Acta Mech 223, 2117–2133 (2012)
DOI 10.1007/s00707-012-0687-z

Djordje Mušicki

Analysis of a class of nonconservative systems reducible to pseudoconservative ones and their energy relations

Received: 27 May 2011 / Revised: 11 February 2012 / Published online: 28 June 2012

Abstract This paper analyzes a class of nonconservative systems, whose Lagrangian equations can be reduced to Euler–Lagrangian equations by introducing a new Lagrangian, which is equal to a product of some function of time $f(t)$ and the primary Lagrangian. These equations formally have the same form as for the systems with potential forces, while the influence of nonconservative forces is contained in the factor $f(t)$, and such systems are called pseudoconservative. It is further shown that the requirement for a nonconservative system to be considered as a pseudoconservative is the existence of at least one particular solution of a system of differential equations with unknown function $f(t)$, or their linear combination with suitably chosen multipliers. Further on, the energy relations and corresponding conservation laws of those systems are analyzed from two aspects: directly, on the basis of the corresponding Lagrangian equations and via modified Emmy Noether's theorem. So, it has been shown, even in two different ways, that there are two types of the integrals of motion, in the form of the product of an exponential factor and the sum of the generalized energy (energy function) and an additional term. For the existence of these integrals of motion, it is necessary and sufficient that there exists at least one particular solution of a partial differential equation, which is in accordance with the Lagrangian equations for the observed problem. The obtained results are equivalent to so-called energy-like conservation laws, obtained via Vujanović-Djukić's generalized Noether's theorem for nonconservative systems (Vujanović and Jones in: Variational Methods in Nonconservative Phenomena (monograph). Acad. Press, Boston, 1989).

1 Reduction of a nonconservative system to the pseudoconservative one

1.1 Lagrangian of the pseudoconservative system

Consider a nonconservative mechanical system, whose position is determined by the set of generalized coordinates q^i $(i = 1, 2, \dots, n)$ and whose differential equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i^* \quad (i = 1, 2, \dots, n), \tag{1.1}$$

where $L(q^i, \dot{q}^i, t) = T - U$ is the Lagrangian of the system, T being the kinetic and U the potential energy, and Q_i^* the generalized nonpotential forces. Now, define the following problem: find a new Lagrangian in the form

$$\tilde{L}\left(q^i, \dot{q}^i, t\right) = f(t) L\left(q^i, \dot{q}^i, t\right), \tag{1.2}$$

D. Mušicki (✉)
Faculty of Physics, University of Belgrade, and Mathematical Institute SANU,
Knez Mihajlova 36, Belgrade, Serbia
E-mail: draganu@mi.sanu.ac.rs

so that the system of Lagrangian equations (1.1) can be transformed into the system of Euler–Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^i} - \frac{\partial \tilde{L}}{\partial q^i} = 0 \quad (i = 1, 2, \ldots, n), \tag{1.3}$$

which is same or equivalent to the system of the primary Lagrangian equations (1.1), meaning it can be reduced to (1.3) using only elementary transformations. When it is possible to find a new Lagrangian in the form for systems (1.2), which enables the transformation from (1.1) to (1.3), we shall call this system as pseudoconservative.

If we insert the expression (1.2) into the Lagrangian equations (1.3)

$$\frac{\partial L}{\partial \dot{q}^i}\frac{\mathrm{d}f}{\mathrm{d}t} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right) f = 0$$

and substitute the variational derivative by a corresponding expression from the Lagrangian equations (1.1), we obtain

$$\frac{\partial L}{\partial \dot{q}^i}\frac{\mathrm{d}f}{\mathrm{d}t} + Q_i^* f = 0 \quad (i = 1, 2, \ldots, n). \tag{1.4}$$

This system of n differential equations determines the function $f(t)$, and let us remark that it does not depend on the nature of the potential forces.

Accordingly, in order for a nonconservative system to be determined by the Lagrangian (1.2), it is necessary and sufficient that there is at least one particular solution for the function $f(t)$, which results from the system of differential equations (1.4).

1.2 Analysis of this solution

This conclusion does not mean that the obtained function $f(t)$ must satisfy each individual equation (1.4). Namely, in the general case, the transformation from (1.1) to (1.3) represents the transformation of the starting system of mutually dependent Lagrangian equations. Therefore, Eq. (1.4) represents the set of mutually dependent requirements, each of which does not need to be satisfied by this function $f(t)$. Only if each Lagrangian equation (1.1) is individually transformed into the corresponding equation (1.3), independently from the others, Eq. (1.4) represents the set of mutually independent requirements and in that case, this function $f(t)$ must satisfy each individual equation (1.4).

Let us emphasize that the only requirement in the so formulated problem is that the Lagrangian (1.2) with the chosen function $f(t)$ gives the system of the transformed Lagrangian equations (1.3), which is same or equivalent to the system of the original Lagrangian equations (1.1). Therefore, the only criterion for the correctness of the chosen function $f(t)$ is the equivalence or even the identity of the so obtained system of Lagrangian equations with the original system of these equations.

1.3 Direct determination of function $f(t)$

In the general case, there is not a single solution for all the differential equations (1.4), which satisfies all these equations. Namely, if any of these equations is solved by separation of variables,

$$\frac{\mathrm{d}f}{f} = -\frac{Q_i^*}{\partial L/\partial \dot{q}^i}\mathrm{d}t,$$

and then by integration, it follows that

$$f(t) = Ce^{-\int \frac{Q_i^*}{\partial L/\partial \dot{q}^i}\mathrm{d}t}. \tag{1.5}$$

In the general case, this expression will not be a function only of time and does not satisfy all other equations (1.4), except in the following cases: for $n = 1$ if the expression $Q^*/(\partial L/\partial \dot{q})$ is constant, and for $n > 1$ if this expression is same for all the generalized coordinates, $Q_i^*/(\partial L/\partial \dot{q}^i) = \text{const} \equiv -2k$. In both cases, this expression (1.5) simplifies, and for $C = 1$ becomes

$$f(t) = e^{2kt} \quad (\text{for } n \geqslant 1), \tag{1.6}$$

and represents one particular solution of the system of equations (1.4). Note that in the second case all the equations (1.4) in fact are reduced only to one.

Example 1 A nonlinear oscillator in a resisting medium with linear damping, when the forces $F_1 = -m\omega^2 x$, $F_2 = -mbx^n$ and $F^* = -2mk\dot{x}$ act on this oscillator. In this case, the Lagrangian and nonpotential force are

$$L(x, \dot{x}, t) = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{mb}{n+1}x^{n+1}, \quad Q^* = -2mk\dot{x}, \tag{1.7}$$

and the corresponding equation (1.4) for $q = x$ has the form

$$m\dot{x}\frac{\mathrm{d}f}{\mathrm{d}t} - 2mk\dot{x}f = 0. \tag{1.8}$$

Hence, as in the case when the nonlinear force is absent, one particular solution of this equation is $f(t) = e^{2kt}$, and the corresponding new Lagrangian according to (1.2) is

$$\tilde{L}\left(q^i, \dot{q}^i, t\right) = f(t)L\left(q^i, \dot{q}^i, t\right) = e^{2kt}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{mb}{n+1}x^{n+1}\right). \tag{1.9}$$

For $b = 0$, it is reduced to the Lagrangian of the usual linear oscillator in a resisting medium with linear damping (see [1], p. 51).

1.4 Hamiltonian formalism

If there is at least one particular solution for the function $f(t)$, one can switch from Lagrangian to Hamiltonian formalism by introducing the generalized momenta

$$\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i} = f(t)\frac{\partial L}{\partial \dot{q}^i} \quad (i = 1, 2, \ldots, n). \tag{1.10}$$

Then, the corresponding Hamiltonian (or canonical) equations can be obtained in a usual manner (see e.g., Goldstein [2] or Whittaker [3]). For example, starting from

$$\delta\tilde{L} = \frac{\partial \tilde{L}}{\partial q^i}\delta q^i + \frac{\partial \tilde{L}}{\partial \dot{q}^i}\delta \dot{q}^i,$$

where summation over the repeated indices is understood, substituting $\partial L/\partial q^i$ by the corresponding expression from the Lagrangian equations (1.3) and putting $\tilde{p}_i\delta\dot{q}^i = \delta\left(\tilde{p}_i\dot{q}^i\right) - \dot{q}^i\delta\tilde{p}_i$, it follows that

$$\delta(\tilde{p}_i\dot{q}_i - \tilde{L}) = -\dot{\tilde{p}}_i\delta q^i + \dot{q}^i\delta\tilde{p}_i. \tag{1.11}$$

Then, by comparing it with the variation of the expression in the brackets, denoted by $\tilde{H}$, we obtain

$$\dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial q^i}, \quad \dot{q}^i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i} \quad (i = 1, 2, \ldots, n), \tag{1.12}$$

where a new Hamiltonian is defined as

$$\tilde{H}\left(q^i, \tilde{p}_i, t\right) = \tilde{p}_i\dot{q}^i - \tilde{L}\left(q^i, \dot{q}^i, t\right) = f(t)H\left(q^i, p_i, t\right). \tag{1.13}$$

Consequently, if there is one particular solution for function $f(t)$, with the Lagrangian introduced in such manner, the Lagrangian and Hamiltonian equations formally have the same form as for the systems with potential forces (without term Q_i^*), while the influence of the nonconservative forces is contained in the factor $f(t)$. In that sense, the observed nonconservative system can be treated as a pseudoconservative one, and it is completely determined by a new Lagrangian (1.2).

This proof is formally same as in the usual formulation of mechanics, but here $\tilde{L}$, $\tilde{p}_i$ and $\tilde{H}$ take the role of L, p_i and H, multiplied by $f(t)$. However, each step in the usual proof remains valid here too, due to the fact that the function f is a function only of time, and this will be valid for all other proofs that follow.

1.5 The general case of pseudoconservative systems

At the end, let us note that, in the general case, the function f could also depend on the generalized coordinates, and thus, instead of (1.2), we have

$$\tilde{L}\left(q^i, \dot{q}^i, t\right) = f(q^i, t) L\left(q^i, \dot{q}^i, t\right). \tag{1.14}$$

In that case, by inserting this expression into the Lagrangian equations (1.3), the following system of the partial differential equations for determining the function f is obtained:

$$\mathcal{E}_i^k \frac{\partial f}{\partial q^k} + \frac{\partial L}{\partial \dot{q}^i} \frac{\partial f}{\partial t} + Q_i^* f = 0 \quad (i = 1, 2, \ldots, n), \tag{1.15}$$

where

$$\mathcal{E}_i^k = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^k - \delta_i^k L, \tag{1.16}$$

which could include more complex cases, such as when $Q_i^* = k(\dot{q}^i)^2$. Let us note that for $k = i$ the coefficient $\mathcal{E}_i^k$ represents the generalized energy of the system. However, due to the significantly greater complexity of this problem, and due to the absence of these cases in real circumstances, in this paper we shall limit ourselves to those pseudoconservative systems in which function f only depends on time.

2 Indirect determination of function $f(t)$

2.1 Formation of combined equation for the function $f(t)$

When $n > 1$ and when a particular solution for function $f(t)$ cannot be found through direct solution of the system of differential equations (1.4), we shall do the following. Consider a linear combination of Eq. (1.4) in the form

$$\left(\frac{\partial L}{\partial \dot{q}^1} + \lambda_{j-1} \frac{\partial L}{\partial \dot{q}^j}\right) \frac{\mathrm{d}f}{\mathrm{d}t} + (Q_1^* + \lambda_{j-1} Q_j^*) f = 0 \quad (j = 2, 3, \ldots, n), \tag{2.1}$$

where λ_{j-1} $(j = 2, 3, \ldots, n)$ are the corresponding multipliers. Let us choose those multipliers in such a way that this equation is simplified as much as possible and introduce an *additional requirement*: the function $f(t)$ must not depend on the nature of the problem.

For natural mechanical systems, where the kinetic energy is a quadratic function of the generalized velocities, the coefficient multiplying $\mathrm{d}f/\mathrm{d}t$ is their linear function. Hence, one can see that this equation will have the solutions for the function $f(t)$ not dependent on the nature of the problem only if the quantities Q_i^* are of the same type, that is, of the form $Q_i^* = k_{ij}\dot{q}^j$ (necessary requirement). In that case, choose the multipliers λ_{j-1} so that the coefficients associated with the same $\dot{q}^j$ $(j = 2, 3, \ldots, n)$ in the terms multiplying $\mathrm{d}f/\mathrm{d}t$ and f are equal. Thus, $n - 1$ equations are obtained, from which all the multipliers λ_{j-1} can be determined. Further, by inserting the so obtained expressions in Eq. (2.1), it gets the form

$$A\left(q^i, \dot{q}^1, t\right) \frac{\mathrm{d}f}{\mathrm{d}t} + B\left(q^i, \dot{q}^1, t\right) f = 0, \tag{2.2}$$

where A and B are the corresponding coefficients obtained in this way.

2.2 More complex systems

For more complex, unnatural systems, where the Lagrangian L can be any function of variables q^i, $\dot{q}^i$ and t with or without quantities analogous to Q_i^* (see e.g., [1]), the following can be done. In the first case, when the quantities Q_i^* are present, using this method, only the variables $\dot{q}^j$ $(j = 2, 3, \ldots, n)$ can be eliminated, but some more complex terms such as the terms of the type $(\dot{q}^i)^2$ or $(\dot{q}^i)^3$ remain. Then, instead of Eq. (2.2), the result will be the equation of the same form, but with the coefficients that also depend on the variables that could not be eliminated,

$$A\left[q^i, \dot{q}^1, t; (\dot{q}^i)^2, \ldots\right] \frac{\mathrm{d}f}{\mathrm{d}t} + B\left[q^i, \dot{q}^1, t; (\dot{q}^i)^2, \ldots\right] f = 0. \tag{2.3}$$

2.3 Sufficient requirement for pseudoconservative systems

In order to simplify the transformed equation (2.2) or (2.3) as much as possible, and to obtain the solutions that are independent from the nature of the problem, it is sufficient that the coefficient B is proportional to the coefficient A,

$$B\left(q^i, \dot{q}^1, t, \ldots\right) = KA\left(q^i, \dot{q}^1, t, \ldots\right), \tag{2.4}$$

regardless of the values of variables $q^i, \dot{q}^1, t, \ldots$. In that case, the differential equation (2.2) or (2.3) takes a simple form,

$$\frac{\mathrm{d}f}{\mathrm{d}t} + Kf = 0, \tag{2.5}$$

from which for $K = -2k$ one particular solution can be obtained by separation of variables,

$$\frac{\mathrm{d}f}{f} = 2k\mathrm{d}t,$$

and then by integration

$$f(t) = e^{2kt}. \tag{2.6}$$

This result is same as the one obtained directly by solving the system of differential equations (1.4), and it has the form (1.6), and then the new Lagrangian (1.2) is

$$\tilde{L}\left(q^i, \dot{q}^i, t\right) = e^{2kt} L\left(q^i, \dot{q}^i, t\right). \tag{2.7}$$

If the coefficient B is not proportional to the coefficient A, that is, if $B \neq KA$, then the particular solution of Eqs. (2.2) or (2.3), which satisfies the required conditions, does not exist. Therefore, in that case, the observed nonconservative system cannot be reduced to a pseudoconservative one.

Therefore, the sufficient requirement for a nonconservative system to be treated as a pseudoconservative is the existence of at least one particular solution of the system of differential equations (1.4) or of the combined equation (2.1), which does not depend on the nature of the problem. In the second case, this is true only if this equation, transformed into the form (2.2) or (2.3), satisfies the requirement (2.4).

Example 2 A linear oscillator in a resisting medium on an inclined plane, which is moving in the horizontal direction according to the law $x_A = Vt$ (Fig. 1).

In this case, we want to determine the position of this oscillator in a more complete way, namely with respect to the immobile frame of reference $Oxyz$. This is different from the usual employment of the generalized coordinates for the rheonomic systems and corresponds to the theory of the extended Lagrangian formalism for those systems [4,5]. Namely, in the usual way, the generalized coordinates applied to those systems always refer to a moving frame of reference. But, according to this theory, the generalized coordinates must be such that they determine the position of the considered mechanical system with respect to the same frame of reference (an immobile one) to which all the dynamical quantities as well as the energy laws refer to. This implies that

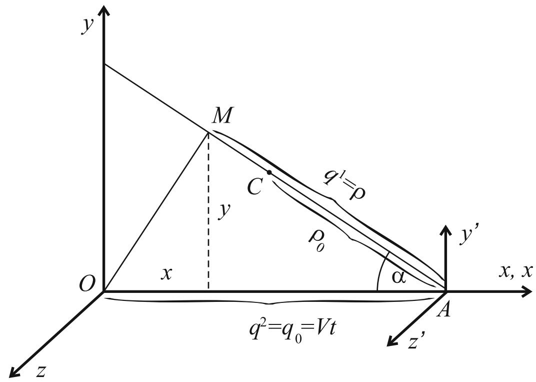

Fig. 1 A linear oscillator in a resisting medium on a moving inclined plane

it is necessary to extend the set of the chosen generalized coordinates by the additional ones, which determine the position of this moving frame of reference with respect to the immobile $Oxyz$. But, they are certain a priori given functions of time, suggested by the considered problem, which limit the motion of this rheonomic system.

These are the fundamental ideas of the extended Lagrangian formalism for the rheonomic systems. Since in this formulation all the quantities refer to the same frame of reference, the corresponding energy laws are more general and more natural than in the standard Lagrangian formulation for such systems, including the influence of the nonstationary constraints on the energy relations.

In this case, we can determine the position of the oscillator on the inclined plane by the generalized coordinate $q^1 = \rho = AM$, presented in the Figure. The position of the corresponding moving frame of reference $Ax'y'z'$ with respect to $Oxyz$ is determined by the quantity $x_A = OA$, and therefore, it needs to be taken as the additional generalized coordinates $q^2 = x_A$. But, according to the formulation of this problem, the motion of this oscillator is limited by the fact that the point A in each instant must satisfy the relation $x_A = Vt$; therefore, this quantity is an a priori given function of time $q^2 \equiv q_0 = x_A = Vt$, so that the complete set of the extended generalized coordinates is $q^\alpha = \{\rho, q_0(t)\}$. However, then, the corresponding Lagrangian equations for ρ and q_0 determine only the first generalized coordinate ρ, while the second one has an a priori given solution $q_0 = Vt$.

According to the figure, we have that

$$\begin{gathered} x = q_0 - \rho\cos\alpha, \quad y = \rho\sin\alpha, \quad z = 0, \\ \vec{r} = \overrightarrow{OM} = q_0\vec{e}_x + \rho\vec{e}_\rho, \end{gathered} \tag{2.8}$$

where $\vec{e}_x$ and $\vec{e}_\rho$ are unit vectors along the x axis and ρ axis, and therefore the kinetic energy of this oscillator is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{\rho}^2 - 2\dot{\rho}\dot{q}_0\cos\alpha + \dot{q}_0^2\right),$$

and its Lagrangian

$$L = T - U = \frac{1}{2}\left(\dot{\rho}^2 - 2\dot{\rho}\dot{q}_0\cos\alpha + \dot{q}_0^2\right) - \frac{1}{2}m\omega^2(\rho - \rho_0)^2. \tag{2.9}$$

In this case, the nonpotential forces are

$$Q^*_\rho = \vec{F}^* \cdot \frac{\partial\vec{r}}{\partial\rho} = -2mk\dot{\rho}, \quad Q^*_{q_0} = \vec{F}^* \cdot \frac{\partial\vec{r}}{\partial q_0} = 2mk\dot{\rho}\cos\alpha, \tag{2.10}$$

and thus the corresponding Lagrangian equations (1.1) have the form

$$\begin{aligned} e_1 &: \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\rho} - m\dot{q}_0\cos\alpha) + m\omega^2(\rho - \rho_0) = -2mk\dot{\rho}, \\ e_2 &: \frac{\mathrm{d}}{\mathrm{d}r}(m\dot{q}_0 - m\dot{\rho}\cos\alpha) = 2mk\dot{\rho}\cos\alpha. \end{aligned} \tag{2.11}$$

Since in this case the system of equations (1.4)

$$
\begin{aligned}
&(m\dot\rho - m\dot q_0\cos\alpha)\frac{\mathrm{d}f}{\mathrm{d}t} - 2mk\dot\rho f = 0,\\
&(m\dot q_0 - m\dot\rho\cos\alpha)\frac{\mathrm{d}f}{\mathrm{d}t} + 2mk\dot\rho\cos\alpha f = 0
\end{aligned}
\tag{2.12}
$$

does not have a solution for the function $f(t)$, which satisfies both of these equations, we shall formulate the corresponding equation (2.1)

$$
\left(\frac{\partial L}{\partial\dot\rho} + \lambda\frac{\partial L}{\partial\dot q_0}\right)\frac{\mathrm{d}f}{\mathrm{d}t} + (Q^*_\rho + \lambda Q^*_{q_0})f = 0,
$$

which, after substituting L, Q^*_P and Q^*_Q by the corresponding expressions (2.9) and (2.10) and some rearrangement, becomes

$$
[m\dot\rho(1-\lambda\cos\alpha) + m\dot q_0(\lambda - \cos\alpha)]\frac{\mathrm{d}f}{\mathrm{d}t} - 2mk\dot\rho(1-\lambda\cos\alpha)f = 0. \tag{2.13}
$$

Now, choose the multiplier λ so that the coefficient associated with $\dot q_0$ be zero: $\lambda - \cos\alpha = 0$, which results in $\lambda = \cos\alpha$, and then the previous equation takes the form

$$
\underbrace{m\dot\rho\sin^2\alpha}_{A(\rho,\dot\rho,t)}\cdot\frac{\mathrm{d}f}{\mathrm{d}t}\ \underbrace{-2\,km\dot\rho\sin^2\alpha}_{B(\rho,\dot\rho,t)}\cdot f = 0.
$$

Hence, it is clear that $B = -2kA$, so the requirement (2.4) is satisfied and the previous equation is reduced to a form which does not depend on the nature of the problem,

$$
\frac{\mathrm{d}f}{\mathrm{d}t} - 2kf = 0. \tag{2.14}
$$

This equation coincides with (2.5), for $K = -2k$, thus its particular solution for the function f is $f(t) = e^{2kt}$, in accordance with (2.6), and the new Lagrangian will be

$$
\tilde L(\rho,\dot\rho,t) = e^{2kt}\left[\frac{1}{2}m(\dot\rho^2 - 2\dot\rho\dot q_0\cos\alpha + \dot q_0^2) - \frac{1}{2}m\omega^2(\rho-\rho_0)^2\right]. \tag{2.15}
$$

The correctness of the obtained solution can be verified by forming a new system of Lagrangian equations (1.3) with this Lagrangian,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde L}{\partial\dot\rho} - \frac{\partial\tilde L}{\partial\rho} = 0,\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde L}{\partial\dot q_0} - \frac{\partial\tilde L}{\partial q_0} = 0,
$$

which, upon introduction of the expression for $\tilde L$ and some rearrangement, results in

$$
\begin{aligned}
&e_1^*: 2k(m\dot\rho - m\dot q_0\cos\alpha) + (m\ddot\rho - m\ddot q_0\cos\alpha) + m\omega^2(\rho-\rho_0) = 0,\\
&e_2^*: 2k(m\dot q_0 - m\dot\rho\cos\alpha) + (m\ddot q_0 - m\ddot\rho\cos\alpha) = 0.
\end{aligned}
\tag{2.16}
$$

This system of equations is equivalent to the primary system of the Lagrangian equations (2.11), since by using only elementary transformations it can be transformed into (2.11). Namely, with the introduced notations for the Eqs. in (2.11) and (2.16), we have $e_1^* + e_2^*\cos\alpha = e_1 + e_2\cos\alpha$ and $e_2^* + e_1^*\cos\alpha = e_2 + e_1\cos\alpha$, which proves that the considered mechanical system indeed can be considered as pseudoconservative.

In this way, it is demonstrated how one can resolve the question whether the considered system is pseudoconservative in the sense of the definition of such systems.

3 Energy laws of pseudoconservative systems

3.1 Energy change law

In further exposition, let us assume that all these requirements are satisfied and that the observed nonconservative system can be treated as a pseudoconservative one. Now, we shall analyze the energy relations of these systems, and to this aim, let us start from the Lagrangian equations (1.3), multiply them by $\mathrm{d}q^i = \dot{q}^i \mathrm{d}t$, and sum them up,

$$\mathrm{d}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\right)\dot{q}^i - \frac{\partial \tilde{L}}{\partial q^i}\mathrm{d}q^i = 0. \tag{3.1}$$

This relation can be transformed in the following way:

$$\mathrm{d}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right) - \frac{\partial \tilde{L}}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i - \frac{\partial \tilde{L}}{\partial q^i}\mathrm{d}q^i = \mathrm{d}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right) - \left(\mathrm{d}\tilde{L} - \frac{\partial \tilde{L}}{\partial t}\mathrm{d}t\right) = 0,$$

which after grouping similar terms and dividing by $\mathrm{d}t$ obtains the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i - \tilde{L}\right) = -\frac{\partial \tilde{L}}{\partial t}. \tag{3.2}$$

This is the general energy change law for pseudoconservative systems, which formally has the same form as for the systems with potential forces.

3.2 Discussion of this result

Since, according to (2.7)

$$\frac{\partial \tilde{L}}{\partial t} = \frac{\partial}{\partial t}(e^{2kt}L) = e^{2kt}\left(\frac{\partial L}{\partial t} + 2kL\right),$$

then, according to the structure of $\partial \tilde{L}/\partial t$, there are three possible options:

(a) If

$$\frac{\partial \tilde{L}}{\partial t} = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial t} + 2kL = 0, \tag{3.3}$$

then the integral of motion is valid in the form

$$\mathcal{E} = \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i - \tilde{L} = e^{2kt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = \text{const}. \tag{3.4}$$

(b) If $\partial \tilde{L}/\partial t$ can be expressed in the form

$$\frac{\partial \tilde{L}}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2kt}\varphi(q^i, \dot{q}^i, t)\right], \tag{3.5}$$

then the relation (3.2) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2kt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L + \varphi\left(q^i, \dot{q}^i, t\right)\right)\right] = 0,$$

which implies that the integral of motion is valid in the form

$$\mathcal{E}^{\text{ext}} = e^{2kt}\left[\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L + \varphi(q^i, \dot{q}^i, t)\right] = \text{const}. \tag{3.6}$$

(c) If $\partial \tilde{L}/\partial t$ is neither zero nor it can be presented in the form (3.5), then there does not exist any integral of motion of this type.

The integrals of motion of the first and second type are specific for the considered systems, and their form justifies the introduction of the name "pseudoconservative system".

3.3 Determination of the function $\varphi(q^i, \dot{q}^i, t)$

Necessary condition for the existence of the integrals of motion in the form (3.6) is the relation (3.5),

$$\frac{\partial}{\partial t}(e^{2kt}L) = \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2kt}\varphi\left(q^i, \dot{q}^i, t\right)\right],$$

which can be represented as

$$2ke^{2kt}L + e^{2kt}\frac{\partial L}{\partial t} = 2ke^{2kt}\varphi + e^{2kt}\frac{\mathrm{d}\varphi}{\mathrm{d}t},$$

and, after canceling e^{2kt}, we obtain the following relation:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + 2k\varphi = \frac{\partial L}{\partial t} + 2kL. \tag{3.7}$$

This equation determines the function φ, and it can be written explicitly as

$$\dot{q}^i\frac{\partial \varphi}{\partial q^i} + \ddot{q}^i\frac{\partial \varphi}{\partial \dot{q}^i} + \frac{\partial \varphi}{\partial t} + 2k\varphi = \frac{\partial L}{\partial t} + 2kL. \tag{3.8}$$

This requirement in the form of a partial differential equation must be satisfied by the function φ, and for each particular solution, there is an integral of motion of the type (3.6). Since for the considered problem no specific requirement has been introduced, the corresponding Lagrangian equations (1.1) must be added to this equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i^* \quad (i = 1, 2, \ldots, n). \tag{3.9}$$

Consequently, the necessary and sufficient requirement for the existence of the integrals of motion of type (3.6) is the existence of at least one particular solution of the partial differential equation (3.8) for the function f, such that is in accordance with the corresponding Lagrangian equations (3.9). Such integrals of motion are equivalent to so-called energy-like conservation laws, obtained by applying Vujanović-Djukić's generalized Noether's theorem ([1], pp. 118–120, [6]), while the so formulated requirement is equivalent to their requirement for the existence of at least one particular solution of the corresponding so-called generalized Killing's equations.

Example 3 For a nonlinear oscillator in a resisting medium with linear damping, defined by (1.7), the equation for determining the function φ (3.8) has the form

$$\dot{x}\frac{\partial \varphi}{\partial x} + \ddot{x}\frac{\partial \varphi}{\partial \dot{x}} + \frac{\partial \varphi}{\partial t} + 2k\varphi = 2kL, \tag{3.10}$$

because of $\partial L/\partial t = 0$, and the Lagrangian equation (3.9) for this oscillator is

$$m\ddot{x} + m\omega^2 x + mbx^n = -2mk\dot{x}.$$

If in the previous relation (3.10) we substitute $\ddot{x}$ by the corresponding expression from this equation and the Lagrangian L by the expression (1.7), it becomes

$$\begin{aligned}&\dot{x}\frac{\partial \varphi}{\partial x} + (-\omega^2 x - bx^n - 2k\dot{x})\frac{\partial \varphi}{\partial \dot{x}} + \frac{\partial \varphi}{\partial t} + 2k\varphi \\ &= 2k\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{mb}{n+1}x^{n+1}\right).\end{aligned} \tag{3.11}$$

A particular solution of this equation must contain a term, which is linear with respect to x and $\dot{x}$, because x^2 and $\dot{x}^2$ appear in the right-hand side of (3.11), and also an additional term, which depends only on x because of the last term in (3.11)

$$\varphi(x, \dot{x}) = Ax\dot{x} + B(x). \tag{3.12}$$

If in the relation (3.11) we substitute φ by this expression, we obtain

$$\dot{x}\left(A\dot{x}+\frac{\mathrm{d}B}{\mathrm{d}x}\right)+(-\omega^2 x - bx^n - 2k\dot{x})\cdot Ax+$$
$$2k\left[Ax\dot{x}+B(x)\right]=mk\dot{x}^2 - mk\omega^2x^2-\frac{2mbk}{n+1}x^{n+1},$$

which will be satisfied if

$$A=mk,\quad \dot{x}\frac{\mathrm{d}B}{\mathrm{d}x}-Abx^{n+1}+2kB=-\frac{2mkb}{n+1}x^{n+1}, \tag{3.13}$$

and the second relation can be written in the form

$$\frac{\mathrm{d}B}{\mathrm{d}t}+2kB=mkb\left(1-\frac{2}{n+1}\right)x^{n+1}. \tag{3.14}$$

Then, a solution of this linear differential equation (with the integration constant zero) is

$$B(t)=e^{-2kt}mkb\left(1-\frac{2}{n+1}\right)\int x^{n+1}(t)e^{2kt}\mathrm{d}t, \tag{3.15}$$

and the corresponding particular solution $\varphi(t)$ is the sum of this expression and the term $mkx\dot{x}$.

Therefore, all the requirements for the existence of the integrals of motion of the type (3.6) are satisfied, and having in mind that

$$\frac{\partial L}{\partial \dot{x}}\dot{x}-L=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+\frac{mb}{n+1}x^{n+1},$$

it follows that the corresponding energy-like conservation law is valid in the form

$$\mathcal{E}^{\text{ext}}=e^{2kt}\left[\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+\frac{mb}{n+1}x^{n+1}+\varphi(x,\dot{x})\right]=\text{const}, \tag{3.16}$$

where

$$\varphi(x,\dot{x})=mkx\dot{x}+e^{-2kt}mkb\left(1-\frac{2}{n+1}\right)\int x^{n+1}(t)e^{2kt}\mathrm{d}t. \tag{3.17}$$

For a special case $b=0$, when this nonlinear oscillator becomes the usual linear oscillator, this result will be reduced to the energy-like conservation law for the usual oscillator with linear damping,

$$\mathcal{E}^{\text{ext}}=e^{2kt}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+mkx\dot{x}\right)=\text{const}, \tag{3.18}$$

(compare with [1], pp. 97–100). But, for $k=0$, when $\varphi=0$ as well, it reduces to the energy conservation law for this nonlinear oscillator in the field of two potential forces,

$$\mathcal{E}=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+\frac{mb}{n+1}x^{n+1}=\text{const}. \tag{3.19}$$

Example 4 A linear oscillator in a resisting medium, on an inclined plane, which is moving in the horizontal direction according to the law $x_A=Vt$ (continuation of Example 2).

Here, Eq. (3.8) for determining the function φ has the following form:

$$\dot{\rho}\frac{\partial\varphi}{\partial\rho}+\dot{q}_0\frac{\partial\varphi}{\partial q_0}+\ddot{\rho}\frac{\partial\varphi}{\partial\dot{\rho}}+\ddot{q}_0\frac{\partial\varphi}{\partial\dot{q}_0}+\frac{\partial\varphi}{\partial t}+2k\varphi=2kL, \tag{3.20}$$

because of $\partial L/\partial t=0$, and the Lagrangian is given by the expression (2.9), that is,

$$L=\frac{1}{2}m(\dot{\rho}^2-2\dot{\rho}\dot{q}_0\cos\alpha+\dot{q}_0^2)-\frac{1}{2}m\omega^2(\rho-\rho_0)^2,$$

which suggests the introduction of the quantity

$$\eta = (\rho - \rho_0) - q_0 \cos\alpha \quad \Rightarrow \quad \dot{\eta} = \dot{\rho} - \dot{q}_0 \cos\alpha. \tag{3.21}$$

Since

$$\dot{\eta}\frac{\partial\varphi}{\partial\eta} = \left(\frac{\partial\eta}{\partial\rho}\dot{\rho} + \frac{\partial\eta}{\partial q_0}\dot{q}_0\right)\frac{\partial\varphi}{\partial\eta} = \dot{\rho}\frac{\partial\varphi}{\partial\rho} + \dot{q}_0\frac{\partial\varphi}{\partial q_0},$$

and a similar relation is valid for $\ddot{\eta}(\partial\varphi/\partial\dot{\eta})$, it follows that the previous relation (3.20) takes the more concise form

$$\dot{\eta}\frac{\partial\varphi}{\partial\eta} + \ddot{\eta}\frac{\partial\varphi}{\partial\dot{\eta}} + \frac{\partial\varphi}{\partial t} + 2k\varphi = 2kL. \tag{3.22}$$

By means of the relation (3.21), putting $\rho - \rho_0 = \eta + q_0\cos\alpha$, this Lagrangian (2.9) can be expressed as a function of η and $\dot{\eta}$ as

$$L = \frac{1}{2}m(\dot{\eta}^2 - \omega^2\eta^2) + \frac{1}{2}m\left(\dot{q}_0^2\sin^2\alpha - 2\omega^2\eta q_0\cos\alpha - \omega_0^2 q_0^2\cos^2\alpha\right).$$

If we insert an a priori given solution $q_0 = Vt$ for the second generalized coordinate in this expression, its second term can be represented in the form $\frac{\mathrm{d}}{\mathrm{d}t}(\ldots)$, where the expression in the brackets is some function only of time. Here, let us note that the Lagrangians that differ only for a term in the form of a time derivative of some function are equivalent, that is, they give the same Lagrangian equations. Therefore, if we omit the last term in the previous expression, we get the equivalent Lagrangian

$$L_{eq} = \frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}m\omega^2\eta^2. \tag{3.23}$$

The results obtained in this way, namely Eq. (3.22) and the Lagrangian (3.23), differ from the corresponding equation (3.10) and the Lagrangian in the previous example for $b = 0$ only by the presence of η instead of x. This suggests that the corresponding generalized nonpotential force, associated with this complex problem and expressed through η, needs to be $Q_\eta^* = -2mk\dot{\eta}$. Namely, if we transform this expression by means of (3.20) as

$$Q_\eta^* = -2mk\dot{\eta} = -2mk\left(\frac{\partial\eta}{\partial\rho}\dot{\rho} + \frac{\partial\eta}{\partial q_0}\dot{q}_0\right) = -2mk\dot{\rho} + 2mk\cos\alpha\ \dot{q}_0,$$

and put an a priori given solution $q_0 = Vt$ in this expression, the second term becomes constant, which is not important for the considered problem. Thus, Q_η^* can be considered as equivalent generalized nonpotential force, which corresponds to the unique given nonpotential force $Q_\rho^* = -2mk\dot{\rho}$, and the corresponding Lagrangian equation for η according to (3.23) will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\eta}} - \frac{\partial L}{\partial\eta} = -2mk\dot{\eta}. \tag{3.24}$$

If we write this expression explicitly,

$$m\ddot{\eta} + m\omega^2\eta = -2mk\dot{\eta},$$

and in the relation (3.22) substitute $\tilde{\eta}$ by the corresponding expression from this equation and L by the expression (3.23), it obtains the form

$$\dot{\eta}\frac{\partial\varphi}{\partial\eta} + (-\omega^2\eta - 2k\dot{\eta})\frac{\partial\varphi}{\partial\dot{\eta}} + \frac{\partial\varphi}{\partial t} + 2k\varphi = 2k\left(\frac{1}{2}m\dot{\eta}^2 - \frac{1}{2}m\omega^2\eta^2\right). \tag{3.25}$$

This relation is identical in the form to the relation (3.11) for $b = 0$, having η instead of x. Therefore, one particular solution of this partial differential equation analogously to (3.17) is

$$\varphi(\eta, \dot{\eta}, t) = mk\eta\dot{\eta}, \tag{3.26}$$

which can be verified also directly by inserting this expression in the previous relation. Thus, the corresponding integral of motion can be immediately obtained by substituting x by η in the relation (3.18),

$$\mathcal{E}^{\text{ext}} = e^{2kt}\left(\frac{1}{2}m\dot{\eta}^2 + \frac{1}{2}m\omega^2\eta^2 + mk\eta\dot{\eta}\right) = \text{const.} \tag{3.27}$$

This is the energy-like conservation law, which coincides with the corresponding integral of motion in the extended Lagrangian formalism for the rheonomic systems (Mušicki [5]).

This example illustrates how one can obtain the energy-like conservation laws, by finding only one particular solution of the partial differential equation (3.8). This method is simpler and more practical than the method based on the Vujanović–Djukić's generalized Noether's theorem.

4 Noether's theorem for pseudoconservative systems

4.1 Total variation in action

Another approach to the analysis of the energy relations is possible by means of Emmy Noether's theorem, which is based on the total variation in the action. In this aim, let us start from its definition

$$\Delta W = \int_{\bar{t}_0}^{\bar{t}_1} \tilde{L}(\bar{q}^i, \dot{\bar{q}}^i, \bar{t})d\bar{t} - \int_{t_0}^{t_1} \tilde{L}(q^i, \dot{q}^i, t)\mathrm{d}t \tag{4.1}$$

where

$$\bar{q}^i = q^i + \Delta q^i, \quad \dot{\bar{q}}^i = \dot{q}^i + \Delta\dot{q}^i, \quad \bar{t} = t + \Delta t, \tag{4.2}$$

and then apply the same procedure as in the usual formulation of mechanics. Namely, we can approximate the first integral by the integral in the interval (t_0, t_1), then develop the function $\tilde{L}(\bar{q}^i, \dot{\bar{q}}^i, \bar{t})$ in Taylor's series and express it as a function of Δq^i and Δt, and afterward ignore all the infinitesimal quantities of higher order. In this way, the total variation in the action obtains the following form (see e.g., Dobronravov [7], pp. 142–146)

$$\begin{aligned}\Delta W = \int_{t_0}^{t_1} \Bigg\{ &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \tilde{L}}{\partial \dot{q}^i}(\Delta q^i - \dot{q}^i\Delta t) + \tilde{L}\Delta t\right] \\ &+ \left(\frac{\partial \tilde{L}}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^i}\right)(\Delta q^i - \dot{q}^i\Delta t)\Bigg\}\,\mathrm{d}t,\end{aligned} \tag{4.3}$$

This formula has the same form as in the usual formulation of mechanics with only one difference that $\tilde{L}$ is present instead of L, but let us emphasize that nevertheless every step in proving this formula in the usual formulation of mechanics in this process is valid too.

4.2 Modified Emmy Noether's theorem

Based on this result and due to the simplified form of the Lagrangian equations (1.3), and following the procedure in the usual formulation, it is possible to formulate the corresponding Emmy Noether's theorem for the pseudoconservative systems as well. Now, take the total variations Δq^i and Δt in the following form:

$$\Delta q^i = \overline{q}i - q^i = \varepsilon^m \xi_m^i(q^k, \dot{q}^k, t), \quad \Delta t = \bar{t} - t = \varepsilon^m \xi_m^0(q^k, \dot{q}^k, t), \tag{4.4}$$

where $\mathcal{E}^m$ $(m = 1, 2, \ldots, r)$ are arbitrary infinitesimal parameters. Here, it is assumed that these functions can be dependent on the generalized velocities too, which is in accordance with Vujanović [1]. Let us choose these transformations so that the Hamilton's action remains invariant ($\Delta W = 0$) or it changes up to the gauge term, in which case

$$\Delta W = \int_{t_0}^{t_1} \dot{\Lambda}\left(q^k, \dot{q}^k, t\right) \mathrm{d}t, \tag{4.5}$$

where Λ, the so-called gauge function, can be any function of q^i, $\dot{q}^i$ and t, which includes the first case as well for $\Lambda = 0$.

If ΔW is substituted by expression (4.3), after some rearrangement, we come to the relation

$$\int_{t_0}^{t_1} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \tilde{L}}{\partial \dot{q}^i}\Delta q^i + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t - \Lambda\right] + \left(\frac{\partial \tilde{L}}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^i}\right)(\Delta q^i - \dot{q}^i \Delta t)\right\} \mathrm{d}t = 0,$$

and if we substitute Δq^i and Δt by (4.4) and put $\Lambda = \varepsilon^m \Lambda_m$, we obtain

$$\int_{t_0}^{t_1} \varepsilon^m \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \tilde{L}}{\partial \dot{q}^i}\xi^i_m + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right)\xi^0_m - \Lambda_m\right] + \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^i}\right)\left(\xi^i_m - \dot{q}^i \xi^0_m\right)\right\} \mathrm{d}t = 0. \tag{4.6}$$

This relation implies the following conclusion: If the Lagrangian equations for the pseudoconservative systems (1.3) are satisfied, then the second term in this relation vanishes, and because of the arbitrariness of the time interval (t_0, t_1) and the parameters ε^m, it follows that

$$I_m = \frac{\partial \tilde{L}}{\partial \dot{q}^i}\xi^i_m + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right)\xi^0_m - \Lambda_m = \text{const} \quad (m = 1, 2, \ldots, r). \tag{4.7}$$

Therefore, for every transformation of the generalized coordinate and time (4.4), which preserves the action invariant or changes it up to the gauge term, there are r independent integrals of motion of the form (4.7). In this way, by introducing the Lagrangian $\tilde{L} = e^{2kt}L$, Noether's theorem becomes applicable to the pseudoconservative systems in the same form as for the systems with potential forces.

In order to find the transformations of the generalized coordinates and time (4.4) that satisfy the requirements of Noether's theorem, B. Vujanović (see [1], pp. 80–83) transformed the requirement for the existence of the integrals of motion (4.5) and expressed it in terms of the functions ξ^i_m and ξ^0_m (so-called basic Noether's identity). So, this problem was reduced to finding at least one particular solution for those functions that satisfies the so formulated requirement or from it implied the generalized Killing's equations.

The described procedure can be applied in this case as well, by generalizing it to multi-parameter transformations and by using only and consistently the total variations. To this aim, let us start form the requirement (4.5) and transform the left-hand side in the following way:

$$\Delta W = \int_{t_0}^{t_1} \Delta(\tilde{L}\mathrm{d}t) = \int_{t_0}^{t_1} \left[\Delta\tilde{L} + \tilde{L}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t,$$

and, after writing explicitly the term $\Delta\tilde{L}$, this requirement (4.5) becomes

$$\int_{t_0}^{t_1} \left[\frac{\partial \tilde{L}}{\partial q^i}\Delta q^i + \frac{\partial \tilde{L}}{\partial \dot{q}^i}\Delta \dot{q}^i + \frac{\partial \tilde{L}}{\partial t}\Delta t + \tilde{L}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) - \dot{\Lambda}\right]\mathrm{d}t = 0. \tag{4.8}$$

If we apply the relation between the operations Δ and $\mathrm{d}/\mathrm{d}t$

$$\Delta\dot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q^i) - \dot{q}^i \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t),$$

and after some rearrangement, the previous relation can be written in the form

$$\int_{t_0}^{t_1} \left[\frac{\partial \tilde{L}}{\partial q^i} \Delta q^i + \frac{\partial \tilde{L}}{\partial \dot{q}^i} \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q^i) + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{q}^i \right) \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t) + \frac{\partial \tilde{L}}{\partial t} \Delta t - \dot{\Lambda} \right] \mathrm{d}t = 0. \tag{4.9}$$

Now, if we substitute Δq^i and Δt by (4.4) and put $\Lambda = \varepsilon^m \Lambda_m$, we obtain

$$\int_{t_0}^{t_1} \varepsilon^m \left[\frac{\partial \tilde{L}}{\partial q^i} \xi_m^i + \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{\xi}_m^i + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{q}^i \right) \dot{\xi}_m^0 + \frac{\partial \tilde{L}}{\partial t} \xi_m^0 - \dot{\Lambda}_m \right] \mathrm{d}t = 0, \tag{4.10}$$

from which, due to the arbitrariness of the time interval (t_0, t_1) and the parameters ε^m, it follows

$$\frac{\partial \tilde{L}}{\partial q^i} \xi_m^i + \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{\xi}_m^i + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{q}^i \right) \dot{\xi}_m^0 + \frac{\partial \tilde{L}}{\partial t} \xi_m^0 - \dot{\Lambda}_m = 0 \quad (m = 1, 2, \ldots, r). \tag{4.11}$$

Therefore, if there is at least one particular solution of the system of equations (4.11) for the functions ξ_m^i, ξ_m^0 and Λ_m, then the requirement for the existence of the integrals of motion is satisfied and the corresponding integrals of motion are given by the formula (4.7).

4.3 Discussion of the obtained result

Now, we shall analyze the possible forms of the integrals of motion (4.7), based on the requirements for their existence (4.11). For this purpose, let us write this requirement in a more suitable form. Transform its first term using the Lagrangian equations (1.3) in the following way:

$$\frac{\partial \tilde{L}}{\partial q^i} \xi_m^i = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} \right) \xi_m^i = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} \xi_m^i \right) - \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{\xi}_m^i,$$

and instead of function ξ_m^i introduce new functions $\tilde{\varphi}_m(q^k, \dot{q}^k, t)$ by

$$\frac{\partial \tilde{L}}{\partial \dot{q}^i} \xi_m^i = -A \tilde{\varphi}_m(q^k, \dot{q}^k, t). \tag{4.12}$$

Hence, relation (4.11) takes the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(-A\tilde{\varphi}_m) + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i} \dot{q}^i \right) \dot{\xi}_m^0 + \frac{\partial \tilde{L}}{\partial t} \xi_m^0 - \dot{\Lambda}_m = 0, \tag{4.13}$$

and represents the requirement for the existence of the integrals of motion in the equivalent form, expressed in terms of $\tilde{\varphi}_m$ (instead of functions ξ_m^i), ξ_m^0 and Λ_m.

Let us choose the functions ξ_m^0 and Λ_m in the form

$$\xi_m^0 = A = \text{const}, \quad \Lambda_m = 0. \tag{4.14}$$

Then, the previous relation (4.13) is simplified and reduced to

$$-\frac{\mathrm{d}\tilde{\varphi}_m}{\mathrm{d}t} + \frac{\partial \tilde{L}}{\partial t} = 0, \tag{4.15}$$

from which we see that with such choice of ξ_m^0 and Λ_m all the functions $\tilde{\varphi}_m$ are equal ($\varphi_m = \varphi$). If we write this relation in the usual formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{2kt}\varphi) - \frac{\partial}{\partial t}(e^{2kt}L) = 0,$$

and represent the term $\mathrm{d}\varphi/\mathrm{d}t$ explicitly, after canceling e^{2kt}, it obtains the form

$$\dot{q}^i\frac{\partial \varphi}{\partial q^i} + \ddot{q}^i\frac{\partial \varphi}{\partial \dot{q}^i} + \frac{\partial \varphi}{\partial t} + 2k\varphi = \frac{\partial L}{\partial t} + 2kL. \tag{4.16}$$

This is the equation that determines the function φ, and it coincides with the previously obtained Eq. (3.8).

Accordingly, if there is at least one particular solution of this equation, then the requirement for the existence of integrals of motion is satisfied. In this case, the corresponding integrals of motion (4.7), based on (4.12) and (4.14), have the form

$$I_m = \frac{\partial \tilde{L}}{\partial \dot{q}^i}\xi_m^i + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i\right)\xi_m^0 - \Lambda_m = -A\tilde{\varphi} - \left(\frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i - \tilde{L}\right)A,$$

and this expression can be represented in the usual formulation as well, for example for $A = -1$ as

$$I_m = \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i - \tilde{L} + \tilde{\varphi} = e^{2kt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L + \varphi\right) = \text{const}. \tag{4.17}$$

In the special case when $\partial\tilde{L}/\partial t = 0$, from Eq. (4.15) it follows that $\mathrm{d}\tilde{\varphi}/\mathrm{d}t = 0$ and hence $\tilde{\varphi} = C = \text{const}$, so the integral of motion (4.17) has the following form:

$$I_m = \frac{\partial \tilde{L}}{\partial \dot{q}^i}\dot{q}^i - \tilde{L} + C = \text{const},$$

that is,

$$I'_m = \frac{\partial \tilde{L}}{\partial q^i}\dot{q}^i - \tilde{L} = e^{2kt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = \text{const}. \tag{4.18}$$

If the partial differential equation (4.16) does not have any particular solution for the function φ, the observed pseudoconservative system does not have any integral of motion.

All these results are in complete accordance with the results from the analysis of the energy relations, obtained by means of the corresponding Lagrangian equations for pseudoconservative systems (Chapter 3).

Example 5 For a nonlinear oscillator in a resisting medium with linear damping, defined by (1.7), the requirement for the existence of the integrals of motion can be taken in the form (4.11), but it is better to take it in the equivalent and simpler form (4.13) for $q = x$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(-A\tilde{\varphi}_m) + \left(\tilde{L} - \frac{\partial \tilde{L}}{\partial \dot{x}}\dot{x}\right)\dot{\xi}_m^0 + \frac{\partial \tilde{L}}{\partial t}\xi_m^0 - \dot{\Lambda}_m = 0. \tag{4.19}$$

If again we take $\xi_m^0 = A$ and $\Lambda_m = 0$, this relation is reduced to Eq. (4.16), and using the corresponding equation of motion of this oscillator, it obtains the form

$$\begin{aligned}&\dot{x}\frac{\partial \varphi}{\partial x} + (-\omega^2 x - bx^n - 2k\dot{x})\frac{\partial \varphi}{\partial \dot{x}} + \frac{\partial \varphi}{\partial t} + 2k\varphi\\ &= 2k\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{mb}{n+1}x^{n+1}\right).\end{aligned} \tag{4.20}$$

This relation is identical to Eq. (3.11), where one particular solution is obtained in the form (3.17). Thus, the corresponding integral of motion (4.17) will be given by (3.16) and (3.17),

$$\begin{aligned}\mathcal{E}^{\text{ext}} &= e^{2kt}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\omega^2x^2+\frac{mb}{n+1}x^{n+1}+mkx\dot{x}\right)\\&\quad+mkb\left(1-\frac{2}{n+1}\right)\int x^{n+1}(t)e^{2kt}\mathrm{d}t=\text{const}.\end{aligned}\tag{4.21}$$

This energy-like conservation law for the considered nonlinear oscillator with linear damping has a similar form as the corresponding ones in a special class of the pseudoconservative systems, where there appears some time integral and which were elaborated by Vujanović and Jones by application of the Noether's theorem ([1], pp. 144–149).

Namely, in this monograph, they considered such systems for which it is possible, by suitable choice of the function ξ_m^i, ξ_m^0 and Λ_m, that the relation (4.11) is reduced to the form $\tilde{L}(q^i,\dot{q}^i,t)-\dot{\Lambda}_m=0$ (with our notation and terminology). Since in this case $\Lambda_m=\int\tilde{L}(q^i,\dot{q}^i,t)\mathrm{d}t$, the corresponding integrals of motion (4.7) are

$$\mathcal{E}^{\text{ext}}=\frac{\partial\tilde{L}}{\partial\dot{q}^i}\xi_m^i+\left(\tilde{L}-\frac{\partial\tilde{L}}{\partial\dot{q}^i}\dot{q}^i\right)\xi_m^0-\int\tilde{L}(q^i,\dot{q}^i,t)\mathrm{d}t=\text{const},\tag{4.22}$$

and for the nonlinear oscillator with linear damping this integral of motion was (for $m=1$)

$$I=e^{2kt}\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}\omega^2x^2+\frac{b}{n+1}x^{n+1}\right)+2k\int\tilde{L}(x,\dot{x},t)\mathrm{d}t=\text{const}.\tag{4.23}$$

Our result (4.21) differs from (4.23) only by the sum of the term $mkx\dot{x}$ and this time integral, but this is not contradictory, since in general a partial differential equation does not have unique particular solutions. The first integrand of motion (4.21) is obtained by the method of this theory, by finding a particular solution of the partial differential equation (3.8), and the second one, the Vujanović–Jones's integral of motion by means of the modified Noether's theorem. Although our integral is obtained by a quite different method, it is equally valid as the second one. This proves the correctness and significance of this theory, and even an advantage and applicability of our solution.

In addition, according to our opinion, the integral of motion (4.21) has some advantage, since it demonstrates directly, through the time integral, the influence of the nonlinear force on the integral of motion. In addition, for $b=0$, the time integral in (4.21) vanishes and this energy-like conservation law immediately reduces to (3.18), that is, to the corresponding one for the linear oscillator with linear damping. Contrary to this, for $b=0$, the time integral in (4.23) does not vanish and this energy-like conservation law will not be reduced to (3.18), although it can be obtained in an indirect way, by means of two particular solutions of this type. However, for $k=0$, both integrals of motion, (4.21) and (4.23), reduce to the same form (4.19), that is, to the energy conservation law for this nonlinear oscillator in the field of the potential forces.

5 Conclusions

(i) The pseudoconservative mechanical systems are defined as such systems for which it is possible to find a new Lagrangian in the form $\tilde{L}(q^i,\dot{q}^i,t)=f(t)L(q^i,\dot{q}^i,t)$, so that the system of their Lagrangian equations (1.1) can be transformed into the equivalent system of Euler–Lagrangian equations (1.3). It immediately implies that the form of their Lagrangian and Hamiltonian equations is same as for the systems with potential forces, and the influence of the nonconservative forces is contained in the factor $f(t)$.

(ii) The resolution of the problem whether a nonconservative system can be treated as a pseudoconservative one is reduced to finding at least one particular solution for the function $f(t)$, which results from the system of n differential equations (1.4). This can be realized by solving either directly this system of equations or their linear combination (2.1), reduced to the form (2.2).

(iii) The corresponding integrals of motion, valid under certain conditions and obtained by means of their Lagrangian equations or using the modified Noether's theorem, have an unusual form (3.4) or (3.6), in which there exists a certain function $\varphi(q^i, \dot{q}^i, t)$. The problem of the existence of such integrals of motion is reduced to finding at least one particular solution of the partial differential equation (3.8) for this function φ. So obtained integrals of motion are equivalent to so-called energy-like conservation laws, which have been obtained by Vujanović and Djukić by means of the generalized Noether's theorem for the nonconservative systems [1].

(iv) The main advantage of this formulation of such nonconservative systems, which can be treated as pseudoconservative, is that all the corresponding relations and equations formally have the same form as for the systems with potential forces. Due to this property, the study and representation of such systems are simplified, as in the procedure of finding their integrals of motion, and some general relations and laws can be represented in the form similar to the usual one, like the general principles of mechanics. In addition, for such systems, it is possible to employ some methods that are not applicable to the nonconservative systems, like the canonical transformations and Hamilton–Jacobi's method, as it can be shown.

References

1. Vujanović, B., Jones, S.: Variational Methods in Nonconservative Phenomena (monograph). Acad. Press, Boston (1989)
2. Goldstein, H.: Classical Mechanics. 2nd edn. Addison-Wesley, Massachusetts (1980)
3. Whittaker, E.: A Treatise of the Analytical Dynamics of Particles and Rigid Bodies. 4th edn. Univ. Press, Cambridge (1952)
4. Mušicki, Dj.: Extended Lagrangian formalism and the corresponding energy relations. Eur. J. Mech. A/Solids **24**, 227–242 (2005)
5. Mušicki, Dj.: A Contribution to the theory of the extended Lagrangian formalism for rheonomic systems. Theor. Appl. Mech. (Belgrade) **36**, 47–83 (2009)
6. Djukić, Dj., Vujanović, B.: Noether's theory in classical nonconservative mechanics. Acta Mech. **23**, 17–27 (1975)
7. Dobronravov, B.: Osnovi analitičeskoj mehaniki. Višaja škola, Moscow (in Russian) (1976)