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Form-finding of tensegrity structures with multiple states of self-stress

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Abstract A numerical method is presented for form-finding of tensegrity structures with multiple states of self-stress. At the first stage, the range of feasible sets of the nodal coordinates and the force densities are iteratively calculated by the only known information of the topology and the types of members until the required rank deficiencies of the force density and equilibrium matrices are satisfied, respectively. The linear constraints on the force densities which are derived from the obtained configuration's symmetry properties and/or directly assigned by designers are then utilized to define a single integral feasible force density vector in the second stage. An explanation on the null space of the force density matrix that generates the configurations of the tensegrities is rigorously given. Several numerical examples are presented to demonstrate the efficiency and robustness in searching new self-equilibrium stable configurations of tensegrity structures with multiple states of self-stress.

1 Introduction

Over the past few decades, tensegrity structures first proposed by Fuller [1] have attracted considerable attention in a wide variety of fields including aerospace [2], architecture [3,4], civil engineering [5,6], biology [7–9], mathematics [10,11] and robotics [12,13]. They belong to a class of free-standing, prestressed, pin-jointed, cable-strut system where contacts are allowed among the struts [14].

As a pioneering work of numerical form-finding, the so-called force density method was proposed by Schek [15] for form-finding of tensile structures. Motro et al. [16] then presented the dynamic relaxation that has been reliably applied to tensile structures [17] and many other nonlinear problems. Recently, Masic et al. [18], Zhang and Ohsaki [19] and Estrada et al. [20] developed new numerical methods using the force density formulation. Pagitz and Tur [21] suggested a form-finding algorithm that is based on the finite element method. Zhang et al. [22] employed a refined dynamic relaxation method for form-finding of nonregular tensegrity systems. Micheletti and Williams [23] used a marching procedure for finding stable placements of irregular tensegrities based on their given regular counterparts. Rieffel et al. [24] and Xu and Luo [25] introduced an evolutionary algorithm for producing irregular tensegrity structures. Most recently, Tran and Lee [26] proposed an advanced form-finding for tensegrity structures based only on the given topology and the types of members, i.e., either tension or compression. They also introduced an approach [27] for the determination of a unique configuration of free-form tensegrity structures. Li et al. [28] suggested a method by incorporating a

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Monte Carlo type approach into tensegrity form-finding. A review of the existing methods for form-finding of tensegrity structures can be found in Tibert and Pellegrino [29], Juan and Tur [30] and Sultan [31]. However, most of the available methods have dealt with the form-finding of tensegrities for only the case of a single self-stress state. Thus, an efficient numerical form-finding procedure for tensegrities with multiple states of self-stress is the motivation of this study.

In this paper, a numerical method is presented for form-finding of tensegrity structures with multiple states of self-stress. The topology and the types of members, i.e., either compression or tension, are the required information. The eigenvalue decomposition (EVD) of the force density matrix and the singular value decomposition (SVD) of the equilibrium matrix are iteratively executed to find the range of feasible sets of the nodal coordinates and the force densities until the required rank deficiencies of the force density and equilibrium matrices for the case of multiple states of self-stress are satisfied, respectively. Since the obtained geometry at the first stage has multiple self-stress states, the linear constraints on the force densities which are derived from its symmetry properties and/or directly assigned by designers are then utilized to define a single integral feasible force density vector in the second stage. An explanation on the null space of the force density matrix that generates the configurations of tensegrities is rigorously given.

2 Self-equilibrium equations and rank deficiency conditions

2.1 Basic assumptions

In this study, the following assumptions are made in tensegrity structures:

- The topology of the structure in terms of nodal connectivity is known.
- Members are connected by pin joints.
- No external load is applied and the self-weight of the structure is neglected during the form-finding procedure.
- There are no dissipative forces acting on the system.
- Both local and global buckling are not considered.

2.2 Self-equilibrium equations

For a d-dimensional ($d = 2$ or 3) tensegrity structure with b members, n free nodes and n_f fixed nodes (supports), its topology can be expressed by a connectivity matrix $\mathbf{C}_s$ ($\in \mathbb{R}^{b\times(n+n_f)}$) as discussed in Tran and Lee [26]. Suppose member k connects nodes i and j ($i < j$), then the ith and jth elements of the kth row of $\mathbf{C}_s$ are set to 1 and -1, respectively, as follows:

$$\mathbf{C}_{s(k,p)} = \begin{cases} 1 & \text{for } p = i, \\ -1 & \text{for } p = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If the free nodes are numbered first, and then the fixed nodes, $\mathbf{C}_s$ can be divided into two parts as

$$\mathbf{C}_s = [\mathbf{C}\ \mathbf{C}_f], \tag{2}$$

where $\mathbf{C}$ ($\in \mathbb{R}^{b\times n}$) and $\mathbf{C}_f$ ($\in \mathbb{R}^{b\times n_f}$) describe the connectivities of the members to the free and fixed nodes, respectively. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ ($\in \mathbb{R}^n$) and $\mathbf{x}_f, \mathbf{y}_f, \mathbf{z}_f$ ($\in \mathbb{R}^{n_f}$) denote the nodal coordinate vectors of the free and fixed nodes, respectively, in x-, y- and z-directions.

The equilibrium equations of the free nodes in each direction of a general pin-jointed structure given by Schek [15] can be stated as

$$\mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{x} + \mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}_f\mathbf{x}_f = \mathbf{p}_x, \tag{3a}$$

$$\mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{y} + \mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}_f\mathbf{y}_f = \mathbf{p}_y, \tag{3b}$$

$$\mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}\mathbf{z} + \mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}_f\mathbf{z}_f = \mathbf{p}_z, \tag{3c}$$

where $\mathbf{p}_x$, $\mathbf{p}_y$ and $\mathbf{p}_z$ ($\in \mathbb{R}^n$) are the vectors of external loads applied at the free nodes in x-, y- and z-directions, respectively. The symbol $(\cdot)^T$ denotes the transpose of a matrix or vector, and diag($\mathbf{q}$) ($\in \mathbb{R}^{b\times b}$) is a diagonal square matrix of $\mathbf{q}$ ($\in \mathbb{R}^b$) which is the force density vector as suggested in Schek [15], defined by

$$\mathbf{q} = \{q_1,\ q_2,\ \ldots,\ q_b\}^T, \tag{4}$$

in which each component of this vector is the force f_k to length l_k ratio $q_k = f_k/l_k$ ($k = 1, 2, \ldots, b$), known as force density or self-stressed coefficient in Vassart and Motro [32].

When external load and self-weight are ignored, the tensegrity system does not require any fixed nodes (supports). Its geometry can be defined by the relative position of the nodes. That is, the system can be considered as a free-standing rigid body in space [26,33]. In this context, Eq. (3) becomes

$$\mathbf{D}\mathbf{x} = \mathbf{0}, \tag{5a}$$

$$\mathbf{D}\mathbf{y} = \mathbf{0}, \tag{5b}$$

$$\mathbf{D}\mathbf{z} = \mathbf{0}, \tag{5c}$$

where $\mathbf{D}$ ($\in \mathbb{R}^{n\times n}$), known as force density matrix [20,29], or stress matrix [34–36], is given by

$$\mathbf{D} = \mathbf{C}^T \mathrm{diag}(\mathbf{q})\mathbf{C}. \tag{6}$$

For simplicity, Eq. (5) can be reorganized as

$$\mathbf{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] = [\mathbf{0}\ \mathbf{0}\ \mathbf{0}], \tag{7}$$

where $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ ($\in \mathbb{R}^{n\times d}$) is a matrix of nodal coordinates for a d-dimensional tensegrity structure.

On the other hand, by substituting Eq. (6) into (5), the self-equilibrium equations of the tensegrity structures can also be reorganized as

$$\mathbf{A}\mathbf{q} = \mathbf{0}, \tag{8}$$

where $\mathbf{A}$ ($\in \mathbb{R}^{dn\times b}$) is known as the equilibrium matrix in [26,33], defined by

$$\mathbf{A} = \begin{pmatrix} \mathbf{C}^T\mathrm{diag}(\mathbf{C}\mathbf{x}) \\ \mathbf{C}^T\mathrm{diag}(\mathbf{C}\mathbf{y}) \\ \mathbf{C}^T\mathrm{diag}(\mathbf{C}\mathbf{z}) \end{pmatrix}. \tag{9}$$

Equation (7) presents the relationship between the force densities and the nodal coordinates, while Eq. (8) shows the relationship between the projected lengths in x-, y- and z-directions, respectively, and the force densities. Both Eqs. (7) and (8) are linear homogeneous systems of the self-equilibrium equations with respect to the nodal coordinates and the force densities, respectively.

2.3 Rank deficiency conditions for multiple states of self-stress

Let $\mathbf{q}$ be the vector of force density and $\mathbf{C}$ be the incidence matrix of a d-dimensional tensegrity structure in self-equilibrium state. It is well known that the set of all solutions to the linear homogeneous system of Eq. (7) is the null space of $\mathbf{D}$. The dimension of this null space or rank deficiency of $\mathbf{D}$ is defined as

$$n_{\mathbf{D}} = n - r_{\mathbf{D}}, \tag{10}$$

where $r_{\mathbf{D}} = \mathrm{rank}(\mathbf{D})$. Note that $\mathbf{D}$ is always square, symmetric and singular with nullity at least 1 since the sum of all components in any row or column is zero for any tensegrity structure [26,29]. The general solution of Eq. (7) [37] has the following form:

$$\mathbf{x} = \sum_{i=1}^{n_{\mathbf{D}}} \alpha_i^x \mathbf{g}_i, \tag{11a}$$

$$\mathbf{y} = \sum_{i=1}^{n_{\mathbf{D}}} \alpha_i^y \mathbf{g}_i, \tag{11b}$$

$$\mathbf{z} = \sum_{i=1}^{n_{\mathbf{D}}} \alpha_i^z \mathbf{g}_i, \tag{11c}$$

where the coefficients α_i^x, α_i^y and α_i^z are arbitrary values and $\mathbf{g}_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, n_\mathbf{D})$ represent particular solutions of Eq. (7). In other words, $\mathbf{g}_i$ are in the null space of $\mathbf{D}$ such that $\mathbf{D}\mathbf{g}_i = \mathbf{0}$. As the coefficients α_i^x, α_i^y and α_i^z range over all possible values, the general solution generates all possible ones. From Eq. (11), the following cases occur based on the dimension of the null space of $\mathbf{D}$:

- If $n_\mathbf{D} = 1$, the basic solution of Eq. (7) is a uniform vector $\mathbf{g}_1 = \{1, 1, \ldots, 1\}^T$ $(\in \mathbb{R}^n)$, since the sum of the elements of a row or a column of $\mathbf{D}$ is always equal to zero [26,29]. Hence, Eq. (11) becomes

$$[\mathbf{x}_1\ \mathbf{y}_1\ \mathbf{z}_1] = \begin{bmatrix} \alpha_1^x & \alpha_1^y & \alpha_1^z \\ \alpha_1^x & \alpha_1^y & \alpha_1^z \\ \vdots & \vdots & \vdots \\ \alpha_1^x & \alpha_1^y & \alpha_1^z \end{bmatrix}_{n\times 3}. \tag{12}$$

  This means that n nodes of the structure possess the same nodal coordinates. In other words, these n nodes degenerate into one node $(\alpha_1^x,\ \alpha_1^y,\ \alpha_1^z)$, named as original node in this paper. Consequently, Eq. (11) can be expressed as

$$\mathbf{x} = \mathbf{x}_1 + \sum_{i=2}^{n_\mathbf{D}} \alpha_i^x \mathbf{g}_i, \tag{13a}$$

$$\mathbf{y} = \mathbf{y}_1 + \sum_{i=2}^{n_\mathbf{D}} \alpha_i^y \mathbf{g}_i, \tag{13b}$$

$$\mathbf{z} = \mathbf{z}_1 + \sum_{i=2}^{n_\mathbf{D}} \alpha_i^z, \mathbf{g}_i, \tag{13c}$$

  or in matrix form

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] = [\mathbf{x}_1\ \mathbf{y}_1\ \mathbf{z}_1] + [\mathbf{g}_2\ \mathbf{g}_3\ \cdots\ \mathbf{g}_{n_\mathbf{D}}] \begin{bmatrix} \alpha_2^x & \alpha_2^y & \alpha_2^z \\ \alpha_3^x & \alpha_3^y & \alpha_3^z \\ \vdots & \vdots & \vdots \\ \alpha_{n_\mathbf{D}}^x & \alpha_{n_\mathbf{D}}^y & \alpha_{n_\mathbf{D}}^z \end{bmatrix}_{(n_\mathbf{D}-1)\times 3}. \tag{14}$$

- If $n_\mathbf{D} = 2$, Eq. (13) defines a line that goes through the original node.
- If $n_\mathbf{D} = 3$, Eq. (13) forms a two-dimensional space holding the original node.
- If $n_\mathbf{D} = 4$, Eq. (13) generates a three-dimensional space containing the original node.
- In general, d eigenvector bases can be selected from $(n_\mathbf{D} - 1)$ ones except $\mathbf{g}_1$ (due to degenerating geometry of the tensegrity structure [26]) in the null space of $\mathbf{D}$ to form a d-dimensional tensegrity structure. Hence, if $n_\mathbf{D} > 4$, Eq. (13) gives in total

$$\mathbf{C}_d^{n_\mathbf{D}-1} = \frac{(n_\mathbf{D} - 1)!}{d!(n_\mathbf{D} - 1 - d)!}, \tag{15}$$

  distinct options that one can choose to generate a d-dimensional tensegrity structure.

There are two rank deficiency conditions for tensegrities with multiple states of self-stress need to be considered. The first one related to the semi-definite matrix $\mathbf{D}$ of Eq. (7) is defined by

$$n_\mathbf{D} \geq d + 1. \tag{16}$$

This condition forces Eq. (7) to yield at least d useful particular solutions that exclude the above vector $\mathbf{g}_1$. Therefore, the minimum rank deficiency or nullity of $\mathbf{D}$ must be $(d + 1)$ for configuration of any tensegrity structure with multiple states of self-stress embedding into $\mathbb{R}^d$.

Let $n_\mathbf{A}$ denote the dimension of the null space of the equilibrium matrix $\mathbf{A}$, which is computed by

$$n_\mathbf{A} = b - r_\mathbf{A}, \tag{17}$$

where $r_{\mathbf{A}} = \text{rank}(\mathbf{A})$. Similarly, the general solution $\bar{\mathbf{q}} (\in \mathbb{R}^b)$ of Eq. (8) that lies in the null space of $\mathbf{A}$ is calculated as

$$\bar{\mathbf{q}} = \sum_{i=1}^{n_{\mathbf{A}}} c_i \mathbf{q}_i, \tag{18}$$

where the coefficients c_i are arbitrary values and $\mathbf{q}_i \in \mathbb{R}^b$ $(i = 1, 2, \ldots, n_{\mathbf{A}})$ are the particular solutions of Eq. (8). The second rank deficiency condition related to matrix $\mathbf{A}$ of Eq. (8), which ensures the existence of at least two states of self-stress, can be stated as

$$s = n_{\mathbf{A}} \geq 2, \tag{19}$$

where s is known as the number of independent states of self-stress. It is clear that Eq. (19) allows Eq. (8) to create at least two useful particular solutions.

3 Form-finding process

The proposed form-finding procedure only needs to know the topology of the structure, in terms of the incidence matrix $\mathbf{C}$ and the type of each member, i.e., either cable or strut which is under tension or compression, respectively. There are two stages in the form-finding process: (i) the first one is to find the range of feasible sets of the nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ and the force densities $\mathbf{q}$ that satisfy Eqs. (16) and (19), respectively, and (ii) the other is to determine a single integral feasible set of force densities. Based on the member type, the initial force density coefficients of cables (tension) are automatically assigned as +1, and those of the struts (compression) as −1, respectively, as follows:

$$\mathbf{q}^0 = \{\underbrace{+1\ +1\ \cdots\ +1}_{\text{cables}}\ \ \underbrace{-1\ -1\ \cdots\ -1}_{\text{struts}}\}^T. \tag{20}$$

Subsequently, the force density matrix $\mathbf{D}$ is calculated from $\mathbf{q}^0$ by Eq. (6). After that, the nodal coordinates are selected from the EVD of the matrix $\mathbf{D}$, which is discussed in Sect. 3.1. These nodal coordinates are substituted into Eq. (8) to define the force density vector $\mathbf{q}$ by the SVD of the equilibrium matrix $\mathbf{A}$, which is presented in Sect. 3.2. The force density matrix $\mathbf{D}$ is then updated by Eq. (6). The process is iteratively calculated for searching the range of feasible sets of the nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ and the force densities $\mathbf{q}$ until the rank deficiencies of Eqs. (16) and (19) are satisfied at the first stage, which forces Eqs. (7) and (8) become true. In this context, at least two states of self-stress can be created, $s \geq 2$. In this study, based on the required rank deficiencies from Eqs. (16) and (19), the form-finding process is stopped as

$$n_{\mathbf{D}} \geq n_{\mathbf{D}}^* = d + 1, \tag{21a}$$

$$n_{\mathbf{A}} \geq n_{\mathbf{A}}^* = 2, \tag{21b}$$

where $n_{\mathbf{D}}^*$ and $n_{\mathbf{A}}^*$ are the minimum required rank deficiencies of the force density and equilibrium matrices for the tensegrities with at least two states of self-stress, respectively. The symmetry properties of the obtained configuration are then employed to derive the linear constraints on the force densities in the second stage for the determination of a single integral feasible force density vector. Furthermore, linear relations between some specific force densities directly assigned by designers can also be included in the formulation for the purpose of obtaining the single integral feasible force density vector.

3.1 Determination of feasible set of nodal coordinates

The square, symmetric force density matrix $\mathbf{D}$ can be factorized as follows by using the EVD [37]:

$$\mathbf{D} = \Phi \Lambda \Phi^T, \tag{22}$$

where $\Phi (\in \mathbb{R}^{n \times n})$ is the orthogonal matrix ($\Phi\Phi^T = \mathbf{I}_n$, in which $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ is the unit matrix) whose ith column is the eigenvector basis $\boldsymbol{\phi}_i (\in \mathbb{R}^n)$ of $\mathbf{D}$ and $\Lambda (\in \mathbb{R}^{n \times n})$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e., $\Lambda_{ii} = \lambda_i$. The eigenvector $\boldsymbol{\phi}_i$ of Φ corresponds to eigenvalue

λ_i of Λ. The eigenvalues are in ascending order as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. It is clear that the number of zero eigenvalues of $\mathbf{D}$ is equal to the dimension of its null space. Let p be the number of zero and negative eigenvalues of $\mathbf{D}$. There are two cases need to be considered. The first one is $p \leq n^*_{\mathbf{D}}$, and the other is $p > n^*_{\mathbf{D}}$. However, since initial force densities $\mathbf{q}^0$, which consist of unitary entries $+1$ and -1 for members in tension and compression, respectively, usually result in $p \leq n^*_{\mathbf{D}}$, only this case is considered in this paper.

The first $n^*_{\mathbf{D}}$ orthonormal eigenvectors of Φ are directly taken as potential nodal coordinates

$$[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] \in \bar{\Phi} = [\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2\ \cdots\ \boldsymbol{\phi}_{n^*_{\mathbf{D}}}]. \tag{23}$$

Let $\mathbf{G}$ ($\in \mathbb{R}^{b\times d(d+1)/2}$) denote the potential geometry matrix of the structure formed by any combination of d eigenvectors from $\bar{\Phi}$, determined by the connectivity and potential nodal coordinates as

$$\mathbf{G} = \begin{cases} \begin{bmatrix} \mathbf{L}^x\mathbf{l}^x & \mathbf{L}^y\mathbf{l}^y & \mathbf{L}^z\mathbf{l}^z & \mathbf{L}^x\mathbf{l}^y & \mathbf{L}^x\mathbf{l}^z & \mathbf{L}^y\mathbf{l}^z \end{bmatrix} & \text{if } d = 3 \\ \begin{bmatrix} \mathbf{L}^x\mathbf{l}^x & \mathbf{L}^y\mathbf{l}^y & \mathbf{L}^x\mathbf{l}^y \end{bmatrix} & \text{if } d = 3 \end{cases} \tag{24}$$

where $\mathbf{L}^x(=\text{diag}(\mathbf{l}^x))$, $\mathbf{L}^y(=\text{diag}(\mathbf{l}^y))$ and $\mathbf{L}^z(=\text{diag}(\mathbf{l}^z))$ ($\in \mathbb{R}^{b\times b}$) are diagonal square matrices of $\mathbf{l}^x$, $\mathbf{l}^y$ and $\mathbf{l}^z$, respectively, and $\mathbf{l}^x(=l^x_k)$, $\mathbf{l}^y(=l^y_k)$ and $\mathbf{l}^z(=l^z_k) \in \mathbb{R}^b$ ($k = 1, 2, \ldots, b$) denote the coordinate difference vectors of the b members in x-, y- and z-directions, respectively, which are calculated from

$$\begin{aligned} \mathbf{l}^x &= \mathbf{C}\boldsymbol{\phi}_i, \\ \mathbf{l}^y &= \mathbf{C}\boldsymbol{\phi}_j \quad (\boldsymbol{\phi}_i, \boldsymbol{\phi}_j, \boldsymbol{\phi}_k \in \bar{\Phi}), \\ \mathbf{l}^z &= \mathbf{C}\boldsymbol{\phi}_k. \end{aligned} \tag{25}$$

A vector $\boldsymbol{\phi}_i$ of $\bar{\Phi}$ which linearly depends on the vector $\mathbf{g}_1$ (i.e., $\mathbf{C}\boldsymbol{\phi}_i = \mathbf{0}$) or causes a zero length to any member among the b members of the structure (whose lengths are defined by $l_k = \sqrt{(l^x_k)^2 + (l^y_k)^2 + (l^z_k)^2}$, assuming $d = 3$) can be identified and eliminated. If there is no $\boldsymbol{\phi}_i$ found, the first three eigenvectors of $\bar{\Phi}$ are chosen as nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ for 3-dimensional tensegrity structure. And the potential geometry matrix $\mathbf{G}$ formed by the chosen nodal coordinates should satisfy the following condition:

$$r_{\mathbf{G}} = \frac{d(d+1)}{2}, \tag{26}$$

where $r_{\mathbf{G}} = \text{rank}(\mathbf{G})$. Equation (26) indicates that the rank of the potential geometry matrix $\mathbf{G}$ must be $d(d+1)/2$ for a d-dimensional structure, i.e., it must equal to three or six for two- or three-dimensional structure, respectively; if it is less than $d(d+1)/2$, then there exist nontrivial affine motions [27,38] in the null space of the geometrical stiffness matrix $\mathbf{K}_G$ ($= \mathbf{I}_d \otimes \mathbf{D}$); $\mathbf{I}_d$ ($\in \mathbb{R}^{d\times d}$) and $\otimes$ are the unit matrix and the tensor product, respectively.

The force density vector $\mathbf{q}$ that is repeatedly approximated from Eq. (31) is in fact the least-square solution of the linear homogeneous Eq. (8) solved by the SVD technique of the equilibrium matrix $\mathbf{A}$ as presented in the next section. In other words, the algorithm iteratively modifies the force density vector $\mathbf{q}$ as little as possible to make the first $n^*_{\mathbf{D}}$ eigenvalues of $\mathbf{D}$ become null as

$$\lambda_i = 0\ (i = 1, 2, \ldots, n^*_{\mathbf{D}}). \tag{27}$$

Accordingly, $\mathbf{D}$ will finally have at least the minimum required rank deficiency $n^*_{\mathbf{D}}$ without any negative eigenvalue. It implies $\mathbf{D}$ is positive semi-definite and there is no nontrivial affine motion in the null space of the geometrical stiffness matrix $\mathbf{K}_G$. Any tensegrity structure falling into this case is super stable regardless of material properties and level of self-stress coefficients [27,36,38].

In short, the best scenario of configuration in 3-dimensional space is formed by three best candidate eigenvectors selected from the first $n^*_{\mathbf{D}}$ eigenvector bases which corresponding to the first $n^*_{\mathbf{D}}$ smallest eigenvalues, respectively. These eigenvalues will be gradually modified to be zero by the proposed iterative form-finding algorithm. In other words, the proposed form-finding procedure has repeatedly approximated equilibrium configuration such that

$$\mathbf{D}[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] \approx [\mathbf{0}\ \mathbf{0}\ \mathbf{0}]. \tag{28}$$

In essence, the nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ repeatedly approximated from Eq. (28) are the least-square solutions of the linear homogeneous Eq. (7), which is solved by the EVD technique of the force density matrix $\mathbf{D}$. This is since they are selected from the first four eigenvector bases corresponding to the first four smallest eigenvalues.

3.2 Determination of feasible set of force densities

The equilibrium matrix $\mathbf{A}$ is computed by substituting the set of approximated nodal coordinates $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]$ from Eq. (28) into (9). In order to solve the linear homogeneous Eq. (8), the SVD [37] is carried out on the equilibrium matrix $\mathbf{A}$:

$$\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{W}^T, \tag{29}$$

where $\mathbf{U}\ (\in \mathbb{R}^{dn\times dn}) = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_{dn}]$ and $\mathbf{W}\ (\in \mathbb{R}^{b\times b}) = [\mathbf{w}_1\ \mathbf{w}_2\ \cdots\ \mathbf{w}_b]$ are the orthogonal matrices and $\mathbf{V}\ (\in \mathbb{R}^{dn\times b})$ is a diagonal matrix with non-negative singular values of $\mathbf{A}$ in descending order as

$$\sigma_1\ \geq\ \sigma_2\ \geq\ \cdots\ \geq\ \sigma_b\ \geq\ 0. \tag{30}$$

As indicated in Eq. (21b), the iterative form-finding algorithm for the case of multiple states of self-stress is successful in case of $s \geq n^*_{\mathbf{A}} = 2$. Accordingly, there are also two cases for s during the procedure:
Case 1: (i) $s = 0$ and (ii) $s = 1$. Regarding $s = 0$, there exists no null space of $\mathbf{A}$. That is, the structure is not in self-equilibrium with its current approximated nodal coordinates, which is the usual case of the structure generated from the incidence matrix $\mathbf{C}$ and the initial assigned force density vector $\mathbf{q}^0$. In particular, the right singular value (σ_b) of $\mathbf{A}$ in $\mathbf{V}$ is not equal to zero. It denotes that Eq. (8) has no nonzero force density vector $\mathbf{q}$ as a solution. The form-finding procedure defines the approximated $\mathbf{q}$ [39] such that

$$\mathbf{A}\mathbf{q} \approx \mathbf{0}. \tag{31}$$

Regarding $s = 1$, the vector $\mathbf{q}_1\ (\in \mathbb{R}^b)$ in Eq. (32b) matching in sign with $\mathbf{q}^0$ is indeed the single state of self-stress which satisfies the homogeneous Eq. (8). However, the proposed form-finding algorithm keeps iterating until Eq. (21b) is satisfied.

Case 2: $s \geq 2$, it is known [40] that the vector space bases of mechanisms and force densities of any tensegrity structure are calculated from the null space of the equilibrium matrix. In this case, the matrices $\mathbf{U}$ and $\mathbf{W}$ from Eq. (29) can be expressed, respectively, as

$$\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_{r_{\mathbf{A}}} | \mathbf{m}_1\ \cdots\ \mathbf{m}_m], \tag{32a}$$
$$\mathbf{W} = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_{r_{\mathbf{A}}} | \mathbf{q}_1\ \cdots\ \mathbf{q}_s], \tag{32b}$$

where the vectors $\mathbf{m}_i\ \in \mathbb{R}^{dn}\ (i = 1, 2, \ldots, m)$ denote $m\ (= dn - r_{\mathbf{A}})$ inextensional mechanisms including both possible infinitesmal mechanisms and rigid body motions, while the vectors $\mathbf{q}_j\ \in \mathbb{R}^b\ (j = 1, 2, \ldots, s)$ are s independent states of self-stress which satisfy the linear homogeneous equation (8).

3.3 Determination of a single integral feasible set of force densities

Let $\mathbf{c} = \{c_1, c_2, \ldots, c_s\}^T\ (\in \mathbb{R}^s)$ denote the coefficient vector of s independent self-stress states. Equation (18) can be simplified as

$$\bar{\mathbf{q}} = \mathbf{S}\mathbf{c}, \tag{33}$$

where $\mathbf{S}\ (\in \mathbb{R}^{b\times s})$ is the matrix of self-stress basis states defined by

$$\mathbf{S} = [\mathbf{q}_1\ \mathbf{q}_2\ \cdots \mathbf{q}_s]. \tag{34}$$

The general solution $\bar{\mathbf{q}}$ is considered as a single integral feasible self-stress state [27] if and only if it satisfies the three following conditions: (i) symmetry properties of the structure (i.e., members at symmetrical positions are considered to belong to the same group and have the same force density); (ii) unilateral behaviors of members (i.e., cables and struts must be under tension and compression, respectively); (iii) and it is uniquely defined. The self-stress basis states $\mathbf{S}$ resulting from the null space of the equilibrium matrix $\mathbf{A}$ cannot be employed directly since every independent state itself does not satisfy the unilateral behavior of members. This is because the equilibrium matrix $\mathbf{A}$ does not take into account member characteristics.

In practical situations, the tensegrity system usually has symmetric properties, i.e., invariance conditions to reflection with respect to some planes and/or rotation around some axes. Therefore, the same force densities should be assigned to the symmetrically located members. In other words, based on the geometric symmetry of the tensegrity structure, members at symmetrically positions are considered to belong to the same group and have the same force density. Let h denote the number of groups of members in tensegrity structure, $\bar{\mathbf{q}}$ can then be written as

$$\bar{\mathbf{q}} = \{q_1\ q_1\ q_1\ \cdots\ q_i\ q_i\ q_i\ \cdots\ q_h\ q_h\ q_h\}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \vdots & 1 & 0 & 0 & 0 \\ \vdots & 1 & 0 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 \\ 0 & \vdots & 1 & 0 & 0 \\ 0 & 0 & \vdots & 1 & \vdots \\ 0 & 0 & \vdots & 1 & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}_{(b\times h)} \begin{Bmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_h \end{Bmatrix}_{(h\times 1)}. \tag{35}$$

Equation (35) can be rewritten as

$$\bar{\mathbf{q}} = [\mathbf{e}_1\ \cdots\ \mathbf{e}_i\ \cdots\ \mathbf{e}_h]\{q_1\ \cdots\ q_i\ \cdots\ q_h\}^T, \tag{36}$$

where q_i is the force density of members in the ith group; $\{q_1\ \cdots\ q_i\ \cdots\ q_h\}^T\ (= \mathbf{q}_h) \in \mathbb{R}^h$ is the force density vector of h groups; and $\mathbf{e}_i\ (\in \mathbb{R}^b)$ is the basis vector composed of a unit in the ith group and zero in the other $(h-1)$ groups.

Equation (18) is substituted by (36) and re-arranged into the following form:

$$\mathbf{q}_1 c_1 + \mathbf{q}_2 c_2 + \cdots + \mathbf{q}_s c_s + (-\mathbf{e}_1 q_1) + \cdots (-\mathbf{e}_i q_i) \cdots + (-\mathbf{e}_h q_h) = \mathbf{0}. \tag{37}$$

Let $\bar{\mathbf{c}}\ (\in \mathbb{R}^{(s+h)})$ is the vector of the coefficients of s independent self-stress states and the force densities of h groups denoted as $\bar{\mathbf{c}} = \{c_1, c_2, \ldots, c_s, q_1, \ldots, q_i, \ldots, q_h\}^T$. Equation (37) can be rewritten in matrix form:

$$\bar{\mathbf{S}}\bar{\mathbf{c}} = \mathbf{0} \tag{38}$$

where $\bar{\mathbf{S}}\ (\in \mathbb{R}^{b\times(s+h)})$ is a matrix computed by

$$\bar{\mathbf{S}} = [\mathbf{q}_1\ \mathbf{q}_2\ \cdots\ \mathbf{q}_s\ -\mathbf{e}_1\ \cdots\ -\mathbf{e}_i\ \cdots\ -\mathbf{e}_h]. \tag{39}$$

It should be noted that linear relations on the force densities between two groups can be directly assigned as

$$q_i = \alpha q_j\ (i \neq j), \tag{40}$$

where q_i, q_j and α are the force density coefficients of member groups i, j and a specified value, respectively. In this case, substituting Eq. (40) into (37), Eq. (38) will be updated by a new version of vector $\bar{\mathbf{c}}$ as well as matrix $\bar{\mathbf{S}}$, and the force density variables of member groups can be reduced by these linear relations. Similarly, the set of all solutions to the linear homogeneous system of Eq. (38) lies in the null space of $\bar{\mathbf{S}}$. Let $n_{\bar{\mathbf{S}}}$ denote the dimension of this null space which is computed by

$$n_{\bar{\mathbf{S}}} = (s+h) - r_{\bar{\mathbf{S}}},\ (s+h \leq b), \tag{41}$$

where $r_{\bar{\mathbf{S}}} = \text{rank}(\bar{\mathbf{S}})$. Both the number of independent self-stress states s and the number of groups h are dependent on the obtained geometry of the tensegrity. However, usually, their sum is less or equal to the number of members $(s+h \leq b)$. Therefore, only this case is considered in this paper. Depending on the value of $n_{\bar{\mathbf{S}}}$, there are also the following two cases:

Case 1. $n_{\bar{\mathbf{S}}} = 1$, Eq. (38) has a single nontrivial solution ($\bar{\mathbf{c}} \neq \mathbf{0}$). That is, the tensegrity structure possesses a single integral self-stress state. It satisfies not only the self-equilibrium condition but also the condition of symmetry derived by the geometric shape of the structure. Moreover, this single integral self-stress state automatically satisfies the condition of cables under tension and struts under compression, which can be considered as the benefit of the single solution of the linear homogeneous Eq. (38) for the tensegrity structure. Hence, it is a single integral feasible self-stress state.

Case 2. (i) $n_{\bar{\mathbf{S}}} = 0$ and (ii) $n_{\bar{\mathbf{S}}} > 1$. For $n_{\bar{\mathbf{S}}} = 0$, Eq. (38) has no nontrivial solution. It implies that group division for members does not consist with the obtained configuration of the tensegrity structure. In other words, the tensegrity is not in self-equilibrium with such geometric symmetry. In order to get a unique nontrivial solution as in Case 1, the number of member groups should be correctly increased until $n_{\bar{\mathbf{S}}}$ becomes 1. For $n_{\bar{\mathbf{S}}} > 1$, Eq. (38) has more than one nontrivial solution, i.e., there are multiple integral self-stress states for the structure. In this case, a single nontrivial solution (Case 1) may exist if the number of member groups is appropriately decreased until $n_{\bar{\mathbf{S}}}$ becomes 1. Alternatively, together with the constraints on geometric symmetry, some linear relations on force densities between some specific groups can be directly introduced by designers through Eq. (40) to impose the unique solution (i.e., $n_{\bar{\mathbf{S}}} = 1$).

To solve the linear homogeneous equation (38), the second application of the SVD on matrix $\bar{\mathbf{S}}$ is employed:

$$\bar{\mathbf{S}} = \bar{\mathbf{U}}\bar{\mathbf{V}}\bar{\mathbf{W}}^T. \tag{42}$$

The matrices $\bar{\mathbf{U}}$ ($\in \mathbb{R}^{b\times b}$), $\bar{\mathbf{V}}$ ($\in \mathbb{R}^{b\times(s+h)}$) and $\bar{\mathbf{W}}$ ($\in \mathbb{R}^{(s+h)\times(s+h)}$) are similarly defined as the matrices $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ described above. If the null space of the matrix $\bar{\mathbf{S}}$ exists, i.e., $n_{\bar{\mathbf{S}}} \geq 1$, the following equation can be obtained:

$$\bar{\mathbf{S}}\bar{\mathbf{w}}_i = 0 \ (i = r_{\bar{\mathbf{S}}} + 1, r_{\bar{\mathbf{S}}} + 2, \ldots, s + h). \tag{43}$$

Equation (43) also means $\bar{\mathbf{w}}_i$ are the solutions of $\bar{\mathbf{c}}$. The number of integral independent self-stress states of the tensegrities which satisfy the symmetric shape condition is $\bar{s} = n_{\bar{\mathbf{S}}}$. Note that the $(s + 1)$th to $(s + h)$th component of $\bar{\mathbf{c}}$ are the force densities of h groups which satisfy the condition that the members in the same group have the same force density.

In this paper, only Case 1 ($\bar{s} = n_{\bar{\mathbf{S}}} = 1$) is considered. If the obtained tensegrity structure at the first stage falls into Case 2, it will be converted into Case 1 by appropriately increasing or decreasing its number of member groups and/or directly assigning some linear relations on force densities between some specific groups. Once $\bar{\mathbf{c}}$ is known, the single integral feasible self-stress force density vector $\bar{\mathbf{q}}$ can be obtained from the last h variables of $\bar{\mathbf{c}}$ or from Eq. (33).

In summary, the EVD of the force density matrix $\mathbf{D}$ and the SVD of the equilibrium matrix $\mathbf{A}$ are performed iteratively to find the feasible sets of the nodal coordinates [$\mathbf{x}$ $\mathbf{y}$ $\mathbf{z}$] and the force densities $\mathbf{q}$ at the first stage by selecting the appropriate singular vector bases in each decomposition as the least-square solutions until the minimum required rank deficiencies of these two matrices for the case of multiple states of self-stress are satisfied, respectively, as presented in Eq. (21). Then, the symmetry properties of the obtained configuration are employed to derive the linear constraints on the force densities in the second stage for the determination of a single integral feasible force density vector. Furthermore, linear relations between some specific force densities directly assigned by designers can also be included in the formulation for the purpose of obtaining the single integral feasible force density vector. The final nodal coordinates [$\mathbf{x}$ $\mathbf{y}$ $\mathbf{z}$] and the single integral feasible force density vector $\bar{\mathbf{q}}$ may be not unique for the structure with given topology and the types of members. Other two sets of the nodal coordinates and the force densities may exist for the same topology and the same types of members.

Since the tensegrity structure should satisfy the self-equilibrium conditions, the vector of unbalanced forces ε_f ($\in \mathbb{R}^{dn}$) defined as follows can be used for evaluating the accuracy of the results:

$$\varepsilon_f = \mathbf{A}\mathbf{q}. \tag{44}$$

The Euclidean norm of ε_f is used to define the design error ϵ as

$$\epsilon = \sqrt{\varepsilon_f(\varepsilon_f)^T}. \tag{45}$$

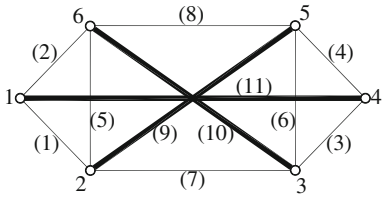

Fig. 1 A two-dimensional three-strut tensegrity structure. The thick and thin lines represent the struts and cables, respectively

3.4 Form-finding procedure for tensegrity structures with multiple states of self-stress

The feasible sets of parameters which are the nodal coordinates and the single integral feasible force densities of the tensegrity structures can be simultaneously defined by the proposed form-finding through the following procedure.

Form-finding procedure

- *Step 1*: Define $\mathbf{C}$ by Eq. (1) for the given topology of tensegrity structure.
- *Step 2*: Specify the types of members to generate initial force density vector $\mathbf{q}^0$ by Eq. (20). Set $i = 0$.
- *Step 3*: Calculate $\mathbf{D}^i$ using Eq. (6)
- *Step 4*: Carry out Eq. (22) to define $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}]^i$ through Eq. (28).
- *Step 5*: Determine $\mathbf{A}^i$ by Eq. (9).
- *Step 6*: Perform Eq. (29) to define $\mathbf{q}^{i+1}$ through Eq. (31).
- *Step 7*: Define $\mathbf{D}^{i+1}$ with $\mathbf{q}^{i+1}$ by Eq. (6). If Eq. (21) is satisfied, the solutions exist. Otherwise, set $i = i+1$ and return to *Step 4*.
- *Step 8*: Based on the symmetry of the obtained geometry, calculate $\bar{\mathbf{S}}$ from Eq. (39) and $n_{\bar{\mathbf{S}}}$ from Eq. (41).
- *Step 9*:
  1. If $n_{\bar{\mathbf{S}}} = 1$, go to *Step 10*.
  2. If $n_{\bar{\mathbf{S}}} = 0$, increase the number of member groups h and define $\bar{\mathbf{S}}$ from Eq. (39), $n_{\bar{\mathbf{S}}}$ from (41) repeatedly until $n_{\bar{\mathbf{S}}}$ becomes 1.
  3. If $n_{\bar{\mathbf{S}}} > 1$, decrease the number of member groups h and/or directly assign some linear relations on force densities between some specific groups, and define $\bar{\mathbf{S}}$ from Eq. (39), $n_{\bar{\mathbf{S}}}$ from (41) repeatedly until $n_{\bar{\mathbf{S}}}$ becomes 1.
- *Step 10*: Define $\bar{\mathbf{c}}$ by Eq. (42). Extract $\mathbf{c}$ and $\mathbf{q}_h$ from $\bar{\mathbf{c}}$. Compute $\bar{\mathbf{q}}$ by Eq. (33) or (36).
- *Step 11*: The process is terminated until Eq. (45) has been checked. The final coordinates and the single integral feasible force density vector $\bar{\mathbf{q}}$ are the solutions. Otherwise, set $i = i + 1$ and return to *Step 4*.

4 Numerical examples

Numerical examples are presented for several tensegrity structures using Matlab Version 7.4 (R2007a) [41]. Based on the algorithm developed, both the nodal coordinates and the single integral feasible force density vector are simultaneously defined with limited information of the nodal connectivity and the type of each member. A two-dimensional three-strut tensegrity, a three-strut octahedral cell tensegrity, a three-dimensional six-strut tensegrity and an expanded octahedron tensegrity are then illustrated to demonstrate the capability of the proposed method.

4.1 Two-dimensional tensegrity structures

4.1.1 Three-strut tensegrity

The initial topology of the three-strut tensegrity structure comprising three struts and eight cables is shown in Fig. 1. Initial nodal coordinates, member lengths and force density coefficients are unknown in advance. The only basic information is the incidence matrix $\mathbf{C}$ and the type of each member which is employed to

automatically assign the initial force density vector by the proposed form-finding procedure as

$$\mathbf{q}^0 = \{q_1\text{–}q_8 = 1,\ q_9\text{–}q_{11} = -1\}^T. \tag{46}$$

The obtained self-equilibrium configuration of the structure is plotted in Fig. 2. The form-finding procedure converges in two iterations at the first stage with the design error $\epsilon = 2.135 \times 10^{-15}$ defined by Eq. (45). The structure achieved has two self-stress states ($s = 2$) and no infinitesimal mechanism ($m = 0$) after constraining its three rigid body motions, indicating it is statically indeterminate and kinematically determinate [42].

Two independent self-stress states $\mathbf{q}_1 - \mathbf{q}_2$ are now available in the null space of the equilibrium matrix $\mathbf{A}$. However, they cannot be utilized directly because they do not satisfy the unilateral behavior of elements as mentioned in Sect. 3.3. In order to find the single integral feasible self-stress state, the symmetry properties of the obtained structure are employed to impose the linear constraints on the force densities in the second stage. Table 1 generally shows a relationship between the number of member groups (h) and the dimension of the null space of $\bar{\mathbf{S}}$ ($n_{\bar{\mathbf{S}}}$) in the three-strut tensegrity. For $h = 2$, all cables and struts belong to their own group, respectively. For all the other cases $2 < h \leq 9$ (note that $h \leq b - s = 11 - 2 = 9$ as mentioned in Eq. (41)), the members are grouped based on the symmetry of the structure. From Table 1, it can be seen that the single integral feasible prestressed mode exists ($n_{\bar{\mathbf{S}}} = 1$) if the number of member groups is four. That is, if the structure is divided into four groups, it satisfies not only the equilibrium but also the unilateral behavior of elements.

However, according to the symmetry of the structure, the members can be appropriately divided into five groups, which results $n_{\bar{\mathbf{S}}} = 2$, as listed in Table 2. In order to obtain a single integral feasible prestressed mode, a linear relation between member groups 2 and 3 is imposed as $q_3 = 1.5q_2$. Accordingly, the vector of the coefficients of two independent prestressed modes and force densities of five groups ($\bar{\mathbf{c}}$) can be obtained from

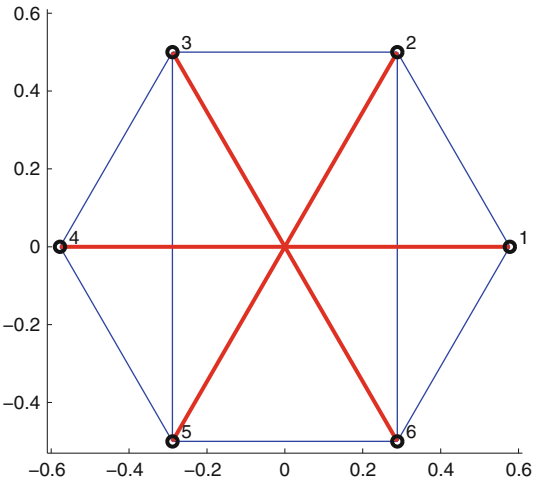

Fig. 2 The obtained geometry of the two-dimensional three-strut tensegrity structure

Table 1 A relationship between the number of groups (h) and the dimension of the null space of $\bar{\mathbf{S}}$ ($n_{\bar{\mathbf{S}}}$) in the 2-D three-strut tensegrity

h	2	3	4	5	6	7	8	9
$n_{\bar{\mathbf{S}}}$	0	1	1	2	2	2	2	2

Table 2 Single integral feasible self-stress state of the 2-D three-strut tensegrity structure

Member group	Member	q_h
1	1–4	1.0
2	5–6	2.0
3	7–8	3.0
4	9–10	−2.5
5	11	−0.5

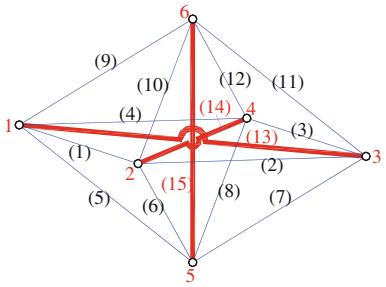

Fig. 3 A three-strut octahedral cell tensegrity structure

Eq. (38) by using SVD of $\bar{\mathbf{S}}$ [Eq. (42)] as follows:

$$\bar{\mathbf{S}}\bar{\mathbf{c}} = \begin{bmatrix} 0.1954 & -0.3902 & -1 & 0 & 0 & 0 \\ 0.1954 & -0.3902 & -1 & 0 & 0 & 0 \\ 0.1954 & -0.3902 & -1 & 0 & 0 & 0 \\ 0.1954 & -0.3902 & -1 & 0 & 0 & 0 \\ -0.4627 & 0.0123 & 0 & -1 & 0 & 0 \\ -0.4627 & 0.0123 & 0 & -1 & 0 & 0 \\ -0.2673 & -0.3779 & 0 & -1.5 & 0 & 0 \\ -0.2673 & -0.3779 & 0 & -1.5 & 0 & 0 \\ 0.3650 & 0.1828 & 0 & 0 & -1 & 0 \\ 0.3650 & 0.1828 & 0 & 0 & -1 & 0 \\ -0.0977 & 0.1951 & 0 & 0 & 0 & -1 \end{bmatrix}_{(11\times 6)} \begin{Bmatrix} c_1 \\ c_2 \\ q_1 \\ q_2 \\ q_4 \\ q_5 \end{Bmatrix}_{(6\times 1)} = \mathbf{0}. \tag{47}$$

Equation (47) yields a single solution since the dimension of the null space of the matrix $\bar{\mathbf{S}}$ equals to one ($n_{\bar{\mathbf{S}}} = 1$):

$$\bar{\mathbf{c}} = \{-0.6041,\ -0.6504,\ 0.1358,\ 0.2715,\ -0.3394,\ -0.0679\}^T. \tag{48}$$

The coefficient vector $\mathbf{c}$ is obtained by extracting the first two components of $\bar{\mathbf{c}}$, and its remaining components are the force density coefficients of four groups. The single integral self-stress state $\mathbf{q}_h$ (force density vector of h groups) normalized with respect to the force density coefficient of member group 1 is presented in Table 2. It is also the single integral feasible self-stress state of the tensegrity since it satisfies the condition of cables in tension and struts in compression. Equation (29) in Tran and Lee [27] is fulfilled: the rank of the geometry matrix $\mathbf{G}$ is three ($r_{\mathbf{G}} = 3$); the required rank deficiency $n_{\mathbf{D}}$ of the force density matrix $\mathbf{D}$ is three ($n - r_{\mathbf{D}} = 3$); and $\mathbf{D}$ is positive semi-definite. Accordingly, the structure is certainly super stable regardless of materials and prestress levels as given in [27, Eq. (29)].

4.2 Three-dimensional tensegrity structures

4.2.1 Three-strut octahedral cell

As the next example, consider the structure shown in Fig. 3 has three struts and eight cables. Similarly, the only known information is the incidence matrix $\mathbf{C}$ and the type of each member which is employed to automatically generate the initial force density vector by the proposed form-finding procedure as

$$\mathbf{q}^0 = \{q_1{=}q_{12} = 1,\ q_{13}{=}q_{15} = -1\}^T. \tag{49}$$

Figure 4 shows the obtained self-equilibrium configuration. The form-finding procedure converges in only one iteration at the first stage with the design error $\epsilon = 1.633 \times 10^{-14}$. The structure achieved has three self-stress states ($s = 3$) and no infinitesimal mechanism ($m = 0$) excluding its six rigid body motions. Accordingly, it belongs to statically indeterminate and kinematically determinate [42].

Three independent self-stress states $\mathbf{q}_1 - \mathbf{q}_3$ can be found by the SVD of the equilibrium matrix $\mathbf{A}$. Based on the symmetry of the obtained geometry (Fig. 4), the members can be pertinently divided into four groups

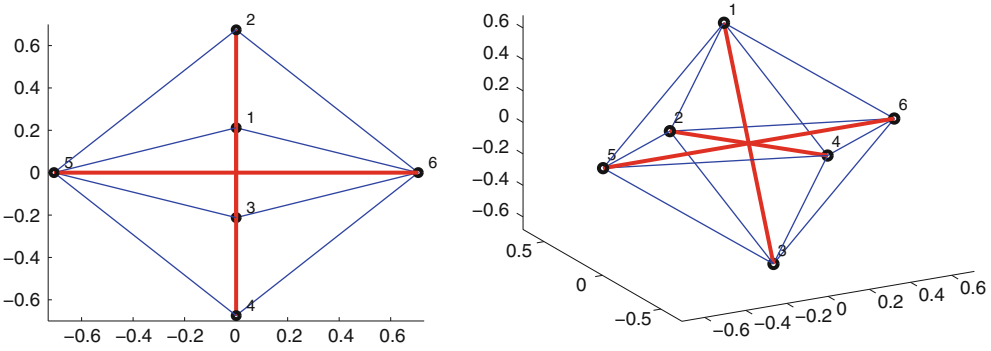

Fig. 4 The obtained geometry of the three-strut octahedral cell tensegrity structure, **a** top view, **b** perspective view

Table 3 Single integral feasible self-stress state of the three-strut octahedral cell tensegrity structure

Member group	Member	q_h
1	1–4	1.0
2	5–12	0.5
3	13–14	−1.5
4	15	−1.0

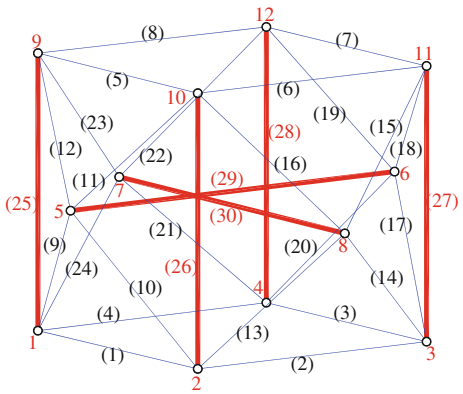

Fig. 5 A three-dimensional six-strut tensegrity structure

as shown in Table 3, which results in $n_{\bar{\mathbf{S}}} = 2$. In this case, to obtain a single integral feasible self-stress state ($n_{\bar{\mathbf{S}}} = 1$), a linear relation between member groups 1 and 2 is additionally imposed as $q_1 = 2q_2$. The calculated coefficient vector of three independent self-stress states is:

$$\mathbf{c} = \{-0.2683,\ 0.4225,\ -0.7185\}^T. \tag{50}$$

The single integral feasible self-stress state (normalized with respect to the force density coefficient of member group 1) is listed in Table 3.

4.2.2 Six-strut tensegrity

A three-dimensional six-strut tensegrity (Fig. 5) has 6 struts and 24 cables. Similarly, the obtained configuration of the structure is plotted in Fig. 6. The design error is $\epsilon = 2.614 \times 10^{-15}$ achieved after two iterations at the first stage. The structure obtained has two states of self-stress ($s = 2$) and two infinitesimal mechanisms ($m = 2$) after constraining its six rigid body motions, indicating it is statically and kinematically indeterminate [42].

There are two independent self-stress states $\mathbf{q}_1 - \mathbf{q}_2$ in the null space of the equilibrium matrix $\mathbf{A}$. From the obtained configuration's symmetry (Fig. 6), the members can be appropriately divided into four groups

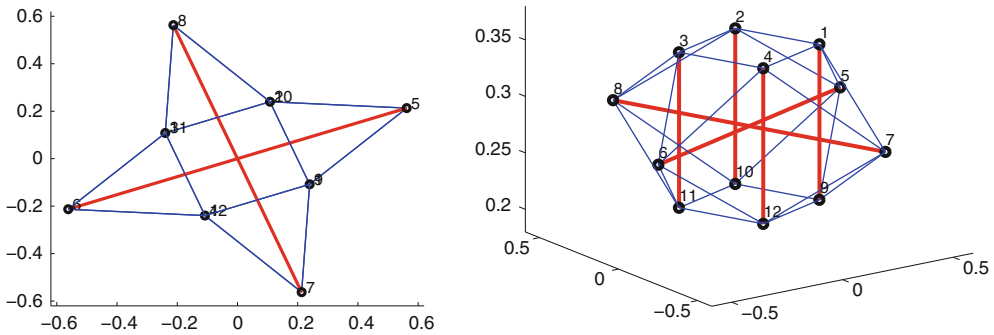

Fig. 6 The obtained geometry of the three-dimensional six-strut tensegrity structure, **a** top view, **b** perspective view

Table 4 Single integral feasible self-stress state of the 3-D six-strut tensegrity structure

Member group	Member	q_h
1	1–8	1.0000
2	9–24	1.6180
3	25–28	−1.6180
4	29–30	−2.2361

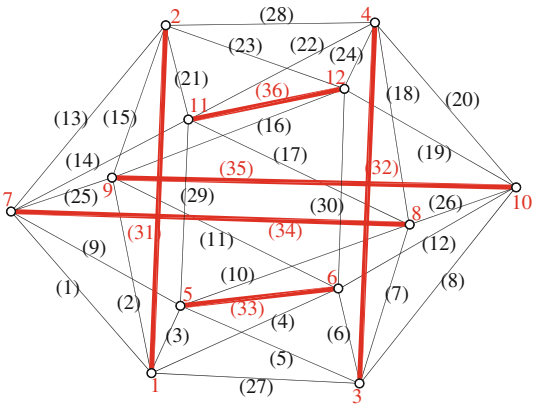

Fig. 7 An expanded octahedron tensegrity structure

($h = 4$) as listed in Table 4. The calculated coefficient vector of two independent self-stress states is:

$$\mathbf{c} = \{0.7810,\ 0.5024\}^T. \tag{51}$$

The single integral feasible self-stress state (normalized with respect to the force density coefficient of member group 1) is shown in Table 4. Equation (29) in Tran and Lee [27] is satisfied: the rank of the geometry matrix $\mathbf{G}$ is six ($r_{\mathbf{G}} = 6$); the required rank deficiency $n_{\mathbf{D}}$ of the force density matrix $\mathbf{D}$ is four ($n - r_{\mathbf{D}} = 4$); and $\mathbf{D}$ is positive semi-definite. Hence, the structure is certainly super stable regardless of materials and prestress levels. It is clear that the introduction of a single integral feasible prestress stiffens infinitesimal mechanisms to make the structure stable.

4.2.3 Expanded octahedron

The expanded octahedron consisting of 6 struts and 36 cables as shown in Fig. 7 is originated from the classic expandable octahedron [26,29]. It has a similar connectivity as the classic expandable octahedron, but with 6 additional cables pulling 6 pairs of nodes closer together. Similarly, the obtained configuration of the structure is plotted in Fig. 8. The design error (Fig. 9) is $\epsilon = 3.806 \times 10^{-15}$ achieved after eighteen iterations at the first stage. The structure obtained has six states of self-stress ($s = 6$) and no infinitesimal mechanism

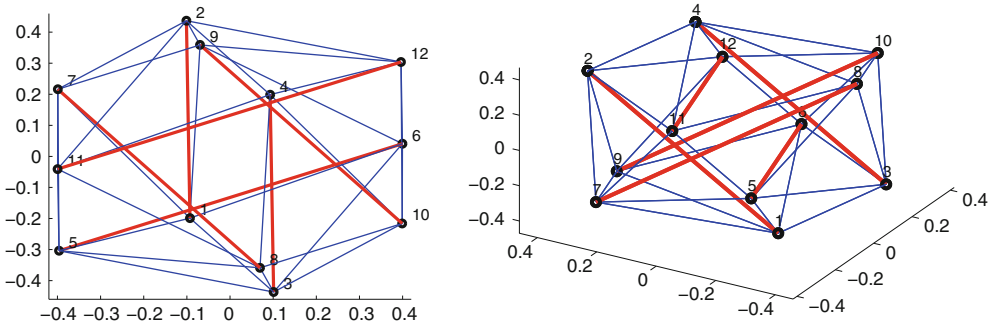

Fig. 8 The obtained geometry of the three-dimensional expanded octahedron tensegrity structure, **a** top view, **b** perspective view

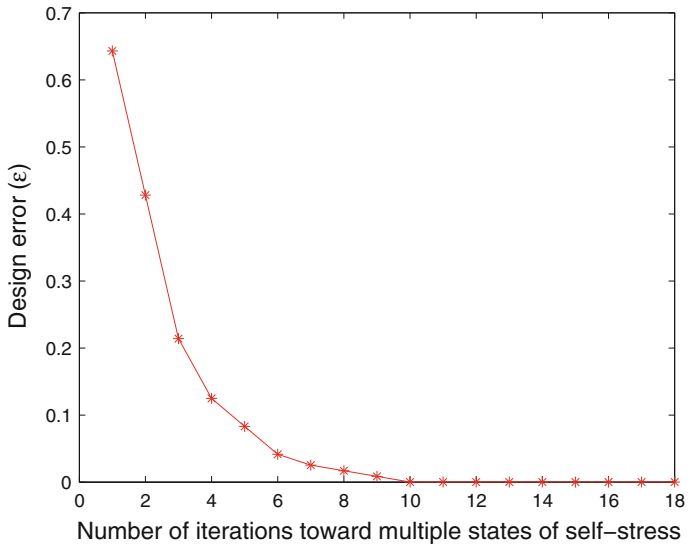

Fig. 9 The convergence of the proposed iterative algorithm for the expanded octahedron

Table 5 Single integral feasible self-stress state of the expanded octahedron tensegrity structure

Member group	Member	q_h
1	1–24	1.0000
2	25–30	0.4143
3	31–36	−1.5857

($m = 0$) after constraining its six rigid body motions, indicating it is statically indeterminate and kinematically determinate [42].

There are six independent self-stress states $\mathbf{q}_1 - \mathbf{q}_6$ in the null space of the equilibrium matrix $\mathbf{A}$. From the obtained configuration's symmetry (Fig. 8), the members can be appropriately divided into three groups ($h = 3$) as listed in Table 5. The calculated coefficient vector of six independent self-stress states is:

$$\mathbf{c} = \{-0.2828,\ -0.2506,\ -0.3373,\ 0.3490,\ -0.7074,\ -0.1923\}^T. \tag{52}$$

The single integral feasible self-stress state (normalized with respect to the force density coefficient of member group 1) is shown in Table 5. The force density matrix $\mathbf{D}$ is positive semi-definite, which leads to a super stable structure regardless of materials and prestress levels [10,34–36].

5 Concluding remarks

The numerical form-finding procedure for tensegrity structures with multiple states of self-stress has been proposed. The topology and the initial force density vector assigned based on the types of members are the required information. The components of this vector consist of unitary entries $+1$ and -1 for members in tension and compression, respectively. At the first stage, the eigenvalue decomposition of the force density matrix and the singular value decomposition of the equilibrium matrix are iteratively executed to find the range of feasible sets of the nodal coordinates and the force densities which satisfy the required rank deficiencies of the force density and equilibrium matrices for the case of multiple states of self-stress, respectively. Since the obtained geometry at the first stage has multiple self-stress states, its symmetry properties are employed to derive the linear constraints on the force densities in the second stage for the determination of a single integral feasible force density vector. Furthermore, linear relations between some specific force densities directly assigned by designers can also be included in the formulation for the purpose of obtaining the single integral feasible force density vector. It is interesting to note that the existence of multiple states of self-stress allows the optimization of self-stress design for the tensegrities. An explanation on the null space of the force density matrix that generates the configurations of the tensegrities is rigorously given. A very good performance of the proposed method has been shown in the numerical examples for two- and three-dimensional tensegrity structures. As a natural extension of this research, form-finding with multiple states of self-stress subject to more complicated constraints awaits further attention.

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