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Oscillator with non-integer order nonlinearity and time variable parameters

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Abstract In this paper, the vibrations of the oscillator with nonlinearity of integer or non-integer order and with mass variable parameters are considered. New appreciative analytical procedures are developed: first, based on the generating solution that is the exact analytic solution of the system with constant parameters and the second, based on the approximate solution in the form of a trigonometric function with exact period of vibration of the system with constant parameters. For the both methods, the assumed trial solutions represent the perturbed versions of the solutions of the equations with constant parameters, where the amplitude and phase of vibration are supposed to be time variable. The amplitude and phase functions are determined using the averaging procedure over the period of vibration. The obtained approximate analytic solutions are compared with numerical ones. It is shown that the developed methods are accurate for the monotone slow time variable systems. The example of mass variable oscillator is considered. The influence of mass variation, small linear viscous damping and of the reactive force is investigated, too.

1 Introduction

The oscillators with time variable parameters are usually divided into two groups: first, with parameters that are time periodical functions and the oscillations are called parametrically excited vibrations, and the second, where the parameters are monotone time functions and the vibrations are usually called unstationary vibrations (see for example [1,2]). In this paper, only the second type of oscillators will be considered. The parameter variation is assumed to be slow as is the function of the slow variable time. This type of oscillators is widely applied in textile, carpet and cable industry but also in various machines like sieves, separators and centrifuges. In general, the mathematical model of the one-degree-of-freedom oscillator with slow time variable parameter is

$$\ddot{x} + \omega^2(\tau)x\,|x|^{\alpha-1} = \varepsilon f\,(\tau, x, \dot{x}) \tag{1}$$

where α is the order of nonlinearity (an integer or non-integer), ω is the time variable parameter, $\tau = \varepsilon t$ is the slow time, $\varepsilon \ll 1$ is a small parameter, and εf is the additional small nonlinear function.

For the special case when $\alpha = 1$, a significant number of analytical solving procedures of the differential equations with slow time variable parameters and weak nonlinearity are developed. Let us mention some of them: the Mitropolski–Bogolubov method [3], Krylov–Bogolubov method [4–6], the multiple-scale method [1], etc. For all of them, it is common that the solution is based on the exact solution of the linear differential equation with constant parameters where, obviously, the nonlinearity is neglected. The trial solution is adopted as the perturbed version of the generating solution. The accuracy of the obtained approximate solution depends on the level of the nonlinearity and the velocity of mass variation: the smaller the nonlinearity and the

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slower the parameter variation, the difference between the analytical and numerical solution of the problem is smaller. Yuste [7] was the first to extend the Krylov–Bogolubov method for solving of the differential equations with strong cubic nonlinearity ($\alpha = 3$) and time variable parameters. The solution is assumed in the form of the Jacobi elliptic function, where the amplitude, modulus and frequency of the function are supposed to be time functions. The lack of the procedure is that it is applicable only for the oscillator with the strong cubic nonlinearity.

The aim of the paper is to consider the oscillator (1) where the parameters are time variable functions and the order α of the nonlinearity is an integer or a non-integer. Equation (1) represents the generalization of the previously considered oscillators with slow time variable parameters. In this paper, the procedure developed for strong nonlinear differential equations with constant parameters [8]

$$\ddot{x} + \omega_0^2 x\, |x|^{\alpha-1} = 0 \tag{2}$$

with initial conditions

$$x(0) = x_0, \ \ \dot{x}(0) = 0 \tag{3}$$

where $\omega_0^2 \equiv \omega^2(0) = const.$, is adopted for solving of the differential equation (1). Two methods based on the procedure are developed: in Sect. 2, the trial solution of (1) is introduced in the form [9] of the exact solution of (2), but with time variable amplitude and phase, and in Sect. 3, the approximate solution is assumed as a trigonometric function with time variable period function whose profile corresponds to the exact analytical period [10] of vibration of the oscillator (2). For both procedures, the time variable amplitude, frequency and phase have to satisfy the averaged version of the differential equation of motion (1). In Sect. 4, the solutions obtained with the two methods are compared. Two examples are considered: the linear oscillator and the non-integer order nonlinear oscillator with time variable parameter. To prove the accuracy of the approximate analytical solution of the non-integer order oscillator with time variable parameter, it is compared with the numerical solution of the corresponding differential equation. In Sect. 5, the nonlinear oscillator with linear damping and time variable mass is investigated. The reactive force is of Levi-Civita type [11]. The approximate analytical solution is obtained by applying the aforementioned methods. As the special case, the linear oscillator with time variable parameter is considered, and the obtained solutions are compared with previous ones (see [12–23]). In Sect. 6, the conclusions are given.

2 Solving procedure based on the exact solution of the oscillator with constant parameters

The exact analytical solution of Eq. (2) and its first time derivative have the form of the cosine (ca) and sine (sa) Ateb functions [24–26], that is,

$$x = Aca(\alpha, 1, \psi) \tag{4}$$

and

$$\dot{x} = -\frac{\sqrt{2}}{\sqrt{\alpha+1}}\, |\omega_0|\, A^{\frac{\alpha+1}{2}} sa(1, \alpha, \psi) \tag{5}$$

where

$$\psi = \frac{\sqrt{\alpha+1}}{\sqrt{2}}\, |\omega_0|\, A^{\frac{\alpha-1}{2}} t + \theta. \tag{6}$$

A and θ are arbitrary constants, and the time derivatives of the cosine and sine Ateb functions are $\frac{\mathrm{d}}{\mathrm{d}z} ca(\alpha, 1, \psi) = -\frac{2}{\alpha+1} sa(1, \alpha, \psi)$ and $\frac{\mathrm{d}}{\mathrm{d}z} sa(1, \alpha, \psi) = ca^{\alpha}(\alpha, 1, \psi)$, respectively. Substituting (4) and the time derivative of (5) into (2), the equation is identically satisfied.

The solving procedure suggested in this paper assumes the approximate analytical solution of (1) and its first time derivative in the forms (4) and (5), respectively, but with time variable amplitude, $A(t)$, and time variable phase, $\psi(t)$, that is,

$$x = A(t) ca(\alpha, 1, \psi(t)) \tag{7}$$

and

$$\dot{x} = -\frac{\sqrt{2}}{\sqrt{\alpha+1}} |\omega(\tau)| \, A(t)^{\frac{\alpha+1}{2}} sa(1, \alpha, \psi(t)), \tag{8}$$

where, according to (6), the first time derivative of the time variable phase is

$$\dot{\psi}(t) = \frac{\sqrt{\alpha+1}}{\sqrt{2}} \omega(\tau) A(t)^{\frac{\alpha-1}{2}} + \dot{\theta}(t). \tag{9}$$

Let us find the time derivative of the approximate solution (6)

$$\dot{x} = -\frac{\sqrt{2}}{\sqrt{\alpha+1}} |\omega(\tau)| \, A(t)^{\frac{\alpha+1}{2}} sa(1, \alpha, \psi(t)) + \dot{A} ca(\alpha, 1, \psi(t)) - \frac{2}{\alpha+1} A(t)\dot{\theta}(t) sa(1, \alpha, \psi(t), \tag{10}$$

and compare with (8). The relation (10) is equal to (8), if the following relation is satisfied:

$$\dot{A} ca - \frac{2A\dot{\theta}}{\alpha+1} sa = 0, \tag{11}$$

where $A \equiv A(t)$, $\theta \equiv \theta(t)$, $\omega \equiv \omega(\tau)$, $\psi \equiv \psi(t)$, $ca \equiv ca(\alpha, 1, \psi)$ and $sa \equiv sa(\alpha, 1, \psi)$.

Substituting (7), (8) and the time derivative of (8) into (1), it follows

$$\dot{A} sa + \frac{2}{\alpha+1} ca^{\alpha} A\dot{\theta} = -\frac{\varepsilon}{\omega}\omega' \frac{2}{\alpha+1} A sa - \frac{\sqrt{2}}{\sqrt{\alpha+1}} \frac{1}{\omega A^{\frac{\alpha-1}{2}}} \varepsilon f\left(\tau, Aca, -\frac{\omega\sqrt{2}}{\sqrt{\alpha+1}} A^{\frac{\alpha+1}{2}} sa\right). \tag{12}$$

As we see, the second-order differential equation (1) is replaced with two first-order differential equations (11) and (12). After some modification, (11) and (12) are rewritten as

$$\dot{A} = -\frac{\varepsilon}{\omega}\omega' \frac{2}{\alpha+1} A sa^2 - \frac{\sqrt{2}}{\sqrt{\alpha+1}} \frac{1}{\omega A^{\frac{\alpha-1}{2}}} \varepsilon f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}} \omega A^{\frac{\alpha-1}{2}} sa\right) sa, \tag{13}$$

$$A\dot{\theta} = -\frac{\varepsilon}{\omega}\omega' A saca - \frac{\sqrt{\alpha+1}}{\sqrt{2}} \frac{1}{\omega A^{\frac{\alpha-1}{2}}} \varepsilon f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}} \omega A^{\frac{\alpha-1}{2}} sa\right) ca, \tag{14}$$

where $\omega' = \mathrm{d}/\mathrm{d}\tau$. Usually, to find the exact solution of Eqs. (13) and (14) is very complicated. This is the point where the averaging of Ateb functions over the period $2\Pi_\alpha$ is done. Introducing Π_α as

$$\Pi_\alpha = B\left(\frac{1}{\alpha+1}, \frac{1}{2}\right),$$

where B is the beta function [27], and using (13) and (14), we obtain

$$\begin{aligned} \dot{A} = &-\varepsilon A \frac{\omega'}{\omega} \left(\frac{2}{\alpha+1}\right) \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} sa^2 \mathrm{d}\psi \\ &- \frac{\sqrt{2}}{\sqrt{\alpha+1}} \frac{1}{\omega A^{\frac{\alpha-1}{2}}} \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} \varepsilon f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}} \omega A^{\frac{\alpha-1}{2}} sa\right) sa \mathrm{d}\psi, \end{aligned} \tag{15}$$

$$\dot{\theta} = -\varepsilon \frac{\omega'}{\omega} \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} saca \mathrm{d}\psi - \frac{\sqrt{\alpha+1}}{\sqrt{2}} \frac{1}{2\Pi_\alpha \omega A^{\frac{\alpha+1}{2}}} \int_0^{2\Pi_\alpha} \varepsilon f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}} \omega A^{\frac{\alpha-1}{2}} sa\right) ca \mathrm{d}\psi, \tag{16}$$

that is, according to (9),

$$\dot{\psi}(t) = \frac{\sqrt{\alpha+1}}{\sqrt{2}}\omega A^{\frac{\alpha-1}{2}} - \frac{\varepsilon}{2\Pi_\alpha}\frac{\omega'}{\omega}\int_0^{2\Pi_\alpha} saca\mathrm{d}\psi$$
$$-\frac{\sqrt{\alpha+1}}{2\Pi_\alpha\sqrt{2}}\frac{A^{-\frac{\alpha+1}{2}}}{\omega}\int_0^{2\Pi_\alpha}\varepsilon f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}}\omega A^{\frac{\alpha-1}{2}} sa\right)ca\mathrm{d}\psi. \tag{17}$$

Using the results given by Drogomirecka [25]

$$\int_0^{2\Pi_\alpha} saca\mathrm{d}\psi = 0, \quad \int_0^{2\Pi_\alpha} sa^2\mathrm{d}\psi = 2B\left(\frac{3}{\alpha+1}, \frac{1}{2}\right).$$

We have the averaged differential equations of motion

$$\dot{A} = -\frac{\omega'}{\omega}\left(\frac{2\varepsilon A}{\alpha+1}\right)\frac{1}{\Pi_\alpha}B\left(\frac{3}{\alpha+1}, \frac{1}{2}\right)$$
$$-\frac{\varepsilon\sqrt{2}}{2\sqrt{\alpha+1}}\frac{1}{\omega A^{\frac{\alpha-1}{2}}}\frac{1}{\Pi_\alpha}\int_0^{2\Pi_\alpha} f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}}\omega A^{\frac{\alpha-1}{2}} sa\right)sa\mathrm{d}\psi, \tag{18}$$

$$\dot{\psi} = \frac{\sqrt{\alpha+1}}{\sqrt{2}}\omega A^{\frac{\alpha-1}{2}} - \frac{\varepsilon\sqrt{\alpha+1}}{2\Pi_\alpha\sqrt{2}}\frac{A^{-\frac{\alpha+1}{2}}}{\omega}\int_0^{2\Pi_\alpha} f\left(\tau, Aca, -\frac{\sqrt{2}}{\sqrt{\alpha+1}}\omega A^{\frac{\alpha-1}{2}} sa\right)ca\mathrm{d}\psi. \tag{19}$$

As Eq. (18) is an uncoupled first-order differential equation, it has to be solved, first. Substituting the obtained $A(t)$ function into (7) and (19) and using the initial conditions (3), the approximate solution (7) of the differential equation (1) is determined.

2.1 Purely nonlinear oscillator with time variable parameter

Let us consider the purely nonlinear oscillator, when the function f in the differential equation (1) is zero,

$$\ddot{x} + \omega^2(\tau)x\,|x|^{\alpha-1} = 0. \tag{20}$$

If $f = 0$, the amplitude and phase expressions according to (18) and (19) are

$$A = A_0\left(\frac{\omega_0}{\omega}\right)^{\beta}, \tag{21}$$

$$\psi = \psi_0 + \frac{\sqrt{\alpha+1}}{\sqrt{2}}A_0^{\frac{\alpha-1}{2}}\omega_0\int\left(\frac{\omega}{\omega_0}\right)^{1-\gamma}\mathrm{d}t, \tag{22}$$

where A_0 and ψ_0 are arbitrary constants,

$$\gamma = \left(\frac{\alpha-1}{2}\right)\beta \tag{23}$$

and

$$\beta = \left(\frac{2}{\alpha+1}\right)\frac{1}{\Pi_\alpha}B\left(\frac{3}{\alpha+1}, \frac{1}{2}\right). \tag{24}$$

It can be concluded that the amplitude and phase of vibration depend not only on ω which is the time variable, but also on the order of nonlinearity α. Besides:

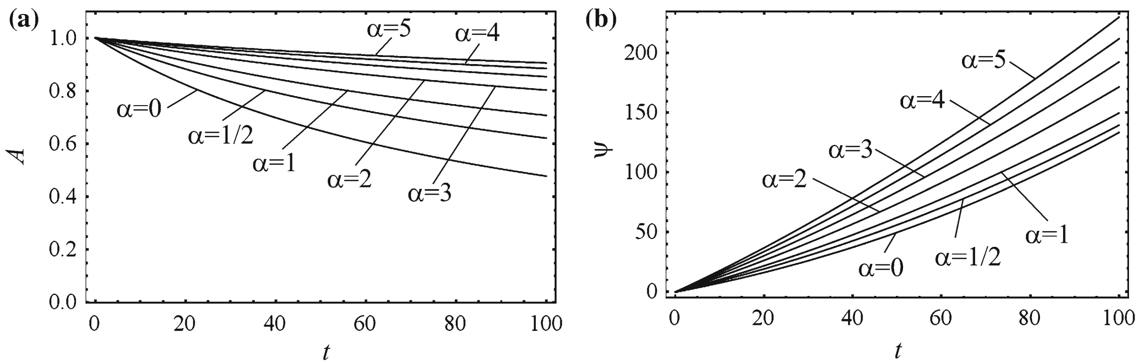

Fig. 1 **a** Amplitude-time and **b** phase-time diagrams for various values of α and increasing mass

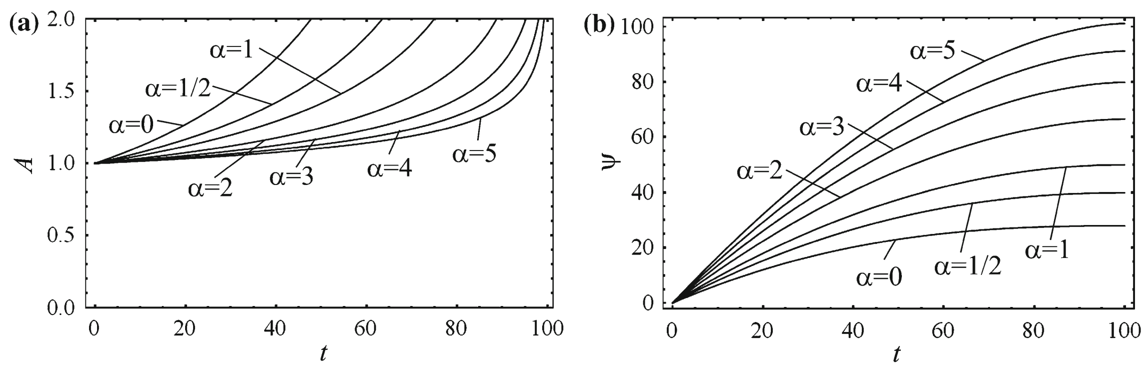

Fig. 2 **a** Amplitude-time and **b** phase-time diagrams for various values of α and decreasing mass

1. Analyzing the relation (24), it can be seen that the parameter β is positive for all values of α and has a tendency to decrease by increasing the parameter α: for $\alpha = 0$ we have $\beta = 16/15$, and for extremely high values of α, β tends to zero. Thereby, the amplitude of vibration (21) decreases in time for the case when ω is a monotone increasing time function and increases in time for ω a monotone decreasing time function independently on the order of nonlinearity α.
2. The parameter γ, given in (23), increases from $(-8/15)$ for $\alpha = 0$ to $\lim_{\alpha\to\infty} \gamma = 1/3$. For $\alpha < 1$ the parameter γ is negative, for $\alpha > 1$ it is positive and for $\alpha = 1$ it is zero. The influence of the nonlinearity order on the phase properties could not be in general discussed, but only for certain problems.

2.2 Purely nonlinear oscillator with linear time function

Let us consider the special case of the purely nonlinear oscillator (20) where ω is a linear time function

$$\omega = \omega_0(1 \pm \varepsilon t). \tag{25}$$

The amplitude-time expression is already given as (21), but the phase-time relation (22) transforms into

$$\psi = \psi_0 \pm \frac{\sqrt{\alpha+1}}{\sqrt{2}} A_0^{\frac{\alpha-1}{2}} \omega_0 \frac{(\omega/\omega_0)^{2-\gamma} - 1}{\varepsilon(2-\gamma)}, \tag{26}$$

where "+" sign in (26) corresponds to the "+"sign in (25) and the "−" sign in (26) to the "−" sign in (25). For the initial amplitude and phase $A_0 = 1$, $\psi_0 = 0$ and parameter values $\omega_0 = 1$ and $\varepsilon = 0.01$, the amplitude-time (21) and phase-time (26) diagrams for various orders of nonlinearity α are plotted (see Figs. 1, 2).

1. For the case when ω increases, the amplitude of vibration decreases in time proportional to α as given in Fig. 1a. The phase angle increases in time: the higher α, the increase is faster (Fig. 1b).
2. If ω decreases with time, the corresponding amplitude of vibration increases: for smaller α, the amplitude increase is faster (Fig. 2a). The phase angle increases for the case when ω is a linear decreasing function of time: the increase of the phase angle is faster for a higher order of nonlinearity α (Fig. 2b).

2.3 Comparison of the purely nonlinear oscillators

Let us compare the purely nonlinear oscillator ($f = 0$) with time variable parameter (20) with the corresponding oscillator with constant parameter (2). Comparing the amplitude and phase properties (21) and (26) for the oscillator with time variable parameter with the relations for the system with constant parameters (4) and (5), the amplitude and phase angle variations are, respectively,

$$A = A_0 k_A, \tag{27}$$

$$\psi = \psi_0 + k_\psi \frac{\sqrt{\alpha+1}}{\sqrt{2}} A_0^{\frac{\alpha-1}{2}} \omega_0 t \tag{28}$$

where the correction terms for the amplitude and phase are

$$k_A = \left(\frac{1}{1 \pm \varepsilon t}\right)^{\beta}, \tag{29}$$

$$k_\psi = \frac{(1 \pm \varepsilon t)^{2-\gamma} - 1}{(\pm)\varepsilon t(2-\gamma)}. \tag{30}$$

The correction terms (29) and (30) give the correlation between the decrement of the parameter variation εt and the order of nonlinearity α.

1. For $\varepsilon t \to 0$, independently on the order of nonlinearity α, both correction parameters, k_A and k_ψ, tend to 1, and the motion of the oscillator is near that with constant parameters.
2. According to (29), for $1 + \varepsilon t \to \infty$,the amplitude correction parameter k_A is zero, while it is one when $\varepsilon t \to 0$. If $1 - \varepsilon t \to 0$, the amplitude correction parameter tends to infinity independently on the order of nonlinearity.
3. The phase correction term k_ψ depends on the values of (εt) and for high values of (εt) corresponds to the limes value of (30), that is,

$$\lim_{\varepsilon t \to \infty} k_\psi \approx \lim_{\varepsilon t \to \infty} (1 \pm \varepsilon t)^{1-\gamma}. \tag{31}$$

4. Using the result 2. in Sect. 2.1, that is, $\gamma \in [-8/15, 1/3)$ for $\alpha \in [0, \infty)$, and analyzing the relation (31), it can be concluded that for $1 + \varepsilon t \to \infty$, independently on the value of α, the phase correction term $k_\psi$tends to infinity: the higher the smaller the value of α, the faster the increase. For $1 - \varepsilon t \to 0$, the phase correction term tends to zero for all values of parameter α: the smaller the parameter α the faster the decrease of k_ψ to zero.

The method suggested in this Section is based on the Ateb function [23] whose application gives very accurate results in spite of the existence of time variable parameters. The main disadvantage of the Ateb functions is their complexity for calculation and that they are not already included into the list of known functions [27].

3 Solving procedure based on the exact period of vibration of the oscillator with constant parameters

To avoid the complication connected to Ateb functions, as it is mentioned that they are not included in the list of known functions, let us suggest the solution of (1) in the form of a trigonometric function that is more appropriate for calculation.

For $\varepsilon = 0$, the generating solution of Eq. (2) with initial conditions (3) is assumed as a cosine function with a period that is specified to be the same as the period of vibration of the oscillator (2),

$$x = A \cos \psi \tag{32}$$

where

$$\psi = \theta + \Omega t. \tag{33}$$

A and θ are arbitrary constants, and

$$\Omega = \Omega_\alpha \omega_0 A^{\frac{\alpha-1}{2}} \tag{34}$$

is the exact frequency of vibration (see [10]). Substituting (32) with (33) and the first and the second time derivatives of (32), respectively,

$$\dot{x} = -A\Omega \sin\psi, \tag{35}$$
$$\ddot{x} = -A\Omega^2 \cos\psi, \tag{36}$$

Eq. (2) is approximately satisfied,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(A\cos\psi) + \omega_0^2 (A\cos\psi)^{\frac{\alpha-1}{2}} \approx 0. \tag{37}$$

According to the generating solution (32) with (33) of Eq. (2) and its first time derivative (35), the trial solution of Eq. (1) and its time derivative are assumed as

$$x = A(t)\cos\psi(t), \tag{38}$$
$$\dot{x} = -A(t)\Omega(\tau, A(t)) \sin\psi \tag{39}$$

with

$$\dot{\psi}(t) = \dot{\theta}(t) + \Omega(A(t), \tau), \tag{40}$$

where $A(t)$, $\theta(t)$ and $\psi(t)$ are time variable functions, and according to (34)

$$\Omega(A(t), \tau) = \Omega_\alpha \omega(\tau)(A(t))^{\frac{\alpha-1}{2}}. \tag{41}$$

Calculating the first time derivative of (38) and equating it with expression (39), it follows

$$\dot{A}(t)\cos\psi(t) - A(t)\dot{\theta}(t)\sin\psi(t) = 0. \tag{42}$$

Substituting (38), (39) and the time derivative of (39) into (1) and using the relation (37), we obtain

$$-\dot{A}\Omega\sin\psi - A\dot{A}\frac{\partial\Omega}{\partial A}\sin\psi - \varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\sin\psi - A\Omega\dot{\theta}\cos\psi = \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi), \tag{43}$$

where $A \equiv A(t)$, $\theta \equiv \theta(t)$, $\psi \equiv \psi(t)$. Hence, two first-order differential equations (42) and (43) replace the second-order differential equation (1). Solving (42) and (43) with respect to $\dot{A}$ and $\dot{\theta}$, we have

$$\dot{A}\left(\Omega - A\frac{\partial\Omega}{\partial A}\sin^2\psi\right) = -\varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\sin^2\psi - \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi)\sin\psi, \tag{44}$$
$$A\dot{\theta}\left(\Omega - A\frac{\partial\Omega}{\partial A}\sin^2\psi\right) = -\varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau}\sin\psi\cos\psi - \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi. \tag{45}$$

Averaging the differential equations in the period 2π, we obtain the following equations:

$$\dot{A} = -\frac{2\varepsilon A}{(5-\alpha)}\frac{1}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau} - \frac{4\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\omega\Omega_\alpha}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\sin\psi\,\mathrm{d}\psi \tag{46}$$

$$A\dot{\theta} = -\frac{2\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\Omega_\alpha\omega}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi\,\mathrm{d}\psi, \tag{47}$$

and from (40) and (41)

$$\dot{\psi} = \Omega_\alpha\omega A^{\frac{\alpha-1}{2}} - \frac{2\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\Omega_\alpha A\omega}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi\,\mathrm{d}\psi. \tag{48}$$

Solving the averaged differential equation (46) and substituting the obtained solution for A into (48), the approximate function ψ is obtained that gives the solution (38).

Remark The averaging procedure is valid for $\alpha \neq 5$, because the expressions (46)–(48) are singular for $\alpha = 5$.

For $f = 0$ and initial conditions A_0 and ψ_0, the solution of (46) is

$$A = A_0 (\frac{\omega_0}{\omega})^{\frac{2}{5-\alpha}} \tag{49}$$

which gives

$$\psi = \psi_0 + \Omega_\alpha A_0^{\frac{\alpha-1}{2}} (\omega_0)^{\frac{\alpha-1}{5-\alpha}} \int \omega^{1-\frac{\alpha-1}{5-\alpha}} \mathrm{d}t. \tag{50}$$

The relation (49) represents qualitatively the same result as (21) obtained by the previous method.

4 Comparison of the methods

Let us compare the suggested analytical methods and also the approximate solutions of Eq. (1) with the numerical one obtained by Runge–Kutta method.

4.1 Linear oscillator

For the linear oscillator, when $\alpha = 1$, the averaged differential equations (46) and (48) are

$$\dot{A} = -\frac{1}{2} \frac{\varepsilon A}{\omega} \frac{\mathrm{d}\omega}{\mathrm{d}\tau} - \frac{\varepsilon}{2\pi\omega} \int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\sin\psi \mathrm{d}\psi, \tag{51}$$

$$\dot{\psi}(t) = \omega(\tau) - \frac{\varepsilon}{2\pi\omega A} \int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi. \tag{52}$$

This result is previously published by Bessonov [13]. For $f = 0$ and initial conditions $A(0) = A_0$ and $\psi(0) = \psi_0$, the solution in the first approximation is

$$x = A_0 \left(\frac{\omega_0}{\omega}\right)^{1/2} \cos\left(\psi_0 + \int \omega \mathrm{d}t\right). \tag{53}$$

Considering the relations (21) and (22) for the linear case, when $\alpha = 1$, the approximate solution (7) is obtained as

$$x = A_0 \left(\frac{\omega_0}{\omega}\right)^{1/2} ca\left(1, 1, \psi_0 + \int \omega \mathrm{d}t\right). \tag{54}$$

Due to the fact that the cosine Ateb function for $\alpha = 1$ corresponds to the trigonometric cosine function (see [26]), the relation (54) transforms into (53). Both approximate methods give the same result, namely the amplitude of vibration is not constant, but depends on the variation of the parameter ω. The same is valid for the phase variation.

4.2 Non-integer order nonlinear oscillator

Let us consider the oscillator

$$\ddot{x} + (1 + 0.01t)^2 x^{4/3} = 0, \tag{55}$$

with initial conditions $x(0) = A_0 = 0.1$ and $\dot{x}(0) = 0$.

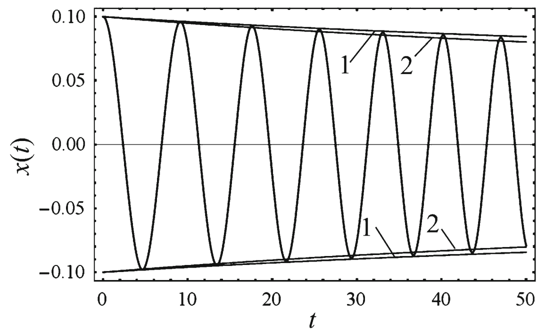

Fig. 3 The $x-t$ diagram obtained numerically, and the $A-t$ diagrams for the first method (1) and the second method (2)

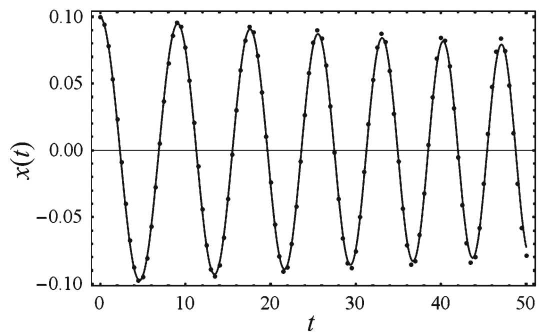

Fig. 4 The $x-t$ diagrams obtained analytically (*full line*) and numerically (*dotted line*)

According to the first procedure given in Sect. 2 (based on the exact solution of the oscillator with constant parameters), the amplitude-time variation (21) is

$$A = A_0\left(\frac{1}{1+0.01t}\right)^{0.42085}. \tag{56}$$

According to the second method given in Sect. 3 of this paper (based on the exact period of vibration of the oscillator with constant parameters), the amplitude-time relation (49) transforms into

$$A = A_0\left(\frac{1}{1+0.01t}\right)^{6/11}. \tag{57}$$

In Fig. 3, the numerical solution of (55) and the amplitude-time diagrams (56) and (57) are plotted. It can be seen that the first analytical procedure gives a very accurate result for the long time period, but the approximate solution (57) obtained using the second method is not so accurate. Due to the second procedure, the phase-time relation (50) is

$$\psi = 52.3809(1+0.01t)^{\frac{21}{11}} \tag{58}$$

where $\Omega_{4/3} = 0.96916$, and the approximate solution (38) has the form

$$x \approx 0.1\left(\frac{1}{1+0.01t}\right)^{6/11}\cos\left(\frac{1100}{21}(1+0.01t)^{\frac{21}{11}}\right). \tag{59}$$

In Fig. 4, the $x-t$ diagrams obtained analytically (59) and numerically by solving (55) are plotted. It can be seen that the difference between the solutions is negligible even for a long time period. The excellent agreement of the solutions is due to the fact that the approximate solution (59) has approximate periods that are very close to the exact ones.

5 Oscillator with time variable mass of Levi-Civita type and with a small damping

One of the mechanical systems with time variable parameters is the mass variable oscillator. If the absolute velocity of mass adding or separating is zero, the mathematical model of the oscillator is in general

$$m(\tau)\ddot{x} + k_\alpha x\,|x|^{\alpha-1} = -\varepsilon \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}\dot{x} - \varepsilon b\dot{x} \tag{60}$$

where the first term on the right side of Eq. (60) represents the reactive force that exists according to mass variation [11] and $\varepsilon b\dot{x}$ is the damping force. As the mass variation is slow and the damping coefficient is small, the reactive and damping force are small in comparison to the elastic force.

Comparing Eq. (60) with (1), we have

$$\omega = \sqrt{\frac{k_\alpha}{m}} \tag{61}$$

and

$$f = -\left(\frac{b}{m} + \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}\tau}\right)\dot{x}. \tag{62}$$

Substituting (61) and (62) into (46) and (48), the differential equation (60) transforms into a system of two averaged first-order differential equations

$$\frac{\dot{A}}{A} = -\frac{\varepsilon}{(5-\alpha)m}\left(\frac{\mathrm{d}m}{\mathrm{d}\tau}\right) - \frac{2\varepsilon b}{(5-\alpha)m}, \tag{63}$$

$$\dot{\psi}(t) = \Omega_\alpha A^{\frac{\alpha-1}{2}}\sqrt{\frac{k_\alpha}{m}}. \tag{64}$$

In general, the averaged amplitude variation is the solution of (63),

$$A = A_0\left(\frac{m_0}{m}\right)^{\frac{1}{5-\alpha}}(\exp)\left(-\frac{2\varepsilon b}{5-\alpha}\int\frac{\mathrm{d}t}{m}\right), \tag{65}$$

which gives the phase angle function

$$\psi = \psi_0 + \Omega_\alpha\sqrt{k_\alpha}\left(A_0 m_0^{\frac{1}{5-\alpha}}\right)^{\frac{\alpha-1}{2}}\int m^{-\frac{(\alpha-1)}{2(5-\alpha)}-\frac{1}{2}}\left((\exp)\left(-\frac{2\varepsilon b}{5-\alpha}\int\frac{\mathrm{d}t}{m}\right)\right)^{\frac{\alpha-1}{2}}\mathrm{d}t. \tag{66}$$

The amplitude and the phase of vibration vary in time due to damping, but also due to mass variation. The order of nonlinearity has a significant influence on the velocity of amplitude and phase increase or decrease.

5.1 Linear mass variation

Let us consider the case when the mass variation is linear, as it is suggested by Yuste [7]

$$m = m_0 + m_1\tau = m_0 + \varepsilon m_1 t, \tag{67}$$

where m_1 is a constant and ε is a small parameter. According to (63), we obtain the differential equation for the amplitude variation

$$\frac{\dot{A}}{A} = -\frac{\varepsilon(2b + m_1)}{(5-\alpha)m}. \tag{68}$$

(a) For the special parameter values, when $m_1/b = -2$, the amplitude of vibration is constant, that is,

$$A = A_0 = const., \tag{69}$$

and the relation (66) transforms into

$$\psi = \psi_0 + \frac{2}{\varepsilon m_1} \Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{k_\alpha} \left(m^{1/2} - m_0^{1/2} \right). \tag{70}$$

For this special case in spite of the action of the linear damping, the amplitude of vibration is constant due to the fact that the linear mass separation makes the compensation to the effect of damping. Using the series expansion of the function ψ, we have

$$\psi = \psi_0 + 2\Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{\frac{k_\alpha}{m_0}} t, \tag{71}$$

and the approximate period of vibration is independent on the mass variation and damping, as follows:

$$T = \frac{2\pi}{2\Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{\frac{k_\alpha}{m_0}}}. \tag{72}$$

The approximate period value depends only on the order of nonlinearity.

(b) For $m_1/b \neq -2$, the amplitude-time and phase-time functions are

$$A = A_0 \left(\frac{m}{m_0} \right)^{-\frac{1}{5-\alpha}\left(1+\frac{2b}{m_1}\right)} \tag{73}$$

and

$$\psi = \frac{\sqrt{k_\alpha}}{\varepsilon m_1} \sqrt{m_0} \frac{A_0^{\frac{\alpha-1}{2}}}{-\frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)+\frac{1}{2}} \left(\left(\frac{m}{m_0} \right)^{-\frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)+\frac{1}{2}} - 1 \right) \Omega_\alpha + \psi_0, \tag{74}$$

which give the approximate solution (38)

$$x = A_0 \left(\frac{m}{m_0} \right)^{-\frac{1}{5-\alpha}\left(1+\frac{2b}{m_1}\right)} \cos \left(\frac{\sqrt{k_\alpha}}{\varepsilon m_1} \sqrt{m_0} \frac{A_0^{\frac{\alpha-1}{2}}}{-\frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)+\frac{1}{2}} \left(\left(\frac{m}{m_0} \right)^{-\frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)+\frac{1}{2}} - 1 \right) \Omega_\alpha + \psi_0 \right). \tag{75}$$

The amplitude and phase variation depend on the relation m_1/b, parameter m_1 and order of nonlinearity α.

Let us analyze the influence of the relation m_1/b on the motion of the oscillator. A numerical example is considered. If the order of nonlinearity is $\alpha = 4/3$, the rigidity $k_{4/3} = 1$ and the mass decrease is $m = 1 - 0.01t$, where $m_0 = 1$, $m_1 = 1$ and $\varepsilon = 0.01$, the differential equation of motion for the damping coefficient b is

$$\ddot{x} + \frac{x\,|x|^{1/3}}{1 - 0.01t} = 0.01(1 - b)\dot{x}. \tag{76}$$

For the initial conditions $x(0) = A_0 = 0.1$ and $\dot{x}(0) = 0$, the analytical solution (75) transforms into

$$x = \frac{0.1}{(1 - 0.01t)^{0.27273(1-2b)}} \cos \left(\frac{66.028}{0.5(1 - 0.0909(1-2b))} \left(1 - (1 - 0.01t)^{0.5(1-0.0909(1-2b))} \right) \right). \tag{77}$$

In Fig. 5, the approximate solution (77) and the numerical solution of (76), obtained by using the Runge–Kutta procedure, are plotted. The $x - t$ diagrams for various values of the damping parameter b are shown.

It can be concluded that for $b = 1/2$, the amplitude of vibration is constant as it is previously stated (see Eq. (66)). For the case when the damping is neglected ($b = 0$), due to mass decrease and existence of the reactive force, the amplitude of vibration increases. For certain damping ($b = 1$) that is higher than the limit value ($b = 1/2$), the amplitude of vibration decreases. The analytical solution is in a very good relation to the numeric one in spite of the long time interval under consideration.

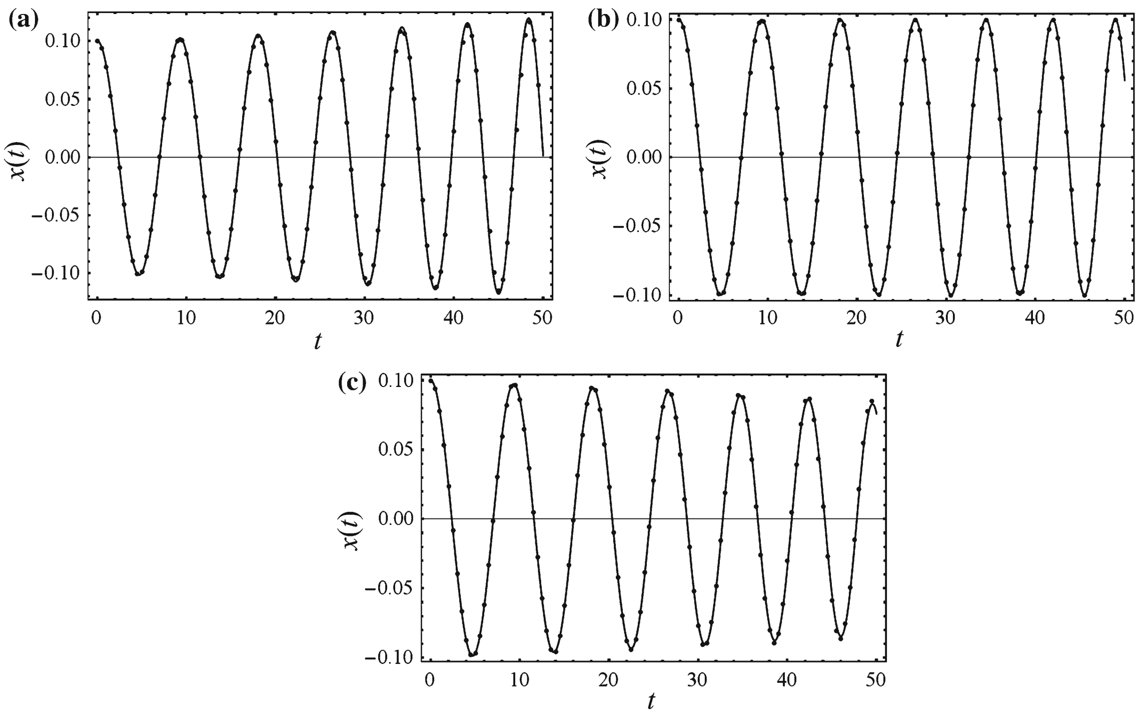

Fig. 5 The $x-t$ diagrams obtained analytically (*a—full line*) and numerically (*n—dot line*) for: **a** $b = 0$, **b** $b = 1/2$ and **c** $b = 1$

5.2 Linear oscillator

For the linear oscillator when $\alpha = 1$

$$A = A_0 \left(\frac{m_0}{m}\right)^{\frac{1}{4}\left(1+\frac{2b}{m_1}\right)} \tag{78}$$

and

$$\psi = \frac{2\sqrt{k_1}}{\varepsilon m_1}\left(\sqrt{m}-\sqrt{m_0}\right) + \psi_0. \tag{79}$$

If the damping parameter is zero, the amplitude variation is $A = A_0(m_0/m)^{1/4}$, as it was previously published in [4]. Using the series expansion of the functions in (79), the approximate frequency of vibration is $\sqrt{k_1/m_0}$ that corresponds to the systems with constant mass and without damping.

6 Conclusions

Due to the previous considerations, it can be concluded:

(i) The vibration of the oscillator with monotone time variable parameter has time variable amplitude and phase. The free vibrations for all of the oscillators with a strong nonlinearity of any order and with the certain monotone slow time variable parameters are qualitatively the same, independently on the order of the nonlinearity. The order of nonlinearity quantitatively changes the amplitude and the phase of vibrations but has no influence on the character of vibrations. Namely, for a certain parameter variation, the higher the order of nonlinearity, the faster or slower is the amplitude and phase increase or decrease. The tendency of increase or decrease of amplitude and phase, that is, frequency of vibration variation, is not directed by the order of nonlinearity but with the type of time parameter variation.

(ii) It is evident that in the oscillator with variable mass for the special relation between the coefficient of damping and parameter of mass variation (which affects the reactive force), the amplitude of vibration is constant, but the phase angle varies independently on the order of the nonlinearity.

(iii) The approximate solution of the nonlinear differential equation with strong nonlinearity of any order (integer or non-integer) and time variable parameter can be obtained analytically.
(iv) The approximate analytic method for solving the differential based on the exact solution of the corresponding differential equation with constant parameters and strong nonlinearity of any order (integer or non-integer) gives very accurate results in comparison with the numerical one.
(v) The solving method based on the approximate solution with exact period of vibration of the corresponding oscillator with constant parameter gives very convenient results for the oscillator with time variable parameters. For technical purpose, the solution is accurate enough and appropriate for practical use. This solution has the form of a trigonometric function and satisfies the requirements for simplicity and usefulness for application in techniques.

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