = \mathrm{tr}\tilde{\boldsymbol{h}}^3,$$

with

$$\tilde{\boldsymbol{h}} = \boldsymbol{h} - \frac{i_1}{3} \boldsymbol{I}.$$

Throughout, $\mathrm{tr}\boldsymbol{T}$ is used to represent the trace of tensor $\boldsymbol{T}$, and $\boldsymbol{I}$ is the second-order identity tensor. Details concerning Hencky strain and its computation may be found in [70].

Let $\boldsymbol{\sigma}$ be the Cauchy stress and $\boldsymbol{\tau}$ the Kirchhoff stress (weighted Cauchy stress), that is,

$$\boldsymbol{\tau} = J \boldsymbol{\sigma},$$

with J the volumetric ratio. For a general case of compressible, isotropic hyper-elastic materials, there exists an isotropic potential

$$W = W(\boldsymbol{h}),$$

in terms of Hencky strain $\boldsymbol{h}$, such that the Kirchhoff stress is directly derivable from this potential, namely (see Hill [38,39], Fetzjerald [26], and also Xiao and Chen [79])

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{h}}. \tag{1}$$

As in [70], Hencky's logarithmic strain [36] with its invariants will play a basic role. Hill [38,39] demonstrated the inherent advantages of Hencky strain and gave prominence to Hencky strain in a general study of constitutive inequalities (see also [40] and [63]). It is widely used in finite deformation problems. Its far-reaching role in the consistent formulation of Eulerian finite elastoplasticity has further been disclosed most recently in, for example, [16,17] and [69–79] (see also [69] and [77] for review). Recent applications of Hencky strain to finite elasticity are given in [1,2,4,16,17,23,24,26,44,76,79] and many others.

As indicated in [70], here Hencky strain stands out not only with its remarkable efficacy in constitutive modeling with sharp accuracy, as evidenced in the studies by Anand [1,2], Criscione et al. [23], Diani and Gillormini [24], Horgan and Murphy [44], and others, but perhaps more essentially with the fact that Hencky strain will supply desirable invariants that will prove crucial to obtaining unified forms of multi-axial elastic potentials via the multi-axial bridging/matching procedures, as will be seen in Sect. 4.

2 Poisson function and compressibility

In this section, with Hencky strain we study the uniaxial extension/compression of a compressible elastic sample, with an intention to capture the essential feature of compressibility behavior.

Let $\boldsymbol{e}$ be a unit vector in the axial direction, and let λ and ρ be the axial stretch and the lateral stretch, respectively. The Hencky strain $\boldsymbol{h}$ and the Cauchy stress $\boldsymbol{\sigma}$ for the uniaxial case are of the form

$$\boldsymbol{h} = (\ln\lambda)\boldsymbol{e}\otimes\boldsymbol{e} + \ln\rho(\boldsymbol{I} - \boldsymbol{e}\otimes\boldsymbol{e}), \tag{2}$$

$$\boldsymbol{\sigma} = \sigma\boldsymbol{e}\otimes\boldsymbol{e}. \tag{3}$$

In a uniaxial test, usually the nominal axial stress, denoted P, is given as a function of the axial stretch λ, that is,

$$P = P(\lambda). \tag{4}$$

The following relation holds true:

$$\tau \equiv J\sigma = \lambda P(\lambda) \quad (\equiv f(\ln\lambda)). \tag{5}$$

In the above, τ is the axial component of the Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ with J the volumetric ratio. Note that the last equality simply indicates that the axial Kirchhoff stress τ is given above as a function of the axial Hencky strain $\ln\lambda$ through the nominal axial stress as a function of the axial stretch, namely $P(\lambda)$.

Significant information of the material characteristics, here rubber elasticity, is embraced in the relation $P = P(\lambda)$ and hence the relation $\tau = f(\ln\lambda)$. Usually, data set for the pair $(\lambda,\ P)$ (and hence for the pair $(\ln\lambda,\ \tau)$) will be supplied. Such data set is sufficient for the characterization of incompressible deformation behavior. However, to characterize compressible deformation behavior, additional information will be necessary. This has been shown in a detailed study by Beatty and Stalnaker [10] with a quantity named Poisson function. This quantity establishes the relation between the axial and the lateral stretch. Inspired by Beatty and Stalnaker's idea, here we introduce the following relation:

$$\ln\rho = -\nu\ln\lambda. \tag{6}$$

In the above, the quantity ν is the ratio of the lateral to the axial Hencky strain. It is just the classical Poisson ratio in the infinitesimal strain case. Generally, the ratio ν may rely on the axial stretch. As a result, in a general case it is referred to as Poisson function. Its significance lies in the fact that essentially it represents the compressibility property of a rubberlike material, as will be seen in the subsequent development. Toward our goal, we regard the ratio ν to be a function of the first Hencky invariant i_1, namely

$$\nu = \alpha(i_1) = \alpha(\ln J). \tag{7}$$

This establishes the relationship between the ratio ν and the volumetric ratio J. For the case of incompressibility, the volumetric ratio is constant, that is, $J \equiv 1$, and accordingly, the Poisson function is also constant and given by 0.5. Namely,

$$\nu = \frac{1}{2} \quad \text{for} \quad J \equiv 1. \tag{8}$$

Both the data for axial stress (cf. Eq. (3) or (4)) and the data for the Poisson function (cf. Eq. (6)) will be of significance for the purpose of characterizing compressible material behavior. From experimental facts, as will be indicated later on, the ratio ν changes with i_1 very slowly, and therefore, its derivative will be treated to be zero in the subsequent study, namely

$$\nu' = \frac{\mathrm{d}\nu}{\mathrm{d}i_1} \approx 0. \tag{9}$$

Other forms of the Poisson function may be introduced by different relations between the lateral and axial stretches ρ and λ. Here, the use of Hencky strain will prove advantageous in treating rubberlike materials with slight compressibility.

Experimental facts and properties of the Poisson function can be found in Beatty and Stalnaker [10].

3 Bridging invariants and matching invariants

As in [70], the main procedures of the explicit, straightforward approach to be proposed will be carried out by means of certain Hencky invariants of desirable properties. Two groups of Hencky invariants will be introduced here. Of them, one group is composed of the following three Hencky invariants:

$$q = \frac{i_1}{1-2\nu}, \tag{10}$$

$$\varphi = \frac{1.5}{1+\nu}\psi = \frac{\sqrt{1.5 j_2}}{1+\nu}, \tag{11}$$

$$\beta = 2\sinh\left(\sqrt{0.5 j_2}\right), \tag{12}$$

and the other group of the following two:

$$\gamma = \sqrt{6}\frac{j_3}{\sqrt{j_2^3}}, \tag{13}$$

$$\chi = \gamma^2 = 6\frac{j_3^2}{j_2^3}. \tag{14}$$

In the above, ν is the Poisson function introduced in the last section, and j_2 and j_3 are the second and third invariants of the deviatoric Hencky strain $\tilde{\boldsymbol{h}}$, as given at the end of the introduction. Moreover, the invariant ψ is given by

$$\psi = \sqrt{\frac{2}{3}j_2}, \tag{15}$$

which has been introduced in Part 1 for the incompressibility case.

The two groups of Hencky invariants as given above will be referred to as *bridging invariants* and *matching invariants* for compressible deformations, respectively. Their meanings and roles will be explained in the next section.

The three invariants β, γ, and χ are identical to their respective counterparts in Part 1. For the case of incompressibility, the following hold:

$$i_1 \equiv 0, \quad \nu = \frac{1}{2}.$$

In this case the invariant φ given here reduces to the invariant ψ, that is, the incompressibility counterpart as given in [70]. Besides, the invariant q is here introduced exclusively for the purpose of capturing the feature of compressibility.

The matching invariants are of the following properties:

$$\begin{cases} -1 \le \gamma \le 1, \\ 0 \le \chi \le 1. \end{cases} \tag{16}$$

Refer to Part 1 for details.

The gradients of the above invariants are given as follows:

$$\frac{\partial q}{\partial \boldsymbol{h}} = \frac{1}{1-2\nu}\boldsymbol{I}, \tag{17}$$

$$\frac{\partial \varphi}{\partial \boldsymbol{h}} = \frac{1}{1+\nu}\frac{1}{\psi}\tilde{\boldsymbol{h}}, \tag{18}$$

$$\frac{\partial \beta}{\partial \boldsymbol{h}} = \frac{1}{3}\sqrt{12+3\beta^2}\,\frac{1}{\psi}\tilde{\boldsymbol{h}} \tag{19}$$

for the bridging invariants φ and β, and

$$\frac{\partial \gamma}{\partial \boldsymbol{h}} = \frac{2}{\psi}\breve{\boldsymbol{h}}, \tag{20}$$

$$\frac{\partial \chi}{\partial \boldsymbol{h}} = 4\frac{\gamma}{\psi}\breve{\boldsymbol{h}} \tag{21}$$

for the matching invariants γ and χ. As in Part 1, here and henceforth we use $\breve{\boldsymbol{h}}$ to designate the following dimensionless tensor:

$$\breve{\boldsymbol{h}} = \frac{1}{j_2^2}\left(3j_2\tilde{\boldsymbol{h}}^2 - 3j_3\tilde{\boldsymbol{h}} - j_2^2\boldsymbol{I}\right), \tag{22}$$

or, alternatively,

$$\breve{\boldsymbol{h}} = 2\left(\frac{\tilde{\boldsymbol{h}}}{\psi}\right)^2 - \gamma\frac{\tilde{\boldsymbol{h}}}{\psi} - \boldsymbol{I}. \tag{23}$$

As in Part 1, this tensor will play a significant role in the ensuing development. Since the magnitude of the tensor $\tilde{\boldsymbol{h}}/\psi$ is given by $\sqrt{1.5}$, the tensor $\breve{\boldsymbol{h}}$ is of finite magnitude for nonvanishing $\psi = 0$, that is, for $\sqrt{j_2} \neq 0$.

A useful property of the tensor $\breve{\boldsymbol{h}}$ is as follows:

$$\breve{\boldsymbol{h}} = \boldsymbol{O} \quad \text{for repeated principal stretches.} \tag{24}$$

The following fact will also be useful: The three tensors $(\boldsymbol{I}, \tilde{\boldsymbol{h}}, \breve{\boldsymbol{h}})$ are three mutually orthogonal tensor generators, and therefore, a complete, orthogonal representation for isotropic stress–strain relations may be given in terms of these three generators. Details may be found in [70].

4 Multi-axial potentials via bridging and matching procedures

In this section, first we shall derive two one-dimensional potentials from test data of the uniaxial and shear cases by means of an interpolating procedure for single-variable functions, and then from these two we obtain two multi-axial potentials using a multi-axial bridging procedure and, eventually, obtain a unified form of multi-axial potential using a multi-axial matching procedure.

4.1 One-dimensional potentials from uniaxial and shear tests

This will be done separately for the two cases at issue.

(a) Uniaxial case

The deformation and the stress of the uniaxial extension/compression is described in Sect. 2.4(a) in [70]. Usually, a data set for the pair (P, λ) is provided in the uniaxial test. Here P is the nominal axial stress, and λ is the axial stretch. As in Part 1, let the data set $(P_\alpha, \lambda_\alpha)$, $\alpha = 0, 1, \ldots, N$, supply $N+1$ values of the nominal stress P corresponding with $N+1$ axial stretches $\lambda_\alpha, \alpha = 0, 1, \ldots, N$. We convert such a data set to a data set $(\tau_\alpha,\ h_\alpha)$ for the axial Kirchhoff stress $\tau = J\sigma$ and the axial Hencky strain $h = \ln\lambda$. This can be done using the following relations:

$$\begin{cases} \tau_\alpha = \lambda_\alpha P_\alpha, \\ h_\alpha = \ln\lambda_\alpha. \end{cases} \tag{25}$$

First we intend to find out a one-dimensional relation between the axial Kirchhoff stress τ and the axial Hencky strain h, that is,

$$\tau = f(h),$$

which exactly matches the data set $(\tau_\alpha,\ h_\alpha),\ \alpha = 0, 1, \ldots, N$. This can be done by choosing a usual interpolating function in a certain class of functions. To this end we may take three classes of functions into consideration, namely polynomial functions, cubic spline functions, and rational functions, as elaborated upon in [70]. Here we record the results below. Details may be found in [70].

(a1) Polynomial interpolating function:

In this case, the one-dimensional stress–strain relation is of the following form:

$$\tau = f(h) = \sum_{\alpha=0}^{N} \sigma_\alpha L_\alpha(h), \tag{26}$$

where the $(N+1)$ base functions are of the form

$$L_\alpha(h) = \prod_{t=0(\neq\alpha)}^{N} \frac{h - h_t}{h_\alpha - h_t}, \quad \alpha = 0, 1, \ldots, N. \tag{27}$$

(a2) Cubic spline functions

In this case, the one-dimensional stress–strain relation is piecewise given by

$$\begin{cases} \tau = f(h) = a_r(h-h_r)^3 + b_r(h-h_r)^2 + c_r(h-h_r) + \sigma_r, \quad h_r \le h \le h_{r+1}, \\ r = 0, 1, \ldots, N-1, \end{cases} \tag{28}$$

where

$$\begin{cases} a_r = \frac{1}{6}\frac{\pi_{r+1}-\pi_r}{h_{r+1}-h_r}, \\ b_r = \frac{1}{2}\pi_r, \\ c_r = \frac{\pi_{r+1}-\pi_r}{h_{r+1}-h_r} - \frac{1}{6}(h_{r+1}-h_r)(\pi_{r+1}+\pi_r), \\ r = 0, 1, \ldots, N-1. \end{cases} \tag{29}$$

In the above, the $(N+1)$ parameters $\pi_0, \pi_1, \ldots, \pi_N$ will be determined by the continuity conditions up to twice differentiation at the $(N-1)$ inner points as well as two conditions at the two endpoints. Different kinds of conditions may be prescribed at the two endpoints, which result in different types of cubic spline functions, for instance, the vanishing curvature, parabolic runout, cubic runout conditions at the two endpoints, etc. Here we take the first as an illustrative example. Thus, we obtain a natural cubic spline function with the $N+1$ parameters $\pi_0,\ \pi_1, \ldots, \pi_N$ given by

$$\pi_0 = \pi_N = 0 \tag{30}$$

and

$$\begin{cases} \pi_N = 0, \\ \pi_{i-1} = p_i\pi_i + q_i \quad \text{for} \quad i = N, \ldots, 2, \end{cases} \tag{31}$$

where

$$\begin{cases} p_1 = 0, \\ p_i = \frac{-1}{4+p_{i-1}}, \end{cases} \tag{32}$$

$$\begin{cases} q_1 = 0, \\ q_i = \frac{\sigma_{i+1}-2\sigma_i+\sigma_{i-1}-q_{i-1}}{4+p_{i-1}}, \end{cases} \tag{33}$$

for $i = 2, \ldots, N$.

(a3) Rational interpolating functions

In this case, in general the stress–strain relation is given by a rational function of the form:

$$\sigma = f(h) = \frac{Q_m(h)}{Q_n(h)}, \tag{34}$$

where the nominator $Q_m(h)$ and the denominator $Q_n(h)$ are polynomial functions of degree m and degree n, respectively. For given positive integers m and n of interest, it is to find out two such polynomials $Q_m(h)$ and $Q_n(h)$ that the rational function $Q_m(h)/Q_n(h)$ exactly matches or best approximates the test data. The typical features of rational functions are the incorporation of certain limiting points known as poles. Whenever the variable, here the axial Hencky strain h, is approaching a pole, the value of the function grows indefinitely. Rational functions with any given number of poles may be constructed. The simplest case of rational functions with two poles h_e and $-h_c$ at extension and compression is as follows:

$$\tau = f(h) = \frac{Eh}{\left(1 - \frac{h}{h_e}\right)\left(1 + \frac{h}{h_c}\right)}. \tag{35}$$

Further discussion will be made in Sect. 6.

With a one-dimensional stress–strain relation as given above, it is then ready to arrive at a one-dimensional potential. As in Part 1, this will be done simply by integrating the stress power.

For compressible deformations, the stress power per unit reference volume is given by

$$\dot{W} = \boldsymbol{\tau} : \boldsymbol{D}. \tag{36}$$

For the case of uniaxial loading, this yields

$$\dot{W} = \tau\dot{h}, \quad h = \ln\lambda.$$

By integrating this, we arrive at a one-dimensional potential for the uniaxial case as follows:

$$w_1 = \int_0^h \tau \, \mathrm{d}h = \int_0^h f(h)\mathrm{d}h \quad (\equiv w_1(h)). \tag{37}$$

It should be noted that, for the uniaxial case, a complete one-dimensional potential $w_1 = w_1(h)$ should be given for both positive and negative values of the axial Hencky strain h. This means that the test data should be available for both extension and compression.

(b) Simple shear case

The deformation and the stress for the simple shear case may be found in Sect. 2.4(d) in [70]. In this case, a data set $(\tau_0, \omega_0), (\tau_1, \omega_1), \ldots, (\tau_N, \omega_N)$ is supplied for the shear stress τ and the shear amount ω. Here we set

$$\omega_0 = 0, \quad \tau_0 = 0;$$
$$0 < \omega_1 < \omega_2 < \cdots < \omega_N.$$

The former corresponds with an undeformed natural state.

Following the procedures as elaborated upon for the uniaxial case, via an interpolating function, we obtain a relation between the shear amount ω and the shear stress τ:

$$\tau = g(\omega). \tag{38}$$

The function $g(\omega)$ is given simply by replacing (τ, h) with (τ, ω) and then replacing f with g in the case of uniaxial loading.

Now the stress power is given by

$$\dot{W} = \boldsymbol{\tau} : \boldsymbol{D} = \tau\dot{\omega},$$

for the simple shear case. Hence, by integrating the above expression, we obtain a one-dimensional potential as follows:

$$w_0 = \int_0^{\omega} g(\omega)\mathrm{d}\omega \quad (\equiv w_0(\omega)). \tag{39}$$

One-dimensional potentials may be obtained from other approaches. Details in this respect may be found in Sect. 4.3 in [70].

4.2 Multi-axial potentials via bridging procedure

The two one-dimensional potentials in the last section are derived exclusively for the uniaxial case and the simple shear case, separately. As has been done in Part 1, we shall obtain two multi-axial potentials by means of a multi-axial bridging procedure based on the Hencky invariants $q,\ \varphi,\ \beta$, and γ as introduced by Eqs. (10)–(13) in Sect. 3. The first three are referred to as bridging invariants because they establish bridging relationships between the uniaxial and shear cases and multi-axial cases. This may be seen from their following properties:

$$q = \ln\lambda, \tag{40}$$

$$\varphi = \begin{cases} +\ln\lambda & \text{for} \quad \lambda \geq 1, \\ -\ln\lambda & \text{for} \quad \lambda \leq 1, \end{cases} \tag{41}$$

for the uniaxial loading case (cf. Eqs. (2) and (6)), and

$$\begin{cases} \psi = \frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{\omega}{2}\right), \\ \beta = \omega, \end{cases} \tag{42}$$

for the simple shear case. Note that for the simple shear case the Hencky strain is of the form (cf. Eq. (53) in [70]):

$$\boldsymbol{h} = \tilde{\boldsymbol{h}} = \frac{\sinh^{-1}\left(\frac{\omega}{2}\right)}{\sqrt{1+\left(\frac{\omega}{2}\right)^2}}\left(\bar{\boldsymbol{e}}\otimes\bar{\boldsymbol{a}} + \bar{\boldsymbol{a}}\otimes\bar{\boldsymbol{e}} + \frac{\omega}{2}(\bar{\boldsymbol{e}}\otimes\bar{\boldsymbol{e}} - \bar{\boldsymbol{a}}\otimes\bar{\boldsymbol{a}})\right). \tag{43}$$

From Eqs. (41)–(42) it may be clear that the Hencky invariants φ and $-\varphi$ yield the axial Hencky strain for uniaxial extension and compression, respectively, while the Hencky invariant β produces the shear amount for simple shear. The relationships between the uniaxial and shear cases and multi-axial cases are thus established via the Hencky invariants $q,\ \varphi$, and β.

Now it is opportune to extend the two one-dimensional potentials $w_0(\omega)$ and $w_1(h)$ to the multi-axial case. At first, the multi-axial extension of the one-dimensional potential $w_0(\omega)$ for the simple shear is straightforward and leads to the following multi-axial potential:

$$W_0 = w_0(\beta) = w_0\left(2\sinh\left(\frac{\sqrt{3}}{2}\psi\right)\right). \tag{44}$$

However, it may not be so directly possible to achieve a multi-axial extension of the one-dimensional potential $w_1(h)$, since here two cases for extension and compression need to be distinguished and, furthermore, the volumetric deformation should be taken into account. The Hencky invariants $q,\ \varphi,\ \gamma$ given by Eqs. (10)–(11) and (13) are introduced just for this purpose. In fact, with Eq. (2) we infer that the invariant γ is of the following property:

$$\gamma = \begin{cases} +1 & \text{for} \quad \lambda \geq 1, \\ -1 & \text{for} \quad \lambda \leq 1, \end{cases} \tag{45}$$

for the uniaxial case. As such, the invariant γ plays a role in matching uniaxial tension and compression.

On the other hand, a bridging invariant characterizing the volumetric deformation should come into play in a general case of compressible deformations. This is just the invariant q (cf. Eq. (10)), which will be used to incorporate into the final unified potential a direct contribution exclusively from the volumetric deformation part, as will be seen in next subsection. Here, with the Hencky invariants $q,\ \psi,\ -\psi$ and γ as well as the Poisson function ν, we obtain a multi-axial extension of the one-dimensional potential for the uniaxial case as follows:

$$\bar{W}_1 = \frac{1-2\nu}{3} w_1(q) + \frac{2}{3}(1+\nu)\left(\frac{1+\gamma}{2} w_1(\varphi) + \frac{1-\gamma}{2} w_1(-\varphi)\right), \tag{46}$$

with the Hencky invariants $q,\ \varphi$ and γ given by Eqs. (10), (11), and (13). From the bridging properties of q and φ and the matching property of γ, as shown in Eqs. (40)–(41) and (45), it follows that the above multi-axial potential $\bar{W}_1$ indeed yields the one-dimensional potential in the uniaxial case. Note that the factor $(1-2\nu)/3$ emerges in the first part above representing a direct contribution exclusively from volumetric deformation. This factor arises from the fact that the hydrostatic stress part in the derived stress–strain relation should be given by $f(q)/3$. Since both $w_1(q)$ and the content in the brackets in the second part above, that is,

$$\frac{1+\gamma}{2} w_1(\varphi) + \frac{1-\gamma}{2} w_1(-\varphi),$$

yield the one-dimensional potential $w_1(h)$ in the uniaxial case, as has been shown in Part 1, it may be clear that the matching factors for $w_1(q)$ and for this content are just the factor $(1-2\nu)/3$ and $1-(1-2\nu)/3$, thus leading to the multi-axial potential $\bar{W}_1$ given by Eq. (46).

4.3 Unified multi-axial potential via matching procedure

Since no correlation need be assumed between the two one-dimensional potentials, the two multi-axial potentials in the last subsection are independent of each other and therefore either of them could not exactly match two data sets for both uniaxial test and shear test, albeit either of them exactly matches one corresponding data set on account of the bridging properties of the four Hencky invariants therein. However, it is possible to obtain a unified form of potential from a simple combination of these two. The idea is to use a multi-axial matching method based on the Hencky invariant χ given by Eq. (14), as will be explained below.

The invariant χ given by Eq. (14) is of the following property (cf. Eq. (2) and (43)):

$$\chi = \begin{cases} 0 & \text{for simple shear,} \\ 1 & \text{for uniaxial case.} \end{cases} \tag{47}$$

It is just owing to the above matching properties that the Hencky invariant χ is referred to as matching invariant. Now with this invariant we combine the two multi-axial potentials W_0 and W_1 given by Eqs. (44) and (46) into the following unified potential:

$$W = \chi \bar{W}_1 + (1-\chi)\left(W_0 + \frac{1-2\nu}{3} w_1(q)\right). \tag{48}$$

In the second part above, the term $(1-2\nu)w_1(q)/3$ is also included as in $\bar{W}_1$. This term vanishes in the simple shear case and hence gives no contribution to the one-dimensional potential in this case. In so doing, we intend to incorporate into the unified potential a direct contribution from the volumetric deformation part as represented by the bridging invariant q and, in the meantime, to simplify the final result. Now we have

$$W = \frac{1-2\nu}{3} w_1(q) + \frac{1+\nu}{3}\chi\left((1+\gamma)w_1(\varphi) + (1-\gamma)w_1(-\varphi)\right) + (1-\chi)w_0(\beta), \tag{49}$$

with the bridging and matching invariants $q,\ \psi,\ \beta,\ \gamma,\ \chi$ given by Eqs. (10)–(14). It is clear that the first part above represents a direct contribution from the volumetric deformation part. The factor $(1-2\nu)/3$ arises from the fact that the derived hydrostatic stress part should be given by $f(q)/3$, as has been indicated before. Another significant role of the bridging invariant q will be indicated shortly at the end of this subsection.

With the expression (49) for the unified potential and with the bridging/matching properties as shown by Eqs. (40)–(42), (45), and (47), the meanings of the bridging invariants ψ and β and the matching invariants γ and χ may become evident. Indeed, thanks to their respective "bridging" and "matching" properties, the unified potential exactly matches data of uniaxial and shear tests.

Now with the unified potential W given by Eq. (49), the multi-axial stress–strain relation is obtainable from Eq. (1) for compressible hyper-elastic materials. We have

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{h}} = \frac{1-2\nu}{3} f(q)\frac{\partial q}{\partial \boldsymbol{h}} + \chi\frac{\partial W_1}{\partial \boldsymbol{h}} + (1-\chi)\frac{\partial W_0}{\partial \boldsymbol{h}} + (W_1 - w_0(\beta))\frac{\partial \chi}{\partial \boldsymbol{h}}, \tag{50}$$

with

$$W_1 = \frac{1+\nu}{3}\left((1+\gamma)w_1(\varphi) + (1-\gamma)w_1(-\varphi)\right). \tag{51}$$

$$\frac{\partial W_0}{\partial \boldsymbol{h}} = g(\beta)\frac{\partial \beta}{\partial \boldsymbol{h}}, \tag{52}$$

$$\frac{\partial W_1}{\partial \boldsymbol{h}} = \frac{1+\nu}{3}\left((1+\gamma)f(\varphi) - (1-\gamma)f(-\varphi)\right)\frac{\partial \varphi}{\partial \boldsymbol{h}} + \frac{1+\nu}{3}\left(w_1(\varphi) - w_1(-\varphi)\right)\frac{\partial \gamma}{\partial \boldsymbol{h}}, \tag{53}$$

and the gradients of the five invariants $q,\ \varphi,\ \beta,\ \gamma$, and χ given by Eqs. (17)–(21). With the above results we arrive at

$$\boldsymbol{\tau} = \frac{1}{3}f(q)\boldsymbol{I} + \tilde{\zeta}\tilde{\boldsymbol{h}} + \breve{\zeta}\breve{\boldsymbol{h}}, \tag{54}$$

where the tensor $\breve{\boldsymbol{h}}$ is given by Eq. (22) or (23) and the invariant coefficients $\tilde{\zeta}$ and $\breve{\zeta}$ are given by

$$\tilde{\zeta} = \frac{\chi}{1+\nu}\left(\frac{1+\gamma}{2}\frac{f(\varphi)}{\varphi} - \frac{1-\gamma}{2}\frac{f(-\varphi)}{\varphi}\right) + \frac{1-\chi}{3}\sqrt{12+3\beta^2}\,\frac{g(\beta)}{\psi}, \tag{55}$$

$$\breve{\zeta} = \frac{\chi}{\varphi}\left(w_1(\varphi) - w_1(-\varphi)\right) + 4\frac{\gamma}{\psi}\left(W_1 - w_0(\beta)\right). \tag{56}$$

From the property of the tensor $\breve{\boldsymbol{h}}$, as shown by Eq. (24), it may be concluded that the last term in the stress–strain relation (54) comes into play only for the case of three distinct principal stretches.

To conclude this subsection, we indicate that the stress–strain relation (54) with Eqs. (55)–(56) reduces to the stress–strain relation derived in Part 1 for incompressible materials. Indeed, with $i_1 = 0$ and $\nu = 0.5$ for the incompressibility case, the three invariant coefficients given by Eqs. (55)–(56) yield $\zeta_0 = 0$ and the two given by Eqs. (88)–(89) in [70], and, furthermore, the invariant q will be given as an indefinite limit, say q_0, namely

$$\frac{i_1}{1-2\nu} \to q_0,$$

with $i_1 \to 0$ and $(1-2\nu) \to 0$. As a result, the first part in Eq. (53) will produce an indefinite hydrostatic stress part $p\boldsymbol{I}$.

It turns out that the invariant q establishes bridging relations in a double sense. One is the relation between the uniaxial case and multi-axial case, and the other is the relation between compressibility and incompressibility.

4.4 Results from the uniaxial case alone

It is possible that the multi-axial potential $\bar{W}_1$ derived from the uniaxial test alone, as given by Eq. (46), may serve as a unified potential in a sense of good approximation. This implies that the potential $\bar{W}_1$ given by Eq. (46) may yield the one-dimensional potential $w_0(\beta)$ in the simple shear case. As a result, $\bar{W}_1$ alone provides a unified potential desired, that is, $W = W_1$. Thus, we have

$$\begin{cases} W = \frac{1-2\nu}{3} w_1(q) + \frac{1+\nu}{3}((1+\gamma)w_1(\varphi) + (1-\gamma)w_1(-\varphi)), \\ \boldsymbol{\tau} = \frac{1}{3} f(q)\boldsymbol{I} + \frac{1}{1+\nu}\left(\frac{1+\gamma}{2}\frac{f(\varphi)}{\varphi} - \frac{1-\gamma}{2}\frac{f(-\varphi)}{\varphi}\right)\tilde{\boldsymbol{h}} + \frac{1}{\varphi}\left(\int_{-\varphi}^{\varphi} f(x)\mathrm{d}x\right)\hat{\boldsymbol{h}}. \end{cases} \tag{57}$$

This multi-axial potential may be attractive, since it is determined solely from the data of uniaxial test. As will be seen in Sect. 6, even such a simple form of stress–strain relation may achieve accurate matches with test data.

4.5 Continuous differentiability

Since $(\boldsymbol{I}, \tilde{\boldsymbol{h}}, \breve{\boldsymbol{h}})$ are three mutually orthogonal tensor generators, as indicated at the end of Sect. 3, Eq. (54) furnishes a complete orthogonal representation for isotropic compressible elastic stress–strain relation. An orthogonal representation in a broad sense was presented first in Criscione et al. [23] and later in Xiao et al. [75]. According to the general result proved in Xiao et al. [75], such an orthogonal representation is continuously differentiable. This fact may be demonstrated by following the procedure for incompressible deformations in Part 1. Indeed, it may be evident that the stress–strain relation (54) with (55)–(56) is continuously differentiable for $\sqrt{j_2} \neq 0$. The same is true for the case when $\sqrt{j_2}$ is going to vanish. In fact, we have the following expansions:

$$\begin{aligned} f(\varphi) &= E\varphi + O(\varphi^2), \\ g(\beta) &= \sqrt{3}G\psi + O(\psi^2), \\ w_1(\varphi) &= \frac{1}{2}E\varphi^2 + O\left(\varphi^3\right), \\ w_0(\beta) &= \frac{3}{2}G\psi^2 + O(\psi^3), \\ W_1 &= \frac{3}{4}\frac{E}{1+\nu}\psi^2 + O(\psi^3), \end{aligned}$$

for an infinitesimal $\sqrt{j_2}$. Here, E and G are Young's modulus and shear modulus at infinitesimal strain and related to each other via the Poisson ratio ν, that is,

$$\begin{aligned} &E = f'(0), \quad G = g'(0), \\ &E = 2G(1+\nu). \end{aligned}$$

Utilizing these we infer that the two invariant coefficients given by Eqs. (55)–(56) may be expressed as follows:

$$\begin{aligned} \tilde{\zeta} &= 2G + O\left(\sqrt{j_2}\right), \\ \breve{\zeta} &= O(j_2), \end{aligned}$$

and, accordingly,

$$J\boldsymbol{\sigma} = \frac{1}{3}f(q)\boldsymbol{I} + 2G\tilde{\boldsymbol{h}} + O(j_2),$$

for an infinitesimal $\sqrt{j_2}$. Thus, from this and the fact that both γ and $\breve{\boldsymbol{h}}$ are finite for nonvanishing $j_2 \neq 0$, it follows that the stress–strain relation (54) with Eqs. (55)–(56) is continuously differentiable whenever $\sqrt{j_2}$ is going to vanish.

5 Predictions for five benchmark tests

In this section we are going to study the predictions of the unified potential given by Eq. (49) for the five cases including the all-round compression, uniaxial, biaxial, plane-strain compression, and simple shear tests. The deformation and stress states for the last four tests have been described in detail in §2.4(a)–(d) in [70]. Results will be derived one by one using the multi-axial stress–strain relation given by Eq. (53) with Eqs. (54)–(56).

(a) All-round compression

This is the simplest case with the following forms of Hencky strain and Cauchy stress:

$$\boldsymbol{h} = \frac{1}{3}(\ln J)\boldsymbol{I}, \quad \boldsymbol{\sigma} = p\boldsymbol{I}. \tag{58}$$

Hence, we have

$$j_2 = 0.$$

From Eqs. (53)–(56) we obtain

$$J\boldsymbol{\sigma} = \frac{1}{3}f(q)\boldsymbol{I}. \tag{59}$$

Hence,

$$p = \frac{1}{3}J^{-1}f\left(\frac{\ln J}{1-2\nu}\right). \tag{60}$$

This result supplies the relation between the volumetric ratio J and the all-around pressure p.

(b) Uniaxial case

The Hencky strain is of the form given by Eq. (2). Since there is a pair of repeated principal stretches, we have

$$\begin{cases} q = \frac{\ln\lambda + 2\ln\rho}{1-2\nu}, \quad \varphi = \frac{|\ln\lambda - \ln\rho|}{1+\nu}, \\ \gamma^2 = 1, \quad \chi = 1, \quad \tilde{\zeta} = \frac{1}{3}\frac{1}{1+\nu}\frac{f(\ln\lambda)}{\ln\lambda - \ln\rho}, \\ \breve{\boldsymbol{h}} = \boldsymbol{O}, \quad \tilde{\boldsymbol{h}} = \frac{1}{2}(\ln\lambda - \ln\rho)(3\boldsymbol{e}\otimes\boldsymbol{e} - \boldsymbol{I}). \end{cases} \tag{61}$$

From these and Eq. (45) as well as Eqs. (53)–(56), we obtain the stress as follows:

$$J\boldsymbol{\sigma} = \frac{1}{3}f\left(\frac{\ln\lambda + 2\ln\rho}{1-2\nu}\right)\boldsymbol{I} + \frac{1}{3}f\left(\frac{\ln\lambda - \ln\rho}{1+\nu}\right)(3\boldsymbol{e}\otimes\boldsymbol{e} - \boldsymbol{I}). \tag{62}$$

Since the stress is of the form given by Eq. (3), we derive

$$f\left(\frac{\ln\lambda + 2\ln\rho}{1-2\nu}\right) = f\left(\frac{\ln\lambda - \ln\rho}{1+\nu}\right), \tag{63}$$

$$\tau = J\sigma = f\left(\frac{\ln\lambda - \ln\rho}{1+\nu}\right). \tag{64}$$

With a function $f(x)$, the former should be met for every stretch λ, thus leading to

$$\frac{\ln\lambda + 2\ln\rho}{1-2\nu} = \frac{\ln\lambda - \ln\rho}{1+\nu},$$

and then to

$$\begin{cases} \ln\lambda = -\nu\ln\rho, \\ \tau = J\sigma = f(\ln\lambda). \end{cases} \tag{65}$$

The above results clearly show that all the features of the uniaxial case are exactly derived; in particular, this is the case for the relation between the axial and the lateral stretches.

(c) Biaxial case

The Hencky strain of the equi-biaxial loading case is of the same form as in the uniaxial loading case and also given by Eq. (2). The difference is that the unit vector $\boldsymbol{e}$ is now the normal to the loading plane and that it is set free in the direction along $\boldsymbol{e}$. Moreover, now ρ is the stretch in the loading direction, while λ is the stretch in the free direction normal to the loading plane. As a result, Eqs. (61)–(62) remain true. Now the Cauchy stress is of the form:

$$\boldsymbol{\sigma} = \bar{\sigma}(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}), \tag{66}$$

with the normal stress $\bar{\sigma}$ along two mutually perpendicular directions. From this and Eq. (62) we infer

$$f\left(\frac{\ln\lambda + 2\ln\rho}{1-2\nu}\right) = -2f\left(\frac{\ln\lambda - \ln\rho}{1+\nu}\right), \tag{67}$$

$$\tau = J\bar{\sigma} = -f\left(\frac{\ln\lambda - \ln\rho}{1+\nu}\right). \tag{68}$$

Let the function $f(\cdot)$ be invertible and its inverse be $f^{-1}(\cdot)$. From the above two we deduce

$$\frac{\ln\lambda - \ln\rho}{1+\nu} = f^{-1}(-\tau),$$
$$\frac{\ln\lambda + 2\ln\rho}{1-2\nu} = f^{-1}(2\tau).$$

These two yield

$$\ln\lambda = \frac{2}{3}(1+\nu)f^{-1}(-\tau) + \frac{1}{3}(1-2\nu)f^{-1}(2\tau), \tag{69}$$

$$\ln\rho = -\frac{1}{3}(1+\nu)f^{-1}(-\tau) + \frac{1}{3}(1-2\nu)f^{-1}(2\tau). \tag{70}$$

For a case of slight compressibility with ν very close to 0.5, the latter two reduce to

$$\begin{cases} \ln\lambda = -2\ln\rho, \\ \bar{\sigma} = -f(\ln\lambda) = -f(-2\ln\rho). \end{cases} \tag{71}$$

This recovers the universal relation between the uniaxial and the biaxial cases, as indicated in Theorem A in a previous study [70].

(d) Plane-strain compression/extension

The Hencky strain is of the form

$$\boldsymbol{h} = (\ln\xi)\boldsymbol{e} \otimes \boldsymbol{e} + (\ln\eta)\boldsymbol{a} \otimes \boldsymbol{a}$$

and the Cauchy stress is of the form

$$\boldsymbol{\sigma} = x\boldsymbol{e} \otimes \boldsymbol{e} + y\boldsymbol{b} \otimes \boldsymbol{b} \tag{72}$$

Here, the orthonormal unit vectors $(\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{b})$ are in the loading, free, and constrained directions, respectively; ξ and η are the stretches in the loading and free directions, respectively; and x and y are the normal stresses in the loading and the constrained direction, respectively.

For the sake of simplicity, here we confine ourselves to a case of near incompressibility, namely

$$\xi\eta = 1$$

nearly holds. Hence, we have the following good approximation:

$$\begin{cases} i_1 = 0,\ j_3 = 0,\ \gamma = 0,\ \chi = 0,\ \beta = \xi - \xi^{-1},\ \psi = \frac{2}{\sqrt{3}}\ln\xi, \\ \tilde{\boldsymbol{h}} = \ln\xi(\boldsymbol{e} \otimes \boldsymbol{e} - \boldsymbol{a} \otimes \boldsymbol{a}) \end{cases} \tag{73}$$

for near incompressibility. Utilizing these and Eqs. (53)–(56), we obtain the following reduction:

$$\boldsymbol{\sigma} = \frac{1}{3} f\left(\frac{i_1}{1-2\nu}\right)\boldsymbol{I} + \frac{1}{3}\sqrt{12+3\beta^2}\, g(\beta)\frac{\tilde{\boldsymbol{h}}}{\psi} \tag{74}$$

for near incompressibility. Now in the above either $i_1 = \ln(\xi\eta)$ or $(1-2\nu)$ nearly but not exactly vanishes.

From this and Eqs. (72) and $(73)_2$ we deduce

$$\begin{cases} \frac{1}{3} f\left(\frac{\ln\xi+\ln\eta}{1-2\nu}\right) = \frac{1}{2}(\xi+\xi^{-1})g(\xi-\xi^{-1}), \\ x = 2y = (\xi+\xi^{-1})g(\xi-\xi^{-1}). \end{cases} \tag{75}$$

The first expression above gives the stretch η in the free (unconstrained) direction in terms of the stretch ξ in the loading direction, namely

$$\ln\eta = -\ln\xi - (1-2\nu)f^{-1}\left(\frac{1}{2}(\xi+\xi^{-1})g(\xi-\xi^{-1})\right), \tag{76}$$

while the second in Eq. (75) gives the normal stress in the loading direction. Note that the above solution is accurate to the first order for near incompressibility. For a general case of compressibility, it does not appear to be possible to derive an explicit solution due to strong coupling between the two deformation quantities and the stresses.

(e) Simple shear

The Hencky strain is given by Eq. (43). In this case we have

$$\begin{cases} i_1 = 0,\ j_3 = 0,\ \gamma = 0,\ \chi = 0,\ \beta = \omega, \\ \psi = \frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{\omega}{2}\right). \end{cases} \tag{77}$$

From these and Eq. (43) again we arrive at Eq. (74) with $i_1 = 0$ and hence the first part vanishing. Thus, we infer that the shear stress component is given by

$$\tau = g(\omega). \tag{78}$$

This is exactly the stress–deformation relation for the simple shear case. For every incompressible elastic material, there is an exact correlation between the plane-strain compression and the simple shear case, as indicated in Theorem B in [70].

(f) A summary

In a sense of first-order accuracy for near incompressibility, the results derived above agree with those derived in [70] for the incompressibility case. According to Theorem C in [70], the unified potential exactly matches data of four tests in the case of incompressibility. For rubberlike materials, the deformation is nearly incompressible; the above results may suggest that the unified form of potential derived may accurately match data for five tests including the all-round compression test.

6 Multi-axial potential with strain limit

From data of uniaxial and simple shear tests, it may be a straightforward matter to obtain a unified multi-axial elastic potential following the explicit procedures as explained in the preceding sections. As indicated in Sect. 3.1, an interpolating or approximating procedure for single-variable functions may be used to match data sets of uniaxial and shear tests, separately. Of them, a choice of either a Lagrange's interpolating polynomial or a cubic spline may be direct, while that may not be the case for the choice of a rational interpolating or approximating function. However, it is known that rubber elasticity exhibits strain stiffening effects. A rational function may be surprisingly powerful and accurate in characterizing limiting behavior with such effects.

Indeed, test data for rubbers in the uniaxial, plane-strain compression, and simple shear cases display the aforementioned strain stiffening effects, namely the stress tends to grow indefinitely as the strain is approaching a limit (see, e.g., [5,14,32,33]; see also [8,9,41,47,48,60], among many others). Mathematically, a one-dimensional stress–strain curve with such extremal property incorporates limit points known as poles. As

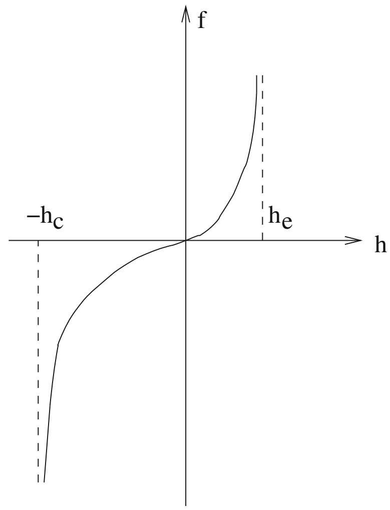

Fig. 1 Uniaxial stress–strain curves with extension and compression limits

is schematically shown in Fig. 1 for a uniaxial stress–strain curve, there are two poles at extension and compression, separately, referred to as extension limit and compression limit and denoted h_e and h_c. It is well known that the incorporation of poles is just the typical feature of a rational function. On account of this, it may be judicious to take into consideration a rational function as an interpolating or approximating function for a one-dimensional stress–strain relation with the foregoing limiting behavior. Following an idea of treating boundary layer effects in perturbation theory, as has been suggested in [70], we propose a simple form of uniaxial stress–strain relation of rational function type with limiting properties as follows:

$$\tau = f(h) = \frac{2Eh}{\left(1 - \frac{h}{h_e}\right)\left(1 + \frac{h}{h_c}\right)} - Eh. \tag{79}$$

In the above expression, the parameters E, h_e and h_c are of direct physical meanings in a phenomenological sense. In fact, E is Young's modulus at infinitesimal strain, and h_e and h_c are the magnitudes of the two limits at extension and compression, as has been indicated before.

With Eq. (79), the one-dimensional potential is given by (cf. Eq. (37))

$$w_1(h) = -\frac{1}{2}Eh^2 - E\frac{2h_e h_c}{h_e + h_c}\left(h_e \ln\left(1 - \frac{h}{h_e}\right) + h_c \ln\left(1 + \frac{h}{h_c}\right)\right). \tag{80}$$

Then, a multi-axial potential via the multi-axial bridging/matching procedure is given by Eq. (46). If we go further to assume that the latter may yield the one-dimensional potential for the simple shear case, then we infer that a unified potential may be given by Eq. $(57)_1$, and hence, we have

$$W = \frac{1-2\nu}{3}w_1(q) - \frac{1+\nu}{3}E\varphi^2 + \frac{4}{3}(1+\nu)\frac{Eh_e h_c}{h_e + h_c}\left(\gamma h_e \ln\left(1 + \frac{\varphi}{h_e}\right)\right.$$
$$\left. + \gamma h_c \ln\left(1 - \frac{\varphi}{h_c}\right) - \frac{1+\gamma}{2}\left(h_e \ln\left(1 - \frac{\varphi^2}{h_e^2}\right) + h_c \ln\left(1 - \frac{\varphi^2}{h_c^2}\right)\right)\right), \tag{81}$$

with the Hencky invariants q, φ and γ given by Eqs. (10), (11) and (13). Accordingly, a multi-axial stress–strain relation is of the form (cf. Eq. $(57)_2$)

$$\boldsymbol{\sigma} = \frac{1}{3}f(q)\boldsymbol{I} + \tilde{\zeta}\tilde{\boldsymbol{h}} + \breve{\zeta}\breve{\boldsymbol{h}}, \tag{82}$$

with

$$f(q) = \frac{2Eq}{\left(1 - \frac{q}{h_e}\right)\left(1 + \frac{q}{h_c}\right)} - Eq, \tag{83}$$

$$\tilde{\zeta} = \frac{E}{1+\nu}\left(\frac{1+\gamma}{\left(1 - \frac{\varphi}{h_e}\right)\left(1 + \frac{\varphi}{h_c}\right)} + \frac{1-\gamma}{\left(1 + \frac{\varphi}{h_e}\right)\left(1 - \frac{\varphi}{h_c}\right)} - 1\right), \tag{84}$$

$$\breve{\zeta} = -2E\frac{h_e h_c}{h_e + h_c}\left(\frac{h_e}{\varphi}\ln\frac{1 - \frac{\varphi}{h_e}}{1 + \frac{\varphi}{h_e}} + \frac{h_c}{\varphi}\ln\frac{1 + \frac{\varphi}{h_c}}{1 - \frac{\varphi}{h_c}}\right). \tag{85}$$

Two perhaps noticeable features of the above multi-axial stress–strain relation are explained below. First, since both the compression and the extension limit, h_c and h_e, are usually very large, Eq. (82) with Eqs. (83)–(85) leads to the following Hencky model (cf. Anand [1]):

$$\boldsymbol{\sigma} = \frac{2G\nu}{1-2\nu}\boldsymbol{I} + 2G\boldsymbol{h}, \tag{86}$$

at small to moderate deformation, that is, $\lambda \in [0.7, 1.3]$ (cf. Anand [1,2]; here $2G = E/(1+\nu)$ is the shear modulus). Next, the strain stiffening effect takes place when the Hencky strain $\boldsymbol{h}$ is approaching a limit prescribed below:

$$\begin{cases} -(1-2\nu)h_c \le \ln J \le (1-2\nu)h_e, \\ -(1+\nu)h_c \le \ln\lambda - \ln\rho \le (1+\nu)h_e \end{cases} \tag{87}$$

for the cases of two coalescent principal stretches, that is,

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = \rho,$$

and

$$\begin{cases} -(1-2\nu)h_c \le \ln J \le (1-2\nu)h_e, \\ \frac{1}{2}j_2 - \frac{1}{3}(1+\nu)^2 h_e^2 \le 0 \end{cases} \tag{88}$$

for three distinct principal stretches, that is,

$$\lambda_1 \ne \lambda_2 \ne \lambda_3 \ne \lambda_1.$$

In Eq. (88), j_2 is the second invariant of the deviatoric Hencky strain and h_e is the extension limit. Here, note that the magnitude of the compression limit is taken to be greater than that of the extension limit, that is, $h_c > h_e$. As in [70], the limit $(88)_2$ for Hencky strain is also the counterpart of the well-known von Mises limit for the Cauchy stress in elastoplasticity as shown below:

$$\frac{1}{2}J_2 - \frac{1}{3}r_0^2 \le 0, \tag{89}$$

where J_2 is the second invariant of the deviatoric Cauchy stress and r_0 is the yield limit.

A detailed study by Anand [1,2] shows that the Hencky model (86) accurately matches test data from small to moderate deformations. Furthermore, with the typical limiting feature of rational function with poles, the equation (82) also accurately matches test data at large deformations. From these it may be expected that the model proposed here accurately matches test data from small to large deformations. To see this clearly, we compare the predictions of the model given by Eq. (82) with the test data from Treloar [66]. Here, uniaxial

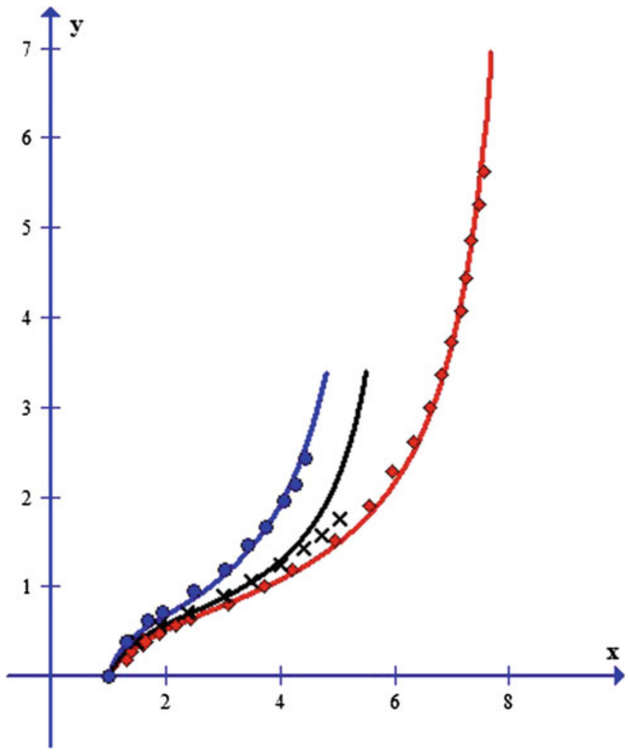

Fig. 2 Model predictions and test data from Treloar [66]: the *upper*, *middle*, *lower curves*, and the associated three types of points for the test data of biaxial, plane-strain, and uniaxial extension, respectively; x and y for the stretch and nominal stress in the loading direction, respectively

extension, biaxial extension, and plane-strain extension are taken into account. The predictions of the model (82) for these three cases are as follows:

$$\frac{P_1}{E} = \left(\frac{2}{\left(1 - \frac{\ln x}{h_e}\right)\left(1 + \frac{\ln x}{h_c}\right)} - 1 \right) \frac{\ln x}{x}, \tag{90}$$

$$\frac{P_2}{E} = \frac{3}{1+\nu} \left(\frac{2}{\left(1 + \frac{3}{1+\nu}\frac{\ln x}{h_e}\right)\left(1 - \frac{3}{1+\nu}\frac{\ln x}{h_c}\right)} - 1 \right) \frac{\ln x}{x}, \tag{91}$$

$$\frac{P_0}{E} = \frac{4}{3} \left(\frac{\frac{2h_c}{h_e+h_c}}{1 - \frac{4}{3}\left(\frac{\ln x}{h_e}\right)^2} + \frac{\frac{2h_e}{h_e+h_c}}{1 - \frac{4}{3}\left(\frac{\ln x}{h_c}\right)^2} - 1 \right) \frac{\ln x}{x}. \tag{92}$$

In the above, x represents the stretch in the loading direction in the cases of uniaxial, biaxial, and plane-strain extension, separately, and P_0, P_1 and P_2 are the nominal stresses in the loading direction in the three cases. The second expression above presents the result of the first-order accuracy for slight compressibility. As in [70], here the extension limit h_e and the compression limit h_c are taken as follows:

$$h_e = \ln 8.48, \quad h_c = \ln 37, \tag{93}$$

and, in addition, the Poisson ratio ν is given below:

$$\nu = 0.499.$$

With these parameters, the P_0/E versus x, P_1/E versus x, P_2/E versus x curves are plotted in Fig. 2. Results are actually the same as those in Part 1. Accurate matches with the test data from Treloar [66] are achieved over the whole range from small to large deformation. In particular, that is the case from small to moderate deformation. In this respect, it may be noted that the usual models to date achieve good agreement

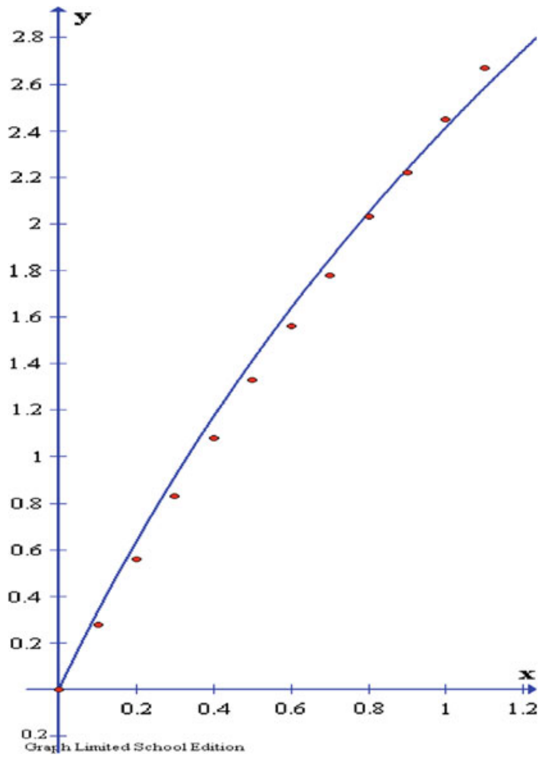

Fig. 3 Model prediction and test data from [61]: x for $\lambda - 1$ and y for $10{,}000(J - 1)$

with data either merely at small deformation or merely at large deformation and that good agreement at small to large deformation necessitates further treatment. On the other hand, test data were presented by Penn [61] in a study of the volumetric deformation in the uniaxial extension. The model prediction with the Poisson ratio ν as introduced in Sect. 2 (cf. Eq. (6)) is as follows:

$$J - 1 = \lambda^{1-2\nu} - 1. \tag{94}$$

This prediction is compared with Penn's data. The result with $\nu = 0.499826$ is shown in Fig. 3.

7 Discussion

Essential features of compressible elastic deformations may be identified in and extracted from test data of uniaxial extension and compression. In addition to the usual well-known relation between the axial stretch and the axial stress, it is realized that the Poisson ratio (function) ν as introduced in Sect. 2 via Hencky strain may be essential for characterizing the compressibility behavior. With an explicit, straightforward approach as has been shown in the preceding sections, a unified multi-axial elastic potential may be obtained by incorporating these material characteristics from the uniaxial test and also from the simple shear test. As has been shown, the multi-axial bridging/matching procedures play substantial roles. The essence of the new approach proposed lies in the fact that these procedures based on the Hencky invariants named bridging and matching invariants may extract and recover information for multi-axial deformation behavior from the information compressed and condensed in one-dimensional data for uniaxial and shear tests and even for uniaxial test alone. It has been demonstrated that simple models with accurate correlations with test data for various modes of deformation may be derived even solely from the case of uniaxial extension and compression.

A compressible model may be free from the need of particular numerical treatment for the incompressibility constraint. However, an incompressible model is attractive with the far-reaching deformation uncoupling resulting from the incompressibility condition. In contrast with this, there exists strong coupling between strain components in a broad case of compressible deformation, that is, the case even in such simple cases as uniaxial and biaxial extension.

Remarks on three types of interpolating functions may be found in [70]. Results with polynomial interpolating functions and cubic splines and other classes of interpolating functions will be presented elsewhere.

Acknowledgments This work was completed under the financial support of the start-up funding from the 211-Project of the Education Committee of China through Shanghai University. This support is gratefully acknowledged.

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