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Hybrid asymptotic-direct approach to the problem of finite vibrations of a curved layered strip

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Abstract A multi-stage approach for the mathematical modeling in the field of nonlinear problems of mechanics of thin-walled structures is the subject of the present paper. A combination of the asymptotic, direct, and numerical methods for consistent and efficient analysis of problems of structural mechanics is presented on the example of plane problem of finite vibrations of a thin curved strip with material inhomogeneity. The method of asymptotic splitting allows for a consistent dimensional reduction of the original two-dimensional continuous problem as the thickness is small: the leading-order solution of the full system of equations of the theory of elasticity results in a one-dimensional formulation of the reduced theory and a problem in the cross-section. The direct approach to a material line extends the results to the geometrically nonlinear range. The appropriate finite element formulation allows for practical applications of the theory; with the numerical solution of the reduced problem, we restore the distributions of stresses, strains, and displacements over the thickness. Numerically and analytically investigated convergence of the solutions of various problems in the original (two-dimensional) and reduced (one-dimensional) models as the thickness tends to zero justifies the analytical conclusion that the curvature and variation of the material properties over the thickness do not require special treatment for classical Kirchhoff's rods. Further terms of the asymptotic expansion lead to a model with shear and extension, in which curvature appears in a nontrivial way.

1 Introduction

1.1 State-of-the-art in the mechanics of thin-walled structures

Theories of thin-walled structures, as they exist in the literature, generally fall into one of the following major categories:

"Method of hypotheses", also known as technical or engineering approach, stands historically at the first place. These theories, which appear nearly in each course on strength of materials all over the world, are based on standard assumptions concerning the distribution of mechanical entities over the cross-section, the negligibility of certain values, etc. Equations of balance and constitutive relations are derived using some of the equations of the theory of elasticity. Results, which are now considered as classical in the theories of rods (ordinary and with an open profile), plates, and shells can be found e.g. in [33,39,46] and other books. Although the resulting formulations are reasonable, and all assumptions have theoretical and practical argumentation,

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some more fundamental and thorough research in the field of reduction of the three-dimensional problem is needed.

Variational methods appear as a subsequent theoretical step. Based on the weak formulation of the three-dimensional problem, in which an appropriate variational principle replaces the field equations, this approach uses the famous Galerkin-Ritz method with an approximation of the unknowns over the cross-section (kinematic assumptions). The variational principle produces equations and natural boundary conditions for the coefficients in the approximations, which can be interpreted in terms of the theory of the dimensionally reduced continuum. Among numerous works concerned with the analytical and numerical application of this method to plates and rods, we mention [6,23,33,37,45].

Being very useful in engineering and numerical applications, the traditional variational method does not reveal the particularity of thin bodies: the resulting theory strongly depends on the chosen set of approximating functions. Advantageous approximations can only be selected based on previous experience, and the correspondence to the true solution of the original three-dimensional problem is estimated by numerical modeling. A different approach is the variational-asymptotic method (VAM), proposed by Berdichevsky [4,5] and further developed by his colleagues [18,48]: stationary points of a functional with a small parameter are determined asymptotically, when the small parameter tends to zero. Clarity and attractivity of the idea of the method are counterbalanced by the absence of a formal mathematical proof of its validity along with a certain dependence of the results on the decisions, made during the analysis [4].

Another known version of the combination of variational and asymptotic methods for linear shells was proposed by Koiter [22]; a mathematical analysis of the variational three-dimensional formulation leads to the principle of minimal strain energy of the reduced shell model in the position of equilibrium. An important advantage of the variational methods, which is especially important in applications, is that the material nonlinearity can be treated relatively easily and consistently, for reference see e.g. [13,19,29,45].

Direct approach to rods and shells uses the concept of material lines or surfaces with a certain set of degrees of freedom of particles, see e.g. [1,10,32] for shells, [3,9,16,34,35] for rods, and [40] for thin-walled rods of open profile; various types of continua are treated e.g. in [12,35], etc. Being free from logical contradictions, this powerful method easily deals with such complicated problems as geometrically nonlinear and electromechanically coupled behavior of curved shells with piezoelectric patches [43].

The practical application of this method requires certain three-dimensional analysis to be performed prior to the solution of the dimensionally reduced problem, which provides explicit constitutive relations and allows to find the stressed state of the actual solid body from the computed force factors of the reduced model; for combined presentations of both the direct and three-dimensional analysis, see e.g. the comprehensive works [3,7,30,49]. Bridging the gap between appropriate three-dimensional analysis and reduced models is the central point of the present study.

The procedure of through-the-thickness integration of the three-dimensional balance relations provides important relations for the stress and other resultants of the reduced theory; see e.g. [26] for purely mechanical or [11] for thermoelastic analysis. However, the complete system of equations requires some kinematic assumptions or hypotheses, which makes the procedure close to the first two methods considered above.

Series expansions applied to the solution of differential equations of mechanics of three-dimensional continuum are broadly discussed in the literature; for a general look of asymptotic methods commonly applied in continuum mechanics, see e.g. [2,31].

There exists a class of works in which the solution is expanded into series with respect to the coordinate along the thickness direction, see e.g. [21]. Writing recurrent relations for the subsequent terms according to the three-dimensional equations and truncating the series, one obtains a hierarchy of reduced models capturing various effects.

More fundamental is the approach in which the three-dimensional solution is sought as a series expansion with respect to a small parameter, which is related to the thickness of the structure. Thus, Goldenveizer [14] obtained two-dimensional equations for the leading-order terms of the series expansion for a homogeneous plate. An advanced study of piezoelectric plates with material inhomogeneity in all three dimensions is reported in [20]; the usage of powerful mathematical techniques, typical for the analysis of periodic structures, makes the latter work quite complicated.

Some authors considered the boundary conditions for a plate within the asymptotic framework, see e.g. [15,27]. The challenging problem of matching of the asymptotic expansions, valid inside the domain and in the edge layer (near the boundary), requires thorough mathematical treatment.

Examples of asymptotic analysis of thin-walled rods of open profile can be found in [8,17,41].

Relevant to the present study are the works [38] for plates and [17] for thin-walled rods of open profile. In the equations for the components of stresses and displacements, written for a particular material structure, a small parameter is introduced by means of nondimensional variables. The analysis proceeds by assigning particular orders of smallness to different components of the stress tensor a priori. The conditions of solvability for the minor terms of the series expansion of the solution with respect to the small parameter take part in the formulation of the reduced equations.

1.2 Hybrid asymptotic-direct approach

The present study uses the procedure of asymptotic splitting in the form suggested by Eliseev [8,9]. The method has already been systematically applied to the dimensional reduction in the theories of thin rods with an inhomogeneous cross-section [47], of thin-walled rods of open profile [8,41] and of thin piezoelectric plates with an inhomogeneous material structure through the thickness [44]. The main idea of the method is that the principal terms of the series expansion of the solution are determined from the conditions of solvability for the minor terms. In comparison to the method of formal asymptotic expansion in its traditional form, the advantages are the following:

- The procedure is formal and straightforward as soon as we have the original three-dimensional problem with a small parameter formulated; no assignment of orders of smallness to different components of stresses and displacements ("scaling"), which is intrinsic to some earlier works (see e.g. [38]), needs to be done a priori.
- Instead of the complete recurrent system of relations between successive terms in the series expansion (see e.g. [27]), we consider only those equations which appear to be relevant for the analysis.
- The procedure has a clear modular structure: we study the stresses, strains, displacements, and electrical (or other nonmechanical) effects almost independently. Each of these stages provides a corresponding part of the complete theory for the reduced continuum. The formulation is completed by the constitutive relations of the reduced model. This particular simplicity of the analysis is provided by the use of the three-dimensional compatibility conditions. Moreover, the analysis of the edge layer for the boundary conditions can be strongly based on the compatibility conditions.

The mathematical aspects are presented in Sect. 2.2.

The key point of the hybrid approach, proposed in the present paper, is the assumption that the mathematical equivalence of the linearized equations of the direct approach and of the asymptotically reduced equations of the three-dimensional linear theory justifies the general geometrically nonlinear theory obtained with the direct approach. This allows for solving challenging problems of nonlinear deformations of thin-walled structures with complex properties. Thus, the method has successfully been applied to the challenging problem of nonlinear and electromechanically coupled deformations of thin shells, equipped with piezoelectric actuators and sensors [43,44]. The problem of the effect of the curvature of the shell on the results of the asymptotic analysis remains yet open and will be addressed in the present paper on the simpler example of deformation of a curved strip.

2 Hybrid approach to a thin curved strip

2.1 Linear plane problem for a curved strip

We will demonstrate the philosophy and the stages of the hybrid approach by applying it to the sample dynamic problem of plane deformation of a curved strip, Fig. 1.

The strip has a constant thickness h, and the position of the points of the middle line is parametrically defined by the vector $\boldsymbol{x}_0(s)$. This results in the basis

$$\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{k}\}, \quad \boldsymbol{t} = \boldsymbol{x}_0', \quad \boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{t}, \quad \boldsymbol{n}' = \alpha \boldsymbol{t}, \quad \boldsymbol{t}' = -\alpha \boldsymbol{n}, \tag{1}$$

where $\boldsymbol{k}$ is the unit out-of-plane vector, $\boldsymbol{t}$ is the tangent vector ($\boldsymbol{t} \cdot \boldsymbol{t} = 1$ because s is the arc coordinate for the middle line), $\boldsymbol{n}$ is the unit normal vector, and α is the curvature. The position vector of a point in the strip

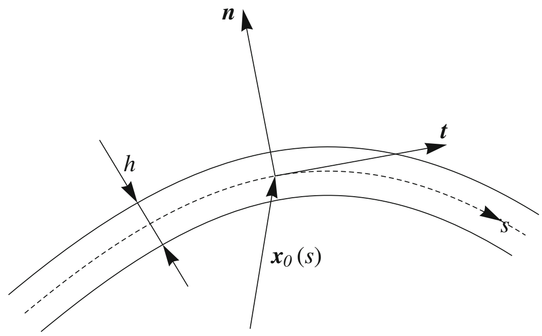

Fig. 1 Geometry of a curved strip with a given *middle line* $\boldsymbol{x}_0(s)$ and thickness h

$$\boldsymbol{x}(s, n) = \boldsymbol{x}_0 + n\boldsymbol{n}, \tag{2}$$

is defined by the arc coordinate s and the thickness coordinate n. The two-dimensional differential operator is

$$\nabla = \boldsymbol{n}\partial_n + (1 + \alpha n)^{-1}\boldsymbol{t}\partial_s, \quad \nabla\boldsymbol{x} = \mathbf{I}; \tag{3}$$

the gradient of the position vector is the two-dimensional identity tensor $\mathbf{I} = \boldsymbol{tt} + \boldsymbol{nn}$.

We consider a plane strain problem with the following linear equations for the stresses $\boldsymbol{\tau}$, the strains $\boldsymbol{\varepsilon}$ and the displacements $\boldsymbol{u}$ [28]:

$$\begin{aligned} &\nabla\cdot\boldsymbol{\tau} + \boldsymbol{f} = \rho\ddot{\boldsymbol{u}}, \quad \boldsymbol{n}\cdot\boldsymbol{\tau}\big|_{n=\pm h/2} = 0, \\ &\boldsymbol{\varepsilon} = \nabla\boldsymbol{u}^S, \\ &\nabla\times(\nabla\times\boldsymbol{\varepsilon})^T = 0, \\ &\boldsymbol{\tau} = {}^4\mathbf{C}\cdot\cdot\boldsymbol{\varepsilon}, \quad {}^4\mathbf{C} = {}^4\mathbf{C}(n), \end{aligned} \tag{4}$$

external forces $\boldsymbol{f} = f_n\boldsymbol{n} + f_t\boldsymbol{t}$ are considered inside the domain, and the boundary is free from traction forces. The fourth-rank tensor of material elastic properties ${}^4\mathbf{C}$ as well as the density ρ may vary over the thickness of the strip. The condition of compatibility $(4)_3$ will play an important role in the subsequent analysis.

A general exact solution of the formulated plane problem of the theory of elasticity cannot be written in a closed form, but for thin strips, one can reduce the problem to a one-dimensional formulation of a rod in the plane, in which all unknowns are functions of the arc coordinate s.

2.2 Asymptotic analysis

General idea of the method. The procedure of asymptotic splitting has a simple basic idea: the equations for the principal terms of the solution are found as the conditions of solvability for the minor terms. It can be illustrated by the following example [9]. Consider a system of linear algebraic equations:

$$(C_0 + \lambda C_1)u = f, \quad \det C_0 = 0, \quad \lambda \to 0. \tag{5}$$

The system becomes singular as the small parameter λ approaches zero. Therefore, we seek the solution u in the form of a series, starting with negative powers of λ:

$$u = \lambda^{-1}\overset{0}{u} + \overset{1}{u} + \lambda\overset{2}{u} + \cdots; \quad C_0\overset{0}{u} = 0 \;\Rightarrow\; \overset{0}{u} = \sum a_k\varphi_k; \quad C_0\varphi_k = 0. \tag{6}$$

At the first step, we find the principal term of the solution $\overset{0}{u}$ as a linear combination of the vectors of the fundamental solutions φ_k. The coefficients a_k follow from the conditions of solvability for $\overset{1}{u}$:

$$C_0\overset{1}{u} = f - C_1\overset{0}{u} \;\Rightarrow\; \psi_i^T\left(f - C_1\sum a_k\varphi_k\right) = 0, \quad C_0^T\psi_i = 0 \;\Rightarrow\; a_k. \tag{7}$$

The right-hand side $f - C_1\overset{0}{u}$ must be orthogonal to all solutions ψ_i of the conjugate system. If the resulting system for the coefficients a_k is again singular, then more terms with negative powers of λ must be included in the expansion of u, and several steps of the procedure need to be taken. It should be noted that this simple procedure cannot be explicitly found in such a classical treatise as [31].

Equations of balance with the small parameter. In the expressions for the position vector of a point (2) and for Hamilton's operator (3), we introduce a formal small parameter λ which indicates the thinness of the strip:

$$\boldsymbol{x}(s,n) = \lambda^{-1}\boldsymbol{x}_0 + n\boldsymbol{n}, \quad \nabla = \boldsymbol{n}\partial_n + \lambda(1+\lambda\alpha n)^{-1}\boldsymbol{t}\partial_s. \tag{8}$$

Now the magnitudes of n and s in (8) have formally the same asymptotic order. These expressions for $\boldsymbol{x}$ and ∇ may be seen as a mathematical definition of a thin strip.

Decomposing the stress tensor into components,

$$\boldsymbol{\tau} = \sigma_t\boldsymbol{tt} + \sigma_n\boldsymbol{nn} + \tau(\boldsymbol{nt}+\boldsymbol{tn}) + \sigma_z\boldsymbol{kk}, \tag{9}$$

and using (8), we write the balance equations $(4)_1$ with the small parameter:

$$\begin{aligned}
&\partial_n\sigma_n + \lambda\alpha(\sigma_n - \sigma_t + n\,\partial_n\sigma_n) + \lambda\,\partial_s\tau + f_n(1+\lambda\alpha n) = 0,\\
&\partial_n\tau + \lambda\alpha(2\tau + n\,\partial_n\tau) + \lambda\,\partial_s\sigma_t + f_t(1+\lambda\alpha n) = 0,\\
&\sigma_n\big|_{n=\pm h/2} = 0, \quad \tau\big|_{n=\pm h/2} = 0
\end{aligned} \tag{10}$$

(we yet restrict ourselves to a static case).

We seek for the unknown field of stresses in the form of a power series in the small parameter:

$$\boldsymbol{\tau} = \lambda^{-2}\overset{0}{\boldsymbol{\tau}} + \lambda^{-1}\overset{1}{\boldsymbol{\tau}} + \lambda^{0}\overset{2}{\boldsymbol{\tau}} + \cdots \tag{11}$$

Our goal is to find those terms in the solution which dominate as the strip is getting thinner and $\lambda \to 0$. It means that we are interested in the convergence of the solution to the principal term $\overset{0}{\boldsymbol{\tau}}$, rather than in the convergence of the series itself, and the role of the rest of the series is to determine the principal term from the conditions of solvability for the minor terms. The formal small parameter allows us to consider not just a thin strip, but rather a special class of "rod" solutions with a particular asymptotic behavior; for an additional discussion, see [41,44,47]. The leading power λ^{-2} in (11) is guessed in advance from other known solutions; otherwise it would have been necessary to estimate it with the help of trial-and-error.

Asymptotic splitting of the equations of balance. At the *first step* of the procedure, we substitute the expansion (11) in (10) and balance the principal terms in the resulting equations and boundary conditions, which have the order of smallness λ^{-2}. The results are:

$$\begin{aligned}
&\partial_n\overset{0}{\sigma}_n = 0, \quad \overset{0}{\sigma}_n\big|_{n=\pm h/2} = 0 \;\Rightarrow\; \overset{0}{\sigma}_n = 0,\\
&\partial_n\overset{0}{\tau} = 0, \quad \overset{0}{\tau}\big|_{n=\pm h/2} = 0 \;\Rightarrow\; \overset{0}{\tau} = 0.
\end{aligned} \tag{12}$$

The principal term in the tangential stress component $\overset{0}{\sigma}_t$ remains undetermined at the present stage.

At the *second step* of the procedure, we proceed to the terms of the order λ^{-1}, taking (12) into account:

$$\begin{aligned}
&\partial_n\overset{1}{\sigma}_n - \alpha\overset{0}{\sigma}_t = 0, \quad \overset{1}{\sigma}_n\big|_{n=\pm h/2} = 0,\\
&\partial_n\overset{1}{\tau} + \partial_s\overset{0}{\sigma}_t = 0, \quad \overset{1}{\tau}\big|_{n=\pm h/2} = 0.
\end{aligned} \tag{13}$$

We conclude that

$$\int\limits_{-h/2}^{h/2}\overset{0}{\sigma}_t\,\mathrm{d}n = 0; \tag{14}$$

for a straight rod with $\alpha = 0$, the asymptotics looks slightly different. Denoting

$$Q_n(s) = w\int\limits_{-h/2}^{h/2}\overset{1}{\tau}\,\mathrm{d}n, \quad M(s) = -w\int\limits_{-h/2}^{h/2}n\overset{0}{\sigma}_t\,\mathrm{d}n \tag{15}$$

(here, w is the out-of-plane size of the strip), integrating the expression for Q_n by parts, using the boundary conditions and applying the relation in the second line of (13), we arrive at

$$M' = -Q_n; \quad (\ldots)' \equiv \partial_s(\ldots). \tag{16}$$

The *third step* completes the procedure. We collect the terms of the order λ^0 and integrate them over the thickness. As

$$\int_{-h/2}^{h/2} \partial_n \overset{2}{\sigma}_n \, \mathrm{d}n = 0, \quad \int_{-h/2}^{h/2} \partial_n \overset{2}{\tau} \, \mathrm{d}n = 0 \tag{17}$$

(these are the conditions of solvability for the minor terms), we obtain

$$\begin{aligned} &\int_{-h/2}^{h/2} \left(f_n + \alpha \left(\overset{1}{\sigma}_n - \overset{1}{\sigma}_t \right) + n\alpha \, \partial_n \overset{1}{\sigma}_n + \partial_s \overset{1}{\tau} \right) \mathrm{d}n = 0, \\ &\int_{-h/2}^{h/2} \left(f_t + 2\alpha \overset{1}{\tau} + n\alpha \, \partial_n \overset{1}{\tau} + \partial_s \overset{1}{\sigma}_t \right) \mathrm{d}n = 0. \end{aligned} \tag{18}$$

Integrating again by parts, using the boundary conditions and denoting

$$q_n = w \int_{-h/2}^{h/2} f_n \, \mathrm{d}n, \quad q_t = w \int_{-h/2}^{h/2} f_t \, \mathrm{d}n, \quad Q_t = w \int_{-h/2}^{h/2} \overset{1}{\sigma}_t \, \mathrm{d}n, \tag{19}$$

we arrive at

$$q_n + Q_n' - \alpha Q_t = 0, \quad q_t + Q_t' + \alpha Q_n = 0. \tag{20}$$

With the vectors of external force, moment, and transversal force

$$\boldsymbol{q} = q_n \boldsymbol{n} + q_t \boldsymbol{t} = w \int_{-h/2}^{h/2} \boldsymbol{f} \, \mathrm{d}n, \quad \boldsymbol{M} = M\boldsymbol{k}, \quad \boldsymbol{Q} = Q_n \boldsymbol{n} + Q_t \boldsymbol{t}, \tag{21}$$

we rewrite (16) and (20) in the known vectorial form of the equations of balance of forces and moments in a rod (see e.g. [3,47]):

$$\boldsymbol{Q}' + \boldsymbol{q} = 0, \quad \boldsymbol{M}' + \boldsymbol{t} \times \boldsymbol{Q} = 0. \tag{22}$$

External distributed moments do not appear in the resulting asymptotic equations. The asymptotic analysis of stresses does not only produce the known balance equations (22) but provides important information for the subsequent analysis of the deformation.

Asymptotic analysis of the field of strains. Since the components of ${}^4\mathrm{C}$ are independent from the small parameter, as in the relation of elasticity $(4)_4$, the strain tensor must have the form

$$\boldsymbol{\varepsilon}(s, n) = \varepsilon_n \boldsymbol{nn} + \varepsilon_t \boldsymbol{tt} + \varepsilon_{nt}(\boldsymbol{nt} + \boldsymbol{tn}) = \lambda^{-2} \overset{0}{\boldsymbol{\varepsilon}} + \lambda^{-1} \overset{1}{\boldsymbol{\varepsilon}} + \lambda^{0} \overset{2}{\boldsymbol{\varepsilon}} + \cdots \tag{23}$$

Balancing the subsequent powers of λ in the condition of compatibility $(4)_3$, we obtain

$$\begin{aligned} \lambda^{-2}: &\quad \partial_n^2 \overset{0}{\varepsilon}_t = 0 \;\Rightarrow\; \overset{0}{\varepsilon}_t = \varepsilon + \kappa n; \\ \lambda^{-1}: &\quad \partial_n^2 \overset{1}{\varepsilon}_t = 2\partial_n \partial_s \overset{0}{\varepsilon}_{nt} + \alpha \, \partial_n \overset{0}{\varepsilon}_n - 2\alpha\kappa; \\ \lambda^{0}: &\quad \ldots \text{ etc.} \end{aligned} \tag{24}$$

The principal term of the axial strain $\overset{0}{\varepsilon}_t$ is linearly distributed over the cross-section, which completes the system of equations for the leading-order solution:

- The elastic relation $(4)_4$ and the results of the first step of the analysis of stresses (12) allow to write $\overset{0}{\varepsilon}_n$, $\overset{0}{\varepsilon}_{nt}$, and $\overset{0}{\sigma}_t$ as functions of n, ε, and κ.
- From (14), we express ε via κ: the only independent parameter in the principal term of strains has the meaning of change of curvature, which is intrinsic to the classical Kirchhoff rod theory with constrained shear and extension [47]; for a straight rod, the asymptotics is different and ε remains independent.
- From the second integral in (15), we explicitly compute the bending stiffness a, which is independent from the curvature of the strip:

$$M = a\kappa. \tag{25}$$

The analysis of the effects of shear and extension requires further treatment of the correction terms in the expansion of $\boldsymbol{\varepsilon}$ with the help of the equation in the second line of (24).

Asymptotics of displacements From the three-dimensional analysis [44,47], we know in advance that the vector of displacements should have the asymptotic order λ^{-4}:

$$\boldsymbol{u}(s,n) = u_t\boldsymbol{t} + u_n\boldsymbol{n} = \lambda^{-4}\overset{0}{\boldsymbol{u}} + \lambda^{-3}\overset{1}{\boldsymbol{u}} + \lambda^{-2}\overset{2}{\boldsymbol{u}} + \cdots \tag{26}$$

From the kinematic relation $(4)_2$ and from (24), we obtain:

$$\begin{aligned} \lambda^{-2}:&\quad \partial_n\overset{0}{u}_n = 0, \quad \partial_n\overset{0}{u}_t = 0 \;\Rightarrow\; \overset{0}{\boldsymbol{u}} = \boldsymbol{U}(s),\\ \lambda^{-1}:&\quad \partial_n\overset{1}{u}_n = 0, \quad \alpha\overset{0}{u}_t = \partial_n\overset{1}{u}_t + \partial_s\overset{0}{u}_n, \quad \alpha\overset{0}{u}_n + \partial_s\overset{0}{u}_t = 0,\\ \lambda^{0}:&\quad \partial_s\overset{1}{u}_t + \alpha\overset{1}{u}_n = \overset{0}{\varepsilon}_t, \quad \ldots \end{aligned} \tag{27}$$

The principal term is the displacement of the cross-section in terms of the rod theory $\boldsymbol{U}(s)$. From the second and third lines in (27), we conclude that

$$\alpha\overset{0}{u}_n + \partial_s\overset{0}{u}_t = 0, \quad \partial_s\left(\alpha\overset{0}{u}_t - \partial_s\overset{0}{u}_n\right) = \kappa \;\Rightarrow\; \boldsymbol{U}'\cdot\boldsymbol{t} = 0, \quad \left(\boldsymbol{U}'\cdot\boldsymbol{n}\right)' = \kappa, \tag{28}$$

which is in full correspondence with the kinematic relations of classical Kirchhoff rods [9,47], written in terms of the vector of small rotation $\boldsymbol{\theta} = \theta\boldsymbol{k}$ and the vector of bending and torsional deformation $\boldsymbol{\kappa} = \kappa\boldsymbol{k}$:

$$\boldsymbol{U}' = \boldsymbol{\theta}\times\boldsymbol{t}, \quad \boldsymbol{\theta}' = \boldsymbol{\kappa}. \tag{29}$$

It remains to say that the first correction term $\overset{1}{\boldsymbol{u}}$ corresponds to the rotation of the cross-section with the angle θ, i.e. the classical Kirchhoff hypothesis of straight normals is fulfilled asymptotically. Without the compatibility conditions (i.e. solving the problem in displacements from the very beginning), the clarity of the approach would be lost, as the solution would require several additional steps.

Dynamical effects. Finally, we are able to consider effects due to the inertial term in the equation of balance $(4)_1$. We consider processes with a particular asymptotic rate over time, which is reflected by a formal change of the time variable:

$$t = \lambda^{-2}\tilde{t}, \quad (\ldots)^{\cdot} = \lambda^2\partial_{\tilde{t}}(\ldots), \quad \ddot{\boldsymbol{u}} = \partial_{\tilde{t}}^2\boldsymbol{U} + \cdots \tag{30}$$

(higher order terms are replaced by an ellipsis in the right-hand side of the last equality). After the asymptotic procedure is completed, we set the formal small parameter λ to 1. This allows to write the inertial term in the equation of balance of forces in the classical form:

$$\boldsymbol{Q}' + \boldsymbol{q} = \ddot{\boldsymbol{U}}\, w \int_{-h/2}^{h/2} \rho\,\mathrm{d}n. \tag{31}$$

Concluding remarks on the asymptotic analysis. Finding the leading-order solution of the original two-dimensional problem requires the following steps:

1. The problem in the cross-section (solution scheme before (25)) needs to be solved, which produces the bending stiffness of the one-dimensional model.
2. The one-dimensional rod model, determined by the relations (22), (25), and (28), needs to be analyzed. In a general case, the boundary conditions for the reduced model should also be obtained by the asymptotic analysis of the two-dimensional fields near the boundary and by matching of the two asymptotic expansions inside the domain and in the edge layer [15,27,44].
3. With the displacements $\boldsymbol{U}$ and strains κ, we restore the principal terms in the corresponding fields of the nonreduced two-dimensional model.

Asymptotics is different for a straight rod, and the axial force Q_t gets its own relation of elasticity. The procedure allows for a relatively simple extension to coupled problems, e.g. piezoelectric [44]. It is advantageous that the procedure does not allow for alternatives as soon as "slow" and "fast" variables are chosen, and the relations of the original nonreduced theory are written with a formal small parameter.

2.3 Direct approach

For the present problem of plane deformation of a rod, we consider a one-dimensional continuum of particles. Following the strategy, presented e.g. in [9,10,24,40], we write the principle of virtual work for a part of the rod:

$$\int_{s_0}^{s_1} \left(\boldsymbol{q} \cdot \delta \boldsymbol{x} + m\, \delta\theta + \delta A^{(i)} \right) \mathrm{d}s + (\boldsymbol{Q} \cdot \delta \boldsymbol{x} + M \delta\theta)\Big|_{s_0}^{s_1} = 0. \tag{32}$$

Here, external force $\boldsymbol{q}$ and moment m are distributed along the arc length, $\boldsymbol{Q}$ and M are the force and the moment of interaction with the remaining parts of the rod ($s < s_0$ and $s > s_1$), $\boldsymbol{x}(s)$ and $\theta(s)$ are the position vector and the angle of rotation of particles, and $\delta A^{(i)}$ is the virtual work of the internal forces on the virtual deformation $\delta\boldsymbol{x}$ and $\delta\theta$: each particle has three mechanical degrees of freedom. In the reference configuration, we have $\boldsymbol{x} = \boldsymbol{x}_0$ and $\theta = 0$. We rewrite the boundary term in (32) as an integral; a local variational relation follows from the arbitrariness of $s_{0,1}$ [9]:

$$\left(\boldsymbol{Q}' + \boldsymbol{q} \right) \cdot \delta \boldsymbol{x} + (M' + m)\delta\theta + \boldsymbol{Q} \cdot \delta \boldsymbol{x}' + M\delta\theta' - \delta A^{(i)} = 0. \tag{33}$$

Now, the internal forces produce no work at rigid body motion; hence,

$$\begin{aligned} &\delta \boldsymbol{x} = \delta \boldsymbol{x}_1 + \delta\theta_1 \boldsymbol{k} \times \boldsymbol{x}, \quad \delta \boldsymbol{x}_1 = \text{const.} \quad \delta\theta_1 = \text{const.} \quad \Leftrightarrow \quad \delta A^{(i)} = 0 \quad \Rightarrow \\ &\boldsymbol{Q}' + \boldsymbol{q} = 0, \quad M' + \boldsymbol{x}' \times \boldsymbol{Q} \cdot \boldsymbol{k} + m = 0. \end{aligned} \tag{34}$$

In contrast to (22), the balance equations (34) allow for external moments and axial extension of the rod; writing the static boundary conditions is simple in the considered case. The following terms remain in (33):

$$M\delta\theta' + \boldsymbol{Q} \cdot (\delta \boldsymbol{x}' - \delta\theta\, \boldsymbol{k} \times \boldsymbol{x}') + \delta A^{(i)} = 0. \tag{35}$$

Instead of the known model of a Cosserat continuum, see e.g. [1,36], we will consider a model with constrained shear deformation. The axial extension is allowed for the sake of an efficient numerical implementation. The kinematical constraint between virtual displacements and rotations of particles, which is similar to (29), follows as

$$\delta(\boldsymbol{n} \cdot \boldsymbol{x}') = 0, \quad \delta\boldsymbol{n} = \delta\theta\, \boldsymbol{k} \times \boldsymbol{n} \quad \Rightarrow \quad \delta\theta\, \boldsymbol{t} \cdot \boldsymbol{x}' = \boldsymbol{n} \cdot \delta\boldsymbol{x}'; \quad \boldsymbol{t} = \boldsymbol{x}' \big/ |\boldsymbol{x}'|; \tag{36}$$

here, the normal vector $\boldsymbol{n}$ is "frozen" into a particle and rotates with it.

For an elastic rod, the work of the internal forces is $\delta A^{(i)} = -\delta\Pi$; Π is the strain energy per unit "length" s. From (35) and (36), we show that the strain energy is a function of the deformation of bending κ (change of curvature of the rod) and of axial extension ε:

$$\begin{aligned} &\delta\Pi = M\delta\theta' + \boldsymbol{Q} \cdot \left(\delta\boldsymbol{x}' - \boldsymbol{n}\boldsymbol{n} \cdot \delta\boldsymbol{x}' \right) = M\delta\kappa + Q_t\delta\varepsilon, \\ &Q_t = \boldsymbol{Q} \cdot \boldsymbol{t}, \quad \kappa = \theta', \quad \boldsymbol{x}' = (1+\varepsilon)\boldsymbol{t}, \quad \varepsilon = |\boldsymbol{x}'| - 1. \end{aligned} \tag{37}$$

The elastic relations read

$$M = \frac{\partial \Pi}{\partial \kappa}, \quad Q_t = \frac{\partial \Pi}{\partial \varepsilon}, \tag{38}$$

the transversal force $Q_n = \boldsymbol{Q} \cdot \boldsymbol{n}$ is determined by the constraint (36). This type of a generalization of the Bernoulli-Euler beam theory has already been discussed in the literature, see e.g. [3,25].

For problems with small local strains (which does not exclude large overall rotations and displacements), a quadratic approximation of the strain energy should be satisfactory:

$$2\Pi = a\kappa^2 + b\varepsilon^2. \tag{39}$$

The stiffnesses a and b cannot be determined in the framework of the direct approach.

2.4 Hybrid asymptotic-direct formulation

As it was shown by the asymptotic analysis, the effect of the axial extension is negligible for thin curved rods, and the second term in (39) can be considered as a penalty, which keeps $\varepsilon \to 0$ as $b \to \infty$. Moreover, in [8], it is shown that the model will be asymptotically equivalent to the classical model of Kirchhoff when the stiffness b is asymptotically larger than a. Axial extension may be important for straight rods or for numerical analysis.

As the direct approach must be equivalent to the asymptotic model in the linear setting, a must be equal to the bending stiffness in (25); the problem in the cross-section, which results from the asymptotic analysis, needs to be solved prior to the analysis of the rod problem. It is also important that the asymptotics determines the minimal set of degrees of freedom of particles, which need to be included in the direct approach.

2.5 Numerical analysis

The rod model, derived in Sect. 2.3 with the help of the principle of virtual work, allows for a very simple implementation in the framework of the finite element method. The balance equations and the static boundary conditions are equivalent to the following variational principle:

$$\begin{aligned}
&\Pi^{(\Sigma)}[\boldsymbol{x}(s)] = \Pi^{(\mathrm{def})} + \Pi^{(\mathrm{ext})} \to \min, \quad \Pi^{(\mathrm{def})} = \int_0^l \Pi(\varepsilon, \kappa)\,\mathrm{d}s, \\
&\Pi^{(\mathrm{ext})} = -\int_0^l (\boldsymbol{q} \cdot \boldsymbol{x} + m\theta)\,\mathrm{d}s - (\boldsymbol{Q} \cdot \boldsymbol{x} + M\theta)\big|_0^l, \\
&\kappa = \boldsymbol{n}' \cdot \boldsymbol{t} - \boldsymbol{n}_0' \cdot \boldsymbol{t}_0 = \frac{1}{|\boldsymbol{x}'|}\boldsymbol{n} \cdot \boldsymbol{x}'' - \boldsymbol{n}_0 \cdot \boldsymbol{x}_0''.
\end{aligned} \tag{40}$$

At a stable equilibrium, the sum of the total strain energy $\Pi^{(\mathrm{def})}$ (integrated over the length of the rod l) and of the potential energy of external loads $\Pi^{(\mathrm{ext})}$ has a minimum. The functional $\Pi^{(\Sigma)}$ is considered on the configurations of the rod $\boldsymbol{x}(s)$, which are kinematically admissible.

The problem of minimization of (40) is well suited for the application of the Ritz method; the approximations of the field $\boldsymbol{x}(s)$ need to be chosen such, that the functions κ and ε have no singularities. A two-node finite element with the following approximation can serve to this purpose:

$$\boldsymbol{x}(\tilde{s}) = \boldsymbol{x}_i N_1(\tilde{s}) + \partial_{\tilde{s}}\boldsymbol{x}_i N_2(\tilde{s}) + \boldsymbol{x}_j N_3(\tilde{s}) + \partial_{\tilde{s}}\boldsymbol{x}_j N_4(\tilde{s}), \quad -1 \le \tilde{s} \le 1. \tag{41}$$

The cubic shape functions N_i are such that the position vector and its derivative in the node k take on the values $\boldsymbol{x}_k$ and $\partial_{\tilde{s}}\boldsymbol{x}_k$, i.e. we have four degrees of freedom in each node. Because the local coordinate on the element $\tilde{s}$ is no longer the arc coordinate in the reference configuration, the coefficient $\mathrm{d}s/\mathrm{d}\tilde{s}$ will enter the formulas for ε in (37) and for κ in (40). This choice of the nodal positions and local derivatives guarantees the necessary continuity in $\boldsymbol{x}$ and $\boldsymbol{n}$.

The element makes use of the so-called absolute nodal coordinate formulation (ANCF, for further details see e.g. [13]); the main benefit is that the mass matrix remains constant during simulation, which allows for efficient transient analysis.

These finite elements were applied in [24]; the version of the numerical method, extended to the nonlinear and electromechanically coupled shell problems, was used in [10,43]; an extension to a three-dimensional geometrically and physically nonlinear rod problem was presented in [45]. Being free from a priori chosen approximation over the cross-section, these numerical schemes represent a good alternative to methods based on nonreduced theories.

3 Numerical example

For the practical validation of the analytical conclusions, we have chosen the problem of force bending of a clampled quarter of a circle, presented in Fig. 2. The radius of the middle line is $r = 1\,m$; material parameters of the two layers, each of them having the thickness $h/2$, are presented in Table 1. The out-of-plane thickness was chosen $w = 1\,m$.

In the following, we will consider solutions of various problems when the thickness tends to zero:

$$h = h_0/k, \tag{42}$$

the reference thickness value is $h_0 = 0.1\,m$, and the thickness factor k will be sequentially increased. Comparing solutions of the problems in the rod formulation and continuous two-dimensional solutions, obtained by the finite element method, we can make conclusions concerning the asymptotic nature of the rod theory as $h \to 0$. The continuous finite element solutions were obtained in ABAQUS. The mesh was chosen such that there were always 60 quadratic quadrilateral elements (CPS8R) over the thickness, and the height to width ratio for all elements was approximately 1.

While the analytic study in Sect. 2.2 was easier to formulate under the plane strain assumption, we consider the particular solutions for the plane stress formulation.

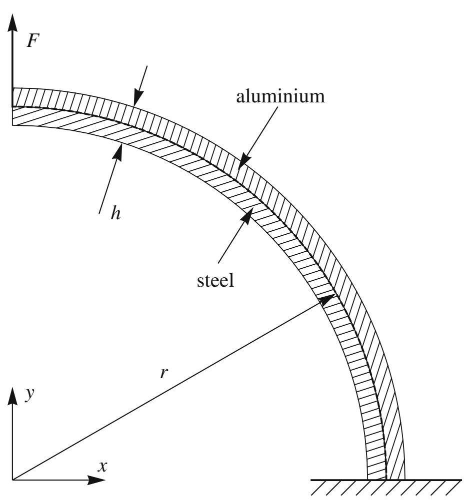

Fig. 2 Force bending of a layered curved strip in the form of a quarter of a *circle*

Table 1 Material parameters of the two layers of the strip

	Layer 1 (inner, steel)	Layer 2 (outer, aluminium)
Young's modulus E	2.1×10^{11} Pa	7×10^{10} Pa
Poisson's ratio ν	0.3	0.34
Density ρ	7,800 kg/m^3	2,700 kg/m^3

3.1 Linear static problem

We started with a simple geometrically linear problem of static bending. Finding an analytical solution for the statically determined rod model is easy with the help of Castigliano's method. Choosing the arc coordinate s with its origin in the clamped point of the rod, we write the bending moment M, the total strain energy Π^Σ, and the vertical displacement of the point under the load as the derivative of the total strain energy with respect to the force:

$$M = -Fr\cos\frac{s}{r}, \quad \Pi^\Sigma = \int_0^{\pi r/2} \frac{M^2}{2a}\mathrm{d}s = \frac{F^2\pi r^3}{8a}, \quad u_y = \frac{\partial \Pi^\Sigma}{\partial F} = \frac{F\pi r^3}{4a}. \tag{43}$$

The bending stiffness a is determined in accordance with the procedure, described before (25). The relation between ε and κ due to the conditions (12) and (14) is equivalent to finding the coordinate n_0 of the neutral layer of the rod:

$$\int_{-h/2}^{h/2} E(n)(n-n_0)\,\mathrm{d}n = 0 \quad \Rightarrow \quad n_0 = \frac{h}{4}\frac{E_2-E_1}{E_2+E_1}$$

$$\Rightarrow \quad a = w\int_{-h/2}^{h/2} E(n)(n-n_0)^2\,dn = \frac{wh^3}{96}\frac{E_1^2+14E_1E_2+E_2^2}{E_1+E_2}. \tag{44}$$

Substituting this a into (43), we obtain an expression for the vertical displacement, which we will call $u_y^{(1)}$. It may also make sense to modify the formula by using the radius of the neutral fiber $r+n_0$ instead of the radius of the geometric middle line r; the corresponding expression will be called $u_y^{(2)}$ in the following. And the third rod solution, which we considered, was obtained by taking axial extension into account according to the model obtained in Sect. 2.3. Instead of (39), we used the expression for the strain energy with coupling between κ and ε due to the unsymmetry of the cross-section:

$$\Pi = \frac{1}{48}w\left(12(E_1+E_2)\varepsilon^2 + 6(E_2-E_1)h\varepsilon\kappa + (E_1+E_2)h^2\kappa^2\right). \tag{45}$$

Writing again the expressions for the bending moment M and for the axial force Q_t, performing Legendre transformation of (45) according to (38), integrating the strain energy over the length, and computing the derivative with respect to F, we arrive at the expression for the vertical deflection

$$u_y^{(3)} = \frac{2Fr\left((E_1+E_2)h^2\pi + 12(E_2-E_1)hr + 12(E_1+E_2)\pi r^2\right)}{w\left(E_1^2+14E_1E_2+E_2^2\right)h^3}. \tag{46}$$

The finite element solutions u_y^* for particular values of k, obtained in ABAQUS, were considered as reference values for computing the relative errors of the rod solutions

$$e^{(i)} = |u_y^{(i)} - u_y^*|/u_y^*. \tag{47}$$

It is interesting to notice that the simple formula predicts displacements larger than the continuous model: $u_y^{(1)} > u_y^*$. Usually, it is assumed that the rod model is too stiff, but the curvature and the material inhomogeneity change the situation; the opposite effect can be achieved by switching the layers. The two other solutions $u_y^{(2)}$ and $u_y^{(3)}$ generally predict smaller displacements than the continuous model.

The dependence of the three relative errors on the thickness factor k in logarithmic scale is presented in Fig. 3. All three errors obviously converge toward zero. Correction of the radius by the offset of the neutral fiber makes the situation worse. Taking axial extension into account, we obtain much more precise solutions. Particularities concerning the last set of results at $k = 2^5$ appear probably due to the numerical problems in the finite element solution: with more than one million finite elements in total, accumulated numerical errors seem to prevent ABAQUS from reaching the required precision of the results, and using less finite elements, we arrive at solutions which are not sufficiently converged.

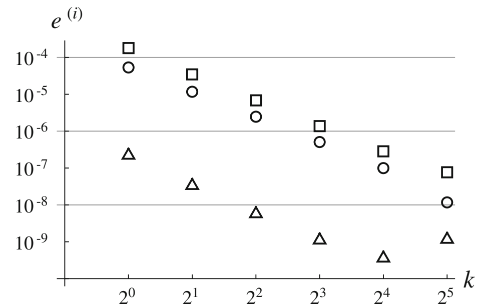

Fig. 3 Convergence of the solutions of the linear static problem to the continuous solution. *Open circle* $u_y^{(1)}$: only bending, r is the radius of the middle line. *Open square* $u_y^{(2)}$: only bending, r is the radius of the neutral layer. *Open triangle* $u_y^{(3)}$: bending and axial extension

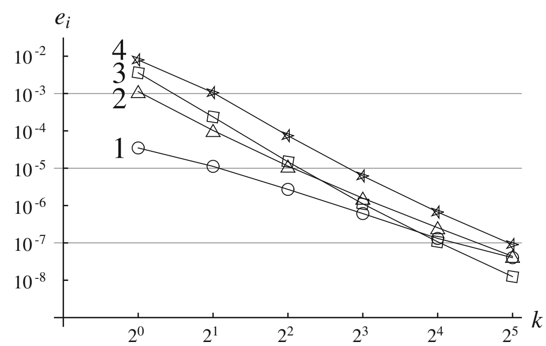

Fig. 4 Convergence of the first four eigenfrequencies of the linear dynamic problem

The noticeable outcome of the results in Fig. 3 is that although the precision of the model with the axial extension is higher, the rate of convergence is obviously the same. Obtaining a model with higher asymptotic precision would require subsequent terms to be considered in the asymptotic analysis of Sect. 2.2, and the starting point would be to consider the correction terms to the strain tensor according to the condition of compatibility (24). The resulting theory should include axial extension together with the shear effects, and it may be expected that the resulting stiffness coefficients will depend on the curvature of the strip.

3.2 Linear eigenvalue problem

For dynamical problems, the correspondence of the eigenfrequencies of the reduced rod model ω_i to the eigenfrequencies of the continuous model ω_i^* is crucial. While the reference values ω_i^* are again computed in ABAQUS, for the rod model, we have also decided to use a numerical method. The model was discretized with finite elements, presented in Sect. 2.5; the equations of motion were linearized in the vicinity of the undeformed state, and the eigenvalue problem was solved. For thicker rods, sufficiently converged values ω_i were obtained with ten one-dimensional finite elements, while at higher k, we went up to twenty (compare with hundreds of thousands of two-dimensional finite elements needed for thin structures!).

The relative errors

$$e_i = |\omega_i - \omega_i^*|/\omega_i^* \tag{48}$$

are presented for the first four eigenfrequencies in Fig. 4 depending on the thickness factor k in logarithmic scale (the number of the eigenfrequency is marked to the left of the points at $k = 1$).

There is no clear understanding, whether numerical problems or some other reason make the third eigenfrequency converge better than the others. But the general convergence of the solutions of the two-dimensional and one-dimensional eigenvalue problems as the thickness tends to zero can clearly be seen.

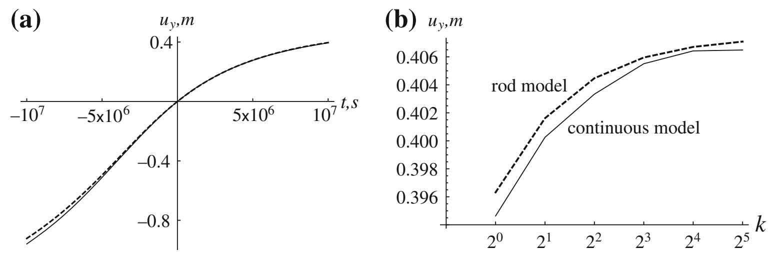

Fig. 5 Nonlinear static analysis with the rod model (*thick dashed line*) and with the continuous model (*solid line*) (**a**), convergence of the solutions (**b**)

Table 2 Relative errors of the eigenfrequencies of the pre-deformed structure

k	e_1	e_2	e_3	e_4
2^0	0.01505	0.01191	0.03239	0.00517
2^1	0.0085	0.00147	0.00606	0.0175
2^2	0.00449	0.00016	0.00021	0.0035
2^3	0.00231	0.0005	0.00066	0.00038
2^4	0.00115	0.00026	0.0005	0.00015
2^5	0.02724	0.01085	0.00301	0.00127

3.3 Nonlinear static problem

As the next step, we considered geometrically nonlinear static deformation of the structure. For thick strip with $k = 1$, the vertical displacement as a function of the dead force F in the range $-F_0 \leq F \leq F_0$, $F_0 = 10^7$ N is presented in Fig. 5a. The solution, obtained with the rod finite element model, is compared to the displacements of the middle node of the loaded edge of the two-dimensional finite element model.

The convergence of the vertical displacement was studied for the force values F, which were scaled according to the thickness factor:

$$F = F_0/k^3. \tag{49}$$

As it is shown in Fig. 5b, the resulting deflections remain in the vicinity of a moderately large value with varying k. The dashed line in Fig. 5b connects the points, obtained with the rod model, and the solid line represents mean vertical displacement of the loaded end in the two-dimensional solution. Fine convergence can be observed except for the last point $k = 2^5$, at which numerical difficulties prevent us from getting a good continuous solution.

3.4 Eigenfrequencies of the pre-deformed structure

Quantitative analysis of the convergence of the nonlinear dynamical properties of the reduced and continuous models is a nontrivial problem, which is largely affected by the chosen criteria and by the precision of the available solutions. In the present work, we studied eigenfrequencies of the structure, pre-deformed by the force F; the force itself was supposed not to bring additional inertia to the structure. The relative errors (48) for the first four eigenfrequencies depending on the thickness factor are presented in Table 2. Although the numerical effects make the situation less clear than in the fully linear case (see Fig. 4), the convergence of the two solutions can again be seen.

3.5 Nonlinear transient simulation

We performed transient analysis of large oscillations due to the instant application of the dead force (49) for $k = 0$ and $k = 4$. The results, which are shown in Fig. 6, speak for themselves. Although dynamic effects lower the reliability of the rod model for thicker structures in comparison to static analysis, for thinner structures, the results are quite satisfactory.

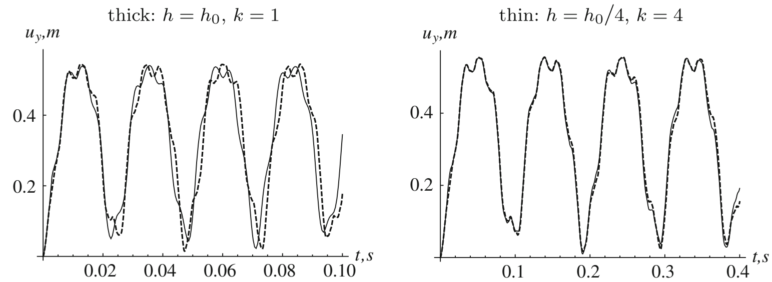

Fig. 6 Nonlinear transient analysis with the rod model (*thick dashed line*) and with the continuous model (*solid line*)

4 Conclusions

The plane problem of finite deformations and dynamics of a thin curved strip is considered with the help of the hybrid asymptotic-direct approach, which leads to a dimensionally reduced one-dimensional rod model. The theoretical results were validated in a series of numerical experiments. Solutions of linear and nonlinear as well as statical and dynamical problems were considered both in the original continuous and in the rod models. As the thickness tends to zero, the equivalence of the solutions can be clearly seen from the results of the simulation.

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