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Hydrodynamic permeability of membranes built up by spherical particles covered by porous shells: effect of stress jump condition

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Abstract This paper concerns the flow of an incompressible, viscous fluid past a porous spherical particle enclosing a solid core, using particle-in-cell method. The Brinkman's equation in the porous region and the Stokes equation for clear fluid are used. At the fluid–porous interface, the stress jump boundary condition for the tangential stresses along with continuity of normal stress and velocity components are employed. No-slip and impenetrability boundary conditions on the solid spherical core have been used. The hydrodynamic drag force experienced by a porous spherical particle enclosing a solid core and permeability of membrane built up by solid particles with a porous shell are evaluated. It is found that the hydrodynamic drag force and dimensionless hydrodynamic permeability depends not only on the porous shell thickness, particle volume fraction γ and viscosities of porous and fluid medium, but also on the stress jump coefficient. Four known boundary conditions on the hypothetical surface are considered and compared: Happel's, Kuwabara's, Kvashnin's and Cunningham's (Mehta–Morse's condition). Some previous results for the hydrodynamic drag force and dimensionless hydrodynamic permeability have been verified.

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1 Introduction

Flow through porous media has been a topic of longstanding interest for researchers due to its numerous applications in bio-mechanics, physical sciences, chemical engineering, etc. An arbitrary flow of viscous, incompressible fluid through a swarm of porous particles has many industrial and engineering applications, such as, in flow through porous beds, in petroleum reservoir rocks, in flow sedimentation, etc. For effective use of a porous medium in the above-mentioned areas, the structure of a porous layer should be viewed from all angles, for example, it is not necessary that the particles always have a smooth homogeneous surface but also have a rough surface or a surface covered by a porous shell. For the medium of high porosity, the sum suggested by Brinkman [1] is more suitable for describing the flow through the porous medium. He evaluated the viscous force exerted by a flowing fluid on a dense swarm of particles by modifying Darcy's equation for a porous medium, which is commonly known as Brinkman equation.

The problem of flow through a swarm of particles becomes complex, if we consider the solution of the flow field over the entire swarm by taking exact positions of particles. In order to avoid the above complication, it is sufficient to obtain the analytical expression by considering the effects of the neighboring particles on the flow field around a single particle of the swarm, which can be used to develop relatively simple and reliable models for heat and mass transfer. This has lead to the development of particle-in-cell models.

Uchida [2] proposed a cell model for a sedimenting swarm of particles, considering a spherical particle surrounded by a fluid envelope with cubic outer boundary. Happel [3,4] proposed cell models in which the particle and outer envelope both are spherical/cylindrical. He solved the problem when the inner sphere/cylinder is solid with respective boundary conditions on the cell surface. The Happel model assumes uniform velocity condition and no tangential stress at the cell surface. The merit of this formulation is that it leads to an axially symmetric flow that has a simple analytical solution in closed form and thus can be used for heat and mass transfer calculations. Analytical solutions of particle-in-cell models discussed earlier are always practically useful to many industrial problems, but the solutions of creeping flow for the above models have not been found in case of complex geometry. Kuwabara [5] proposed again a cell model in which he used the nil vorticity condition on the cell surface to investigate the flow through a swarm of spherical/cylindrical particles. However, the Kuwabara formulation requires a small exchange of mechanical energy with the environment. The mechanical power given by the sphere to the fluid is not all consumed by viscous dissipation in the fluid layer. Apart from this, Kvashnin [6] and Mehta and Morse [7] gave their respective boundary conditions for the outer cell surface. Kvashnin [6] proposed the condition that the tangential component of velocity reaches a minimum at the cell surface with respect to radial distance, signifying the symmetry on the cell. However, Mehta and Morse [7] used Cunningham's [8] approach by assuming the tangential velocity as a component of the fluid velocity, signifying the homogeneity of the flow on the cell boundary. The importance of the Mehta and Morse [7] boundary condition is that since we are interested in the flow behavior on a large scale, we shall average the flow variables on the small scale over a cell volume to obtain large scale behavior.

Beavers and Joseph [9] have proposed an empirical slip flow condition at the interface for a plane boundary, which is given for a rectilinear flow of a viscous fluid through a two-dimensional parallel channel formed by an impermeable upper wall and a permeable lower wall as $\left(\frac{\partial u}{\partial y}\right)_{y=q_+} = \alpha(u_\beta - q)$ where q_+ is a boundary limit point from the exterior fluid, q is the velocity in the porous region, and u_β is the velocity in the interface, α is the adjustable coefficient.

Jones [10] had investigated the solution of the problem of Stokes flow past a porous spherical shell. He used Darcy's law for the porous region and evaluated the drag force experienced by a shell. A Cartesian-tensor solution of the Brinkman equation was investigated by Qin and Kaloni [11], and they also evaluated the drag force on a porous sphere in an unbounded medium. Flow through beds of porous particles was studied by Davis and Stone [12], and they evaluated the overall bed permeability of the swarm by using a cell model. Vasin and Filippov [13], Filippov et al. [14] evaluated the hydrodynamic permeability of a membrane of porous spherical particles using the Mehta–Morse condition on the cell surface. Recently, Vasin et al. [15,16] compared all four cell models to evaluate the permeability of the membrane of porous spherical particles with a permeable shell and discussed the effect of different parameters on the hydrodynamic permeability of the membrane for all the four above-mentioned boundary conditions.

Ochoa-Tapia and Whitaker [17,18] studied the momentum transfer at the boundary between a porous medium and a homogeneous fluid theoretically and experimentally. They developed the appropriate jump condition for momentum transport within the framework of the method of volume averaging and compared the theory with the experimental studies of Beavers and Joseph [9]. They explored the use of a variable porosity model as a substitute for the jump condition.

Many authors have used the stress jump boundary condition in various flow problems and reported significant changes in the results. Kuznetsov [19,20] used the stress jump boundary condition at the fluid–porous interface to discuss flow in channels partially filled with porous medium. The problem of Stokes flow inside a porous spherical shell was solved by Raja-Sekhar and Amarnath [21] by using Darcy's law for the porous outside region and Stokes flow for the inside region. Arbitrary viscous flow past a porous sphere with an impermeable core was studied by Bhattacharya and Raja Shekhar [22]. They reported the dependence of drag and torque on the jump coefficient.

Srivastava and Srivastava [23] studied the Stokes flow through a porous sphere using the stress jump condition at the fluid–porous interface and matched Stoke's and Oseen's solutions far away from the sphere. They concluded that drag on a porous sphere decreases with increasing permeability of the medium. Diffusive mass transfer between a microporous medium and a homogeneous fluid was studied by Valdes-Parada et al [24]. They obtained the jump boundary condition between a fluid and a porous medium, for diffusive and chemical reaction. They also derived an expression for a jump stress boundary condition free of adjustable coefficient using a method of volume averaging. Chandesris and Jamet [25] explained the sensitivity of the jump parameters to the location of the discontinuous interface. Recently, Jamet et al. [26], Chandesris and Jamet [27] and Valdes-Parada et al. [28] discussed various aspects of the jump condition to account for transport phenomena in a fluid–porous domain. The motivations of these papers lead us to discuss the present problem which includes some earlier known results.

This paper concerns the problem of slow viscous flow through a swarm of porous spherical particles. As boundary conditions, continuity of velocity, continuity of normal stress and the stress jump condition at the porous and fluid interface, the impenetrability and no-slip conditions on the solid spherical core are employed. On the hypothetical surface, uniform velocity and consequently each of four known boundary conditions are used (proposed by Happel, Kuwabara, Kvashnin and Cunningham/Mehta and Morse). The drag force experienced by each porous spherical shell in a cell is evaluated. As a particular case, the drag force experienced by a porous sphere in a cell and in an unbounded medium with jump is also investigated. The earlier results reported for the drag force by Davis and Stone [12] for the drag force experienced by a porous sphere in a cell without jump, Happel [3]for a solid sphere in a cell and Qin and Kaloni [11] for a porous sphere in an unbounded medium have been then deduced.

2 Mathematical formulation of the problem

In the mathematical model, we will represent a disperse system (membrane) by a periodic net of identical solid spherical particles of radius $\tilde{R}$ where each of them is covered by a porous spherical shell of thickness $\tilde{\delta}$. Further, we assume that each particle is enveloped by a concentric sphere of radius $\tilde{b}$ (Fig. 1) named as cell surface. These porous spherical shells are randomly and homogeneously distributed in the viscous fluid. The porous medium is assumed to be homogeneous and isotropic. Let us consider that porous spherical shells are stationary, and steady axisymmetric viscous flow has been established around and through it by a uniform velocity $\tilde{\mathbf{U}}\left(\left|\tilde{\mathbf{U}}\right| = \tilde{U}\right)$ directed in the positive z-axis. The radius $\tilde{b}$ of the hypothetical cell is so chosen that the particle volume fraction γ^3in the cell is equal to the volume fraction in the real membrane (Fig. 1), i.e.,

$$1 - \varepsilon = \gamma^3 = \left(\frac{\tilde{a}}{\tilde{b}}\right)^3, \tag{1}$$

where ε is an outer porosity of the membrane and $\tilde{a} = \tilde{R} + \tilde{\delta}$ is the total radius of the particle.

The flow of an incompressible fluid in the outside region of the porous spherical shell ($\tilde{a} \le \tilde{r} \le \tilde{b}$) is governed by the Stokes equation (Happel and Brenner [29]) with the continuity condition:

$$\tilde{\nabla}\tilde{p}^o = \tilde{\mu}^o\tilde{\Delta}\tilde{\mathbf{v}}^o, \quad \tilde{\nabla}.\tilde{\mathbf{v}}^o = 0. \tag{2}$$

For the region inside the porous shell ($\tilde{R} \le \tilde{r} \le \tilde{a}$), we assume that the flow is governed by the Brinkman [1] equation and the continuity condition:

$$\tilde{\nabla}\tilde{p}^i = \tilde{\mu}^i\tilde{\Delta}\tilde{\mathbf{v}}^i - \tilde{k}\tilde{\mathbf{v}}^i, \quad \tilde{\nabla}.\tilde{\mathbf{v}}^i = 0, \tag{3}$$

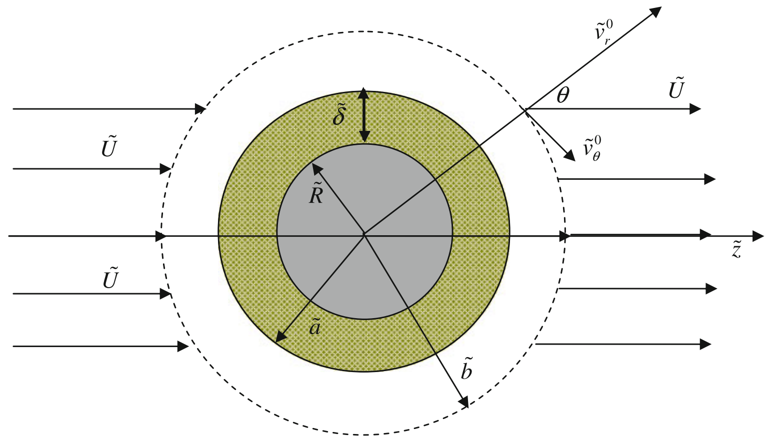

Fig. 1 Spherical shell of radius $\tilde{b}$ with a solid particle of radius $\tilde{R}$covered by a porous shell of thickness $\tilde{\delta}$ in the center, $\tilde{U}$ being the velocity of the uniform flow; $\tilde{v}_r^0$, $\tilde{v}_\theta^0$ are radial and angular velocities on the outer cell surface

where o and i are superscripts, which mark flow in the cell ($\tilde{a} \le \tilde{r} \le \tilde{b}$) and in the porous shell($\tilde{R} \le \tilde{r} \le \tilde{a}$), respectively; a wavy line over a symbols marks dimensional values; $\tilde{\mathbf{v}}^o$, $\tilde{\mathbf{v}}^i$ and $\tilde{p}^o$, $\tilde{p}^i$ are velocity and pressure in the outside and inside of the porous regions; $\tilde{\mu}^o$ and $\tilde{\mu}^i$ are viscosities in the outside and inside of the porous regions, respectively, and $\tilde{k}$ is the hydrodynamic resistance of the porous shell, which is inversely proportional to the hydrodynamic permeability.

3 Solution of the problem

By using the following dimensionless variables:

$$\frac{1}{\gamma} = \frac{\tilde{b}}{\tilde{a}},\ r = \frac{\tilde{r}}{\tilde{a}},\ \nabla = \tilde{\nabla}.\tilde{a},\ \Delta = \tilde{\Delta}.\tilde{a}^2,\ R = \frac{\tilde{R}}{\tilde{a}} = 1 - \delta,\ \delta = \frac{\tilde{\delta}}{\tilde{a}},$$
$$p = \frac{\tilde{p}}{\tilde{p}_o},\ \tilde{p}_o = \frac{\tilde{U}.\tilde{\mu}^o}{\tilde{a}},\ s = \frac{s_o}{m},\ s_o = \frac{\tilde{a}}{\tilde{R}_b},\ m^2 = \frac{\tilde{\mu}^i}{\tilde{\mu}^o},\ v = \frac{\tilde{\mathbf{v}}}{\tilde{U}},\ k = \frac{\tilde{k}}{\tilde{b}^2}, \tag{4}$$

where $\tilde{R}_b = \sqrt{\frac{\tilde{\mu}^o}{\tilde{k}}}$ is the Brinkman's length, which is the characteristic depth of penetration of the flow inside the porous shell, and a spherical co-ordinate system $(\tilde{r}, \theta, \varphi)$ with the origin located at the center of the porous spherical particle, the governing Eqs. (2) and (3) reduce in non-dimensional form to (Vasin et al. [15]):

$$\frac{\partial p^o}{\partial r} = \frac{\partial^2 v_r^o}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r^o}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_r^o}{\partial r} + \frac{ctg\theta}{r^2}\frac{\partial v_r^o}{\partial \theta} - \frac{2}{r^2}\frac{\partial v_\theta^o}{\partial \theta} - \frac{2v_r^o}{r^2} - \frac{2ctg}{r^2}v_\theta^o, \tag{5}$$

$$\frac{1}{r}\frac{\partial p^o}{\partial \theta} = \frac{\partial^2 v_\theta^o}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta^o}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_\theta^o}{\partial r} + \frac{ctg\theta}{r^2}\frac{\partial v_\theta^o}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_\theta^o}{\partial \theta} - \frac{v_\theta^o}{r^2 \sin\theta}, \tag{6}$$

$$\frac{\partial v_r^o}{\partial r} + \frac{1}{r}\frac{\partial v_\theta^o}{\partial \theta} + \frac{2v_r^o}{r^2} + \frac{ctg\theta}{r}v_\theta^o = 0, \tag{7}$$

$$\frac{\partial p^i}{\partial r} = m^2\left(\frac{\partial^2 v_r^i}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r^i}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_r^i}{\partial r} + \frac{ctg\theta}{r^2}\frac{\partial v_r^i}{\partial \theta} - \frac{2}{r^2}\frac{\partial v_\theta^i}{\partial \theta} - \frac{2v_r^i}{r^2} - \frac{2ctg}{r^2}v_\theta^i\right) - s_o^2 v_r^i, \tag{8}$$

$$\frac{1}{r}\frac{\partial p^i}{\partial \theta} = m^2\left(\frac{\partial^2 v_\theta^i}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta^i}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_\theta^i}{\partial r} + \frac{ctg\theta}{r^2}\frac{\partial v_\theta^i}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_\theta^i}{\partial \theta} - \frac{v_\theta^i}{r^2 \sin\theta}\right) - s_o^2 v_r^i, \tag{9}$$

$$\frac{\partial v_r^i}{\partial r} + \frac{1}{r}\frac{\partial v_\theta^i}{\partial \theta} + \frac{2v_r^i}{r^2} + \frac{ctg\theta}{r}v_\theta^i = 0. \tag{10}$$

The values of p^o, p^i, v_r^0, v_r^i, v_θ^o, v_θ^i which are the solutions of the system (5)–(10) are given by (Vasin et al. [15]):

$$v_r^o = \left(\frac{b_1}{r^3} + \frac{b_2}{r} + b_3 + b_4 r^2\right)\cos\theta, \tag{11}$$

$$v_\theta^o = \left(\frac{b_1}{2r^3} - \frac{b_2}{2r} - b_3 - 2b_4 r^2\right)\sin\theta, \tag{12}$$

$$p^o = \left(\frac{b_2}{r^2} + 10 b_4 r\right)\cos\theta, \tag{13}$$

$$v_r^i = \left[c_1\left\{\frac{\cosh(sr)}{r^2s^2} - \frac{\sinh(sr)}{r^3s^3}\right\} + c_2\left\{\frac{\sinh(sr)}{r^2s^2} - \frac{\cosh(sr)}{r^3s^3}\right\} + \frac{c_3}{r^3} + c_4\right]\cos\theta, \tag{14}$$

$$v_\theta^i = \left[c_1\left\{\frac{\cosh(sr)}{2r^2s^2} - \frac{\sinh(sr)}{2r^3s^3} - \frac{\sinh(sr)}{2rs}\right\} + c_2\left\{\frac{\sinh(sr)}{2r^2s^2} - \frac{\cosh(sr)}{2r^3s^3} - \frac{\cosh(sr)}{2rs}\right\} + \frac{c_3}{2r^3} - c_4\right]\sin\theta, \tag{15}$$

$$p^i = m^2s^2\left(\frac{c_3}{2r^2} - c_4 r\right)\cos\theta. \tag{16}$$

Boundary Conditions:

By applying a sophisticated volume averaging technique, Ochoa-Tapia and Whitaker [17,18] have shown that the processes of matching the Brinkman equation to the Stokes equation requires a discontinuity for the tangential stress but retains the continuity of velocity components and normal stress.

The impermeable conditions (no-slip conditions) are imposed on the surface of the solid core of the particle:

$$v_r^i = 0, \quad v_\theta^i = 0 \ \text{ at } \ r = R. \tag{17}$$

To match the solution at the porous-liquid interface ($r = 1$), we use along with the continuity of velocity components and continuity of normal stress the Ochoa-Tapia and Whitaker boundary condition (stress jump) for the tangential stresses, respectively, as follows:

$$v_r^o = v_r^i, \quad v_\theta^o = v_\theta^i, \tag{18}$$

$$-p^o + 2\frac{\partial v_r^o}{\partial r} = -p^i + 2m^2\frac{\partial v_r^i}{\partial r}, \tag{19}$$

$$m^2\left(\frac{1}{r}\frac{\partial v_r^i}{\partial\theta} + \frac{\partial v_\theta^i}{\partial r} - \frac{v_\theta^i}{r}\right) - \left(\frac{1}{r}\frac{\partial v_r^o}{\partial\theta} + \frac{\partial v_\theta^o}{\partial r} - \frac{v_\theta^o}{r}\right) = \frac{\beta}{\sqrt{k}}v_\theta^i. \tag{20}$$

The continuity of the radial component of the fluid velocity on the outer cell $r = 1/\gamma$ implies:

$$v_r^o = \cos\theta. \tag{21}$$

According to Happel [3], the tangential stress vanishes on the outer cell surface $r = 1/\gamma$:

$$\tilde{T}_{r\theta}^o = 0 \quad \text{i.e.} \quad \frac{1}{r}\frac{\partial v_r^o}{\partial\theta} + \frac{\partial v_\theta^o}{\partial r} - \frac{v_\theta^o}{r} = 0, \ \text{at} \ \ r = 1/\gamma. \tag{22}$$

According to Kuwabara [5], the curl of velocity ($\tilde{\mathbf{v}}^o$) vanishes on the outer cell surface $r = 1/\gamma$:

$$rot(\tilde{\mathbf{v}}^o) = 0 \quad \text{i.e.} -\frac{1}{r}\frac{\partial v_r^o}{\partial\theta} + \frac{\partial v_\theta^o}{\partial r} + \frac{v_\theta^o}{r} = 0, \ \text{at} \ \ r = 1/\gamma. \tag{23}$$

According to Kvashnin [6], a symmetry condition is introduced on the outer cell surface $r = 1/\gamma$:

$$\frac{\partial v_\theta^o}{\partial r} = 0, \ \text{at} \ \ r = 1/\gamma. \tag{24}$$

Mehta and Morse [7] assume homogeneity of the flow on the outer cell surface $r = 1/\gamma$:

$$v_\theta^o = -\sin\theta, \quad \text{at } r = 1/\gamma. \tag{25}$$

Substituting the values of p^o, p^i, v_r^0, v_r^i, v_θ^o, v_θ^i from Eqs. (11)–(16) in Eqs. (17)–(21), we have seven equations involving eight arbitrary constants $b_j, c_j, j = 1, 2, 3, 4$. Thus, the values of arbitrary constants mentioned above depend on the selection of the model used, i.e., on the boundary condition (22), (23), (24) or (25). Solving the resulting equations by taking each model, respectively, we find the values of all arbitrary constants b_j's and c_j's.

4 Results and discussion

Integrating the normal and tangential stresses over the porous spherical shell surface of radius $\tilde{a}$ in a cell yields the experienced hydrodynamics drag force by each porous spherical particle as:

$$\tilde{F} = \oiint_S (\tilde{T}_{rr}^{(o)} \cos\theta - \tilde{T}_{r\theta}^{(o)} \sin\theta) ds. \tag{26}$$

Inserting the values of $\tilde{T}_{rr}^{(o)}$ and $\tilde{T}_{r\theta}^{(o)}$ into the above Eq. (26) and integrating, we get

$$\tilde{F} = -4\pi b_2 \tilde{a} \tilde{\mu}^o \tilde{U}. \tag{27}$$

Hydrodynamic permeability of a membrane $\tilde{L}_{11}$ is defined as the ratio of the uniform flow rate $\tilde{U}$ to the cell gradient pressure $\tilde{F}/\tilde{V}$ [29]:

$$\tilde{L}_{11} = \frac{\tilde{U}}{\tilde{F}/\tilde{V}}, \tag{28}$$

where $\tilde{V} = \frac{4}{3}\pi \tilde{b}^3$ is the volume of the cell.

Substituting the value of $\tilde{F}$ from Eq. (27) and the value of $\tilde{V}$ from above, in Eq. (28), we have

$$\tilde{L}_{11} = -\frac{1}{3} \frac{\tilde{a}^2}{b_2 \gamma^3 \tilde{\mu}^o} = L_{11} \frac{\tilde{a}^2}{\tilde{\mu}^o}, \tag{29}$$

where $L_{11} = -1/3b_2\gamma^3$ is the dimensionless hydrodynamic permeability of a membrane. The ratio Ω of drag force $\tilde{F} = -4\pi b_2 \tilde{a} \tilde{\mu}^o \tilde{U}$ and Stokes force $\tilde{F}_S = 6\pi \tilde{a} \tilde{\mu}^o \tilde{U}$ is

$$\Omega = -2b_2/3. \tag{30}$$

Figure 2 represents the dependence of the natural logarithm of dimensionless hydrodynamic permeability of a membrane on the dimensionless thickness of the porous shell. Evidently, the hydrodynamic permeability increases with the thickness for a thin shell (small δ) but when thickness becomes comparable to Brinkman's length, the hydrodynamic permeability turns out to be independent of the dimensionless thickness. Also, due to the introduction of the stress jump boundary condition at the fluid–porous interface, a slightly higher value of hydrodynamic permeability is observed for all four formulations. For the limiting case $\beta = 0$, our results agree with those of Vasin and Filippov [16].

The dimensionless hydrodynamic permeability of a membrane decreases with viscosity ratio, i.e., the flow through the membrane becomes difficult with the increase of inner viscosity. Again the membrane hydrodynamic permeability attains a slightly higher value in all four formulations due to a jump in the shearing stress at fluid–porous interface (Fig. 3).

Figure 4 shows the variation of the natural logarithm of dimensionless hydrodynamic permeability of a membrane with the parameter s_0 representing the penetrability through the porous shell. As the penetrability decreases (i.e., s_0 increases), the membrane hydrodynamic permeability decreases. However, for the more penetrable region (smaller s_0), the hydrodynamic permeability increases with s_0 due to the stress jump boundary condition.

On analyzing the dependence of the natural logarithm of the dimensionless hydrodynamic permeability on the jump coefficient β, we observe that with the increase of β the dimensionless hydrodynamic permeability of a membrane increases and for high $\beta(\beta \to 0.9)$ it increases unboundedly (Fig. 5). The outcome strengthens our claim that the stress jump condition cannot be ignored. The variation of the dimensionless hydrodynamic

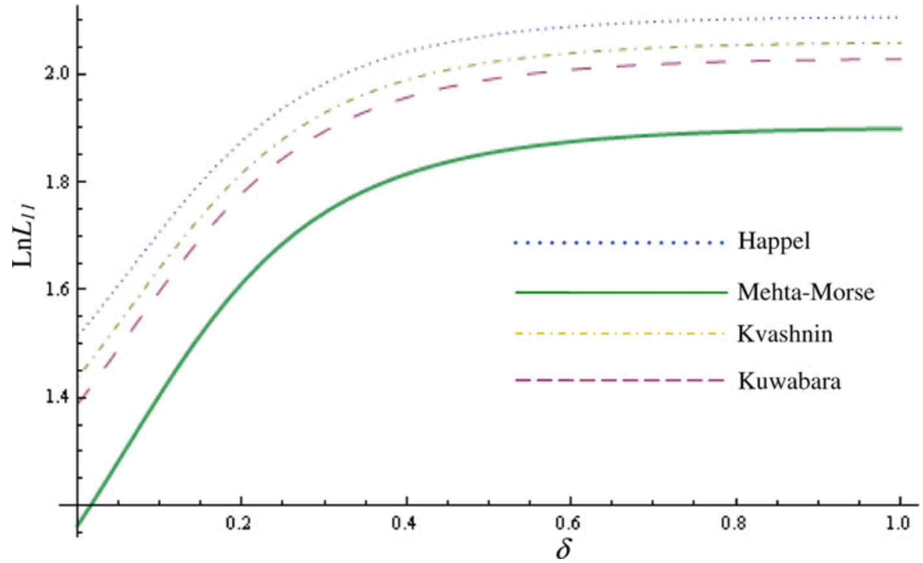

Fig. 2 Variation of natural logarithm of the dimensionless hydrodynamic permeability L_{11}, of a membrane built up with solid particles covered by a porous shell, with the dimensionless thickness of the porous shell, δ, at $m = 1$ $s_0 = 5$, $\gamma = 0.3$ and $\beta = 0.3$

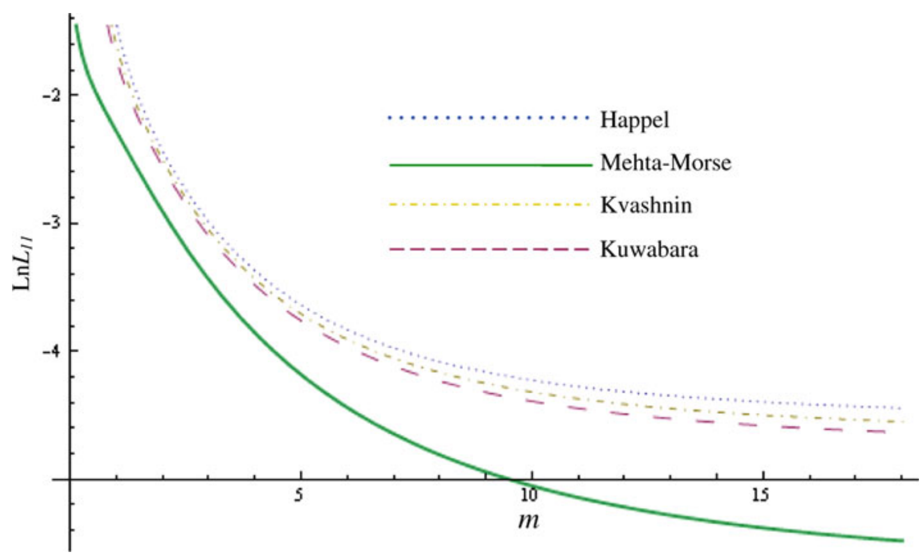

Fig. 3 Variation of natural logarithm of the dimensionless hydrodynamic permeability, L_{11}, of a membrane built up with solid particles covered by a porous shell, with the viscosity ratio m at $\delta = 0.5$, $s_0 = 5$, $\gamma = 0.8$ and $\beta = 0.6$

permeability with volume fraction is the same as reported by Vasin and Filippov [16] except the fact that it assumes a slightly higher value because of non-zero β influence, and all four models agree for low volume fraction (Fig. 6).

The semi-empirical Kozeny–Carman formula for the permeability of a porous medium is given by

$$L_{11} = \frac{\varepsilon \rho^2}{k \tilde{a}^2}, \tag{31}$$

where k is a dimensionless Kozeny constant, ρ is the hydraulic radius which is equal to the ratio of pore volume to the wetting area; ε is the porosity of the medium. The Kozeny constant is given by

$$k = \frac{\varepsilon^3}{9(1-\varepsilon)^2 L_{11}}. \tag{32}$$

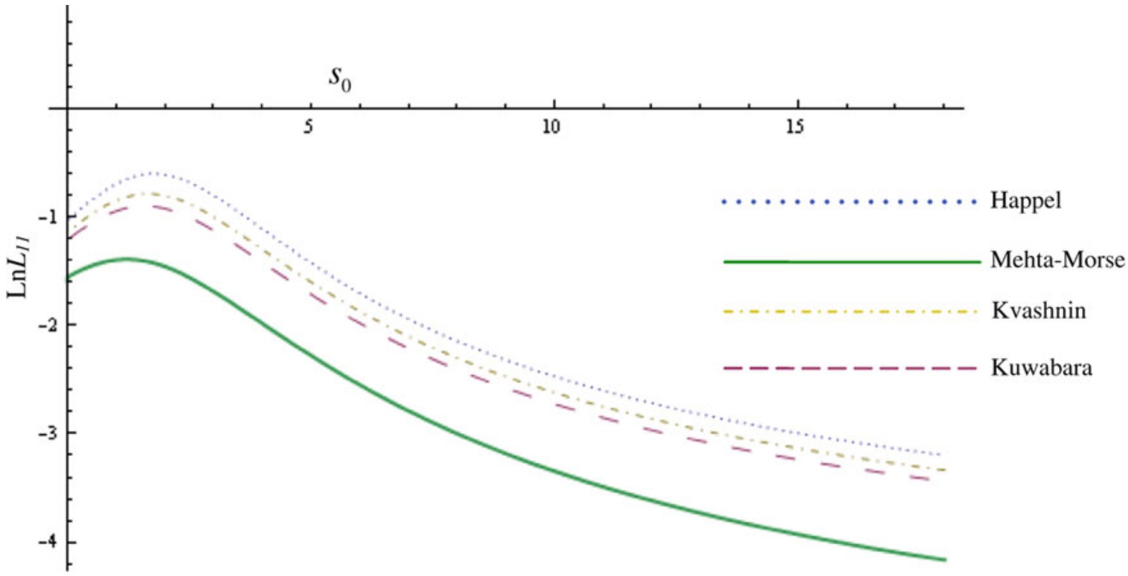

Fig. 4 Variation of the natural logarithm of the dimensionless hydrodynamic permeability, L_{11} of a membrane built up with solid particles covered by a porous shell, with the parameter s_0 at $\delta = 0.5$, $m = 1$, $\gamma = 0.8$ and $\beta = 0.6$

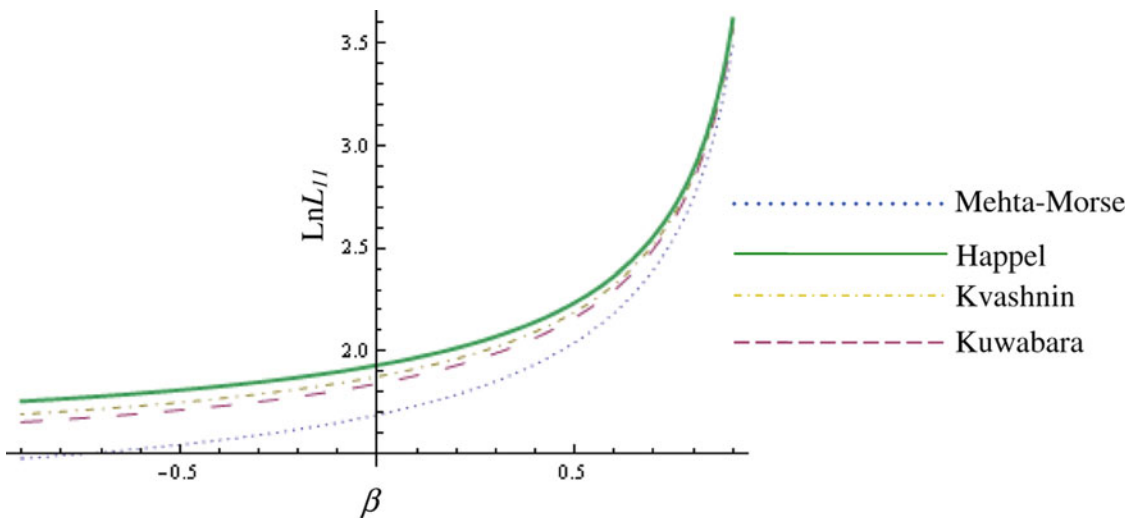

Fig. 5 Variation of the natural logarithm of the dimensionless hydrodynamic permeability, L_{11}, of a membrane built up by solid particles covered by a porous shell, with the parameter β at $\delta = 0.5$, $m = 1$, $s_0 = 5$, $\gamma = 0.3$

From Table 1, we observe that, likewise in the previous case the Kozeny constant is almost the same for Happel, Kuwabara and Kvashnin boundary conditions. However, for the Cunningham boundary condition, the value of the Kozeny constant is quite different from the other three formulations. The Kozeny constant shows a slight decay with the jump parameter for any porosity (Table 1).

5 Particular cases

5.1 Porous spherical shell enclosing a solid core in an unbounded medium

When the hypothetical cell radius $\tilde{b} \to \infty$ i.e., $\gamma \to 0$, then the porous spherical shell lies in an unbounded medium. In this case, the value of the hydrodynamic drag force ratio Ω comes out as

$$\Omega = -2b_2^*/3. \tag{33}$$

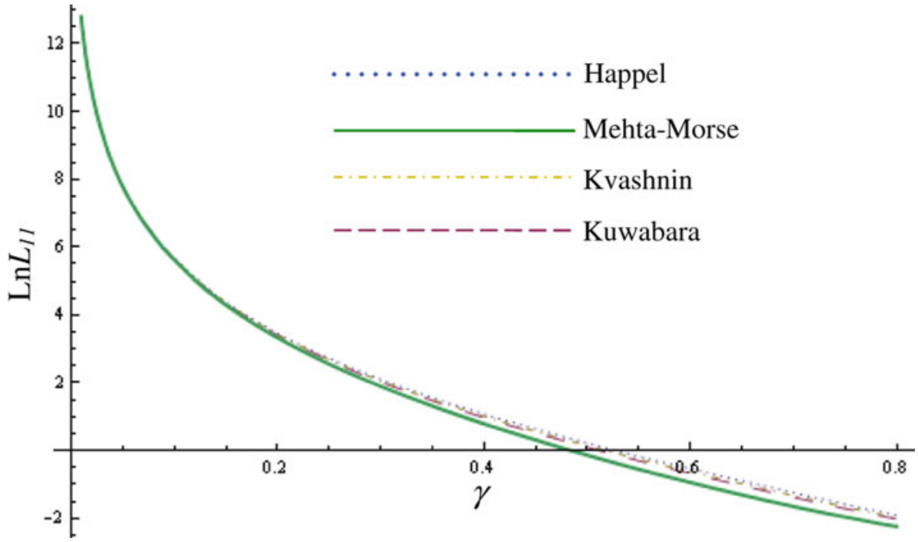

Fig. 6 Variation of the natural logarithm of the dimensionless hydrodynamic permeability, L_{11}, of a membrane built up by completely porous particles with the parameter γ (which is (volume fraction of particles) $^{1/3}$) at $m = 1$, $s_0 = 5$ and $\beta = 0.3$

Table 1 Quantitative values of the Kozeny constant for different porosities and β

Boundary condition	Happel		Kuwabara		Kvashnin		Cunningham	
Porosity	$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$
0.2	0.03421	0.02795	0.03670	0.03063	0.03556	0.02940	0.05043	0.04529
0.6	0.98480	0.83049	1.12122	0.94981	1.06272	0.89874	1.53228	1.30434
0.7	1.67801	1.44334	1.93652	1.66485	1.82701	1.57132	2.65326	2.26615
0.8	2.92088	2.57265	3.38698	2.9695	3.19226	2.80442	4.57061	3.95187
0.9	6.10278	5.52613	6.97312	6.27309	6.61707	5.96885	8.9314	7.91417

If $\beta = 0$, then the value of the hydrodynamic drag force ratio Ω is:

$$\Omega = -2b_3^*/3, \tag{34}$$

where the values of $b_2^*$and b_3^* are given in Appendix A.
When $m = 1$, then the value of the hydrodynamic drag force ratio Ω for a porous spherical shell enclosing a solid core in an unbounded medium comes out as:

$$\Omega = \frac{\mathrm{s}\{3(-1+\delta)\delta + \mathrm{s}^2(-3+3\delta-3\delta^2+\delta^3)\}\cosh(\mathrm{s}\delta) + \{3+3(-1+\mathrm{s}^2)\delta - \mathrm{s}^2\delta^3\}\sinh(\mathrm{s}\delta)}{-6\mathrm{s}(-1+\delta) + \mathrm{s}\{3(-2+\delta)+\mathrm{s}^2(-3+3\delta-3\delta^2+\delta^3)\}\cosh(\mathrm{s}\delta) - 3\{-1+\mathrm{s}^2(-1+\delta)^2\}\sinh(\mathrm{s}\delta)}. \tag{35}$$

The effect of the jump condition is discussed on the ratio of drag forces acting on a porous shell over a solid sphere (Stokes force). As evident, for all cases, the quantity $\Omega < 1$. In comparison with the previous case, Ω decreases more rapidly with dimensionless thickness of the porous shell δ (Fig. 7) and attains a smaller value for more penetrable region (i.e., for low s_0; Fig. 8). An interesting observation is that as β and Ω approach to their peak values more rapidly with an increase in s_0 (Fig. 9). Also, as the inner viscosity increases, Ω approaches toward one (Fig. 10).

5.2 Completely porous particles in a cell

When $\delta = 1$, then the porous spherical shell in a cell reduces to the porous sphere in a cell. Therefore, the value of the hydrodynamics drag force experienced by the porous sphere of radius $\tilde{a}$ comes out as

$$\tilde{F} = -4\pi b_3^* \tilde{a}\tilde{\mu}^o \tilde{U}. \tag{36}$$

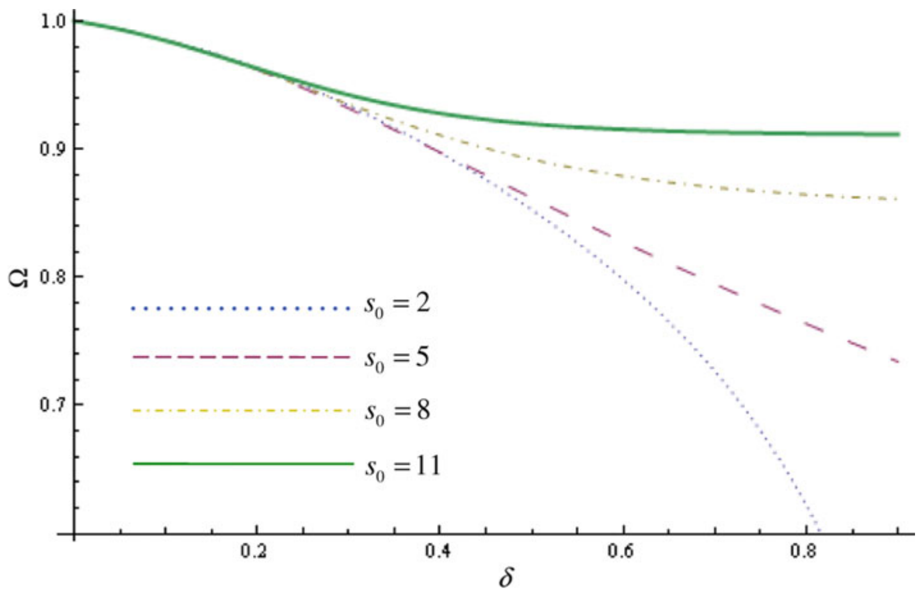

Fig. 7 Variation of the hydrodynamic drag force ratio Ω (the real drag force divided by the Stokes force exerted to the solid non-porous particles of the same radius) exerted to solid particles covered by a porous shell with the dimensionless thickness of the porous shell δ at $m = 3$ for various values of s_0, when $\beta = 0.6$

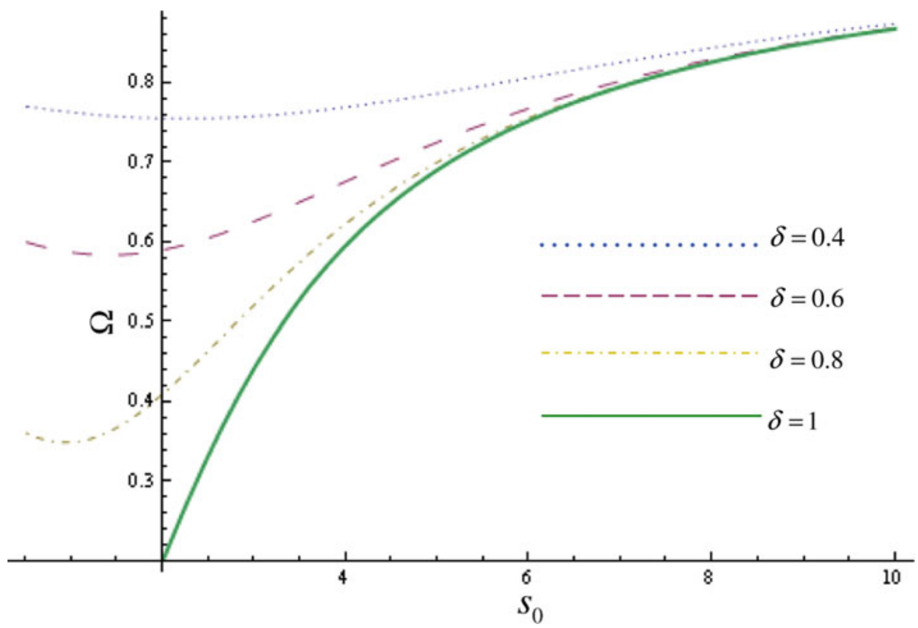

Fig. 8 Variation of the hydrodynamic drag force ratio Ω (the real drag force divided by the Stokes force exerted to the solid non-porous particles of the same radius) exerted to solid particles covered by a porous shell with the parameter s_0 at $m = 3^{1/2}$ for various values of δ, when $\beta = 0.6$

The dimensionless hydrodynamic permeability of a membrane and the value of Ω are

$$L_{11} = -\frac{1}{3b_3^*\gamma^3} \quad \text{and} \quad \Omega = -2b_3^*/3, \tag{37}$$

where the values of b_3^* for all models are given in Appendix B.

If $\beta = 0$, then the values of the dimensionless hydrodynamic permeability of a membrane for all the models are given as:

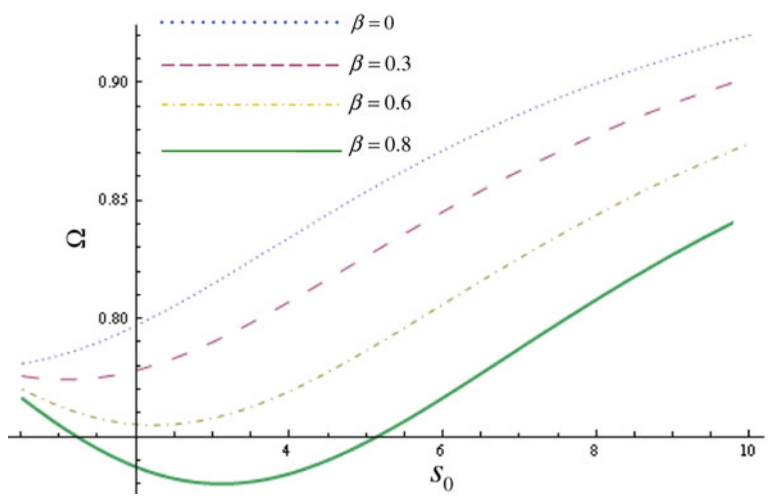

Fig. 9 Variation of the hydrodynamic drag force ratio Ω (the real drag force divided by the Stokes force exerted to the solid non-porous particles of the same radius) exerted to solid particles covered by a porous shell with the parameter s_0 at $m = 3^{1/2}$ for various values of β, when $\delta = 0.4$

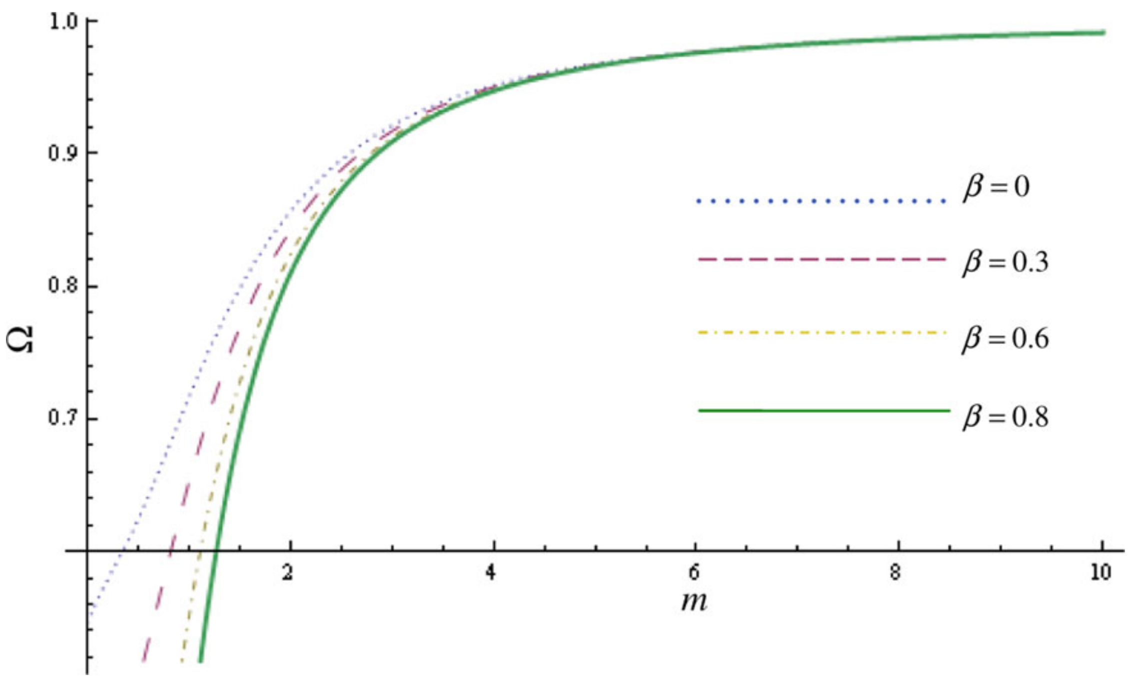

Fig. 10 Variation of the hydrodynamic drag force ratio Ω (the real drag force divided by the Stokes force exerted to the solid non-porous particles of the same radius) exerted to solid particles covered by a porous shell with the viscosity ratio m at $s_0 = 3$ for various values of β, when $\delta = 0.37$

Happel's model:

$$\begin{aligned}
L_{11} = {} & (\mathrm{s}(-\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^3(1+\gamma)(2+\gamma+2\gamma^2) \\
& +54(-1+\gamma^5)+3\mathrm{m}^2(18+12\gamma^5+\mathrm{s}^2(-3+6\gamma+8\gamma^5-6\gamma^6)))\cosh \mathrm{s} \\
& +3(\mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^3(1+\gamma)(2+\gamma+2\gamma^2)-6(3+\mathrm{s}^2)(-1+\gamma^5) \\
& +\mathrm{m}^2(-6(3+2\gamma^5)+2\mathrm{s}^4(1-\gamma-\gamma^5+\gamma^6) \\
& +3\mathrm{s}^2(-1+2\gamma(-1+(-2+\gamma)\gamma^4))))\sinh \mathrm{s})/(3\mathrm{m}^2\mathrm{s}^2\gamma^3(\mathrm{s}(18(-1+\gamma^5)
\end{aligned}$$

$$+ \mathrm{m}^2(6+\mathrm{s}^2)(3+2\gamma^5))\cosh \mathrm{s} - 3(2(3+\mathrm{s}^2)(-1+\gamma^5)$$
$$+ \mathrm{m}^2(2+\mathrm{s}^2)(3+2\gamma^5))\sinh \mathrm{s})). \tag{38}$$

Kuwabara's model:

$$\begin{aligned} L_{11} = &(-\mathrm{s}(270 + 2\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^3(5+6\gamma+3\gamma^2+\gamma^3) \\ &+ 9\mathrm{m}^2(-30+\mathrm{s}^2(5-12\gamma+2\gamma^6)))\cosh \mathrm{s} + (90(3+\mathrm{s}^2) \\ &+ 6\mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^3(5+6\gamma \\ &+ 3\gamma^2+\gamma^3) + 3\mathrm{m}^2(-90+2\mathrm{s}^4(5-6\gamma+\gamma^6) \\ &+ 3\mathrm{s}^2(-5-12\gamma+2\gamma^6)))\sinh \mathrm{s})/(45\mathrm{ms}\gamma^3(\mathrm{s}(-6\mathrm{ms}+\mathrm{m}^3\mathrm{s}(6+\mathrm{s}^2))\cosh \mathrm{s} \\ &- (3\mathrm{m}^3\mathrm{s}(2+\mathrm{s}^2) - 2\mathrm{ms}(3+\mathrm{s}^2))\sinh \mathrm{s})). \end{aligned} \tag{39}$$

Kvashnin's model:

$$\begin{aligned} L_{11} = &[\mathrm{s}(54(-8+3\gamma^5) - \mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma))) \\ &- 18\mathrm{m}^2(-6(4+\gamma^5)+\mathrm{s}^2(4+\gamma(-9+4(-1+\gamma)\gamma^4))))\cosh \mathrm{s} \\ &+ 3(-6(3+\mathrm{s}^2)(-8+3\gamma^5) + 2\mathrm{m}^2(-18(4+\gamma^5) \\ &+ 3\mathrm{s}^2(-4-9\gamma-6\gamma^5+4\gamma^6) + \mathrm{s}^4(8-9\gamma-3\gamma^5+4\gamma^6)) \\ &+ \mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma))))\sinh \mathrm{s}]/[18\mathrm{m}^2\mathrm{s}^2\gamma^3(\mathrm{s}(-24+9\gamma^5 \\ &+ \mathrm{m}^2(6+\mathrm{s}^2)(4+\gamma^5))\cosh \mathrm{s} - (3\mathrm{m}^2(2+\mathrm{s}^2)(4+\gamma^5) + (3+\mathrm{s}^2)(-8+3\gamma^5))\sinh \mathrm{s})]. \end{aligned} \tag{40}$$

Mehta–Morse's/Cunnigham's model:

$$\begin{aligned} L_{11} = &[\mathrm{s}(54(2+3\gamma^5) - \mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^4(4+\gamma(7+4\gamma)) \\ &- 18\mathrm{m}^2(6-6\gamma^5+\mathrm{s}^2(-1+\gamma(3+2(-2+\gamma)\gamma^4))))\cosh \mathrm{s} + 3(-6(3+\mathrm{s}^2)(2+3\gamma^5) \\ &+ \mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^4(4+\gamma(7+4\gamma)) \\ &+ 2\mathrm{m}^2(18-18\gamma^5+\mathrm{s}^4(-2+3\gamma-3\gamma^5+2\gamma^6) \\ &+ 3\mathrm{s}^2(1+\gamma(3+2(-3+\gamma)\gamma^4))))\sinh \mathrm{s}]/[18\mathrm{m}^2\mathrm{s}^2\gamma^3(\mathrm{s}(6+9\gamma^5+\mathrm{m}^2(6+\mathrm{s}^2)(-1+\gamma^5))\cosh \mathrm{s} \\ &- (3\mathrm{m}^2(2+\mathrm{s}^2)(-1+\gamma^5) + (3+\mathrm{s}^2)(2+3\gamma^5))\sinh \mathrm{s})]. \end{aligned} \tag{41}$$

Expressions (38)–(41) agree with the results of Vasin and Filippov [16].

When $m = 1$, the value of the drag force experienced by a porous sphere of radius $\tilde{a}$ for all models comes out as:

Happel's model:

$$\tilde{F} = \frac{4\pi\mu\tilde{a}\tilde{U}[(3s^2+2s^2\gamma^5+30\gamma^5)\cosh s - 3(s^2+4s^2\gamma^5+10\gamma^5)s^{-1}\sinh s]}{\Delta_4}, \tag{42}$$

where

$$\begin{aligned} \Delta_4 = &[2s^2 - 3s^2\gamma + 3s^2\gamma^5 - 2s^2\gamma^6 + 3 + 42\gamma^5 - 30\gamma^6 + 90s^{-2}\gamma^5]\cosh s \\ &- [-3s^2\gamma + 15s^2\gamma^5 - 12s^2\gamma^6 + 3 + 72\gamma^5 - 30\gamma^6 + 90s^{-2}\gamma^5]s^{-1}\sinh s. \end{aligned} \tag{43}$$

This agrees with the result of Davis and Stone [12] for the drag force experienced by a porous sphere in a cell.

Kuwabara's model:

$$\tilde{F} = \frac{60\,\tilde{a}\pi\mathrm{s}^2\tilde{U}\tilde{\mu}\,(\sinh \mathrm{s}\text{-}\mathrm{s}\cosh \mathrm{s})}{\mathrm{s}(2\mathrm{s}^2(-5+9\gamma-5\gamma^3+\gamma^6)+15(-1-4\gamma^3+2\gamma^6))\cosh \mathrm{s} + 3(5+20\gamma^3-10\gamma^6\text{-}2\mathrm{s}^2(3\gamma-5\gamma^3+2\gamma^6))\sinh \mathrm{s}} \tag{44}$$

Kvashnin's model:

$$\tilde{F} = -\frac{24\,\pi\tilde{\mu}\tilde{a}\tilde{U}\mathrm{s}^2(\mathrm{s}(15\gamma^5+\mathrm{s}^2(4+\gamma^5))\cosh\mathrm{s} - (15\gamma^5+\mathrm{s}^2(4+6\gamma^5))\sinh\mathrm{s})}{\mathrm{s}(-270\gamma^5+\mathrm{s}^4(-1+\gamma)^3(16+21\gamma+15\gamma^2+8\gamma^3)+6\mathrm{s}^2(-4-10\gamma^3-21\gamma^5+20\gamma^6))\cosh\mathrm{s}}$$
$$+3(90\gamma^5+\mathrm{s}^2(8+20\gamma^3+72\gamma^5-40\gamma^6)+\mathrm{s}^4(-9\gamma+10\gamma^3+15\gamma^5-16\gamma^6))\sinh\mathrm{s}. \tag{45}$$

Mehta–Morse's/Cunningham's model:

$$\tilde{F} = -\frac{24\,\pi\tilde{\mu}\tilde{a}\tilde{U}\mathrm{s}^2(\mathrm{s}(15\gamma^5+\mathrm{s}^2(-1+\gamma^5))\cosh\mathrm{s} + (-15\gamma^5+\mathrm{s}^2(1-6\gamma^5))\sinh\mathrm{s})}{\mathrm{s}(-270\gamma^5+\mathrm{s}^4(-1+\gamma)^4(4+7\gamma+4\gamma^2)+6\mathrm{s}^2(1+10\gamma^3-21\gamma^5+10\gamma^6))\cosh\mathrm{s}}$$
$$-3(-90\gamma^5+\mathrm{s}^4(-1+\gamma)^3\gamma(3+9\gamma+8\gamma^2)+\mathrm{s}^2(2+20\gamma^3-72\gamma^5+20\gamma^6))\sinh\mathrm{s}. \tag{46}$$

5.3 Completely porous particles in an unbounded medium

When the limit $\tilde{b}\to\infty$, i.e., $\gamma\to 0$, and $\delta = 1$, the physical problems corresponds to the porous sphere of radius $\tilde{a}$ in an unbounded medium. In this case, the value of the hydrodynamic drag force ratio Ω comes out:

$$\Omega = -2b_4^*/3, \tag{47}$$

where the value of b_4^* is given as:

$$\begin{aligned}
\mathrm{b}_4^* = -[&3\mathrm{ms}(\mathrm{s}(-6\mathrm{ms}+\mathrm{m}^3\mathrm{s}(6+\mathrm{s}^2)\\
&+12\beta+\mathrm{m}^2(-12+\mathrm{s}^2)\beta)\cosh\mathrm{s} - (3\mathrm{m}^3\mathrm{s}(2+\mathrm{s}^2)-2\mathrm{ms}(3+\mathrm{s}^2)\\
&+4(3+\mathrm{s}^2)\beta+\mathrm{m}^2(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta)\sinh\mathrm{s})]/[\mathrm{s}(-54-9\mathrm{m}^2(-6+\mathrm{s}^2)\\
&+2\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)+27\mathrm{ms}\beta+2\mathrm{m}^3\mathrm{s}(-12+\mathrm{s}^2)\beta)\cosh\mathrm{s} - (6\mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)\\
&-18(3+\mathrm{s}^2)+\mathrm{m}^2(54+9\mathrm{s}^2-6\mathrm{s}^2)+9\mathrm{ms}(3+\mathrm{s}^2)\beta\\
&+2\mathrm{m}^3\mathrm{s}(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta)\sinh\mathrm{s}].
\end{aligned}$$

If $\beta = 0$, then the value of the hydrodynamic drag force ratio Ω:

$$\Omega = \left[1+\frac{9}{2ms^2}+\frac{1}{2(1-m)-m/Q}\right]^{-1}, \tag{48}$$

where

$$Q = 1+\frac{3}{s^2}-\left(1-\frac{\tanh s}{s}\right)^{-1}. \tag{49}$$

When $m = 1$, then the value of the hydrodynamic drag force ratio Ω for the porous sphere of radius $\tilde{a}$ comes out as:

$$\Omega = \left[\left(1-\frac{\tanh s}{s}\right)^{-1}+\frac{3}{2s^2}\right]^{-1}. \tag{50}$$

These results agree with the previously established results of Vasin and Filippov [16].

The following special cases can be deduced from the present analysis:

5.4 Solid sphere in a cell

If $\delta = 0$ or $s \to \infty$, then the porous sphere in a cell reduces to a solid sphere in a cell. In this case, the expressions for the dimensionless hydrodynamic permeability of a membrane for all the models are:

Happel's model:

$$L_{11} = \frac{-2\gamma^6 + 3\gamma^5 - 3\gamma + 2}{6\gamma^8 + 9\gamma^3}, \tag{51}$$

which agrees with the expression in Ref. [3].

Kuwabara's model:

$$L_{11} = \frac{-2\gamma^6 - 5\gamma^3 + 9\gamma - 5}{45\gamma^3}, \tag{52}$$

which agrees with the expression in Ref. [5].

Kvashnin's model:

$$L_{11} = \frac{-(\gamma - 1)^3(8\gamma^3 + 15\gamma^2 + 21\gamma + 16)}{18\gamma^8 + 72\gamma^3}, \tag{53}$$

which agrees with the expression in Ref. [6].

Mehta–Morse's/Cunningham's model:

$$L_{11} = \frac{(1 - \gamma)^3(4\gamma^2 + 7\gamma + 4)}{18\gamma^3(\gamma^4 + \gamma^3 + \gamma^2 + \gamma + 1)}, \tag{54}$$

which agrees with the expression in Ref. [7].

6 Conclusions

The following conclusions can be drawn from the above analysis:

(i) This study reveals that the jump condition for tangential stresses at the interface between porous medium and clear liquid gives a good possibility to take into account the influence of slipping on the hydrodynamic permeability of membranes having globular structure in comparison. This is very important, especially in the case of hydrophobic nanofiltration membranes.
(ii) The Cunningham (Mehta–Morse) boundary condition shows a poor description of the influence of neighbor particles on the flow in a chosen cell and, hence, cannot be recommended for applications.
(iii) A quantitative analysis of the Kozeny constant is performed for all four formulations, and the deviation in the value of the Kozeny constant for the Cunningham formulation is observed.
(iv) Likewise the previous cases, all the four formulations agree for low particle volume fraction.

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Appendix A

$$\begin{aligned} b_2^* = {} & (m(72(-1 + m^2)s^2(ms - 2\beta)(-1 + \delta) + s(72s\beta(2 - 2\delta + \delta^2) \\ & - 3m^2s\beta(48 - 3(17 + s^2)\delta + 27\delta^2 + s^2\delta^3) \\ & + 18\, m(9(-1 + \delta)\delta + s^2(-4 + 7\delta - 8\delta^2 + 3\delta^3)) \end{aligned}$$

$$
\begin{aligned}
&-3m^3(54(-1+\delta)\delta+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(-8+13\delta-15\delta^2+6\delta^3)))\cosh(s\delta)-(24s\beta\delta(3+s^2(3-3\delta+\delta^2))\\
&+6m(27(-1+\delta)+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(3-8\delta+3\delta^3))-3m^2s\beta(-3+27\delta+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(-1+8\delta-8\delta^2+3\delta^3))-9m^3(18(-1+\delta)+s^4(-2+\delta-2\delta^2+\delta^3)\\
&+s^2(5-15\delta+6\delta^3)))\sinh(s\delta)))/(-12s(18-4m^4s^2+3m^2(-6+s^2)-9ms\beta+8m^3s\beta)(1-\delta)\\
&+2s(54(2-2\delta+\delta^2)-27ms\beta(2-2\delta+\delta^2)+m^3s\beta(48-3(17+s^2)\delta+27\delta^2+s^2\delta^3)\\
&-9m^2(6(2-3\delta+2\delta^2)+s^2(-2+4\delta-5\delta^2+2\delta^3))+m^4(54(-1+\delta)\delta+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(-8+13\delta-15\delta^2+6\delta^3)))\cosh(s\delta)+(-36\delta(3+s^2(3-3\delta+\delta^2))+18ms\beta\delta(3+s^2(3-3\delta+\delta^2))\\
&-2m^3s\beta(-3+27\delta+s^4(-3+3\delta-3\delta^2+\delta^3)+3s^2(-1+8\delta-8\delta^2+3\delta^3))\\
&+6m^2(-18+36\delta+s^4(-3+3\delta-3\delta^2+\delta^3)+3s^2(2+\delta-6\delta^2+4\delta^3))-6m^4(18(-1+\delta)\\
&+s^4(-2+\delta-2\delta^2+\delta^3)+s^2(5-15\delta+6\delta^3)))\sinh(s\delta).
\end{aligned}
$$

$$
\begin{aligned}
b_3^* = {}&[m(72m(-1+m^2)s^3(-1+\delta)+s(18m(9(-1+\delta)\delta+s^2(-4+7\delta-8\delta^2+3\delta^3))\\
&-3m^3(54(-1+\delta)\delta+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(-8+13\delta-15\delta^2+6\delta^3)))\cosh(s\delta)-(6m(27(-1+\delta)+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(3-8\delta+3\delta^3))-9m^3(18(-1+\delta)+s^4(-2+\delta-2\delta^2+\delta^3)\\
&+s^2(5-15\delta+6\delta^3)))\sinh(s\delta)]/[-12s(18-4m^4s^2+3m^2(-6+s^2))(1-\delta)\\
&+2s(54(2-2\delta+\delta^2)-9m^2(6(2-3\delta+2\delta^2)+s^2(-2+4\delta-5\delta^2+2\delta^3))\\
&+m^4(54(-1+\delta)\delta+s^4(-3+3\delta-3\delta^2+\delta^3)+3s^2(-8+13\delta-15\delta^2+6\delta^3)))\cosh(s\delta)+\\
&\times(-36\delta(3+s^2(3-3\delta+\delta^2))+6m^2(-18+36\delta+s^4(-3+3\delta-3\delta^2+\delta^3)\\
&+3s^2(2+\delta-6\delta^2+4\delta^3))-6m^4(18(-1+\delta)+s^4(-2+\delta-2\delta^2+\delta^3)\\
&+s^2(5-15\delta+6\delta^3)))\sinh(s\delta)]
\end{aligned}
$$

Appendix B

Happel:

$$
\begin{aligned}
b_3^* = {}&[2ms(s(18ms(-1+\gamma^5)-36\beta(-1+\gamma^5)+m^3s(6+s^2)(3+2\gamma^5)\\
&+m^2(-12+s^2)\beta(3+2\gamma^5))\cosh s-(6ms(3+s^2)(-1+\gamma^5)-12(3+s^2)\beta(-1+\gamma^5)\\
&+3m^3s(2+s^2)(3+2\gamma^5)\\
&+m^2(-12-3s^2+s^4)\beta(3+2\gamma^5))\sinh s))]/(2s(m^4s^2(6+s^2)(-1+\gamma)^3(2+3\gamma+3\gamma^2+2\gamma^3)\\
&+m^3s(-12+s^2)\beta(-1+\gamma)^3(2+3\gamma+3\gamma^2+2\gamma^3)-54(-1+\gamma^5)\\
&-9ms\beta(3-4\gamma+2\gamma^5+4\gamma^6)+3m^2(-6(3+2\gamma^5)\\
&+s^2(3-6\gamma-8\gamma^5+6\gamma^6)))\cosh s+(-6m^4s^2(2+s^2)(-1+\gamma)^3(2+3\gamma+3\gamma^2+2\gamma^3)\\
&-2m^3s(-12-3s^2+s^4)\beta(-1+\gamma)^3(2+3\gamma+3\gamma^2+2\gamma^3)+36(3+s^2)(-1+\gamma^5)
\end{aligned}
$$

$$+6\mathrm{ms}(3+\mathrm{s}^2)\beta(3-4\gamma+2\gamma^5+4\gamma^6)-6\mathrm{m}^2(-6(3+2\gamma^5)+2\mathrm{s}^4(1-\gamma-\gamma^5+\gamma^6)$$
$$+3\mathrm{s}^2(-1-2\gamma-4\gamma^5+2\gamma^6)))\sinh \mathrm{s})$$

Kuwabara:

$$\begin{aligned}
\mathrm{b}_3^* = {} & (15\mathrm{ms}(\mathrm{s}(-6\mathrm{ms}+\mathrm{m}^3\mathrm{s}(6+\mathrm{s}^2)+12\beta+\mathrm{m}^2(-12+\mathrm{s}^2)\beta)\cosh \mathrm{s}\\
& -(3\mathrm{m}^3\mathrm{s}(2+\mathrm{s}^2)-2\mathrm{ms}(3+\mathrm{s}^2)+4(3+\mathrm{s}^2)\beta\\
& +\mathrm{m}^2(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta)\sinh \mathrm{s}))/(\mathrm{s}(270+2\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^3(5+6\gamma+3\gamma^2+\gamma^3)\\
& +2\mathrm{m}^3\mathrm{s}(-12+\mathrm{s}^2)\beta(-1+\gamma)^3(5+6\gamma+3\gamma^2+\gamma^3)\\
& -9\mathrm{ms}\beta(15-24\gamma+20\gamma^3+4\gamma^6)\\
& +9\mathrm{m}^2(-30+\mathrm{s}^2(5-12\gamma+2\gamma^6)))\cosh \mathrm{s}\\
& -(90(3+\mathrm{s}^2)+6\mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^3(5+6\gamma+3\gamma^2+\gamma^3)\\
& +2\mathrm{m}^3\mathrm{s}(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta(-1+\gamma)^3(5+6\gamma+3\gamma^2+\gamma^3)\\
& -3\mathrm{ms}(3+\mathrm{s}^2)\beta(15-24\gamma+20\gamma^3+4\gamma^6)+3\mathrm{m}^2(-90+2\mathrm{s}^4(5-6\gamma+\gamma^6)\\
& +3\mathrm{s}^2(-5-12\gamma+2\gamma^6)))\sinh \mathrm{s})
\end{aligned}$$

Kvashnin:

$$\begin{aligned}
\mathrm{b}_3^* = {} & [6\mathrm{ms}(\mathrm{s}(6\beta(8-3\gamma^5)\\
& +\mathrm{m}^3\mathrm{s}(6+\mathrm{s}^2)(4+\gamma^5)+\mathrm{m}^2(-12+\mathrm{s}^2)\beta(4+\gamma^5)\\
& +3\mathrm{ms}(-8+3\gamma^5))\cosh \mathrm{s}-(3\mathrm{m}^3\mathrm{s}(2+\mathrm{s}^2)(4+\gamma^5)\\
& +\mathrm{m}^2(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta(4+\gamma^5)+\mathrm{ms}(3+\mathrm{s}^2)(-8+3\gamma^5)-2(3+\mathrm{s}^2)\\
& \times\beta(-8+3\gamma^5))\sinh \mathrm{s})]/(\mathrm{s}(54(8-3\gamma^5)-18\mathrm{ms}\beta(12-18\gamma+10\gamma^3+3\gamma^5+8\gamma^6)\\
& +\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma)))\\
& +\mathrm{m}^3\mathrm{s}(-12+\mathrm{s}^2)\beta(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma)))+18\mathrm{m}^2(-6(4+\gamma^5)\\
& +\mathrm{s}^2(4+\gamma(-9+4(-1+\gamma)\gamma^4))))\cosh \mathrm{s}+(18(3+\mathrm{s}^2)(-8+3\gamma^5)\\
& +6\mathrm{ms}(3+\mathrm{s}^2)\beta(12-18\gamma+10\gamma^3+3\gamma^5+8\gamma^6)-6\mathrm{m}^2(-18(4+\gamma^5)\\
& +3\mathrm{s}^2(-4-9\gamma-6\gamma^5+4\gamma^6)\\
& +\mathrm{s}^4(8-9\gamma-3\gamma^5+4\gamma^6))-3\mathrm{m}^4\mathrm{s}^2(2+\mathrm{s}^2)(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma)))\\
& -\mathrm{m}^3\mathrm{s}(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta(-1+\gamma)^3(16+\gamma(21+\gamma(15+8\gamma))))\sinh \mathrm{s})
\end{aligned}$$

Mehta–Morse/Cunnigham:

$$\begin{aligned}
\mathrm{b}_3^* = {} & [12\mathrm{ms}(\mathrm{s}(\mathrm{m}^3\mathrm{s}(6+\mathrm{s}^2)(-1+\gamma^5)\\
& +\mathrm{m}^2(-12+\mathrm{s}^2)\beta(-1+\gamma^5)\\
& +3\mathrm{ms}(2+3\gamma^5)-6\beta(2+3\gamma^5))\cosh \mathrm{s}-(3\mathrm{m}^3\mathrm{s}(2+\mathrm{s}^2)(-1+\gamma^5)\\
& +\mathrm{m}^2(-12-3\mathrm{s}^2+\mathrm{s}^4)\beta(-1+\gamma^5)\\
& +\mathrm{ms}(3+\mathrm{s}^2)(2+3\gamma^5)\\
& -2(3+\mathrm{s}^2)\beta(2+3\gamma^5))\sinh \mathrm{s})]/[2\mathrm{s}(\mathrm{m}^4\mathrm{s}^2(6+\mathrm{s}^2)(-1+\gamma)^4(4+7\gamma+4\gamma^2)
\end{aligned}$$

$$+ m^3 s(-12+s^2)\beta(-1+\gamma)^4(4+7\gamma+4\gamma^2) - 54(2+3\gamma^5)$$
$$- 18ms\beta(-3+6\gamma-10\gamma^3+3\gamma^5+4\gamma^6)$$
$$+ 18m^2(6-6\gamma^5+s^2(-1+3\gamma-4\gamma^5+2\gamma^6)))\cosh s$$
$$- 2(3m^4s^2(2+s^2)(-1+\gamma)^4(4+7\gamma+4\gamma^2)$$
$$+ m^3s(-12-3s^2+s^4)\beta(-1+\gamma)^4(4+7\gamma+4\gamma^2)$$
$$- 18(3+s^2)(2+3\gamma^5) - 6ms(3+s^2)\beta(-3+6\gamma-10\gamma^3+3\gamma^5+4\gamma^6)$$
$$+ 6m^2(-18(-1+\gamma^5)+3\,s^2(1+3\gamma-6\gamma^5+2\gamma^6)$$
$$+ s^4(-2+3\gamma-3\gamma^5+2\gamma^6)))\ \sinh s);$$

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