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On the simple and mixed first-order theories for plates resting on elastic foundations

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Abstract This article investigates the bending response of an orthotropic rectangular plate resting on two-parameter elastic foundations. Analytical solutions for deflection and stresses are developed by means of the simple and mixed first-order shear deformation plate theories. The present mixed plate theory accounts for variable transverse shear stress distributions through the thickness and does not require a shear correction factor. The governing equations that include the interaction between the plate and the foundations are obtained. Numerical results are presented to demonstrate the behavior of the system. The results are compared with those obtained in the literature using three-dimensional elasticity theory or higher-order shear deformation plate theory to check the accuracy of the simple and mixed first-order shear deformation theories.

1 Introduction

A plate on an elastic foundation belongs to the problem of mutual action between two media. Early research adopted single-parameter Winkler's model to simulate the foundation. It considered that the displacement on a foundation surface is limited only on the loaded domain, which conflicts with the practical response situation. In some of the analyzes of the plates on elastic foundations, a single-parameter K_1 is used to describe the foundation behavior [1]. In this model, it is assumed that there is a proportional interaction between the external forces and the deflection of the applied point in the foundation. A two-parameter elastic foundation model can reflect the practical deformation of a foundation, so it is widely accepted by investigators.

Plates supported by elastic foundations are commonly encountered technical problems in many engineering applications. The studies of plates resting on elastic foundations have attracted the attention of many researchers [2–5]. Some other researchers have modeled the foundations with two parameters. One of these models is

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the Pasternak's model. This two-parameter model takes into account the effect of shear interaction among the points in the foundation [6], and the well-known Winkler's model is one of its special cases. The response of structural elements resting on the one- and two-parameter foundation is usually analyzed by assuming that the foundation supports compressive as well as tensile stresses, which simplifies the analysis considerably. However, this assumption is questionable or not valid for many supporting media including the soil. For structural elements on soil capable of supporting compressive reactions only, the tensionless foundation model should be adopted for realistic results [7]. Liew et al. [8] have studied the differential quadrature method for Mindlin's plates on Winkler's foundation. Eratll and Akoz [9] have used a new function to examine Mindlin's plate on a Winkler's foundation. Han and Liew [10] have investigated a numerical differential quadrature method for Reissner-Mindlin's plates on two-parameter elastic foundations. Omurtag and Kadioglu [11] have investigated the vibration of Kirchhoff's plates on Winkler's and Pasternak's foundations. Chen et al. [12] have studied the mixed method for bending and free vibration of beams resting on a Pasternak's elastic foundation. Singh et al. [13] have investigated the post buckling response of a laminated composite plate on an elastic foundation with random system properties. Based on a refined sinusoidal plate theory, Zenkour et al. [14] have presented an investigation on the bending response of functionally graded viscoelastic beams resting on elastic foundations.

The usual refined theory considered in the treatment of the dynamic response of flat orthotropic plates is the simple first-order transverse shear deformation plate theory (SFPT) [15–17]. The theory is given for statics by Reissner [15,16] and extended to dynamics by Mindlin [17]. In this theory, the in-plane displacements are expanded up to the first term in the thickness coordinate, and the rotations of normals to the mid-surface are assumed to be independent of the transverse deflection. For SFPT, any variational principle may be used to derive a consistent set of differential equations governing the motion of the plate. The mixed first-order transverse shear deformation plate theory (MFPT) is a modification of the SFPT. In the MFPT, both the displacements and stresses must be considered arbitrary. For this reason, a mixed variational formula should be used [18–23]. The utilization of the mixed variational principles allows one to treat the plate problems by introducing kinematics assumptions with any power of the thickness coordinate. Also, the transverse shear stresses are consistent with the surface conditions. So, the rationale for the shear correction factor required to the SFPT is obviated. In addition, the effect of transverse normal stress is taken into account. Zenkour [21] has investigated the natural vibration of symmetrical cross-ply laminated plates using a mixed variational formulation. A relationship between the simple and mixed first-order transverse shear deformation theories has been presented by Zenkour [22]. Zenkour and Mashat [23] have investigated the bending of a ceramic-metal arched bridge using a mixed first-order theory.

In this paper, an accurate solution for the simply supported rectangular plate resting on elastic foundations is presented. Pasternak's model is used here to describe the two-parameter elastic foundation, and getting a special case of Winkler's foundation model by considering a one-parameter elastic foundation. The interaction between the plate and the elastic foundations is considered and included in the equilibrium equations. A relationship between the simple and mixed first-order transverse shear deformation theories is presented. Numerical results for deflection and stresses of orthotropic plates are given. Comparisons between the results of the simple and mixed first-order theories are made, and some conclusions are formulated.

2 Governing equations

Consider a rectangular plate of length a, width b, and thickness h and resting on elastic foundations (see Fig. 1). The mid-plane of the plate is taken as the xy plane, and the x- and y-axes are directed along the

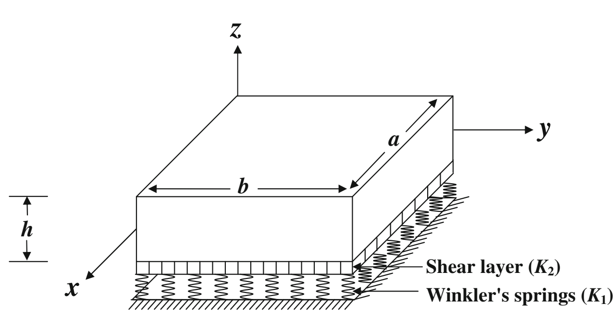

Fig. 1 Geometry and coordinate system for an orthotropic rectangular plate resting on two-parameter elastic foundations

edges. The z-axis is taken perpendicular to the mid-plane. Let the plate be subjected to a distributed transverse load $q(x, y)$, and the load-displacement relation between the plate and the supporting foundations follows the two-parameter Pasternak's model:

$$\Re = K_1 w - K_2 \nabla^2 w, \tag{1}$$

where $\Re$ is the foundation reaction per unit area, K_1 and K_2 are the Winkler's and Pasternak's foundation stiffnesses, respectively, w is the plate deflection, and ∇^2 is the Laplace operator in x and y. This model is simply known as Winkler's type when $K_2 = 0$. We start in the usual way by proposing the first-order displacement field:

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + z\,\psi_x(x, y), \\ u_2(x, y, z) &= v(x, y) + z\,\psi_y(x, y), \\ u_3(x, y, z) &= w(x, y), \end{aligned} \tag{2}$$

where (u, v, w) are the displacements of a point (x, y) on the mid-plane, and ψ_x and ψ_y are the rotations of normals to the mid-plane about the y- and x-axes, respectively. The six strain components are compatible with the displacement field, Eq. (2), as:

$$\begin{aligned} &\varepsilon_{ii} = \varepsilon_i^0 + z\beta_i, \quad \varepsilon_{yz} = \psi_y + \chi_y, \quad (i = x, y), \\ &\varepsilon_{xy} = \varepsilon_{xy}^0 + z\beta_{xy}, \quad \varepsilon_{zx} = \psi_x + \chi_x, \quad \varepsilon_{zz} = 0, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\varepsilon_x^0 = \partial_x u, \quad \beta_x = \partial_x \psi_x, \quad \chi_x = \partial_x w, \quad \varepsilon_{xy}^0 = \partial_y u + \partial_x v, \\ &\varepsilon_y^0 = \partial_y v, \quad \beta_y = \partial_y \psi_y, \quad \chi_y = \partial_y w, \quad \beta_{xy} = \partial_y \psi_x + \partial_x \psi_y. \end{aligned} \tag{4}$$

2.1 Simple shear deformation plate theory

For SFPT, the stress–strain relations of an orthotropic body accounting for transversal shear deformation in rectangular plate coordinates can be expressed as:

$$\begin{Bmatrix} \sigma_{xx}^{s} \\ \sigma_{yy}^{s} \\ \sigma_{yz}^{s} \\ \sigma_{zx}^{s} \\ \sigma_{xy}^{s} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix}. \tag{5}$$

The stress resultants in this case are given by:

$$\left\{N_{ij}^{s}, M_{ij}^{s}\right\} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \{1, z\}\sigma_{ij}^{s}\mathrm{d}z, \quad Q_{iz}^{s} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} k_i \sigma_{iz}^{s}\,\mathrm{d}z, \qquad (i, j = x, y), \tag{6}$$

where k_i ($k_x = k_y = k$) are shear factors to correct for the errors stemming from Eq. (3) that σ_{xz}^{s} and σ_{yz}^{s} are constants over the thickness.

2.2 Mixed shear deformation plate theory

For MFPT, the stress and displacement fields are taken to be arbitrary; then, the non-vanishing stress field is assumed to be in the form [22]

$$\sigma_{ij}^{\mathrm{m}} = G_{ij}^{0}(x,y) + zG_{ij}^{1}(x,y), \quad \sigma_{zz}^{\mathrm{m}} = \sum_{\mathrm{n}=1}^{4} z^{\mathrm{n}-1} G_{zz}^{\mathrm{n}}(x,y),$$
$$\sigma_{iz}^{\mathrm{m}} = G_{iz}^{0}(x,y)\left[1-\left(\frac{z}{h/2}\right)^{2}\right], \quad (i,j=x,y), \tag{7}$$

where the functions G_{ij}^{0}, G_{ij}^{1} and G_{iz}^{0} may be obtained easily from the point that the stresses σ_{ij}^{m} and σ_{jz}^{m} satisfy the following stress resultants:

$$\left\{N_{ij}^{\mathrm{m}}, M_{ij}^{\mathrm{m}}\right\} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \{1, z\}\sigma_{ij}^{\mathrm{m}}\mathrm{d}z, \quad Q_{iz}^{\mathrm{m}} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{iz}^{\mathrm{m}}\,\mathrm{d}z, \qquad (i,j=x,y), \tag{8}$$

where N_{xx}^{α}, N_{yy}^{α}, N_{xy}^{α}, M_{xx}^{α}, M_{yy}^{α} and M_{xy}^{α} are the basic components of stress resultants. Also, the functions G_{zz}^{n} arise from the point that the transverse normal stress σ_{zz}^{m} satisfies the following conditions:

$$\sigma_{zz}^{\mathrm{m}}\,|_{z=-\frac{h}{2}} = -\Re(x,y) \quad \sigma_{zz}^{\mathrm{m}}\,|_{z=\frac{h}{2}} = -q(x,y), \quad \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{zz}^{\mathrm{m}}\,\mathrm{d}z = 0, \quad \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\,\sigma_{zz}^{\mathrm{m}}\,\mathrm{d}z = 0. \tag{9}$$

Therefore, the final expressions for the stress components can be written in terms of their resultants and the thickness coordinate z:

$$\sigma_{ij}^{\mathrm{m}} = \frac{N_{ij}^{\mathrm{m}}}{h} + \frac{12\,M_{ij}^{\mathrm{m}}}{h^{3}}z, \quad \sigma_{iz}^{\mathrm{m}} = \frac{3}{2h}Q_{iz}^{\mathrm{m}}\left[1-\left(\frac{z}{h/2}\right)^{2}\right], \qquad (i,j=x,y),$$
$$\sigma_{zz}^{\mathrm{m}} = \left(\frac{1}{4} - \frac{3z}{2h} - \frac{3z^{2}}{h^{2}} + \frac{10z^{3}}{h^{3}}\right)\Re(x,y) + \left(\frac{1}{4} + \frac{3z}{2h} - \frac{3z^{2}}{h^{2}} - \frac{10z^{3}}{h^{3}}\right)q(x,y). \tag{10}$$

It is to be noted that the transverse shear stresses σ_{xz}^{m} and σ_{yz}^{m} are functions of z and vanish on the bounding planes ($z = \pm\frac{h}{2}$).

The simple and mixed variational formulations based upon Hamilton's principle are given, respectively, by [22]:

$$\iiint_{\mathrm{v}} \delta U^{\mathrm{s}}\mathrm{dv} + \delta V^{\mathrm{s}} = 0, \tag{11}$$

$$\iiint_{\mathrm{v}} \left[\delta\left(\sigma_{ij}^{\mathrm{m}}\varepsilon_{ij}\right) - \delta R^{\mathrm{m}}\right]\mathrm{dv} + \delta V^{\mathrm{m}} = 0, \tag{12}$$

where U^{s} and R^{m} are the strain and complementary energy densities. They are given by:

$$U^{\mathrm{s}} = \frac{1}{2}\left[c_{11}\left(\varepsilon_{xx}^{\mathrm{s}}\right)^{2} + c_{22}\left(\varepsilon_{yy}^{\mathrm{s}}\right)^{2} + c_{44}\left(\varepsilon_{yz}^{\mathrm{s}}\right)^{2} + c_{55}\left(\varepsilon_{xz}^{\mathrm{s}}\right)^{2} + c_{66}\left(\varepsilon_{xy}^{\mathrm{s}}\right)^{2}\right] + c_{12}\varepsilon_{xx}^{\mathrm{s}}\varepsilon_{yy}^{\mathrm{s}}, \tag{13}$$

$$R^{\mathrm{m}} = \frac{1}{2}\left[a_{11}\left(\sigma_{xx}^{\mathrm{m}}\right)^{2} + a_{22}\left(\sigma_{yy}^{\mathrm{m}}\right)^{2} + a_{33}\left(\sigma_{zz}^{\mathrm{m}}\right)^{2} + a_{44}\left(\sigma_{yz}^{\mathrm{m}}\right)^{2} + a_{55}\left(\sigma_{xz}^{\mathrm{m}}\right)^{2} + a_{66}\left(\sigma_{xy}^{\mathrm{m}}\right)^{2}\right]$$
$$+ a_{12}\sigma_{xx}^{\mathrm{m}}\sigma_{yy}^{\mathrm{m}} + a_{23}\sigma_{yy}^{\mathrm{m}}\sigma_{zz}^{\mathrm{m}} + a_{13}\sigma_{xx}^{\mathrm{m}}\sigma_{zz}^{\mathrm{m}}, \tag{14}$$

where the constitutive constants c_{ij} and their compliances a_{ij} may be expressed in terms of the engineering orthotropic characteristics as:

$$c_{11} = \frac{E_{1}}{1-\nu_{12}\nu_{21}}, \quad c_{12} = \frac{\nu_{12}E_{1}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_{2}}{1-\nu_{12}\nu_{21}}, \quad c_{22} = \frac{E_{2}}{1-\nu_{12}\nu_{21}},$$
$$c_{44} = G_{23} = \frac{1}{a_{44}}, \quad c_{55} = G_{13} = \frac{1}{a_{55}}, \quad a_{12} = -\frac{\nu_{12}}{E_{1}} = -\frac{\nu_{21}}{E_{2}},$$
$$c_{66} = G_{12} = \frac{1}{a_{66}}, \quad a_{11} = \frac{1}{E_{1}}, \quad a_{22} = \frac{1}{E_{2}}, \quad a_{33} = \frac{1}{E_{3}}. \tag{15}$$

Here, E_i, G_{ij}, and ν_{ij} stand for Young's moduli, shear moduli, and Poisson's ratios, respectively.

In the case of an isotropic body, the specialized counterparts of Eqs. (15) become:

$$\begin{aligned}
&c_{11}=c_{22}=c_{33}=\frac{E}{1-\nu^2},\quad c_{12}=c_{13}=c_{23}=\frac{\nu E}{1-\nu^2},\\
&c_{44}=c_{55}=c_{66}=G=\frac{E}{2(1+\nu)},\quad a_{11}=a_{22}=a_{33}=\frac{1}{E},\\
&a_{12}=\frac{-\nu}{E},\quad a_{44}=a_{55}=a_{66}=\frac{1}{G}.
\end{aligned}\tag{16}$$

The potential energy $V^\alpha(\alpha=\mathrm{s},\mathrm{m})$ of the applied loads can be defined as a function of the displacement field u_i and the applied loads as

$$\begin{aligned}
V^{\mathrm{s}}&=-\iiint_{\mathrm{v}}B_iu_i\mathrm{dv}-\iint_{\Omega_\sigma}[F_iu_i-\Re(x,y)u_3]\,\mathrm{d}\Omega,\\
V^{\mathrm{m}}&=-\iiint_{\mathrm{v}}B_iu_i\mathrm{dv}-\iint_{\Omega_\sigma}[F_iu_i-\Re(x,y)u_3]\,\mathrm{d}\Omega-\iint_{\Omega_u}n_j\sigma_{ij}^{\mathrm{m}}\left(u_i-u_i^*\right)\mathrm{d}\Omega,
\end{aligned}\tag{17}$$

where n_j are the components of the unit vector along the outward normal to the total surface $\Omega_\sigma+\Omega_u$, B_i are the body forces measured per unit volume of the undeformed body, F_i are the prescribed components of the stress vector per unit area of the surface Ω_σ, and u_i^* are the prescribed components of the displacements of the remaining surface Ω_σ. In the absence of the body forces and the prescribed displacements, we have for the first variation of V^α

$$\delta V^\alpha=-\iint_{\Omega_\sigma}[q(x,y)-\Re(x,y)]\,\delta w\mathrm{d}\Omega.\tag{18}$$

2.3 Equations of equilibrium

The governing equilibrium equations can be derived from substituting Eqs. (1), (2), (3), (5), (10), (13), (14), and (18) into the two variational formulations given in Eqs. (11) and (12) by integrating and setting the coefficients of δu, $\delta\upsilon$, δw, $\delta\psi_x$, and $\delta\psi_y$ to zero, separately. Thus, one can obtain:

$$\begin{aligned}
\delta u^\alpha&:\frac{\partial N_{xx}^\alpha}{\partial x}+\frac{\partial N_{xy}^\alpha}{\partial y}=0,\\
\delta\upsilon^\alpha&:\frac{\partial N_{xy}^\alpha}{\partial x}+\frac{\partial N_{yy}^\alpha}{\partial y}=0,\\
\delta w^\alpha&:\frac{\partial Q_{xz}^\alpha}{\partial x}+\frac{\partial Q_{yz}^\alpha}{\partial y}+q^\alpha-\Re^\alpha=0,\\
\delta\psi_x^\alpha&:\frac{\partial M_{xx}^\alpha}{\partial x}+\frac{\partial M_{xy}^\alpha}{\partial y}-Q_{xz}^\alpha=0,\\
\delta\psi_y^\alpha&:\frac{\partial M_{xy}^\alpha}{\partial x}+\frac{\partial M_{yy}^\alpha}{\partial y}-Q_{yz}^\alpha=0,
\end{aligned}\tag{19}$$

where

$$q^\alpha=\begin{cases}q(x,y) & \text{if}\quad\alpha=\mathrm{s},\\ \left[1+\frac{1}{70}ha_{33}\left(K_1-K_2\nabla^2\right)\right]q(x,y) & \text{if}\quad\alpha=\mathrm{m},\end{cases}\tag{20}$$

$$\Re^\alpha=\begin{cases}\Re(x,y) & \text{if}\quad\alpha=\mathrm{s},\\ \left[1-\frac{3}{35}ha_{33}\left(K_1-K_2\nabla^2\right)\right]\Re(x,y) & \text{if}\quad\alpha=\mathrm{m}.\end{cases}\tag{21}$$

2.4 Boundary conditions

The extremum condition of Eqs. (11) and (12) gives also the form of the geometric and force boundary conditions (Table 1): such that (n_x, n_y) denote the direction cosines of a unit normal to the boundary of the mid-plane. Note that

$$\bar{Q}^{\alpha}_{\gamma z} = \begin{cases} Q^{\mathrm{s}}_{\gamma z} & \text{if} \quad \alpha = \mathrm{s}, \\ Q^{\mathrm{m}}_{\gamma z} - \frac{3}{35} h a_{33} K_2 \left[\frac{\partial \Re}{\partial \gamma} + \frac{1}{6} \frac{\partial q}{\partial \gamma} \right] & \text{if} \quad \alpha = \mathrm{m}, \gamma = (x, y). \end{cases} \tag{22}$$

2.5 Constitutive equations

For the SFPT, the stress resultants are related to the strains by substituting Eqs. (3), (4), and (5) into Eq. (6), while for the MFPT, they will be derived from the extremum condition of the mixed variational formulation, Eq. (12), by setting the coefficients of δN_{xx}, δN_{yy}, δN_{xy}, δM_{xx}, δM_{yy}, δM_{xy}, δQ_{xz}, and δQ_{yz} to zero, separately.

In general, the constitutive equations are given by:

$$\begin{Bmatrix} N^{\alpha}_{xx} \\ N^{\alpha}_{yy} \\ N^{\alpha}_{xy} \end{Bmatrix} = \begin{bmatrix} \Pi^{\alpha}_{11} & \Pi^{\alpha}_{12} & 0 \\ \Pi^{\alpha}_{12} & \Pi^{\alpha}_{22} & 0 \\ 0 & 0 & \Pi^{\alpha}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_x \\ \varepsilon^0_y \\ \varepsilon^0_{xy} \end{Bmatrix}, \tag{23}$$

$$\begin{Bmatrix} M^{\alpha}_{xx} \\ M^{\alpha}_{yy} \\ M^{\alpha}_{xy} \end{Bmatrix} = \begin{bmatrix} \Gamma^{\alpha}_{11} & \Gamma^{\alpha}_{12} & 0 \\ \Gamma^{\alpha}_{12} & \Gamma^{\alpha}_{22} & 0 \\ 0 & 0 & \Gamma^{\alpha}_{66} \end{bmatrix} \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{Bmatrix}, \tag{24}$$

$$\begin{Bmatrix} Q^{\alpha}_{xz} \\ Q^{\alpha}_{yz} \end{Bmatrix} = \begin{bmatrix} \Pi^{\alpha}_{55} & 0 \\ 0 & \Pi^{\alpha}_{44} \end{bmatrix} \begin{Bmatrix} \psi_x + \chi_x \\ \psi_y + \chi_y \end{Bmatrix}, \tag{25}$$

where

$$\begin{aligned} &\begin{bmatrix} \Pi^{\mathrm{m}}_{11} & \Pi^{\mathrm{m}}_{12} \\ \Pi^{\mathrm{m}}_{12} & \Pi^{\mathrm{m}}_{22} \end{bmatrix} = h \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}^{-1}, \\ &\Pi^{\mathrm{s}}_{rr} = hkc_{rr}, \qquad \Pi^{\mathrm{s}}_{pq} = hc_{pq}, \qquad \Gamma^{\alpha}_{pq} = \frac{h^2}{12} \Pi^{\alpha}_{pq}, \\ &\Pi^{\mathrm{m}}_{rr} = \frac{5h}{6a_{rr}}, \qquad \Pi^{\mathrm{m}}_{66} = \frac{h}{a_{66}}, \qquad (r = 4, 5;\ p, q = 1, 2, 6). \end{aligned} \tag{26}$$

Table 1 Geometric and force boundary conditions

Essential	Natural
u^{α}	$N^{\alpha}_{xx} n_x + N^{\alpha}_{xy} n_y$
v^{α}	$N^{\alpha}_{xy} n_x + N^{\alpha}_{yy} n_y$
w^{α}	$\bar{Q}^{\alpha}_{xz} n_x + \bar{Q}^{\alpha}_{yz} n_y$
ψ^{α}_x	$M^{\alpha}_{xx} n_x + M^{\alpha}_{xy} n_y$
ψ^{α}_y	$M^{\alpha}_{xy} n_x + M^{\alpha}_{yy} n_y$
$\frac{\partial \delta w^{\mathrm{m}}}{\partial x}$	$\frac{3}{35} h a_{33} K_2 \left[\Re(x, y) + \frac{1}{6} q(x, y) \right] n_x$
$\frac{\partial \delta w^{\mathrm{m}}}{\partial y}$	$\frac{3}{35} h a_{33} K_2 \left[\Re(x, y) + \frac{1}{6} q(x, y) \right] n_y$

3 Solution procedure

The simple and mixed variational formulations will be extended here in order to analyze the bending response of rectangular plates resting on elastic foundations. The SFPT and MFPT will be established in this Section by suppressing the in-plane displacement degree of freedom. The following simply supported boundary conditions are considered:

$$\begin{aligned} &v = w = \psi_y = N_{xx}^{\alpha} = M_{xx}^{\alpha} = 0 \quad \text{at} \quad x = 0, a, \\ &u = w = \psi_x = N_{yy}^{\alpha} = M_{yy}^{\alpha} = 0 \quad \text{at} \quad y = 0, b. \end{aligned} \tag{27}$$

We are here concerned with the solutions of Eqs. (19) for a simply supported plate. To solve this problem, Navier presented the external force in the form

$$q(x, y) = \sum_{m,n=1,3,5,\ldots}^{\infty} q_{mn} \sin(\lambda x) \sin(\mu y), \tag{28}$$

where the coefficients q_{mn} for the case of a uniformly distributed load are defined as follows:

$$q_{mn} = \frac{16 q_0}{mn\pi^2}, \tag{29}$$

and $\lambda = \frac{m\pi}{a}$, $\mu = \frac{n\pi}{b}$, m and n are mode numbers. For the case of a sinusoidally distributed load, $m = n = 1$ and $q_{mn} = q_0$, where q_0 represents the intensity of the load at the plate center. Following the Navier solution procedure, we assume the following solution form for $(u, v, w, \psi_x, \psi_y)$ that satisfies the simply supported boundary conditions:

$$\begin{Bmatrix} u \\ v \\ w \\ \psi_x \\ \psi_y \end{Bmatrix} = \sum_{m,n=1,3,5,\ldots}^{\infty} \begin{Bmatrix} X_{mn} \cos(\lambda x) \sin(\mu y) \\ Y_{mn} \sin(\lambda x) \cos(\mu y) \\ Z_{mn} \sin(\lambda x) \sin(\mu y) \\ \Psi_{mn} \cos(\lambda x) \sin(\mu y) \\ \Phi_{mn} \sin(\lambda x) \cos(\mu y) \end{Bmatrix}, \tag{30}$$

where X_{mn}, Y_{mn}, Z_{mn}, Ψ_{mn} and Φ_{mn} are arbitrary parameters.

Substituting Eqs. (1), (4), (23), (24), (25) and (30) in (19) yields a system of algebraic equations expressed in a compact form as

$$[P^{\alpha}]\{\Lambda\} = \{f^{\alpha}\}, \tag{31}$$

where $\{f\}$ and $\{\Lambda\}$ denote the columns as:

$$\{f^{\alpha}\} = \begin{cases} \{0, 0, q_0, 0, 0\}^{\mathrm{t}} & \text{if} \quad \alpha = \mathrm{s}, \\ \left\{0, 0, \left[1 + \frac{1}{70} h a_{33} \left(K_1 + K_2(\lambda^2 + \mu^2)\right)\right] q_0, 0, 0\right\}^{\mathrm{t}} & \text{if} \quad \alpha = \mathrm{m}, \end{cases} \tag{32}$$

$$\{\Lambda\} = \{X_{mn}, Y_{mn}, Z_{mn}, \Psi_{mn}, \Phi_{mn}\}^{\mathrm{t}}, \tag{33}$$

in which the superscript t denotes the transpose of the given vectors, and the elements $P_{ij}^{\alpha} = P_{ji}^{\alpha}$ of matrix $[P^{\alpha}]$ are given by:

$$\begin{aligned} P_{11}^{\alpha} &= \lambda^2 \Pi_{11}^{\alpha} + \mu^2 \Pi_{66}^{\alpha}, \\ P_{12}^{\alpha} &= \lambda\mu \left(\Pi_{12}^{\alpha} + \Pi_{66}^{\alpha}\right), \\ P_{22}^{\alpha} &= \mu^2 \Pi_{22}^{\alpha} + \lambda^2 \Pi_{66}^{\alpha}, \\ P_{33}^{\mathrm{s}} &= \lambda^2 \left(\Pi_{55}^{\mathrm{s}} + K_2\right) + \mu^2 \left(\Pi_{44}^{\mathrm{s}} + K_2\right) + K_1, \\ P_{33}^{\mathrm{m}} &= \lambda^2 \Pi_{55}^{\mathrm{m}} + \mu^2 \Pi_{44}^{\mathrm{m}} + \left[1 - \frac{3}{35} h a_{33} \left(K_1 + K_2(\lambda^2 + \mu^2)\right)\right] \left[K_1 + K_2 \left(\lambda^2 + \mu^2\right)\right], \\ P_{34}^{\alpha} &= \lambda \Pi_{55}^{\alpha}, \\ P_{35}^{\alpha} &= \mu \Pi_{44}^{\alpha}, \end{aligned}$$

$$
\begin{aligned}
&P_{44}^{\alpha} = \lambda^2 \Gamma_{11}^{\alpha} + \mu^2 \Gamma_{66}^{\alpha} + \Pi_{55}^{\alpha},\\
&P_{45}^{\alpha} = \lambda\mu \left(\Gamma_{12}^{\alpha} + \Gamma_{66}^{\alpha}\right),\\
&P_{55}^{\alpha} = \mu^2 \Gamma_{22}^{\alpha} + \lambda^2 \Gamma_{66}^{\alpha} + \Pi_{44}^{\alpha},\\
&P_{13}^{\alpha} = P_{14}^{\alpha} = P_{15}^{\alpha} = P_{23}^{\alpha} = P_{24}^{\alpha} = P_{25}^{\alpha} = 0.
\end{aligned} \tag{34}
$$

Moreover, for the SFPT, substituting Eqs. (3), (4), and (30) into Eq. (5) with the help of Eq. (15) and for the MFPT, substituting Eqs. (4), (23), (24), (25), (26), and (30) into Eq. (10) with the help also of Eq. (15), one can obtain the stress components in terms of Young's moduli, shear moduli, and the arbitrary parameters X_{mn}, Y_{mn}, Z_{mn}, Ψ_{mn}, and Φ_{mn} as follows:

$$
\sigma_{xx}^{\alpha} = -\sum_{m,n=1,3,5,\ldots}^{\infty} \frac{E_1}{1-\nu_{12}\nu_{21}} [\lambda\,(X_{mn} + z\,\Psi_{mn}) + \mu\nu_{12}\,(Y_{mn} + z\,\Phi_{mn})] \sin(\lambda\,x)\sin(\mu\,y), \tag{35}
$$

$$
\sigma_{yy}^{\alpha} = -\sum_{m,n=1,3,5,\ldots}^{\infty} \frac{E_2}{1-\nu_{12}\nu_{21}} [\lambda\nu_{21}\,(X_{mn} + z\,\Psi_{mn}) + \mu\,(Y_{mn} + z\,\Phi_{mn})] \sin(\lambda\,x)\sin(\mu\,y), \tag{36}
$$

$$
\sigma_{xy}^{\alpha} = \sum_{m,n=1,3,5,\ldots}^{\infty} G_{12}\,[\mu\,(X_{mn} + z\,\Psi_{mn}) + \lambda\,\,(Y_{mn} + z\,\Phi_{mn})] \cos(\lambda\,x)\cos(\mu\,y), \tag{37}
$$

$$
\sigma_{yz}^{\mathrm{s}} = \sum_{m,n=1,3,5,\ldots}^{\infty} G_{23}\,[\Phi_{mn} + \mu Z_{mn}] \sin(\lambda\,x)\cos(\mu\,y), \tag{38}
$$

$$
\sigma_{yz}^{\mathrm{m}} = \sum_{m,n=1,3,5,\ldots}^{\infty} \frac{5}{4} G_{23}\,[\Phi_{mn} + \mu\,Z_{mn}] \left[1 - \left(\frac{z}{h/2}\right)^2\right] \sin(\lambda\,x)\cos(\mu\,y), \tag{39}
$$

$$
\sigma_{xz}^{\mathrm{s}} = \sum_{m,n=1,3,5,\ldots}^{\infty} G_{13}\,[\Psi_{mn} + \lambda Z_{mn}] \cos(\lambda\,x)\sin(\mu\,y), \tag{40}
$$

$$
\sigma_{xz}^{\mathrm{m}} = \sum_{m,n=1,3,5,\ldots}^{\infty} \frac{5}{4} G_{13}\,[\Psi_{mn} + \lambda\,Z_{mn}] \left[1 - \left(\frac{z}{h/2}\right)^2\right] \cos(\lambda\,x)\sin(\mu\,y), \tag{41}
$$

$$
\sigma_{zz}^{\mathrm{m}} = \sum_{m,n=1,3,5,\ldots}^{\infty} \left\{ \left(\frac{1}{4} - \frac{3z}{2h} - \frac{3z^2}{h^2} + \frac{10z^3}{h^3}\right) [K_1 + K_2(\lambda^2 + \mu^2)] Z_{mn} + \left(\frac{1}{4} + \frac{3z}{2h} - \frac{3z^2}{h^2} - \frac{10z^3}{h^3}\right) q_0 \right\} \sin(\lambda\,x)\sin(\mu\,y). \tag{42}
$$

4 Numerical results

The static behaviors of the simply supported, orthotropic rectangular plates resting on elastic foundations are considered. Table 2 presents the material properties of two samples. In this Table, the two kinds of fiber-reinforced laminates have various orthotropic properties.

Table 2 Material properties of the plate

Materials	E_1	E_2	E_3	G_{12}	G_{13}	G_{23}	ν_{12}	ν_{13}	ν_{32}
Material I (Msi):	20.83	10.94	10	6.1	3.71	6.19	0.44	0.44	0.44
Materials II (GPa):	37.8	13.1	13.1	8	8	5.3	0.25	0.25	0.25

The numerical results of deflections and stresses are presented for plates with four edges simply supported resting on elastic foundations. The various non-dimensional parameters used are:

$$\begin{aligned}
&\kappa_1=\frac{a^4}{h^3}K_1,\quad \kappa_2=\frac{a^2}{h^3}K_2,\quad \bar{w}=\frac{10^2h^3}{q_0a^4}w\left(\frac{a}{2},\frac{b}{2}\right),\\
&\sigma_1=\frac{h^2}{q_0a^2}\sigma_{xx}^{\alpha}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right),\quad \sigma_2=\frac{h}{q_0a}\sigma_{yy}^{\alpha}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right),\\
&\sigma_3=\frac{1}{q_0}\sigma_{zz}^{\mathrm{m}}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right),\quad \sigma_4=\frac{1}{q_0}\sigma_{yz}^{\alpha}\left(\frac{a}{2},0,\frac{z}{h}\right),\\
&\sigma_5=\frac{1}{q_0}\sigma_{xz}^{\alpha}\left(0,\frac{b}{2},\frac{z}{h}\right),\quad \sigma_6=\frac{h}{q_0a}\sigma_{xy}^{\alpha}\left(0,0,\frac{z}{h}\right).
\end{aligned}\tag{43}$$

The results for isotropic square plates resting on elastic foundations using SFPT and MFPT are compared with the corresponding results available in the literature [24–26]. The dimensionless deflection and foundations for this case are given by

$$w^*=\frac{10^2D}{q_0a^4}w\left(\frac{a}{2},\frac{b}{2}\right),\quad \bar{\kappa}_1=\frac{a^4}{D}K_1,\quad \bar{\kappa}_2=\frac{a^2}{D}K_2,\quad D=\frac{Eh^3}{12(1-\nu^2)}.\tag{44}$$

Table 3 gives the deflection w^* of an isotropic square plate under uniformly distributed load and resting on Winkler's one-parameter elastic foundation while Table 4 gives the deflection w^* of an isotropic square plate on two-parameter elastic foundations. It is clear that the present theories give results that coincide well with the available ones in the literature.

The results for orthotropic rectangular plates under sinusoidally distributed load and resting on elastic foundations using SFPT and MFPT are reported in Tables 5 and 6 as well as in Figs. 2 and 3. For the sake of comparison, the results for the higher-order shear deformation plate theory (HPT) are also included in Tables 5, 6, and Figs. 2, 3. However, Figs. 4, 5, 6, 7, 8, 9 have been displayed by using MFPT for orthotropic rectangular plates under sinusoidally distributed load and resting on elastic foundations. The shear correction factor of SFPT is chosen to be $k=5/6$ (except otherwise stated). Tables 5 and 6 show results for orthotropic plates

Table 3 Comparison of the deflection w^* of an isotropic square plate on one-parameter elastic foundations under uniformly distributed load ($\nu=0.3,\ \bar{\kappa}_2=0,\ h/a=0.05$)

$\bar{\kappa}_1$	Ref. [24]	Ref. [25]	Present	
	Finite element method	Analytical solution	SFPT	MFPT
0	0.40624	0.41197	0.41150	0.41150
1^4	0.40517	0.41088	0.41040	0.41040
3^4	0.33472	0.33855	0.33814	0.33814
5^4	0.15060	0.15114	0.15094	0.15094
10^4	0.01115	0.01096	0.01108	0.01109

Table 4 Comparison of the deflection w^* of an isotropic square plate on two-parameter elastic foundations under uniformly distributed load ($\nu=0.3,\ h/a=0.01$)

$\bar{\kappa}_1$	$\bar{\kappa}_2$	Ref. [26]	Present	
			SFPT	MFPT
1	1	0.3853	0.38550	0.38550
	3^4	0.0763	0.07630	0.07629
	5^4	0.0115	0.01153	0.01153
3^4	1	0.3210	0.32108	0.32108
	3^4	0.0732	0.07317	0.07317
	5^4	0.0115	0.01145	0.01145
5^4	1	0.1476	0.14765	0.14765
	3^4	0.0570	0.05704	0.05704
	5^4	0.0109	0.01095	0.01095

Table 5 Effects of elastic foundations parameters and side-to-thickness ratio on the dimensionless deflection $\bar{w}$ of rectangular plates ($b = 3a$)

a/h	κ_1	κ_2	MFPT	HPT	SFPT				
					$k = 1$	$k = \frac{\pi^2}{12}$	$k = \frac{5}{6}$	$k = \frac{3}{4}$	$k = \frac{2}{3}$
5	0	0	0.567871	0.567518	0.548631	0.569396	0.567871	0.580692	0.596710
	10	0	0.537373	0.537040	0.520097	0.538722	0.537356	0.548822	0.563109
	0	10	0.350236	0.349812	0.342543	0.350524	0.349949	0.354773	0.360688
	10	10	0.338431	0.337988	0.331198	0.338654	0.338114	0.342618	0.348131
10	0	0	0.481195	0.481173	0.476373	0.481578	0.481195	0.484410	0.488428
	10	0	0.459104	0.459083	0.454711	0.459451	0.459103	0.462029	0.465683
	0	10	0.314998	0.314973	0.312909	0.315146	0.314982	0.316357	0.318065
	10	10	0.305381	0.305355	0.303415	0.305518	0.305364	0.306655	0.308261
20	0	0	0.459487	0.459485	0.458280	0.459582	0.459487	0.460291	0.461296
	10	0	0.439301	0.439300	0.438198	0.439389	0.439301	0.440037	0.440955
	0	10	0.305534	0.305533	0.304999	0.305576	0.305533	0.305889	0.306333
	10	10	0.296476	0.296475	0.295972	0.296515	0.296475	0.296809	0.297228
50	0	0	0.453405	0.453405	0.453212	0.453421	0.453405	0.453534	0.453695
	10	0	0.433739	0.433739	0.433563	0.433753	0.433739	0.433857	0.434004
	0	10	0.302833	0.302833	0.302746	0.302839	0.302833	0.302890	0.302962
	10	10	0.293931	0.293931	0.293850	0.293938	0.293931	0.293985	0.294053

Table 6 Effects of elastic foundations parameters and side-to-thickness ratio on the dimensionless stress σ_5 of rectangular plates ($b = 2a$)

a/h	κ_1	κ_2	MFPT	HPT	SFPT			
					$k = \frac{5}{6}$	$k = \frac{\pi^2}{12}$	$k = \frac{3}{4}$	$k = \frac{2}{3}$
5	0	0	1.995048	1.988545	1.596038	1.616957	1.771832	1.991167
	10	0	1.904300	1.898096	1.523396	1.543168	1.689397	1.896019
	0	10	1.257244	1.252322	1.004882	1.017008	1.106019	1.229792
	10	10	1.220729	1.215835	0.975592	0.987315	1.073326	1.192822
10	0	0	4.014607	4.011310	3.211685	3.254027	3.567702	4.012491
	10	0	3.859613	3.856445	3.087685	3.128292	3.429034	3.855241
	0	10	2.684688	2.682387	2.147637	2.175354	2.380215	2.669284
	10	10	2.614493	2.612239	2.091471	2.118433	2.317685	2.598768
20	0	0	8.042079	8.040426	6.433664	6.518618	7.148088	8.040998
	10	0	7.745646	7.744053	6.196516	6.278288	6.884142	7.743428
	0	10	5.462823	5.461688	4.370244	4.427643	4.852689	5.454871
	10	10	5.324408	5.323299	4.259512	4.315439	4.729583	5.316279
50	0	0	20.114319	20.113662	16.091455	16.304037	17.879226	20.113884
	10	0	19.382778	19.382144	15.506222	15.711052	17.228786	19.381887
	0	10	13.724089	13.723640	10.979271	11.124191	12.197923	13.720879
	10	10	13.379547	13.379110	10.703637	10.844912	11.891635	13.376264

made of material I, while Figs. 2, 3, 4, 5, 6, 7, 8, 9 are for orthotropic plates made of material II. Note that one needs a value of the shear correction factor k. Two commonly used values of the shear correction factor are 5/6 from Reissner's work [16] and $\pi^2/12$ from Mindlin's work [17]. The results of SFPT based on these two and other shear correction factor values are also included in Tables 5, 6 and in Figs. 2, 3.

Table 5 gives the effects of elastic foundations parameters and side-to-thickness ratio on the dimensionless deflection $\bar{w}$ of simply supported plates ($b = 3a$) with different values for the shear correction factors k of SFPT. It is shown that the deflections of SFPT and MFPT are identical when $k = 5/6$.

Table 6 displays the effects of elastic foundations parameters and side-to-thickness ratio a/h on the dimensionless transverse shear stress σ_5. The stress increases as the side-to-thickness ratio a/h increases and the elastic foundation parameters decrease. The MFPT gives very accurate transverse shear stress when compared with HPT. In addition, the transverse shear stress of SFPT may be compared well with MFPT and HPT when $k = 2/3$ only.

Figures 2, 3, 4 depict the distributions of transverse shear stresses σ_4 and σ_5 through the thickness of the plates resting on elastic foundations. The shear stresses σ_4 and σ_5 are parabolic through the thickness with

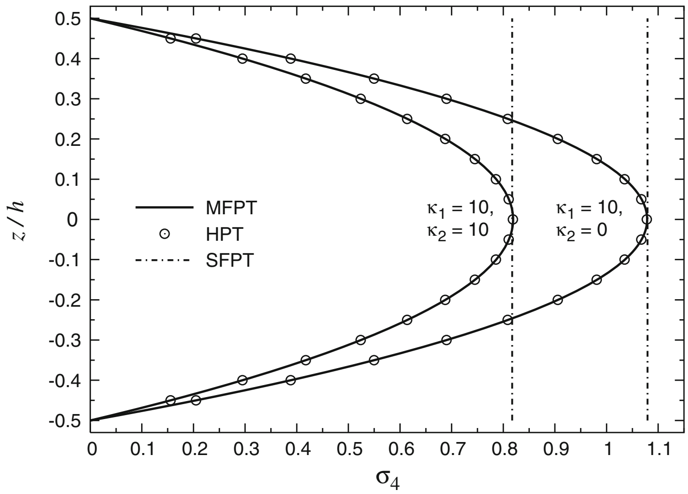

Fig. 2 Dimensionless shear stress σ_4 through the thickness of rectangular plates on elastic foundations ($k = 2/3$)

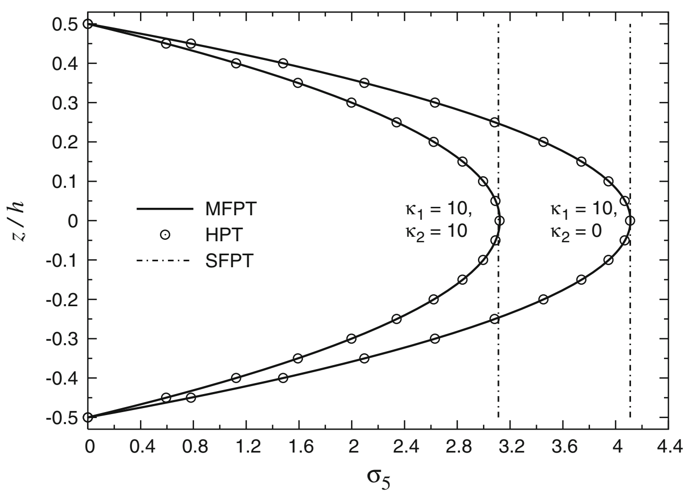

Fig. 3 Dimensionless shear stress σ_5 through the thickness of rectangular plates on elastic foundations ($k = 2/3$)

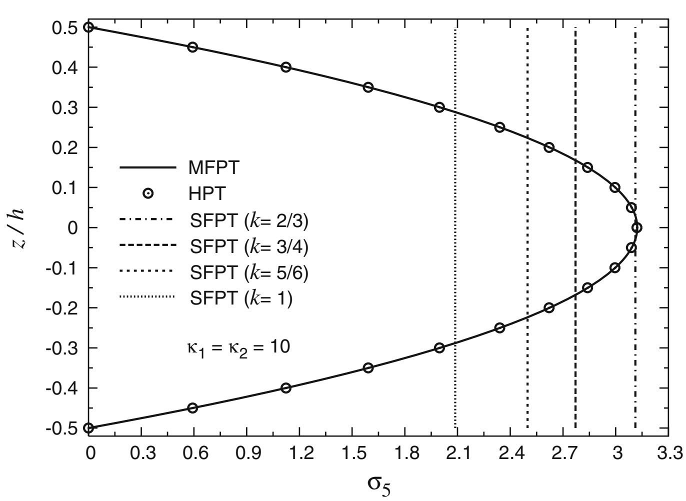

Fig. 4 Dimensionless shear stress σ_5 through the thickness of rectangular plates on elastic foundations

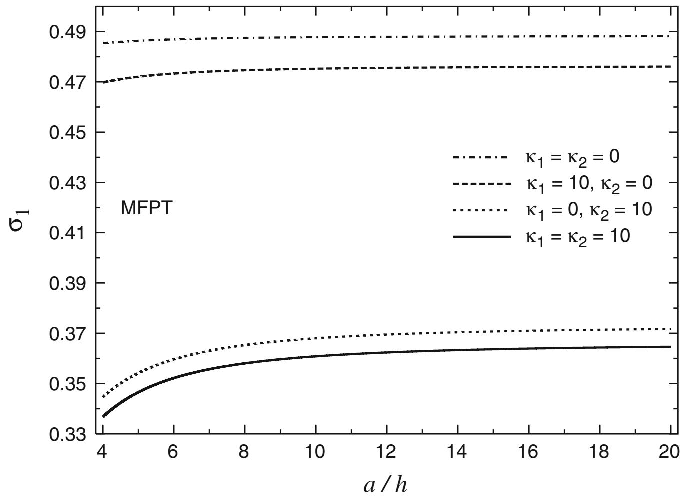

Fig. 5 Dimensionless in-plane normal stress σ_1 versus the side-to-thickness ratio a/h for rectangular plates on elastic foundations

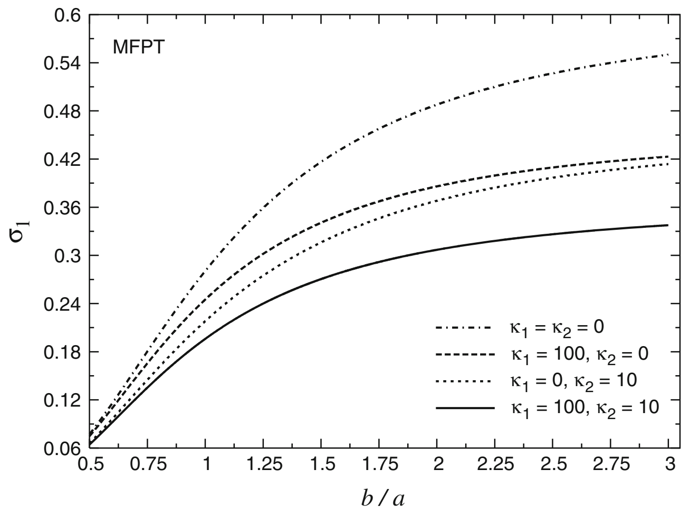

Fig. 6 Dimensionless in-plane normal stress σ_1 versus the aspect ratio b/a of plates on elastic foundations

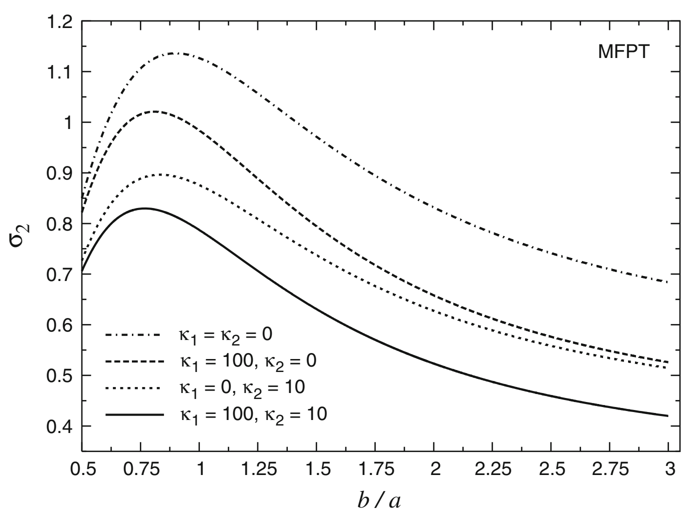

Fig. 7 Dimensionless in-plane normal stress σ_2 versus the aspect ratio b/a of plates on elastic foundations

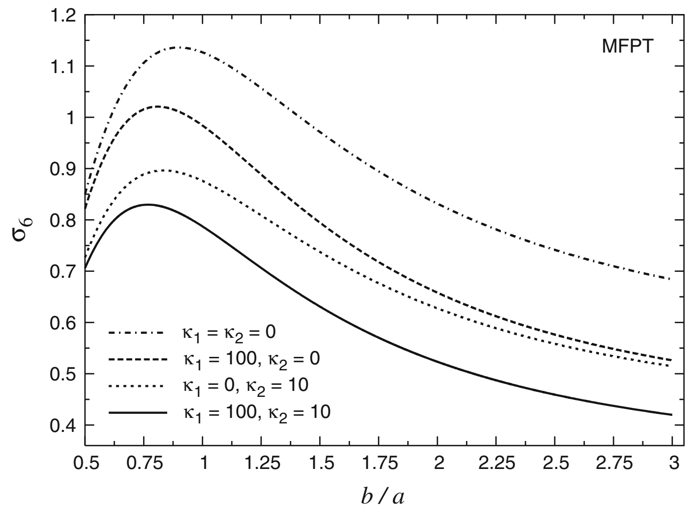

Fig. 8 Dimensionless in-plane tangential stress σ_6 versus the aspect ratio b/a of plates on elastic foundations

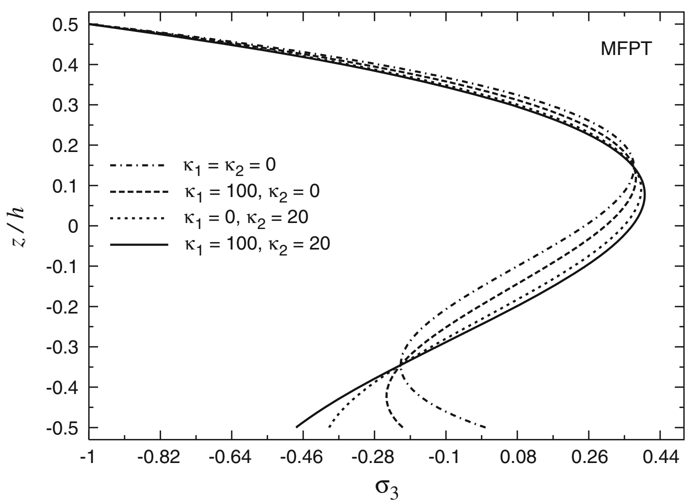

Fig. 9 Dimensionless out-of-plane normal stress σ_3 through the thickness of rectangular plates on elastic foundations

respect to the MFPT and HPT, but the SFPT result, as expected, is constant through the thickness of the plate. The shear correction factor value $k = 2/3$ of SFPT gives good agreement with the results of MFPT and HPT only at the center of the plate. From this point of view, the deflection obtained by SFPT (with $k = 5/6$) and MFPT is extremely close to those obtained by HPT. However, the SFPT fails to represent the correct behavior of the transverse shear stresses through the plate thickness.

Figure 5 contains the plot of σ_1 versus the side-to-thickness ratio a/h of the rectangular plate ($b/a = 2$) for different values of foundation stiffnesses κ_1 and κ_2. It slightly increases to reach a constant value with the increase of a/h ratio. The in-plan normal stresses σ_1, σ_2 and in-plane tangential stress σ_6 versus the aspect ratio b/a are exhibited graphically in Figs. 6, 7, 8, respectively. It is shown that σ_1 increases rapidly as b/a increases, whereas the stresses σ_2 and σ_6 are no longer increasing in b/a.

Finally, the distribution of the transverse normal stress σ_3 through the thickness is illustrated in Fig. 9. The sensitivity of this stress through the thickness of the orthotropic plate is clearly shown.

5 Conclusions

The static response of orthotropic rectangular plates resting on elastic foundations is described and discussed using SFPT and MFPT. The plate is subjected to simply supported boundary conditions, and the interaction between the plate and the foundations is included in the formulations. The results of our calculations for different parameters a/h, b/a, κ_1, and κ_2 are investigated. Non-dimensional stresses and deflection are computed and compared with those given in the literature and by using the higher-order shear deformation plate theory (HPT). The MFPT includes the effect of transverse normal stress which is ignored by SFPT. The SFPT needs a shear correction factor to give reliable results. It fails to represent the transverse shear stresses through the plate thickness. The present MFPT dose not need a shear correction factor as SFPT and gives, with little effort, accurate results that agree well with results of the three-dimensional elasticity theory or of the higher-order plate theory that needs great effort.

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