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Transition of boundary layer flow past a stretching sheet due to a step change in applied constant mass flux

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Abstract The problem of transition of boundary layer flow from the initial unsteady flow to the final steady flow past a permeable stretching sheet is considered in this paper. Two cases are considered: (1) Case-I deals with the transition of the unsteady flow due to a sudden application of constant suction/injection at the surface of the stretching sheet, from the initially prevailing steady flow; (2) Case-II deals with unsteady flow transition due to a sudden removal of constant suction/injection at the surface of the stretching sheet, from the initially prevailing steady flow. Numerical results are obtained using the implicit finite difference method of Crank-Nicholson type. The velocity and local skin friction for different values of injection/suction parameter are graphically presented and discussed. It is found that for the same magnitude of applied mass flux in Case-I and mass flux removal in Case-II, the time to reach steady-state flow is different.

1 Introduction

The problem of flow past a stretching sheet has received much attention due to its applications in polymer industry, fibre industry, chemical drying, etc. Research on the boundary layer flow over a stretching sheet was initiated by Crane [1], with an analytic solution for the problem of flow over a sheet, stretching in its own plane with a velocity proportional to the distance along the surface of the sheet. An investigation into the steady boundary layer fluid flow and heat transfer past a permeable stretching sheet has been made by Gupta and Gupta [2]. Magyari and Keller [3] have obtained exact solutions for self-similar boundary layer flows induced by permeable stretching walls. In their study, the wall stretches with power-law velocity and is subjected to a suitable power-law cross flow. Steady boundary layer flow of a power-law fluid over a stretching sheet was studied by Andersson and Kumaran [4] which gives good estimates of skin friction both for shear-thinning and shear-thickening fluids. Analysis relating to unsteady boundary layer flow due to time- dependent stretching rate was carried out by Chen [5]. In this study, convective heat transfer in a thin non-Newtonian liquid film on an unsteady stretching sheet along with viscous dissipation is analyzed. Dandapat et al. [6] carried out an analysis to study the influence of thermocapillarity on the flow and heat transfer in a thin liquid film on a horizontal stretching sheet. Abel et al. [7] analyzed the MHD flow and heat transfer in a laminar liquid film from a horizontal stretching surface. Unsteady laminar boundary layer flow over a continuously stretching permeable surface was investigated by Ishak et al. [8], where the effects of the unsteadiness parameter, suction/injection parameter and Prandtl number on the heat characteristics are examined. Andersson et al. [9]

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studied the momentum and heat transfer in a laminar liquid film on a horizontal stretching sheet. The homotopy analysis method has been used by Kumari and Nath [10] to obtain the approximate analytical solution of unsteady 3-dimensional MHD boundary layer flow and heat transfer due to an impulsively stretched plane surface. Unsteady flow past an impulsively stretching sheet has been studied by Wang et al. [11] and Liao [12]. The reader can refer to Ali and Magyari [13] for a comprehensive literature survey. Thus the causes of unsteadiness in the above studies are either time dependent stretching rate or impulsive stretching.

Recently, Kumaran et al. [14] studied the unsteady transition phenomena of a viscous and electrically conducting fluid past a stretching sheet due to a step change in the applied magnetic field. The present paper deals with another new unsteady transition phenomenon of unsteady boundary layer flow past a permeable stretching sheet due to a step change in the applied transpiration using the finite difference method. In Sect. 2, the Case-I, which deals with the flow transition from the initial steady-state boundary layer flow past a stretching sheet due to a sudden application of constant mass flux of fluid at the stretching sheet is treated. The step change in constant mass flux is from zero to non-zero level. Case-II, when the step change in the constant mass flux is from non-zero to zero level, is considered in Sect. 3. To the best of our knowledge, this problem has not been considered before.

2 Case-I

2.1 Formulation

We consider a two-dimensional incompressible boundary layer flow of viscous fluid with kinematic viscosity ν, past a stretching horizontal sheet issued from a slit at the origin $x' = 0$, $y' = 0$ of a Cartesian coordinate system. The coordinates x' and y' are measured along the surface of the stretching sheet and normal to it, respectively. We denote by u' and v' the velocity components along the x' and y' directions, respectively. Initially, the fluid flow is steady and flows over the stretching sheet with a velocity $u' = \beta x'$, where $\beta (> 0)$ is the constant stretching rate. This initial steady fluid flow past the stretching sheet at time $t' \leq 0$ is modified at time $t' > 0$ by the application of a constant mass flux v_0 at the permeable stretching sheet. This transient flow attains a steady state when $t' \to \infty$.

2.2 Initial steady-state flow ($t' \leq 0$)—the solution of Crane [1]

Assuming the boundary layer approximations, the initial flow is governed by the equations for a steady-state flow past a stretching sheet given by

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{1}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = \nu\frac{\partial^2 u'}{\partial y'^2}, \tag{2}$$

along with boundary conditions

$$t' \leq 0 : u' = \beta x',\ \ v' = 0 \ \text{at} \ y' = 0 \quad \text{for} \ x' \geq 0, \tag{3}$$

$$u' \to 0 \ \text{as} \ y' \to \infty \quad \text{for} \ x' \geq 0. \tag{4}$$

Using the following dimensionless variables,

$$x = x'\sqrt{\frac{\beta}{\nu}}, \quad y = y'\sqrt{\frac{\beta}{\nu}}, \quad t = \beta t', \quad u_1 = \frac{u'}{\sqrt{\nu\beta}}, \quad v_1 = \frac{v'}{\sqrt{\nu\beta}}, \tag{5}$$

Equations (1)–(4) reduce to

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{6}$$

$$u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2}, \tag{7}$$

$$t \leq 0 : u_1 = x, v_1 = 0 \ \text{at} \ y = 0, x \geq 0, \tag{8}$$

$$u_1 \to 0 \ \text{as} \ y \to \infty,\ x \geq 0. \tag{9}$$

Equations (6)–(9) admit a closed form solution (Crane [1]), given by

$$u_1(x, y) = xe^{-y} \quad \text{and} \quad v_1(x, y) = -1 + e^{-y}. \tag{10}$$

2.3 Unsteady-state flow ($t' > 0$)

Assuming that the initial steady-state flow ($t' = 0$) is given by Eq. (10), a sudden application of a constant mass flux, $v_0(\neq 0)$, of fluid at the sheet for $t' > 0$, changes the flow field into a transient one and is governed by the following boundary layer equations in dimensionless form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \; (t > 0), \tag{12}$$

along with the boundary conditions, see Gupta and Gupta [2]

$$\begin{aligned} t > 0 : u = x, v = -s + \frac{1}{s} \text{ at } y = 0 \quad \text{for } x \geq 0, \\ u \to 0 \text{ as } y \to \infty \quad \text{for } x \geq 0, \end{aligned} \tag{13}$$

where s is the positive root of the equation $-s + \frac{1}{s} = \frac{v_0}{\sqrt{\nu\beta}}$. Note that $s = 1$ corresponds to $v_0 = 0$, $s > 1$ corresponds to $v_0 < 0$ (fluid suction at the sheet) and $s < 1$ corresponds to $v_0 > 0$ (injection of fluid at the sheet). Also, it is instructive to see that s and $\frac{1}{s}$ correspond to the same magnitude of mass flux, $|v_0|$. The Eqs. (11)–(13) along with the initial conditions given by Eq. (10) for $t = 0$ constitute the transient problem.

The skin friction $\tau'_{x'}$ in dimensionless form is given by

$$\tau_x = -\left.\frac{\partial u}{\partial y}\right|_{y=0}, \tag{14}$$

where μ is the dynamic viscosity of the fluid and $\tau_x = \tau'_{x'}/(\mu\beta)$.

2.4 Final steady-state flow ($t' \to \infty$)—the solution of Gupta and Gupta [2]

The exact solution for the steady-state fluid flow ($t' \to \infty$) of the Eqs. (11)–(13) is given by Gupta and Gupta [2] as

$$u_s = xe^{-sy} \quad \text{and} \quad v_s = -s + \frac{1}{s}e^{-sy}. \tag{15}$$

2.5 Numerical solution

The unsteady Eqs. (11)–(13) along with the initial condition (10) are solved, as in Muthucumaraswamy and Ganesan [15] and Muthucumaraswamy et al. [16], using the Crank-Nicholson finite difference method. The step sizes of Δt, Δx and Δy are taken as 0.01, 0.002 and 0.005 respectively. The domain of calculation is taken as $0 \leq x \leq 1$ and $0 \leq y \leq 10$. The convergence criterion is $|u^{n+1}_{i,j} - u^{n}_{i,j}| \leq 5 \times 10^{-5}$, where i, j and n are the discretized parameters along the x, y and t directions, respectively. The computed values for the steady-state velocity and local skin friction and the exact values are found to be in good agreement. We are, therefore, confident that the finite difference method works very efficiently.

Table 1 Values of time t^* to reach the steady-state flow for different values of s (Case-I)

s	t^*	s	t^*
4/8	4.83	8/4	2.75
5/8	3.88	8/5	2.72
6/8	3.00	8/6	2.50
7/8	2.01	8/7	1.81

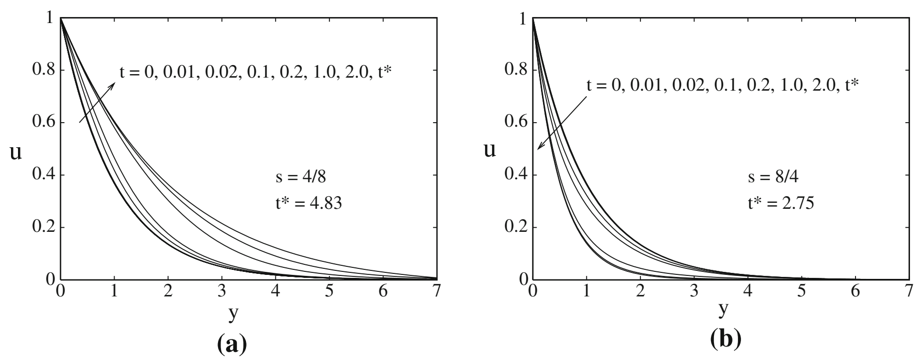

Fig. 1 Profiles of u at $x = 1$ (Case-I): **a** $s = \frac{4}{8}$, **b** $s = \frac{8}{4}$

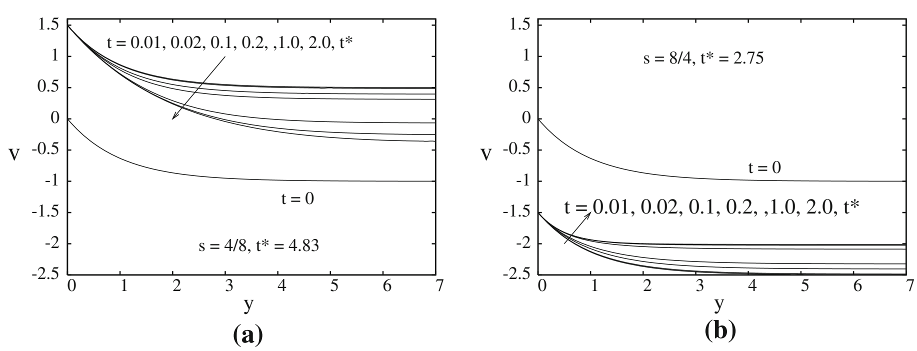

Fig. 2 Profiles of v (Case-I): **a** $s = \frac{4}{8}$, **b** $s = \frac{8}{4}$

2.6 Results and discussion

It is seen from Table 1 that the time (t^*) to reach steady-state flow increases with an increase in both injection of fluid into the flow or suction of fluid from the flow. It is observed from Fig. 1a and b that for small times t, the dimensionless velocity profiles of u at $x = 1$ become closer far away from the sheet. Further, Fig. 2a and b describe the transient dimensionless velocity profiles v. It is seen that the profiles show a sudden jump above the profile at time $t = 0$ in the case of injection and show a sudden jump below the profile at time $t = 0$ in the case of suction. The flow becomes stable with time and reaches the steady state. The magnitude of difference between the dimensionless vertical velocity v at $t = 0$ and that of the steady-state flow gets larger with an increase in the injection/suction of fluid. Figure 5a presents the profiles of dimensionless transient local skin friction $-\tau_x$ at $x = 1$. We can see that τ_x decreases/increases rapidly at small times with an increase in suction/injection, respectively.

3 Case-II

3.1 Formulation

In this case, the initially prevailing steady-state boundary layer flow past a permeable stretching sheet with constant mass flux becomes transient, due to the removal of mass flux at the sheet when $t > 0$. Thus, the initial steady-state flow of the present case is the corresponding steady state of Case-I, given by Eq. (15).

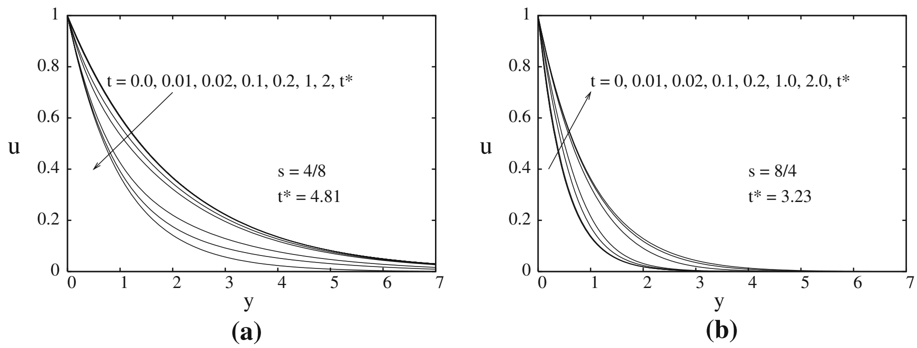

Fig. 3 Profiles of u at $x = 1$ (Case-II): **a** $s = \frac{4}{8}$, **b** $s = \frac{8}{4}$

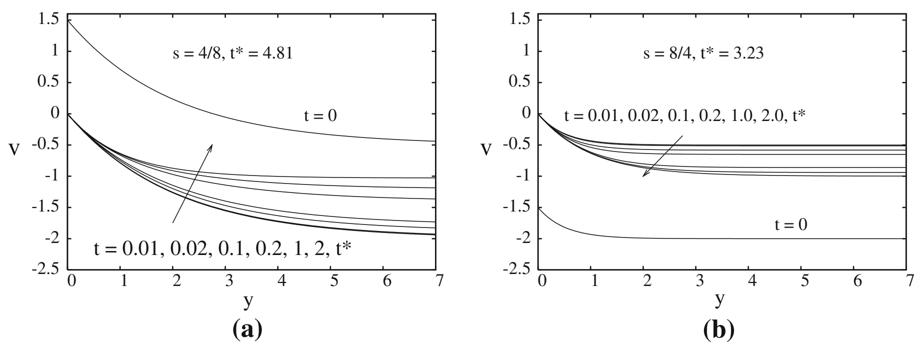

Fig. 4 Profiles of v at $x = 1$ (Case-II): **a** $s = \frac{4}{8}$, **b** $s = \frac{8}{4}$

Therefore, the dimensionless governing transient boundary layer equations are given by Eqs. (11) and (12) subject to the initial and boundary conditions,

$$\begin{aligned} t > 0 : u = x, v = 0 \text{ at } y = 0, \quad &\text{for } x \geq 0, \\ u \to 0 \text{ as } y \to 0 \quad &\text{for } x \geq 0, \end{aligned} \tag{16}$$

$$t \leq 0 : u = u_s, v = v_s \quad \text{for } y \geq 0, x \geq 0. \tag{17}$$

The steady-state ($t \to \infty$) exact solution of this case, satisfying time-independent form of Eqs. (11) and (12) satisfying (16) is given by $u = u_1$ and $v = v_1$.

3.2 Results and discussion

The computations of Eqs. (11), (12), (16) and (17) are carried out as in Sect. 2.5. Transient effects of injection or suction removals are described in Fig. 3a and b for the dimensionless horizontal velocity u at $x = 1$. Sudden removal of injection leads to a continuous drop in u as time t progresses. In the case of sudden suction removal, the velocity u tends to increase as the time progresses as can be seen from Fig. 3b. Figure 4a and b show that the dimensionless steady vertical velocity profiles v are the same, which implies that there is no influence of injection/suction at steady time ($t \to \infty$). For $t > 0$, the velocity v increases/decreases with time t for injection/suction removals. The profiles of the transient skin friction $-\tau_x$ at $x = 1$ are shown in Fig. 5b. It is observed that $-\tau_x$ increases/decreases with an increase in injection removal/suction removal, respectively. In particular, the rate of increase/decrease of $-\tau_x$ is more for small times t. Finally, it is seen from Table 2 that the time (t^*) required to attain the steady-state flow for different values of the parameter s increases with increase in the magnitude of suction/injection removals.

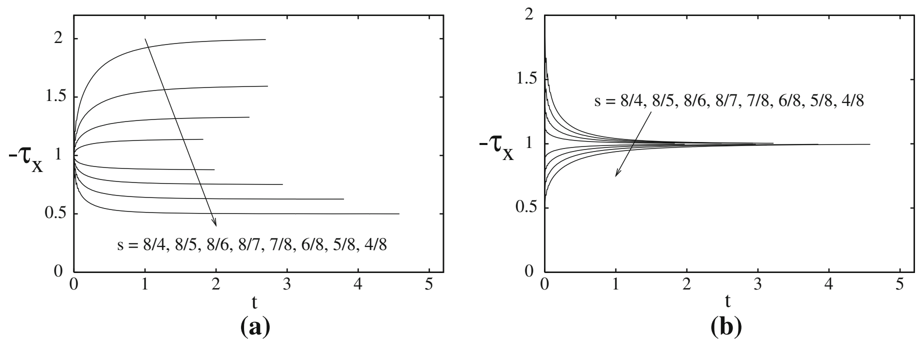

Fig. 5 Profiles of local skin friction τ_x at $x = 1$ for various values of s: **a** Case-I, **b** Case-II

Table 2 Values of time t^* to reach the steady-state flow for different values of s (Case-II)

s	t^*	s	t^*
4/8	4.81	8/4	3.23
5/8	3.82	8/5	2.98
6/8	2.94	8/6	2.50
7/8	1.92	8/7	1.81

4 Conclusions

In this study, the transition of boundary layer flow past a stretching permeable sheet due to a step change in applied constant mass flux at the sheet is considered. Two different cases, namely, Case-I: no mass flux at times $t \leq 0$ and constant mass flux when $t > 0$, and Case-II: constant mass flux when dimensionless time $t \leq 0$ and no mass flux when $t > 0$ are analyzed. The significant observations are as follows: (1) The time to reach steady state increases with the magnitude of mass flux in both cases. In particular, the time is longer for injection when compared with suction of equal magnitude; (2) Different steady times for injection/suction (Case-I) and injection removal/suction removal (Case-II) of the same magnitude imply that the phenomena of Case-I and Case-II are not reversible. (3) For small times t, transient changes in the streamwise velocity component are significant near the sheet, whereas the transient changes are significant away from the sheet for large times.

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