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Reciprocal and variational principles in micropolar thermoelasticity of type II

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Abstract In the present paper, in the context of thermoelasticity of type II (or thermoelasticity without energy dissipation), we establish reciprocal and variational principles of convolutional type for inhomogeneous and anisotropic micropolar thermoelastic materials with a center of symmetry. The results obtained in this work tend to generalize other variational principles (previously proved by the authors) not completely characterizing the initial-boundary value problem in concern.

1 Introduction

Eringen [1–3] and Eringen and Kafadar [4] established the micropolar elasticity theory in order to treat cases for which the classical theory of elasticity did not represent an adequate modeling tool. An extension of such a model that considers also thermal processes is due to Chandrasekharaiah [5], Ciarletta [6], Ciarletta and Ieşan [7] and Passarella [8]. In this context, various theories are available in the literature to describe heat propagating as waves of finite speed, thus avoiding the Fourier paradox (see Caviglia et al. [9], Quintanilla and Racke [10], Vadasz [11], Vadasz et al. [12]). In particular, a fundamental step in the evolution of heat propagation theory is due to Green and Naghdi [13] who presented a thermomechanical theory for deformable media using a general entropy balance postulated in 1977 (Green and Naghdi [14]); such a theory is illustrated in the context of heat flow in a rigid solid, with particular reference to the propagation of thermal waves at finite speed. Green and Naghdi [15–21] introduced the thermoelasticity of type I, II and III based on an entropy balance law, instead of the usual entropy inequality. Said T the temperature, they defined a thermal displacement τ, such that $\dot{\tau} = T$ (where the superposed dot stands for time derivative), whose presence in some sense is linked to a thermal memory effect and enhances heat propagation as thermal displacement waves. While the linear field equations characterizing the thermoelasticity of type I agree with the equations of classical theory, the thermoelasticity of type II, also said without energy dissipation, satisfies the conservation of energy (and represents a limiting case of the thermoelasticity of type III).

Starting from the results of Green and Naghdi [13,16], a theory for micropolar thermoelasticity without energy dissipation was presented by Ciarletta [22]. Moreover, he established a solution of Galerkin type of the field equations for homogeneous and isotropic bodies using this solution to determine the effect of a concentrated heat source in an unbounded domain. Finally, he investigated the continuous dependence of the solution with respect to body loads and initial data. Again, in the context of thermoelasticity without energy dissipation

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for micropolar materials, Passarella and Zampoli [23] stated two variational characterizations of the initial-boundary value problem of Hamilton and Biot types and derived a reciprocity principle of Betti-Rayleigh type.

The present paper is organized as follows: in Sect. 2, we recall the linear theory of thermoelasticity without energy dissipation for materials characterized by a center of symmetry, as shown by Ciarletta [22]. Further, we reformulate the mixed problem in an alternative way, incorporating the initial conditions explicitly into the field of basic equations, following Carlson [24], Chiriţă and Ciarletta [25], Gurtin [26], Lebon [27]; in Sect. 3, we derive a reciprocity relation of Graffi type for the problem in concern avoiding the use of Laplace transform; in Sect. 4, we state a variational characterization of the initial-boundary value problem.

The obtained results tend to generalize the work of Passarella and Zampoli [23], dealing in particular with inhomogeneous and anisotropic micropolar thermoelastic materials without energy dissipation. In fact, the variational principles proved in Passarella and Zampoli [23] do not completely characterize the initial-boundary value problem, since they fail to take into account the initial velocity distribution and presuppose the knowledge of the displacements at a later time.

2 Statement of the problem

We refer the motion of the considered medium to a fixed system of rectangular Cartesian axes Ox_k $(k = 1, 2, 3)$, employing the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $\{1, 2, 3\}$, summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding material Cartesian coordinate. A superposed dot denotes time differentiation. We deal with a body that, at the time t_0, occupies a regular region B of the Euclidean three-dimensional space, bounded by the surface ∂B. In what follows, we will consider functions of space and time having as their domain of definition the Cartesian product $\bar{B} \times [0, \infty)$, where $\bar{B}$ is the closure of B. Bold letters stand for tensors of an order $p \geq 1$ and, if v has the order p, we shall write $v_{ij\ldots k}$ (p subscripts) for the components of v in the Cartesian coordinate frame. We also recall that a function f is said of class $C^{M,N}$ on $\bar{B} \times [0, \infty)$ (with M and N non-negative integers) if f is continuous on $\bar{B} \times [0, \infty)$ and the function $\partial^m / \left(\partial x_i \partial x_j \ldots \partial x_s\right) \left(\partial^n f / \partial t^n\right)$ (for $m \in \{0, 1, 2, \ldots, M\}$, $n \in \{0, 1, 2, \ldots, N\}$, $m + n \leq \max\{M, N\}$) exists and is continuous on $\bar{B} \times [0, \infty)$. In order to describe the model in concern, we introduce, as done by Ciarletta [22] and Green and Naghdi [13,14,16], the displacement vector field u_i, the microrotation vector field φ_i and the thermal displacement τ. Moreover, $\dot{\tau} = T$ represents the temperature variation from the uniform reference temperature T_0. We remark once again that the thermal displacement introduces a thermal memory and enhances heat propagation as thermal displacement waves at finite speed (see Green and Naghdi [16]).

In the context of the linear theory for thermoelasticity without energy dissipation established by Ciarletta [22], the behavior of homogeneous micropolar materials with a center of symmetry is governed by the following differential equations:

$$\begin{aligned} & t_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\ & m_{ji,j} + \varepsilon_{ijk} t_{jk} + \rho_0 g_i = I_{ij} \ddot{\varphi}_j, \qquad \text{in } B \times (0, \infty) \\ & T_0 \rho_0 \dot{\eta} = T_0 \Phi_{i,i} + \rho_0 S, \end{aligned} \tag{1}$$

the constitutive equations

$$\begin{aligned} t_{ij} &= A_{ijrs} e_{rs} + B_{ijrs} \kappa_{rs} - D_{ij} \dot{\tau}, \\ m_{ij} &= B_{rsij} e_{rs} + C_{ijrs} \kappa_{rs} - E_{ij} \dot{\tau}, \qquad \text{in } \bar{B} \times [0, \infty) \\ \rho_0 \eta &= D_{ij} e_{ij} + E_{ij} \kappa_{ij} + a \dot{\tau}, \\ \Phi_i &= K_{ij} \beta_j, \end{aligned} \tag{2}$$

and the geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \qquad \kappa_{ij} = \varphi_{j,i}, \qquad \beta_j = \tau_{,j} \qquad \text{in } \bar{B} \times [0, \infty), \tag{3}$$

where ρ_0 is the (strictly positive) density in the reference configuration, t_{ji} is the stress tensor, f_i is the body force density, m_{ji} is the couple stress tensor, g_i is the body couple density, I_{ij} is the (symmetric) micro-inertia tensor $(I_{ij} = I_{ji})$, η is the entropy per unit mass and unit time, Φ_i is the entropy flux vector, S is the external rate of heat supply per unit mass, and ε_{ijk} is the alternating symbol. The constitutive coefficients are characterized by the following symmetries:

$$A_{ijrs} = A_{rsij}, \qquad C_{ijrs} = C_{rsij}, \qquad K_{ij} = K_{ji}. \tag{4}$$

Throughout this paper, we assume that

(i) T_0 is a strictly positive constant;
(ii) $A_{ijrs},\ B_{ijrs},\ C_{ijrs},\ D_{ij},\ E_{ij}$ and K_{ij} are smooth on $\bar{B}$, while a is continuous on $\bar{B}$;
(iii) $A_{ijrs},\ C_{ijrs}$ and K_{ij} satisfy the relation (4).

As already done by Passarella and Zampoli [23], one could easily prove that the field equations, expressed in terms of functions $u_i,\ \varphi_i$ and T for homogeneous and isotropic materials, are

$$\begin{aligned}
&(\mu+\kappa)\Delta u_i + (\lambda+\mu)u_{j,ji} + \kappa\varepsilon_{irs}\varphi_{s,r} - mT_{,i} + \rho_0 f_i = \rho_0 \ddot{u}_i,\\
&\gamma\Delta\varphi_i + (\alpha+\beta)\varphi_{j,ji} + \kappa\varepsilon_{irs}u_{s,r} - 2\kappa\varphi_i + \rho_0 g_i = I\ddot{\varphi}_i,\\
&k\Delta T - aT_0\ddot{T} - mT_0\ddot{u}_{j,j} = -\rho_0\dot{S},
\end{aligned}$$

and that this system is fully hyperbolic under the following hypotheses:

$$\begin{aligned}
\rho_0 > 0,\quad I > 0,\quad a > 0,\quad k > 0,\quad 3\lambda + 2\mu + \kappa > 0,\quad 2\mu + \kappa > 0,\\
\kappa > 0,\quad 3\alpha + \beta + \gamma > 0,\quad \beta + \gamma > 0,\quad \gamma - \beta > 0.
\end{aligned}$$

(The coefficients $\lambda,\ \mu,\ \kappa,\ \alpha,\ \beta,\ \gamma,\ a$ and k are constant and I is a coefficient of inertia).

Denoting

$$t_i = t_{ji}n_j, \qquad m_i = m_{ji}n_j, \qquad \Phi = \Phi_j n_j, \tag{5}$$

we can adjoin to the field equations (1)–(3) the following initial and boundary conditions:

$$\begin{aligned}
&u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \qquad \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}), \qquad \varphi_i(\mathbf{x},0) = \varphi_i^0(\mathbf{x}),\\
&\dot{\varphi}_i(\mathbf{x},0) = \dot{\varphi}_i^0(\mathbf{x}), \qquad \tau(\mathbf{x},0) = 0, \qquad \dot{\tau}(\mathbf{x},0) = T^0, \qquad \mathbf{x}\in\bar{B},
\end{aligned} \tag{6}$$

$$u_i = u_i^* \quad \text{on } \bar{\Sigma}_1\times[0,\infty), \qquad t_i = t_i^* \quad \text{on } \Sigma_2\times[0,\infty), \tag{7.1}$$

$$\varphi_i = \varphi_i^* \quad \text{on } \bar{\Sigma}_3\times[0,\infty), \qquad m_i = m_i^* \quad \text{on } \Sigma_4\times[0,\infty), \tag{7.2}$$

$$\tau = \tau^* \quad \text{on } \bar{\Sigma}_5\times[0,\infty), \qquad \Phi = \Phi^* \quad \text{on } \Sigma_6\times[0,\infty), \tag{7.3}$$

where $\{\Sigma_i, \Sigma_{i+1}\}$ $(i = 1, 3, 5)$ are a subset of ∂B such that $\bar{\Sigma}_i \cup \Sigma_{i+1} = \partial B$ and $\Sigma_i \cap \Sigma_{i+1} = \emptyset$. We can remark that the choice of null initial condition for the thermal displacement τ implies no loss of generality: the performed analysis works also in the case of a non-homogeneous initial condition assigned for τ.

All right-hand terms in Eqs. (6) and (7), along with $f_i,\ g_i$ and S, are the given data of the considered mixed problem. These (prescribed) functions are such that

(i) $f_i,\ g_i,\ S \in C^{0,0}$ on $\bar{B}\times[0,\infty)$;
(ii) $u_i^0,\ v_i^0,\ \varphi_i^0,\ \dot{\varphi}_i^0,\ T^0 \in C^{0,0}$ on $\bar{B}$;
(iii) $u_i^* \in C^{0,0}$ on $\bar{\Sigma}_1\times[0,\infty)$, $\varphi_i^* \in C^{0,0}$ on $\bar{\Sigma}_3\times[0,\infty)$, $\tau^* \in C^{0,0}$ on $\bar{\Sigma}_5\times[0,\infty)$;
(iv) $t_i^*,\ m_i^*,\ \Phi^*$ are piecewise regular and continuous in time respectively on $\Sigma_2\times[0,\infty)$, $\Sigma_4\times[0,\infty)$, $\Sigma_6\times[0,\infty)$.

Moreover, let us define an ordered array of functions $\pi = (u_i, \varphi_i, \tau, e_{ij}, \kappa_{ij}, \beta_i, t_{ij}, m_{ij}, \Phi_i, \eta)$ as an admissible process on $\bar{B}\times[0,\infty)$ with the following properties:

(i) $u_i,\ \varphi_i,\ \tau \in C^{1,2}$ on $\bar{B}\times[0,\infty)$;
(ii) $e_{ij},\ \kappa_{ij},\ \beta_i \in C^{0,0}$ on $\bar{B}\times[0,\infty)$;
(iii) $t_{ij},\ m_{ij},\ \Phi_i \in C^{1,0}$ on $\bar{B}\times[0,\infty)$;
(iv) $\eta \in C^{0,1}$ on $\bar{B}\times[0,\infty)$.

The set $\mathcal{V}$ of all *admissible processes* on $\bar{B} \times [0, \infty)$ as above defined can be organized as a vector space such that scalar multiplication and addition are understood as follows:

$$\lambda\pi = (\lambda u_i, \lambda\varphi_i, \lambda\tau, \lambda e_{ij}, \lambda\kappa_{ij}, \lambda\beta_i, \lambda t_{ij}, \lambda m_{ij}, \lambda\Phi_i, \lambda\eta) \in \mathcal{V} \quad \forall\pi \in \mathcal{V}, \forall\lambda \in \mathbb{R},$$
$$\pi + \pi' = (u_i + u'_i, \varphi_i + \varphi'_i, \tau + \tau', e_{ij} + e'_{ij}, \kappa_{ij} + \kappa'_{ij},$$
$$\beta_i + \beta'_i, t_{ij} + t'_{ij}, m_{ij} + m'_{ij}, \Phi_i + \Phi'_i, \eta + \eta') \in \mathcal{V} \qquad \forall\pi, \pi' \in \mathcal{V}.$$

Moreover, we say that $\pi = (u_i, \varphi_i, \tau, e_{ij}, \kappa_{ij}, \beta_i, t_{ij}, m_{ij}, \Phi_i, \eta)$ is a *thermoelastic process* corresponding to the supply terms $(f_i,\ g_i,\ S)$ if π is an admissible process that satisfies the fundamental system of field equations (1)–(3) on $B \times [0, \infty)$. Then, if a thermoelastic process π satisfies also the initial conditions (6) and the boundary conditions (7), we identify it as a *solution of the mixed initial-boundary value problem* (1)–(3), (6), (7). In this context, we give an alternative formulation of the problem (1)–(3) in which the initial conditions (6) are incorporated into the field equations.

To this end, for any continuous functions a_1 and a_2 on $\bar{B} \times [0, \infty)$, we introduce the time convolution product as follows:

$$a_1 * a_2(\mathbf{x}, t) = \int_0^t a_1(\mathbf{x}, s) a_2(\mathbf{x}, t - s) \mathrm{d}s$$

satisfying the following properties for any $a_1, a_2, a_3 \in C^{0,0}$ on $\bar{B} \times [0, \infty)$:

$$\begin{aligned} a_1 * a_2 &= a_2 * a_3, \\ a_1 * (a_2 * a_3) &= (a_1 * a_2) * a_3 = a_1 * a_2 * a_3, \\ a_1 * (a_2 + a_3) &= a_1 * a_2 + a_1 * a_3, \\ a_1 * a_2 &= 0 \quad \Rightarrow \quad a_1 = 0 \ \text{ or } \ a_2 = 0. \end{aligned} \tag{8}$$

Now, we have to define

$$\begin{aligned} \mathcal{F}_i &= \rho_0 \left(\xi * f_i + t v_i^0 + u_i^0\right), \\ \mathcal{L}_i &= \rho_0 \xi * g_i + I_{ij} \left(t\dot{\varphi}_i^0 + \varphi_i^0\right), \\ \mathcal{R} &= \xi * (\rho_0 / T_0)\, S + \rho_0 t \eta^0, \end{aligned} \tag{9}$$

where ξ is such that $\xi(\mathbf{x}, t) = t$ on $\bar{B} \times [0, \infty)$ and η^0 is defined by

$$\rho_0\eta^0 = D_{ij} \left(u_{j,i}^0 + \varepsilon_{jik}\varphi_k^0\right) + E_{ij}\varphi_{j,i}^0 + aT^0. \tag{10}$$

On the basis of the procedure shown by Carlson [24], Chiriţă and Ciarletta [25], Gurtin [26], and Lebon [27], the following theorem can be proved.

Theorem 1 *Let* $\pi = (u_i, \varphi_i, \tau, e_{ij}, \kappa_{ij}, \beta_i, t_{ij}, m_{ij}, \Phi_i, \eta)$ *be an admissible process on* $\bar{B} \times [0, \infty)$. *Then* π *satisfies the fundamental system of field equations* (1), (2) and the initial conditions (6) if and only if

$$\xi * t_{ji,j} + \mathcal{F}_i = \rho_0 u_i, \tag{11.1}$$
$$\xi * m_{ji,j} + \xi * \varepsilon_{ijk} t_{jk} + \mathcal{L}_i = I_{ij}\varphi_j, \tag{11.2}$$
$$l * \rho_0\eta = \xi * \Phi_{i,i} + \mathcal{R}, \tag{11.3}$$

$$\xi * t_{ij} = \xi * A_{ijrs} e_{rs} + \xi * B_{ijrs}\kappa_{rs} - l * D_{ij}\tau, \tag{12.1}$$
$$\xi * m_{ij} = \xi * B_{rsij} e_{rs} + \xi * C_{ijrs}\kappa_{rs} - l * E_{ij}\tau, \tag{12.2}$$
$$l * \rho_0\eta = l * D_{ij} e_{ij} + l * E_{ij}\kappa_{ij} + a\tau, \tag{12.3}$$
$$\xi * \Phi_i = \xi * K_{ij}\beta_j, \tag{12.4}$$

where function l is such that $l(\mathbf{x}, t) = 1$ on $\bar{B} \times [0, \infty)$.

Proof From the definition of the time convolution product, it is easy to prove that

$$\begin{aligned} \xi * \ddot{u}_i &= u_i - t v_i^0 - u_i^0, & \xi * \ddot{\varphi}_i &= \varphi_i - t\dot{\varphi}_i^0 - \varphi_i^0, \\ l * \rho_0 \dot{\eta} &= \rho_0 \eta - \rho_0 \eta^0, & \xi * \rho_0 \dot{\eta} &= l * \rho_0 \eta - t \rho_0 \eta^0. \end{aligned} \tag{13}$$

Moreover, from the homogeneous initial condition for the thermal displacement τ, we have

$$l * \dot{\tau} = l * \tau. \tag{14}$$

Firstly, we assume that the field equations (1), (2) and the initial conditions (6) are satisfied. Considering the convolution product of Eqs. (1) and (2) by ξ and using Eqs. (13), (14) and the definitions (9), we arrive at Eqs. (11) and (12). On the other hand, starting from Eqs. (11) and (12), we use the definitions (9) and, through Eqs. (8), (13), (14), we obtain Eqs. (1), (2) and (6). The theorem is then proved. □

We conclude this Section underlining that a thermoelastic process π is a solution of the mixed initial-boundary value problem in concern if it satisfies Eqs. (11), (12), the geometrical equations (3) and the boundary conditions (7).

3 Reciprocity relation

This Section is devoted to a reciprocity relation which involves two processes at different instants; the proof of the reciprocal theorem in concern will avoid the use of the Laplace transform. Now, we consider the investigated body subjected to two different sets of external data,

$$\Gamma^{(\alpha)} = \left(f_i^{(\alpha)}, g_i^{(\alpha)}, S^{(\alpha)}, u_i^{0(\alpha)}, v_i^{0(\alpha)}, \varphi_i^{0(\alpha)}, \dot{\varphi}_i^{0(\alpha)}, T^{0(\alpha)}, u_i^{*(\alpha)}, t_i^{*(\alpha)}, \varphi_i^{*(\alpha)}, m_i^{*(\alpha)}, \tau^{*(\alpha)}, \Phi^{*(\alpha)} \right),$$

and denote the corresponding solutions of the mixed initial boundary problem as

$$\pi^{(\alpha)} = (u_i^{(\alpha)}, \varphi_i^{(\alpha)}, \tau^{(\alpha)}, e_{ij}^{(\alpha)}, \kappa_{ij}^{(\alpha)}, \beta_i^{(\alpha)}, t_{ij}^{(\alpha)}, m_{ij}^{(\alpha)}, \Phi_i^{(\alpha)}, \eta^{(\alpha)}) \qquad \text{with } \alpha = 1, 2.$$

Defining $\mathcal{F}_i^{(\alpha)}$, $\mathcal{L}_i^{(\alpha)}$, $\mathcal{R}^{(\alpha)}$, $t_i^{(\alpha)}$, $m_i^{(\alpha)}$, $\Phi^{(\alpha)}$ and $\eta^{0(\alpha)}$ by means of Eqs. (5), (9) and (10), we can prove the following theorem.

Theorem 2 *Let $\pi^{(\alpha)}$ be the solutions corresponding to different sets of data $\Gamma^{(\alpha)}$ ($\alpha = 1, 2$), assuming that the symmetry relations (4) hold and that I_{ij} is a symmetric tensor. Then, the following relation is valid*:

$$\mathcal{I}_{\alpha\beta} = \mathcal{I}_{\beta\alpha} \qquad \text{for } \alpha, \beta = 1, 2 \tag{15}$$

where

$$\begin{aligned} \mathcal{I}_{\alpha\beta} = & \int_B \left(\mathcal{F}_i^{(\beta)} * u_i^{(\alpha)} + \mathcal{L}_i^{(\beta)} * \varphi_i^{(\alpha)} - \mathcal{R}^{(\beta)} * \tau^{(\alpha)} \right) \mathrm{d}V \\ & + \int_{\Sigma_1} \xi * t_i^{(\beta)} * u_i^{*(\alpha)} \mathrm{d}a + \int_{\Sigma_2} \xi * t_i^{*(\beta)} * u_i^{(\alpha)} \mathrm{d}a + \int_{\Sigma_3} \xi * m_i^{(\beta)} * \varphi_i^{*(\alpha)} \mathrm{d}a \\ & + \int_{\Sigma_4} \xi * m_i^{(\beta)} * \varphi_i^{*(\alpha)} \mathrm{d}a - \int_{\Sigma_5} \xi * \Phi^{(\beta)} * \tau^{*(\alpha)} \mathrm{d}a - \int_{\Sigma_6} \xi * \Phi^{*(\beta)} * \tau^{(\alpha)} \mathrm{d}a. \end{aligned}$$

Proof First of all, by Eqs. (11) and (12.3), we are able to write

$$\xi * t_{ji,j}^{(\beta)} + \mathcal{F}_i^{(\beta)} = \rho_0 u_i^{(\beta)}, \tag{16.1}$$

$$\xi * m_{ji,j}^{(\beta)} + \xi * \varepsilon_{ijk} t_{jk}^{(\beta)} + \mathcal{L}_i^{(\beta)} = I_{ij} \varphi_j^{(\beta)}, \tag{16.2}$$

$$-\xi * \Phi_{i,i}^{(\beta)} - \mathcal{R}^{(\beta)} = -l * D_{ij} e_{ij}^{(\beta)} - l * E_{ij} \kappa_{ij}^{(\beta)} - a\tau^{(\beta)}. \tag{16.3}$$

Considering the convolution of Eq. (16.1) by $u_i^{(\alpha)}$, Eq. (16.2) by $\varphi_i^{(\alpha)}$ and Eq. (16.3) by $\tau^{(\alpha)}$, we obtain

$$\begin{aligned}
&\left[\xi * t_{ji}^{(\beta)} * u_i^{(\alpha)} + \xi * m_{ji}^{(\beta)} * \varphi_i^{(\alpha)} - \xi * \Phi_j^{(\beta)} * \tau^{(\alpha)}\right]_{,j} + \mathcal{F}_i^{(\beta)} * u_i^{(\alpha)} + \mathcal{L}_i^{(\beta)} * \varphi_i^{(\alpha)} - \mathcal{R}^{(\beta)} * \tau^{(\alpha)} \\
&\quad = \xi * t_{ji}^{(\beta)} * u_{i,j}^{(\alpha)} + \xi * m_{ji}^{(\beta)} * \varphi_{i,j}^{(\alpha)} - \xi * \varepsilon_{ijk} t_{jk}^{(\beta)} * \varphi_i^{(\alpha)} - \xi * \Phi_j^{(\beta)} * \tau_{,j}^{(\alpha)} \\
&\qquad - l * D_{ij} e_{ij}^{(\beta)} * \tau^{(\alpha)} - l * E_{ij} \kappa_{ij}^{(\beta)} * \tau^{(\alpha)} + \rho_0 u_i^{(\beta)} * u_i^{(\alpha)} + I_{ij} \varphi_j^{(\beta)} * \varphi_i^{(\alpha)} - a\tau^{(\beta)} * \tau^{(\alpha)}.
\end{aligned} \tag{17}$$

Now, we introduce the following functions:

$$\begin{aligned}
\mathcal{J}_{\alpha\beta} = {} & \mathcal{F}_i^{(\beta)} * u_i^{(\alpha)} + \mathcal{L}_i^{(\beta)} * \varphi_i^{(\alpha)} - \mathcal{R}^{(\beta)} * \tau^{(\alpha)} \\
& + \left[\xi * t_{ji}^{(\beta)} * u_i^{(\alpha)} + \xi * m_{ji}^{(\beta)} * \varphi_i^{(\alpha)} - \xi * \Phi_j^{(\beta)} * \tau^{(\alpha)}\right]_{,j}
\end{aligned} \tag{18}$$

and

$$\mathcal{N}_{\alpha\beta} = \xi * t_{ji}^{(\beta)} * e_{ji}^{(\alpha)} + \xi * m_{ji}^{(\beta)} * \kappa_{ji}^{(\alpha)} - \xi * \Phi_j^{(\beta)} * \beta_j^{(\alpha)} - l * D_{ij} e_{ij}^{(\beta)} * \tau^{(\alpha)} - l * E_{ij} \kappa_{ij}^{(\beta)} * \tau^{(\alpha)}. \tag{19}$$

Therefore, using the geometrical equations (3) and Eqs. (17), (18), (19), we get

$$\mathcal{J}_{\alpha\beta} = \mathcal{N}_{\alpha\beta} + \rho_0 u_i^{(\beta)} * u_i^{(\alpha)} + I_{ij} \varphi_j^{(\beta)} * \varphi_i^{(\alpha)} - a\tau^{(\beta)} * \tau^{(\alpha)}. \tag{20}$$

In view of Eqs. (12.1, 2, 4) and (4), we have

$$\begin{aligned}
\mathcal{N}_{\alpha\beta} = {} & \xi * \left[A_{jirs} e_{rs}^{(\beta)} * e_{ji}^{(\alpha)} + B_{jirs}\left(\kappa_{rs}^{(\beta)} * e_{ji}^{(\alpha)} + e_{ji}^{(\beta)} * \kappa_{rs}^{(\alpha)}\right) + C_{jirs} \kappa_{rs}^{(\beta)} * \kappa_{ji}^{(\alpha)}\right] \\
& - l * \left[D_{ji}\left(\tau^{(\beta)} * e_{ji}^{(\alpha)} + e_{ji}^{(\beta)} * \tau^{(\alpha)}\right) + E_{ji}\left(\tau^{(\beta)} * \kappa_{ji}^{(\alpha)} + \kappa_{ji}^{(\beta)} * \tau^{(\alpha)}\right)\right] \\
& - \xi * K_{ji} \tau_{,i}^{(\beta)} * \tau_{,j}^{(\alpha)}
\end{aligned}$$

and, obviously, it is

$$\mathcal{N}_{\alpha\beta} = \mathcal{N}_{\beta\alpha}. \tag{21}$$

Taking into account that I_{ij} is a symmetric tensor, Eqs. (20) and (21) imply that

$$\mathcal{J}_{\alpha\beta} = \mathcal{J}_{\beta\alpha}. \tag{22}$$

On the other hand, if we integrate Eq. (18) on B and use the divergence theorem and Eqs. (7), we obtain

$$\mathcal{I}_{\alpha\beta} = \int_B \mathcal{J}_{\alpha\beta} \mathrm{d}V. \tag{23}$$

Hence, Eqs. (22) and (23) imply that Eq. (15) holds. The theorem is therefore proved. □

4 Variational principle

In this Section, we formulate a variational principle for the considered model of micropolar thermoelasticity without energy dissipation. To this aim, assumed that Eq. (4) holds and that coefficient $a \neq 0$, we define for each $t \in [0, \infty)$ the functional Λ_t on the space of all admissible processes $\mathcal{V}$ as follows:

$$\begin{aligned}
\Lambda_t(\pi) = \int_B \Big\{ & \left(-\xi * t_{ji,j} - \mathcal{F}_i + \rho_0 u_i\right) * u_i + \left(-\xi * m_{ji,j} - \xi * \varepsilon_{ijk} t_{jk} - \mathcal{L}_i + I_{ij}\varphi_j\right) * \varphi_i \\
& - \left(l * \rho_0 \eta - \xi * \Phi_{i,i} - \mathcal{R}\right) * \tau - \xi * t_{ij} * e_{ij} - \xi * m_{ij} * \kappa_{ij} \\
& + (1/2)\xi * A_{ijrs} e_{rs} * e_{ij} + \xi * B_{ijrs} \kappa_{rs} * e_{ij} + (1/2)\xi * C_{ijrs} \kappa_{rs} * \kappa_{ij} \\
& + (1/2a)\xi * \left(\rho_0 \eta - D_{ij} e_{ij} - E_{ij} \kappa_{ij}\right) * \left(\rho_0 \eta - D_{ij} e_{ij} - E_{ij} \kappa_{ij}\right)
\end{aligned}$$

$$-(1/2)\xi * K_{ij}\beta_i * \beta_j + \xi * \Phi_j * \beta_j - (1/2)\rho_0 u_j * u_j - (1/2) I_{ij}\varphi_i * \varphi_j \,\}\,\mathrm{d}V$$
$$+\int_{\Sigma_1} \xi * t_j * u_j^* \mathrm{d}a + \int_{\Sigma_2} \xi * (t_j - t_j^*) * u_j \mathrm{d}a + \int_{\Sigma_3} \xi * m_j * \varphi_j^* \mathrm{d}a$$
$$+\int_{\Sigma_4} \xi * (m_j - m_j^*) * \varphi_j \mathrm{d}a - \int_{\Sigma_5} \xi * \Phi * \tau^* \mathrm{d}a - \int_{\Sigma_6} \xi * \left(\Phi - \Phi^*\right) * \tau \mathrm{d}a, \tag{24}$$

for every $\pi = (u_i, \varphi_i, \tau, e_{ij}, \kappa_{ij}, \beta_i, t_{ij}, m_{ij}, \Phi_i, \eta) \in \mathcal{V}$.

Theorem 3 *For fixed $t \in [0, \infty)$, the variation $\delta\Lambda_t\{\pi\}$ of functional Λ_t corresponding to an admissible process π is zero over $\mathcal{V}$ (or, in other words, Λ_t has a stationary point) if and only if π represents a solution of the considered mixed initial-boundary value problem* (3), (6), (7), (11) and (12).

Proof At the beginning, let us formally recall that the variation of functional Λ_t is zero over $\mathcal{V}$,

$$\delta\Lambda_t\{\pi\} = 0,$$

if and only if $\delta_{\pi'}\Lambda_t\{\pi\}$ exists and is equal to zero for all $\pi' \in \mathcal{V}$, where

$$\delta_{\pi'}\Lambda_t\{\pi\} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Lambda\left\{\pi + \lambda\pi'\right\}\Big|_{\lambda=0}.$$

Using the definitions of Λ_t and $\delta_{\pi'}\Lambda_t$ and with the aid of the divergence theorem, we obtain the following expression for $\delta_{\pi'}\Lambda\{\pi\}$:

$$\begin{aligned}\delta_{\pi'}\Lambda_t(\pi) = \int_B \Big\{&\left(-\xi * t_{ji,j} - \mathcal{F}_i + \rho_0 u_i\right) * u_i' + \left(-\xi * m_{ji,j} - \xi * \varepsilon_{ijk} t_{jk} - \mathcal{L}_i + I_{ij}\varphi_j\right) * \varphi_i' \\
&- \left(l * \rho_0\eta - \xi * \Phi_{i,i} - \mathcal{R}\right) * \tau' + \left[(1/a)\xi * \left(\rho_0\eta - D_{ij}e_{ij} - E_{ij}\kappa_{ij}\right) - l * \tau\right] * \rho_0\eta' \\
&+ \left[-\xi * t_{ij} + \xi * A_{ijrs}e_{rs} + \xi * B_{ijrs}\kappa_{rs} - (1/a)\xi * D_{ij}\left(\rho_0\eta - D_{ij}e_{ij} - E_{ij}\kappa_{ij}\right)\right] * e_{ij}' \\
&+ \left[-\xi * m_{ij} + \xi * B_{rsij}e_{rs} + \xi * C_{ijrs}\kappa_{rs} - (1/a)\xi * E_{ij}\left(\rho_0\eta - D_{ij}e_{ij} - E_{ij}\kappa_{ij}\right)\right] * \kappa_{ij}' \\
&+ \left(\xi * u_{j,i} + \xi * \varepsilon_{jik}\varphi_k - \xi * e_{ij}\right) * t_{ij}' + \left(\xi * \varphi_{j,i} - \xi * \kappa_{ij}\right) * m_{ij}' \\
&+ \left(\xi * \Phi_j - \xi * K_{ij}\beta_i\right) * \beta_j' + \left(\xi * \tau_{,j} - \xi * \beta_j\right) * \Phi_j'\Big\}\,\mathrm{d}V \\
&+\int_{\Sigma_1} \xi * t_j' * \left(u_j^* - u_j\right)\mathrm{d}a + \int_{\Sigma_2} \xi * (t_j - t_j^*) * u_j' \mathrm{d}a + \int_{\Sigma_3} \xi * m_j' * \left(\varphi_j^* - \varphi_j\right)\mathrm{d}a \\
&+\int_{\Sigma_4} \xi * (m_j - m_j^*) * \varphi_j' \mathrm{d}a - \int_{\Sigma_5} \xi * \Phi' * \left(\tau^* - \tau\right)\mathrm{d}a - \int_{\Sigma_6} \xi * \left(\Phi - \Phi^*\right) * \tau' \mathrm{d}a,\end{aligned} \tag{25}$$

where t_j, m_j and Φ are defined in Eqs. (5).

For the first part of the proof, let us suppose that π is a solution of the considered initial-boundary value problem. Then, for every $\pi' = (u_i', \varphi_i', \tau', e_{ij}', \kappa_{ij}', \beta_i', t_{ij}', m_{ij}', \Phi_i', \eta') \in \mathcal{V}$ and considering Eqs. (11) and (12) together with Eqs. (3) and (7), we obtain that $\delta_{\pi'}\Lambda_t(\pi) = 0$ and that the variation $\delta\Lambda_t\{\pi\}$ is consequently equal to zero.

The second part of the proof starts from the hypothesis that the functional Λ_t admits a stationary point corresponding to an admissible process π, i.e. $\delta\Lambda_t\{\pi\} = 0$. Of course, it is also true that

$$\delta_{\pi'}\Lambda_t\{\pi\} = 0 \qquad \forall\, \pi' \in \mathcal{V}. \tag{26}$$

If we consider $\pi' = (u'_i, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{V}$ with u'_i a class C^∞ function vanishing on $\partial B \times [0, \infty)$, then Eqs. (25) and (26) imply that

$$\int\limits_B \left(-\xi * t_{ji,j} - \mathcal{F}_i + \rho_0 u_i\right) * u'_i \mathrm{d}V = 0 \qquad t \in [0, \infty). \tag{27}$$

As shown by Gurtin [26], Eq. (27) leads to Eq. (11.1).

On the other hand, if we choose $\pi' = (u'_i, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in \mathcal{V}$ with u'_i a class C^∞ function vanishing on $\Sigma_1 \times [0, \infty)$, we obtain

$$\int\limits_{\Sigma_2} \xi * (t_j - t_j^*) * u'_j \mathrm{d}a = 0 \qquad t \in [0, \infty) \tag{28}$$

and consequently we arrive at Eq. (7.2). Then, iterating this procedure for other suitable choices of π', we can conclude that π represents a solution of the mixed initial-boundary value problem (3), (6), (7), (11) and (12). The proof is then complete. □

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