Acta Mech 223, 645–654 (2012)
DOI 10.1007/s00707-011-0582-z

N. D. Anh · N. N. Hieu · N. N. Linh

A dual criterion of equivalent linearization method for nonlinear systems subjected to random excitation

Received: 26 December 2010 / Revised: 10 August 2011 / Published online: 9 December 2011

Abstract A dual criterion of equivalent linearization method is suggested. The mean-square responses of Duffing, Van der Pol and Lutes-Sarkani oscillators subjected to random excitation are considered. The obtained results are compared with the numerical calculations of original systems and approximate solutions obtained by three different methods including the conventional linearization technique, energy method and regulation linearization method. The results show that in those nonlinear systems the accuracy of the mean-square response is significantly improved by the proposed criterion.

1 Introduction

The equivalent linearization method of stochastic dynamical system is one of the common approaches to the approximate analysis. The original version of the method considered in this paper was proposed by Caughey [1,2]. This method is based on the replacement of a nonlinear oscillator under Gaussian excitation by a linear one under the same excitation for which the coefficients of linearization can be found from a mean-square criterion. The method is also investigated by many different criteria such as moment criteria, higher-order moment criteria, energy criteria, criteria in probability density space (see [3–5]). Caughey [1] investigated the Van der Pol oscillator under random excitation in 1956, and vibration of a system with bilinear hysteresis in 1960 [2]. The method then was generalized to multi-degree-of-freedom systems by Foster [6] and next by Atalik and Utku [7]. In 1986, Brückner and Lin [8] generalized the method of equivalent linearization such that the response of a nonlinear oscillator subjected to both parametric and external random white noise excitations can be determined approximately. The objective of that work was to obtain a closed system of differential equations for certain statistical moments of the response. The accuracy of the mean-square amplitudes was improved considerably compared with the original equivalent linearization. The method of equivalent linearization was reviewed by Socha and Soong [9] in 1991. In 1993, Casciati and Faravelli [10] discussed a new philosophy for stochastic equivalent linearization and then they studied Duffing and hysteretic oscillators in detail. Some extensions of equivalent linearization were proposed in [11–14]. In 2006, Crandall's work [15]

N. D. Anh (✉) · N. N. Hieu
Institute of Mechanics, 264, Doi Can, Ba Dinh, Hanoi, Vietnam
E-mail: ndanh@imech.ac.vn
Tel.: +84-43-8326134
Fax: +84-43-7622039

N. N. Hieu
E-mail: nhuhieu1412@gmail.com

N. N. Linh
Construction Technical College No. 1, Trung Van, Tu Liem, Hanoi, Vietnam
E-mail: linhcmc@gmail.com

described a number of interesting episodes in the history of the linearization technique that have arisen in the past half century. Recently, the work of Elishakoff et al. [16] developed a version of equivalent linearization suggested by Anh and Di Paola [12]. The authors applied this method to some nonlinear oscillators.

The main purpose of our paper is to give a dual criterion of the equivalent linearization method. This criterion is based on an idea that the original nonlinear system can be replaced by an equivalent linearization system and then this equivalent system is replaced by another nonlinear system which belongs to the same class of the original nonlinear system. We then investigate the mean-square response of the Duffing, Van der Pol and Lutes-Sarkani oscillators using three methods, namely, the energy method, regulation linearization method and proposed method. The obtained comparison shows a good agreement between these methods.

2 A dual criterion of equivalent linearization

Content of conventional linearization was described in the works of Caughey [1,2] in detail. Here, we consider a system which is a single degree of freedom with the nonlinear function only depending on two arguments of displacement and velocity,

$$\ddot{z} + 2h\dot{z} + \omega_0^2 z + g\left(z, \dot{z}\right) = f\left(t\right), \tag{1}$$

where h and ω_0 are constants, $g\left(z,\ \dot{z}\right)$ is a nonlinear function of two arguments $z, \dot{z}$, the function $f(t)$ is a zero mean Gaussian stationary process with correlation function and spectral density given by, respectively,

$$R_f\left(\tau\right) = E\left[f\left(t\right) f\left(t+\tau\right)\right], \tag{2}$$

$$S_f\left(\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_f\left(\tau\right) e^{-\omega\tau} \mathrm{d}\tau, \tag{3}$$

where the notation $E\left[\cdot\right]$ denotes the mathematical expectation operation. For the sake of simplicity, we restrict to the case of stationary response of Eq. (1), if it exists.

Equation (1) is linearized to become an equation in the following linear form:

$$\ddot{x} + \left(2h + b\right)\dot{x} + \left(\omega_0^2 + k\right)x = f\left(t\right), \tag{4}$$

where the coefficients of linearization b, k are found by an optimal criterion. There are some criteria for determining the coefficients b, k (see [3,4]). The most extensively used criterion is the mean-square error criterion which requires that the mean square of the error $e\left(x\right) = g\left(x,\ \dot{x}\right) - b\dot{x} - kx$ between Eq. (1) and its linearization (4) be a minimum,

$$E\left[e^2\left(x\right)\right] = E\left[\left(g\left(x,\ \dot{x}\right) - b\dot{x} - kx\right)^2\right] \to \min_{b,\ k}. \tag{5}$$

Although in general the mean-square criterion (5) gives a quite good prediction, it was shown, however, by many authors, that in the case of major nonlinearity, the solution error may be unacceptable [12]. In order to reduce the solution error, one may use the dual approach to the equivalent linearization method as proposed in [17]. The conventional linearization method is based on replacing the original nonlinear system by a linear one, which is equivalent to the original system in some probabilistic sense. Using the dual conception, one needs to balance this "forward" replacement by considering the "backward" replacement of the obtained equivalent linear system by another nonlinear one that belongs to the same class as the original nonlinear system. Combining those two steps, one may consider the following dual criterion:

$$e_1 = E\left[\left(g\left(x,\ \dot{x}\right) - b\dot{x} - kx\right)^2\right] + E\left[\left(b\dot{x} + kx - \lambda g\left(x,\ \dot{x}\right)\right)^2\right] \to \min_{b,\ k,\ \lambda}, \tag{6}$$

where it is seen that the first term describes the conventional replacing, while the second term describes its dual replacing. The coefficients b, k, λ of this criterion are determined by the following system:

$$\frac{\partial e_1}{\partial b} = 0, \tag{7}$$

$$\frac{\partial e_1}{\partial k} = 0, \tag{8}$$

$$\frac{\partial e_1}{\partial \lambda} = 0. \tag{9}$$

Noting $E\left[x\,\dot{x}\right] = 0$, Eqs. (7), (8), (9) lead to the following equations, respectively,

$$b = \frac{1+\lambda}{2}\frac{E\left[\dot{x}\,g\right]}{E\left[\dot{x}^2\right]}, \tag{10}$$

$$k = \frac{1+\lambda}{2}\frac{E\left[xg\right]}{E\left[x^2\right]}, \tag{11}$$

$$\lambda = b\frac{E\left[\dot{x}\,g\right]}{E\left[g^2\right]} + k\frac{E\left[xg\right]}{E\left[g^2\right]}. \tag{12}$$

By solving the system of three equations (10), (11), (12) with respect to three unknowns b, k, λ, we get

$$b = \frac{1}{2-\mu}\frac{E\left[\dot{x}\,g\right]}{E\left[\dot{x}^2\right]}, \tag{13}$$

$$k = \frac{1}{2-\mu}\frac{E\left[xg\right]}{E\left[x^2\right]}, \tag{14}$$

$$\lambda = \frac{\mu}{2-\mu}, \tag{15}$$

where

$$\mu = \frac{\left(E\left[\dot{x}\,g\right]\right)^2}{E\left[\dot{x}^2\right]E\left[g^2\right]} + \frac{\left(E\left[xg\right]\right)^2}{E\left[x^2\right]E\left[g^2\right]}. \tag{16}$$

Thus, using the dual criterion (6), the nonlinear system (1) is replaced by the equivalent linear one (4) where the linearization coefficients b, k are given by (13), (14), (16). In order to elucidate the above criterion of equivalent linearization, in the next sections, three oscillators (Duffing, Van der Pol and Lutes-Sarkani) are considered with white noise excitation.

3 Duffing oscillator

We consider a Duffing oscillator of the form

$$\ddot{z} + 2h\,\dot{z} + \omega_0^2 z + \gamma z^3 = \sigma\xi\,(t)\,, \tag{17}$$

where $h, \omega_0, \gamma, \sigma$ are positive real constants, and the function $\xi(t)$ is a Gaussian white noise process of unit intensity with the correlation function $R_\xi(\tau)$

$$R_\xi\,(\tau) = E\left[\xi\,(t)\,\xi\,(t+\tau)\right] = \delta\,(\tau)\,. \tag{18}$$

Based on the criterion (6) and using the formulas (13), (14), (15) with $g = \gamma x^3$, we obtain three coefficients b, k, λ of the linearized equation of (17) as follows:

$$b = 0, \tag{19}$$

$$k = \frac{15}{7}\gamma E\left[x^2\right], \tag{20}$$

$$\lambda = \frac{3}{7}, \tag{21}$$

where we have utilized the following formula for the normal process $x(t)$:

$$E\left[x^{2n}\right] = (2n-1)!!\left(E\left[x^2\right]\right)^n \quad (n = 1, 2, 3, ...). \tag{22}$$

Then, the linearized equation of the nonlinear equation (17) takes the following form:

$$\ddot{x} + 2h\dot{x} + \left(\omega_0^2 + \frac{15}{7}\gamma E\left[x^2\right]\right)x = \sigma\xi(t). \tag{23}$$

From (23), the mean-square response $E\left[x^2\right]$ can be determined by the relation (see [3])

$$E\left[x^2\right] = \frac{\sigma^2}{4h\left(\omega_0^2 + \frac{15}{7}\gamma E\left[x^2\right]\right)}. \tag{24}$$

Solving Eq. (24) with respect to the unknown $E\left[x^2\right]$, we get

$$E\left[x^2\right] = \frac{7}{30\gamma}\left(-\omega_0^2 + \sqrt{\omega_0^4 + \frac{15\gamma\sigma^2}{7h}}\right). \tag{25}$$

The solution (25) is an approximate value of the mean-square response of Eq. (17) with the input parameters $h, \omega_0, \gamma, \sigma$.

In this section, we also present results of two other methods including the energy method as shown in [11] and the regulation linearization method (one-step and two-step) as shown in [16]. For the energy method [11], the authors required that the mean-square error between the potential energy, associated with the original nonlinear equation (17), and its equivalent linear counterpart, to be a minimum. That means

$$E\left[\left(U - \frac{1}{2}\left(\omega_0^2 + k\right)x^2\right)^2\right] \to \min_k, \tag{26}$$

where U is the potential energy of the nonlinear system (17):

$$U = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\gamma x^4. \tag{27}$$

Thus, the coefficient k of the linearized equation of (17) is determined as

$$k = \frac{5}{2}\gamma E\left[x^2\right]. \tag{28}$$

Similarly to Eq. (24), the approximate mean-square response $E\left[x^2\right]$ corresponding to the energy method is found to be

$$E\left[x^2\right]_{\text{energy}} = \frac{1}{5\gamma}\left[-\omega_0^2 + \sqrt{\omega_0^4 + \frac{5}{2}\frac{\gamma\sigma^2}{h}}\right]. \tag{29}$$

For the one-step regulation linearization [12,16], the nonlinear term γx^3 is replaced by a higher-order term, $\gamma_1 x^5$, and then this term is replaced by $\gamma_2 x^3$, and $\gamma_2 x^3$ is replaced by a linear one, $k_{\text{I}} x$, in the final step. Hence, the procedure can be presented schematically as follows:

$$\gamma x^3 \to \gamma_1 x^5 \to \gamma_2 x^3 \to k_{\text{I}} x. \tag{30}$$

The coefficients $\gamma_1, \gamma_2, k_{\text{I}}$ are determined by the mean-square error criterion in each replacing step. Note that the coefficient k_{I} of this linearized scheme is found to be

$$k_{\text{I}} = \frac{7}{3}E\left[x^2\right]. \tag{31}$$

The result of the approximate mean-square response of the one-step regulation linearization is determined by the following expression:

$$E\left[x^2\right]_{\text{regulated,I}} = \frac{3}{14\gamma}\left[-\omega_0^2 + \sqrt{\omega_0^4 + \frac{7}{3}\frac{\gamma\sigma^2}{h}}\right]. \tag{32}$$

Table 1 The mean-square responses $E\left[x^2\right]$ of the Duffing oscillator versus the parameter γ ($h = 0.5$, $\omega = 1$, $\sigma^2 = 2$)

γ	$E\left[x^2\right]_{ex}$	$E\left[x^2\right]_{co}$	Error (%)	$E\left[x^2\right]_{en}$	Error (%)	$E\left[x^2\right]_{re,I}$	Error (%)	$E\left[x^2\right]_{re,II}$	Error (%)	$E\left[x^2\right]_{pr}$	error (%)
0.1	0.8176	0.8054	1.4876	0.8284	1.3290	0.8367	2.3366	0.8555	4.6404	0.8465	3.5352
0.5	0.5792	0.5486	5.2861	0.5798	0.1028	0.5916	2.1464	0.6202	7.0851	0.6062	4.6670
1.0	0.4679	0.4343	7.1938	0.4633	0.9820	0.4745	1.4162	0.5022	7.3158	0.4885	4.4082
5.0	0.2543	0.2270	10.7384	0.2457	3.4055	0.2530	0.5053	0.2716	6.8072	0.2624	3.1708
10	0.1889	0.1667	11.7721	0.1810	4.1858	0.1867	1.1686	0.2011	6.4822	0.1939	2.6697
50	0.0904	0.0784	13.2721	0.0855	5.3635	0.0884	2.1954	0.0957	5.8974	0.0921	1.8539
100	0.0650	0.0561	13.6491	0.0613	5.6675	0.0634	2.4648	0.0687	5.7302	0.0660	1.6331

ex exact solution, *co* conventional linearization, *en* energy method, *re,I* one-step regulation, *re,II* two-step regulation, *pr* present method

For the two-step regulation linearization [16], the authors required the following regulation scheme:

$$\gamma x^3 \to \gamma_1 x^5 \to \gamma_2 x^7 \to \gamma_3 x^5 \to \gamma_4 x^3 \to k_{\mathrm{II}} x. \tag{33}$$

We obtain the coefficient k_{II} and the mean-square response of regulation, respectively, as

$$k_{\mathrm{II}} = \frac{77}{39} E\left[x^2\right], \tag{34}$$

$$E\left[x^2\right]_{\text{regulated,II}} = \frac{39}{154\gamma}\left[-\omega_0^2 + \sqrt{\omega_0^4 + \frac{77}{39}\frac{\gamma\sigma^2}{h}}\right]. \tag{35}$$

Coming back to Eq. (17), its exact density function reads (see [15])

$$p(z) = A \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 z^2 + \frac{1}{4}\gamma z^4\right)\right\}, \tag{36}$$

where A is a normalization constant evaluated from

$$A^{-1} = \int_{-\infty}^{\infty} \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 z^2 + \frac{1}{4}\gamma z^4\right)\right\} \mathrm{d}z. \tag{37}$$

The exact value of the mean-square response $E\left[z^2\right]$ of (17) is evaluated by

$$E\left[z^2\right] = \frac{\int_{-\infty}^{\infty} z^2 \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 z^2 + \frac{1}{4}\gamma z^4\right)\right\} \mathrm{d}z}{\int_{-\infty}^{\infty} \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\omega_0^2 z^2 + \frac{1}{4}\gamma z^4\right)\right\} \mathrm{d}z}. \tag{38}$$

It is interested in comparing the results of mean-square response of the present method with the other approximate ones. In Table 1, the values of the approximate solutions including the solutions of the conventional linearization method, energy method (29), one-step regulation (32), two-step regulation (35) and present method which is evaluated by the expression (25) are compared with the exact solutions (38) for various values of nonlinearity parameter γ. The error between the results of the approximate methods and the exact solution is defined as

$$\text{error} = \frac{\left|E\left[x^2\right]_{\text{approx}} - E\left[x^2\right]_{\text{exact}}\right|}{E\left[x^2\right]_{\text{exact}}} \times 100\%. \tag{39}$$

It is seen that, in general, for middle and large values of γ, the errors of the methods including the energy method, one-step and two-step regulation linearization methods and the present method give results better than the conventional linearization method. However, the errors of the present method (25) are smallest for large values of the parameter γ, for example, $\gamma = 50$, $\gamma = 100$.

4 Van der Pol oscillator

In this section, a Van der Pol oscillator subjected to random excitation of white noise is considered:

$$\ddot{z} - \left(\alpha - \gamma z^2\right)\dot{z} + \omega_0^2 z = \sigma\xi\left(t\right), \tag{40}$$

where α, γ, ω_0, σ are positive real constants, the function $\xi(t)$ is a Gaussian white noise process of unit intensity with the correlation function $R_\xi(t)$ satisfying (18). In this Van der Pol oscillator, the nonlinear function is $g\left(x, \dot{x}\right) = \gamma x^2 \dot{x}$. The linearization equation of (40) takes the following form:

$$\ddot{x} + (-\alpha + b)\dot{x} + \left(\omega_0^2 + k\right)x = \sigma\xi\left(t\right), \tag{41}$$

where the coefficients b, k are found using the formulas (13)–(16). Calculations are performed similar to Duffing oscillator, and we obtain the expressions of b, k as follows:

$$b = \frac{3}{5}\gamma E\left[x^2\right], \tag{42}$$

$$k = 0. \tag{43}$$

Therefore, using the linearized equation (41) and expressions (42), (43), the mean-square response of the Van der Pol oscillator is determined by the following relation (see [3]):

$$E\left[x^2\right] = \frac{\sigma^2}{2\omega_0^2\left(-\alpha + \frac{3}{5}\gamma E\left[x^2\right]\right)}. \tag{44}$$

Solving Eq. (44) with respect to the unknown $E\left[x^2\right]$, we obtain

$$E\left[x^2\right] = \frac{5}{6\gamma}\left(\alpha + \sqrt{\alpha^2 + \frac{6\gamma\sigma^2}{5\omega_0^2}}\right). \tag{45}$$

Since the energy method is not available for the Van der Pol oscillator, therefore, in this section, we only consider the method of regulation linearization (including one-step and two-step regulation procedures) [16], and apply it to the nonlinear system (40). For the one-step regulation procedure, the nonlinear term $\gamma x^2\dot{x}$ is replaced by a higher-order term, $\gamma_1 x^4\dot{x}$, and then this nonlinear term is replaced by other nonlinear one, $\gamma_2 x^2\dot{x}$, and in the final step, the nonlinear term $\gamma_2 x^2\dot{x}$ is replaced by a linear one, $b_{\mathrm{I}}\dot{x}$. This linearization procedure is illustrated by the following scheme:

$$\gamma x^2\dot{x} \to \gamma_1 x^4\dot{x} \to \gamma_2 x^2\dot{x} \to b_{\mathrm{I}}\dot{x}. \tag{46}$$

The coefficients γ_1, γ_2 and b_{I} are determined by using the mean-square error criterion for each replacing step. In this scheme, we obtain the coefficient b_{I} as follows:

$$b_{\mathrm{I}} = \frac{5}{7}\gamma E\left[x^2\right]. \tag{47}$$

Thus, we get the following expression for $E\left[x^2\right]$:

$$E\left[x^2\right]_{\text{regulated,I}} = \frac{7}{10\gamma}\left[\alpha + \sqrt{\alpha^2 + \frac{10}{7}\frac{\gamma\sigma^2}{\omega_0^2}}\right]. \tag{48}$$

For the two-step regulation procedure, the following scheme is applied to the Van der Pol oscillator:

$$\gamma x^2\dot{x} \to \gamma_1 x^4\dot{x} \to \gamma_2 x^6\dot{x} \to \gamma_3 x^4\dot{x} \to \gamma_4 x^2\dot{x} \to b_{\mathrm{II}}\dot{x}. \tag{49}$$

Table 2 The mean-square responses $E\left[x^2\right]$ of Van der Pol oscillator versus the parameter σ^2 ($\alpha = 0.2, \omega = 1, \gamma = 2$)

σ^2	$E\left[x^2\right]_{\mathrm{MC}}$	$E\left[x^2\right]_{\mathrm{co}}$	Error (%)	$E\left[x^2\right]_{\mathrm{re,I}}$	Error (%)	$E\left[x^2\right]_{\mathrm{re.II}}$	Error (%)	$E\left[x^2\right]_{\mathrm{pr}}$	Error (%)
0.02	0.2081	0.1366	34.3573	0.1791	13.9418	0.2116	1.6592	0.2069	0.5590
0.20	0.3638	0.2791	23.2741	0.3437	5.5309	0.3903	7.2870	0.3838	5.4964
1.00	0.7325	0.5525	24.5742	0.6657	9.1147	0.7452	1.7303	0.7342	0.2304
2.00	1.0255	0.7589	25.9998	0.9096	11.3034	1.0145	1.0759	1.0000	2.4866
4.00	1.4525	1.0512	27.6248	1.2553	13.5776	1.3964	3.8592	1.3770	5.1969

MC Monte-Carlo simulation, *co* conventional linearization, *en* energy method, *re,I* one-step regulation, *re,II* two-step regulation, *pr* present method

The coefficient b_{II} and the approximate mean-square response are found to be, respectively,

$$b_{\mathrm{II}} = \frac{45}{77}\gamma E\left[x^2\right], \tag{50}$$

$$E\left[x^2\right]_{\mathrm{regulated,II}} = \frac{77}{90\gamma}\left[\alpha + \sqrt{\alpha^2 + \frac{90}{77}\frac{\gamma\sigma^2}{\omega_0^2}}\right]. \tag{51}$$

The results of the approximate mean-square response of the Van der Pol oscillator (40) obtained by the present method (45), conventional linearization method, one-step (48) and two-step (51) regulation linearization procedures are compared in Table 2 with input parameters $\alpha = 0.2$, $\omega_0 = 1$, $\gamma = 2$ and various values of σ^2. In the second column of this table, the values of a Monte-Carlo simulation of $E\left[x^2\right]_{\mathrm{MC}}$ are displayed (see[3,12]). The errors are computed by the formula (39) but the symbol $E\left[x^2\right]_{\mathrm{exact}}$ is replaced by the symbol $E\left[x^2\right]_{\mathrm{MC}}$. It is seen that the errors of the two-step regulation procedure and the present method are less than the errors of the conventional linearization method and the one-step regulation procedure. In this table, the greatest error of the present method is 5.4964%, whereas the smallest error of the conventional method is 23.2741% and the smallest error of the one-step regulation procedure is 5.5309%.

5 Lutes-Sarkani oscillator

Consider the nonlinear oscillator described by the following equation [18]

$$\dot{z} + \gamma\,|z|^a\,\mathrm{sgn}(z) = f(t), \tag{52}$$

where a is a positive integer, $f(t)$ is a zero mean, stationary Gaussian white noise with spectral density $S_0 = \mathrm{const}$. By using the criterion (6) with the nonlinear function $g = \gamma\,|x|^a\,\mathrm{sgn}(x)$, g is replaced by a linear function in the following form:

$$b\dot{x} + kx = b\,|\dot{x}|\,\mathrm{sgn}(\dot{x}) + k\,|x|\,\mathrm{sgn}(x), \tag{53}$$

where b, k are determined from the formulas (13)–(16). We notice that the general expression for $E\left[|x|^a\right]$ is (see [16]):

$$E\left[|x|^a\right] = \sigma_x^a \int_{-\infty}^{\infty} \frac{|u|^a}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} \mathrm{d}u. \tag{54}$$

Using (13)–(16) and (53), (54), we obtain

$$b = 0, \tag{55}$$

$$k = \frac{1}{2-\mu}\sigma_x^{a-1} \int_{-\infty}^{\infty} \frac{|u|^{a+1}}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} \mathrm{d}u, \tag{56}$$

$$\mu = \left(\int_{-\infty}^{\infty} \frac{|u|^{a+1}}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} \mathrm{d}u\right)^2 \left(\int_{-\infty}^{\infty} \frac{|u|^{2a}}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} \mathrm{d}u\right)^{-1}, \tag{57}$$

where $\sigma_x^2 = E\left[x^2\right]$ is the variance of the response (because $E[x] = 0$). We use the integration formula

$$\int_0^\infty u^{s-1} \exp\left\{-\alpha u^r\right\} \mathrm{d}u = \frac{1}{r}\alpha^{-\frac{s}{r}} \Gamma\left(\frac{s}{r}\right), \tag{58}$$

where $\Gamma(v)$ denotes the Gamma function of the variable v:

$$\Gamma(v) = \int_0^\infty u^{v-1} \exp\{-u\} \mathrm{d}u. \tag{59}$$

The expressions (56) and (57) are evaluated to be, respectively,

$$k = \frac{1}{\sqrt{2\pi}} \cdot \frac{\gamma 2^{\frac{a}{2}} a \Gamma\left(\frac{a}{2}\right)}{2-\mu} \sigma_x^{a-1}, \tag{60}$$

$$\mu = \frac{1}{2\sqrt{\pi}} a^2 \left[\Gamma\left(\frac{a}{2}\right)\right]^2 \left[\Gamma\left(a+\frac{1}{2}\right)\right]^{-1}. \tag{61}$$

Substituting (61) into (60), we get the following expression for k:

$$k = \frac{1}{\sqrt{2\pi}} \cdot \frac{\gamma 2^{\frac{a}{2}} a \Gamma\left(\frac{a}{2}\right)}{2 - \frac{1}{2\sqrt{\pi}} a^2 \left[\Gamma\left(\frac{a}{2}\right)\right]^2 \left[\Gamma\left(a+\frac{1}{2}\right)\right]^{-1}} \sigma_x^{a-1}. \tag{62}$$

The variance σ_x^2 is determined by the following relation which is obtained from the linearized equation of Eq. (52) (see [3])

$$\sigma_x^2 = \frac{\pi S_0}{k}. \tag{63}$$

Substituting (62) into (63) and solving this equation with respect to the unknown σ_x^2, we obtain

$$\sigma_x^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} \left[2 - \frac{1}{2\sqrt{\pi}} a^2 \left[\Gamma\left(\frac{a}{2}\right)\right]^2 \left[\Gamma\left(a+\frac{1}{2}\right)\right]^{-1}\right]^{\frac{2}{a+1}} \left[\frac{\sqrt{2\pi}}{2^{\frac{a}{2}} a \Gamma\left(\frac{a}{2}\right)}\right]^{\frac{2}{a+1}}. \tag{64}$$

In the following calculations, we also present results of two additional methods including the energy method and regulation linearization method (one-step and two-step) for determining the approximate variance of the Lutes-Sarkani oscillator. Similarly to the Duffing oscillator, the following criterion is utilized:

$$E\left[\left(U - \frac{1}{2}kx^2\right)^2\right] \to \min_k, \tag{65}$$

where

$$U = \frac{1}{a+1} |x|^{a+1}. \tag{66}$$

The criterion (65) leads to the following expression for the coefficient k of the linearized equation of the Lutes-Sarkani oscillator (52):

$$k = \frac{2^{\frac{a+1}{2}} \gamma}{a+1} \sigma_x^{a-1} \frac{\Gamma\left(\frac{a+4}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}. \tag{67}$$

Thus, we obtain the approximate variance of the system as

$$\sigma_{x,\text{energy}}^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} \left[(a+1)\, 2^{-\frac{a+1}{2}} \Gamma\left(\frac{5}{2}\right) \left[\Gamma\left(\frac{a+4}{2}\right)\right]^{-1}\right]^{\frac{2}{a+1}}. \tag{68}$$

Table 3 The variance σ_x^2 of the responses of the Lutes-Sarkani oscillator with input parameters $S_0 = 1$, $\gamma = 1$ and various values of a

a	$E\left[x^2\right]_{\text{ex}}$	$E\left[x^2\right]_{\text{co}}$	Error (%)	$E\left[x^2\right]_{\text{en}}$	Error (%)	$E\left[x^2\right]_{\text{re,I}}$	Error (%)	$E\left[x^2\right]_{\text{re,II}}$	Error (%)	$E\left[x^2\right]_{\text{pr}}$	Error (%)
1	3.1416	3.1416	0.0000	3.1416	0.0000	3.1416	0.0000	3.1416	0.0000	3.1416	0.0000
2	1.6655	1.5708	5.6877	1.6991	2.0165	1.6784	0.7713	1.7599	5.6693	1.7254	3.5927
3	1.1981	1.0233	14.5904	1.1210	6.4385	1.1603	3.1546	1.2614	5.2820	1.2108	1.0580
4	0.9761	0.7531	22.8490	0.8234	15.6460	0.8884	8.9861	0.9881	1.2229	0.9116	6.6135
5	0.8474	0.5939	29.9225	0.6458	23.7991	0.7201	15.0206	0.8134	4.0131	0.7173	15.3605
6	0.7635	0.4894	35.8981	0.5290	30.7191	0.6056	20.6846	0.6917	9.4038	0.5843	23.4754
7	0.7045	0.4159	40.9630	0.4469	36.5606	0.5226	25.8224	0.6019	14.5548	0.4895	30.5200

ex exact solution, *co* conventional linearization, *en* energy method, *re,I* one-step regulation, *re,II* two-step regulation, *pr* present method

On the other hand, according to results presented by Elishakoff et al. [16] the approximate variances are also determined by using the one-step and two-step regulation linearization procedures, respectively, as follows:

$$\sigma_{x,\text{I}}^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} \left[\sqrt{\pi}\, 2^{-\frac{a+1}{2}} \Gamma\left(\frac{4a-1}{2}\right) \Gamma\left(\frac{2a+1}{2}\right) \left[\Gamma\left(\frac{3a}{2}\right)\right]^{-2} \left[\Gamma\left(\frac{a+2}{2}\right)\right]^{-1}\right]^{\frac{2}{a+1}}, \tag{69}$$

$$\sigma_{x,\text{II}}^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} \left\{\sqrt{\pi} 2^{-\frac{a+1}{2}} \frac{\Gamma\left(\frac{2a+1}{2}\right) \Gamma\left(\frac{6a-3}{2}\right) \left[\Gamma\left(\frac{4a-1}{2}\right)\right]^2}{\Gamma\left(\frac{a+2}{2}\right) \left[\Gamma\left(\frac{3a}{2}\right)\right]^2 \left[\Gamma\left(\frac{5a-2}{2}\right)\right]^2}\right\}^{\frac{2}{a+1}}. \tag{70}$$

Returning to Eq. (52), the exact probability density of the response was derived by [18]

$$p(z) = A \exp\left\{-\frac{\gamma}{(a+1)\pi S_0} |z|^{a+1}\right\}, \tag{71}$$

where

$$A = \left(\frac{\gamma}{(a+1)\pi S_0}\right)^{\frac{1}{a+1}} \frac{a+1}{2} \left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{72}$$

The exact variance of the response is

$$\sigma_{x,\text{exact}}^2 = 2A \int_0^\infty u^2 \exp\left\{-\frac{\gamma}{(a+1)\pi S_0} u^{a+1}\right\} \mathrm{d}u. \tag{73}$$

Using the formula (59), we get (see [16])

$$\sigma_{x,\text{exact}}^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} (a+1)^{\frac{2}{a+1}} \Gamma\left(\frac{3}{a+1}\right) \left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{74}$$

The approximate response via the conventional linearization technique are also derived by [18] as

$$\sigma_{x,\text{conventional}}^2 = \left(\frac{\pi S_0}{\gamma}\right)^{\frac{2}{a+1}} \left[\frac{\sqrt{2\pi}}{2^{\frac{a}{2}} a \Gamma\left(\frac{a}{2}\right)}\right]^{\frac{2}{a+1}}. \tag{75}$$

The percentage error due to the approximate nature of the solutions (including the conventional linearization technique (75), energy method (68), one-step regulation linearization (69), two-step regulation linearization (70), and the present method (64)) is presented in Table 3 in comparison to the exact solution provided by Eq. 74), for different values of a. In general, the regulation linearization method (one-step and two-step) and the present method yield better results than the conventional linearization and energy methods in the case of the Lutes-Sarkani oscillator. However, for the middle and large values of a, for example, $a = 4, a = 5, a = 6, a = 7$, the errors of the two-step regulation are smallest. This excellent property was first shown by Elishakoff et al. [16]. It is seen that, for $a = 3$, whereas the error of the conventional linearization is 14.5904%, the present method has an error which is over 13 times less, namely 1.0580%. Thus, the present method gives the best result for the Lutes-Sarkani oscillator.

6 Conclusions

Although the method of equivalent linearization has remained a popular tool over many years and several criteria have been suggested to obtain the expressions of the equivalent stiffness and equivalent damping, new criteria need to be suggested to improve the accuracy and to understand the essence of the method. In this paper, a criterion is considered using the dual approach to the "replacing" problem. Three typical nonlinear systems are examined by using three methods, namely, the energy method, the regulation linearization method and the present method. The obtained comparison shows a good agreement between these methods. The approach has a large potential and it ought to be explored for wider nonlinear classes.

Acknowledgments The paper was supported by the Vietnam National Foundation for Science and Technology Development. The authors express their deep thanks to the reviewers for helpful suggestions.

References

1. Caughey, T.K.: Response of Van der Pol's oscillator to random excitations. Trans. ASME J. Appl. Mech. **26**(1), 345–348 (1956)
2. Caughey, T.K.: Random excitation of a system with bilinear hysteresis. Trans. ASME J. Appl. Mech. **27**(1), 649–652 (1960)
3. Roberts, J.B., Spanos, P.D.: Random Vibration and Statistical Linearization. Wiley, New York (1990)
4. Socha, L.: Linearization methods for stochastic dynamic system. Lecture Notes in Physics. Springer, Berlin (2008)
5. Proppe, C., Pradlwarter, H.J., Schüller, G.I.: Equivalent linearization and Monte-Carlo simulation in stochastic dynamics. J. Probab. Eng. Mech. **18**(1), 1–15 (2003)
6. Foster, E.: Semi-linear random vibrations in discrete systems. Trans. ASME J. Appl. Mech. **35**, 560–564 (1968)
7. Atalik, T., Utku, S.: Stochastic linearization of multi-degree-of-freedom nonlinear system. J. Earthq. Eng. Struct. Dyn. **4**, 411–420 (1976)
8. Brückner, A., Lin, Y.K.: Generalization of the equivalent linearization method for the nonlinear random vibration problems. Int. J. Nonlinear Mech. **22**(4), 227–235 (1987)
9. Socha, L., Soong, T.T.: Linearization in analysis of nonlinear stochastic systems. Appl. Mech. Rev. **44**, 399–422 (1991)
10. Casciati, F., Faravelli, L.: A new philosophy for stochastic equivalent linearization. J. Probab. Eng. Mech. **8**, 179–185 (1993)
11. Zhang, X.T., Elishakoff, I., Zhang, R.C.: A stochastic linearization technique based on minimum mean-square deviation of potential energies. In: Lin, Y.K., Elishakoff, I. Stochastic Structural Dynamics–New Theoretical Developments, pp. 327–338. Springer, Berlin (1990)
12. Anh, N.D., Di Paola, M.: Some extensions of Gaussian equivalent linearization. International Conference on Nonlinear Stochastic Dynamics, pp. 5–16. Hanoi, Vietnam (1995)
13. Anh, N.D., Schiehlen, W.: A technique for obtaining approximate solutions in Gaussian equivalent linearization. Comput. Methods Appl. Mech. Eng. **168**, 113–119 (1999)
14. Anh, N.D., Hung, L.X.: An improved criterion of Gaussian equivalent linearization for analysis of nonlinear stochastic systems. J. Sound Vib. **268**(1), 177–200 (2003)
15. Crandall, S.H.: A half-century of stochastic equivalent linearization. Struct. Control Health Monit. **13**, 27–40 (2006)
16. Elishakoff, I., Andrimasy, L., Dolley, M.: Application and extension of the stochastic linearization by Anh and Di Paola. Acta Mech. **204**, 89–98 (2009)
17. Anh, N.D.: Duality in the analysis of responses to nonlinear systems. Vietnam J. Mech. VAST. **32**(4), 263–266 (2010)
18. Lutes, L.D., Sarkani, S.: Random Vibration: Analysis of Structural and Mechanical Systems, pp. 423–424, 437–438. Elsevier, Amsterdam (2004)