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The investigation of the nonlocal longitudinal stress waves with modified couple stress theory

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Abstract In the present work, the propagation of longitudinal stress waves is investigated using a modified couple stress theory. The analysis of wave motion is based on a Love rod model including the effects of lateral deformation. The present analysis also considers the effect of shear stress components. By applying Hamilton's principle, the explicit nonlocal elasticity solution is obtained, and the effects of shear stress and length scale parameter are discussed.

1 Introduction

Recently, a modified couple stress theory, having only one internal length scale parameter, developed by Yang et al. [1] has attracted interest. A new model for the bending of a Bernoulli-Euler beam was developed [2] using the modified couple stress theory. A variational formulation based on the principle of minimum total potential energy was presented [3] for the modified couple stress theory. A microstructure-dependent Timoshenko beam model was developed [4] using the modified couple stress theory. A Kirchhoff plate model was developed [5] with the modified couple stress theory for the static analysis of isotropic micro-plates having arbitrary shapes. Again, a Kirchhoff plate model was developed [6] using the modified couple stress theory for the dynamic analysis of micro scale plates. In another work, torsional vibration of nanotubes was examined [7] with help of the modified couple stress theory. Based on the modified couple stress theory, a Timoshenko beam model was established [8] to address the size effects of microtubules. On other hand, a notable review on the stress gradient elasticity theory can be found in [9]. Furthermore, recently, a fundamental model for wave propagation in a nonlocal elastic material was developed by Challamel et al. [10].

In the present work, a nonlocal elasticity solution for the longitudinal stress waves of bars is presented based on the modified couple stress theory [1] and Love rod model [11]. Therefore, in the present analysis, lateral inertia effects are also taken into account. However, unlike from the solution of Love, the contribution of the shear stress components on the elastic strain energy is considered as in [12].

This work is dedicated to Herrn Prof. Dr. h. c. M. Cengiz Dokmeci on occasion of his 75th birthday, wishing him good health, happiness, and many more years of creative activity.

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2 The modified couple stress theory and Love rod model

According to the modified couple stress theory of Young et al. [1], the elastic strain energy U in a deformed isotropic linear elastic material occupying a volume V is given by

$$U = \frac{1}{2}\int \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right) dV \qquad (i, j = 1, 2, 3), \tag{1}$$

where ε_{ij} is the strain tensor, σ_{ij} is the Cauchy (local or classical) stress tensor (conjugated with the strain tensor), χ_{ij} is the symmetric curvature tensor, and m_{ij} is the deviatoric part of the couple stress tensor (conjugated with the curvature tensor). The tensors ε_{ij} and χ_{ij} satisfy the following geometrical relations:

$$\varepsilon_{ij} = \frac{1}{2}\left[\nabla u_i + (\nabla u_i)^T\right], \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left[\nabla \theta_i + (\nabla \theta_i)^T\right], \tag{3}$$

where ∇ is the gradient operator, u_i denotes the components of the displacement vector, and θ_i denotes the components of the rotation vector. The relationship between u_i and θ_i is given by

$$\theta_i = \frac{1}{2}\mathrm{curl}(u_i). \tag{4}$$

The constitutive relations for σ_{ij} and m_{ij} are expressed as

$$\sigma_{ij} = \lambda \,\mathrm{tr}\left(\varepsilon_{ij}\right) I + 2\mu\varepsilon_{ij}, \tag{5}$$

$$m_{ij} = 2l^2\mu\chi_{ij}, \tag{6}$$

where λ and μ are Lame' constants, l is the internal length scale parameter, $\mathrm{tr}\left(\varepsilon_{ij}\right)$ is the sum of the diagonal components of the strain tensor, and I is the unit tensor.

According to the Love rod model, the displacement field is expressed as

$$u = u\left(x, t\right), \quad v = -\nu y\frac{\partial u}{\partial x}, \quad w = -\nu z\frac{\partial u}{\partial x}, \tag{7}$$

where ν and w are the x, y, and z components of the displacement vector, respectively, and ν is Poisson's ratio.

By using Eq. (7) in the Eqs. (2) and (5), the strains and stresses are obtained as

$$\begin{aligned}
&\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -\nu\frac{\partial u}{\partial x}, \quad \varepsilon_z = \frac{\partial w}{\partial z} = -\nu\frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\nu y\frac{\partial^2 u}{\partial x^2},\\
&\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -\nu z\frac{\partial^2 u}{\partial x^2}, \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0, \quad \sigma_{xx} = E\varepsilon_x, \quad \sigma_{yy} = \sigma_{zz} = 0,\\
&\tau_{xy} = -\frac{E}{2\left(1+\nu\right)}\nu y\frac{\partial^2 u}{\partial x^2},\\
&\tau_{xz} = -\frac{E}{2\left(1+\nu\right)}\nu z\frac{\partial^2 u}{\partial x^2}, \quad \tau_{yz} = 0,
\end{aligned} \tag{8}$$

where σ_{xx}, σ_{yy} and σ_{zz} are the normal stresses, τ_{xy}, τ_{xz} and τ_{yz} are the shear stresses, γ_{xy}, γ_{xz} and γ_{yz} are the shear strains, and E is the elasticity modulus.

Substituting Eq. (7) into Eq. (4), the components of the rotation vector are obtained as

$$\theta_x = 0, \quad \theta_y = -\frac{1}{2}\frac{\partial w}{\partial x} = \frac{1}{2}\nu z\frac{\partial^2 u}{\partial x^2}, \quad \theta_z = \frac{1}{2}\frac{\partial v}{\partial x} = -\frac{1}{2}\nu y\frac{\partial^2 u}{\partial x^2}. \tag{9}$$

Placing Eq. (9) into Eq. (3) results in

$$\chi_{xy} = \frac{1}{4}\nu z\frac{\partial^3 u}{\partial x^3}, \qquad \chi_{xz} = -\frac{1}{4}\nu y\frac{\partial^3 u}{\partial x^3}, \qquad \chi_{yz} = 0. \tag{10}$$

3 Hamilton's principle and the governing equation of wave equation

Inserting Eq. (8) and Eq. (10) into Eq. (1), the elastic strain energy U is obtained as

$$U = \frac{1}{2}\int_0^L \left[EA\left(\frac{\partial u}{\partial x}\right)^2 + \frac{EI_p\nu^2}{2(1+\nu)}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + l^2\frac{EI_p\nu^2}{16(1+\nu)}\left(\frac{\partial^3 u}{\partial x^3}\right)^3\right]dx, \tag{11}$$

where A is the perpendicular cross section of the bar, and I_p is the second polar moment of area. It must be noted that the macroscopic axial model is equivalent to a fourth-order gradient elasticity model. When the couple stress parameter is vanishing (l=0), the macroscopic model is equivalent to a second-grade model.

The kinetic energy T of the bar is given by

$$T = \frac{1}{2}\rho\int_0^L \left[A\left(\frac{\partial u}{\partial t}\right)^2 + \nu^2 I_p\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2\right]dx, \tag{12}$$

where ρ is the mass density.

By applying Hamilton's principle, the governing equation of wave motion is obtained in the following form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\nu^2 r^2}{2(1+\nu)}\frac{\partial^4 u}{\partial x^4} + l^2 \quad \frac{\nu^2 r^2}{16(1+\nu)}\frac{\partial^6 u}{\partial x^6} + \frac{1}{c_o^2}\nu^2 r^2\frac{\partial^4 u}{\partial x^2\partial t^2} - \frac{1}{c_o^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{13}$$

where $c_o^2 = \frac{E}{\rho}$ and $r^2 = \frac{I_p}{A}$. In Eq. (13), the second term $\left(\text{i.e., } \frac{\nu^2 r^2}{2(1+\nu)}\right)$ denotes the contribution of the shear stress components on the elastic strain energy. The case $l = 0$ corresponds to a gradient elasticity model. Such wave equation has been already obtained in a different way by Challamel et al. (see p. 593, Eq. (7)) [10]. It must be noted that for a finite bar, such a model has to be completed with higher-order boundary conditions that can be obtained from application of variational principles.

4 Dispersion relation and numerical illustration

A harmonic longitudinal wave propagating along the axial direction can be expressed in complex form as

$$u = \tilde{u}e^{ik(x-ct)}, \tag{14}$$

where $\tilde{u}$ is the wave amplitude, k is the wave number, c is the phase velocity, and t is the time. Inserting Eq. (14) into Eq. (13), the corresponding solution for a rod with radius a is obtained as

$$C^* = \frac{c}{c_o} = \sqrt{\frac{1 + \frac{\nu^2(ka)^2}{4(1+\nu)} + \left(\frac{l}{a}\right)^2\frac{\nu^2(ka)^4}{32(1+\nu)}}{1 + \frac{1}{2}\nu^2(ka)^2}}, \tag{15}$$

where C^* is the dimensionless phase velocity, and ka is the dimensionless wave number.

The above relation derived may be regarded as the nonlocal Kecs solution based on the modified couple stress theory. When the shear stress contribution to the elastic strain energy is ignored $\left(\text{i.e., } \frac{\nu^2(ka)^2}{4(1+\nu)} = 0\right)$, Eq. (15) is reduced to the following form:

$$C^* = \frac{c}{c_o} = \sqrt{\frac{1 + \left(\frac{l}{a}\right)^2\frac{\nu^2(ka)^4}{32(1+\nu)}}{1 + \frac{1}{2}\nu^2(ka)^2}}. \tag{16}$$

The above relation may be regarded as the nonlocal Love solution based on the modified couple stress theory.

When the scale parameter l is zero, the Eq. (15) is reduced to the local Kecs solution:

$$C^* = \frac{c}{c_o} = \sqrt{\frac{1 + \frac{\nu^2 (ka)^2}{4(1+\nu)}}{1 + \frac{1}{2}\nu^2 (ka)^2}}. \quad (17)$$

When the shear stress contribution to the elastic strain energy is ignored, Eq. (17) is reduced to the well-known local Love solution:

$$C^* = \frac{c}{c_o} = \sqrt{\frac{1}{1 + \frac{1}{2}\nu^2 (ka)^2}}. \quad (18)$$

As can be understood from the above dispersion relations, the local phase velocities always decrease with increasing wave number. However, the nonlocal phase velocities increase with increasing supercritical wave number. In this work, the extreme values of nonlocal dispersion curves are called critical values of the dimensionless wave number. The critical value of the dimensionless wave number for the present nonlocal longitudinal wave dispersion solution is found from derivation of Eq. (15) (i.e., $\frac{dc^*}{d(ka)} = 0$) in the following form:

$$(ka)_{cr} = \frac{\sqrt{2}}{\nu}\sqrt{\frac{a}{l}\sqrt{\left(\frac{l}{a}\right)^2 + 4\nu^2 (1 + 2\nu)} - 1}. \quad (19)$$

The critical value of the dimensionless wave number for the present nonlocal Love dispersion case is found from derivation of Eq. (16) in the following form:

$$(ka)_{cr} = \frac{\sqrt{2}}{\nu}\sqrt{\frac{a}{l}\sqrt{\left(\frac{l}{a}\right)^2 + 8\nu^2 (1 + \nu)} - 1}. \quad (20)$$

To best explain the size effect with respect to the modified couple stress theory, the dimensionless internal material length scale parameter $\frac{l}{a}$ is taken to be 0.7, in numerical illustration. Figure 1 shows that the effect of shear stress on the longitudinal wave dispersion becomes significant except for small values of the dimensionless wave number. This effect becomes more significant with increasing dimensionless wave number. Poisson's ratio ν is taken as 0.35. Similarly, it can be seen that the effect of the scale parameter l on the longitudinal wave dispersion is significant, but in particular for $(ka) > (ka)_{cr}$, this effect becomes more significant. The critical values of the dimensionless wave numbers for the present solution (15) and the nonlocal Love solution (16) are found from Eqs. (19) and (20) to be 3.240 and 3.883, respectively. It can also be seen from Fig. 1 that the present nonlocal solutions give upper bound solutions with respect to local solutions. An alternative illustration for same dispersion curves as in [10] is shown in Fig. 2.

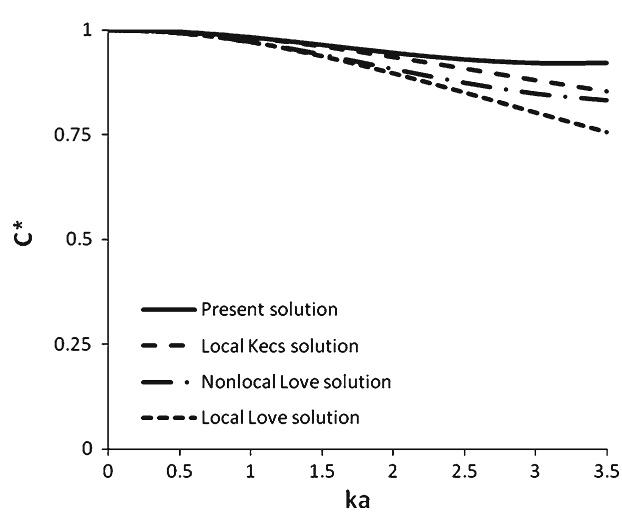

Fig. 1 Nonlocal and local dispersion curves

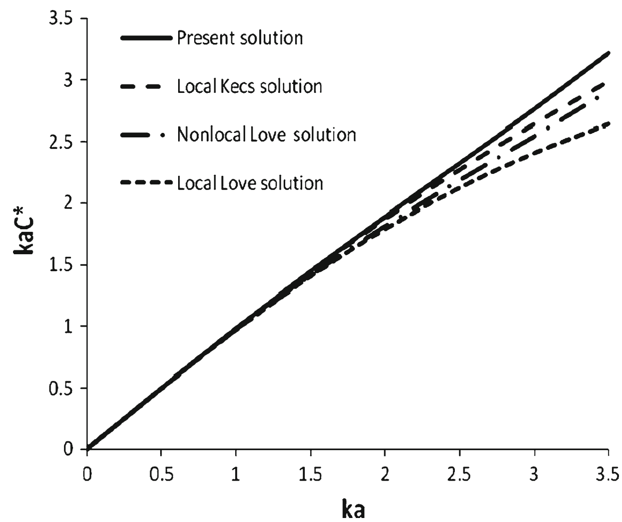

Fig. 2 Alternative illustration for dispersion curves

5 Conclusions

In the present work, an explicit solution for the propagation of longitudinal waves is developed using the modified couple stress theory and Love rod model. The solution presented may be called the generalized nonlocal Kecs solution because it contains the contribution of shear stress. The difference between the nonlocal and local Kecs solutions becomes meaningful when the value of the dimensionless wave number is bigger than a critical value. Also, it can be deduced from the present work that neglecting the shear stress effect will not give the true results at all times, in the local and nonlocal longitudinal wave dispersions investigations.

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