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# The exact theory of one-dimensional quasicrystal deep beams

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**Abstract** Without employing ad hoc assumptions, various equations and solutions for quasicrystal beams are deduced systematically and directly from the plane problem of one-dimensional quasicrystals. These equations and solutions can be used to construct the exact theory of deep beams for extension or compression and bending deformation forms. A method for the solution of two-dimensional equations is presented, and with the method the exact theory can now be explicitly established from the general solution of quasicrystals and the Lur'e method. The exact governing equations for beams under transverse loadings are derived directly from the exact beam theory. In three illustrative examples of quasicrystal beams it is shown that the exact or accurate solutions can be obtained by use of the exact theory.

## 1 Introduction

Quasicrystals (QCs)—solids with a long-range quasiperiodic translational order and a long-range orientational order—as a new structure of solid matter were first discovered around in 1984 [1,2]. The discovery has brought a significant breakthrough for condensed matter physics in recent years, because it challenges long-held beliefs about the nature of translational order. The electronic structure and the optic, magnetic, thermal and mechanical properties of the material have been extensively investigated in experimental and theoretical analyses [3–6], which show their complex structure and unusual properties. The significance of QCs, in theory and practice, has created a great deal of attention by researchers in a range of fields, such as solid state physics, crystallography, materials science, applied mathematics, and solid mechanics.

Elasticity is one of the important properties of QCs. Within the framework of the Landau–Lifshitz phenomenological theory of elementary excitation of condensed matter, the elastic energies of QCs have been formulated by Bak [7,8]. Hydrodynamics equations of motion have been suggested by Lubensky et al. [9]. In particular, the field of linear elastic theory of QCs has been formulated for many years [10–13]. For a comprehensive review in this field, the readers are referred to the works by Hu et al. [14] and Fan and Mai [15].

Since the publication of the excellent work of Cheng [16] on deducing the plate theory directly from the three-dimensional theory of elasticity, several extensions have been found in the plate theory [17–19]. Moreover, Gao and coauthors indicated that applications of Cheng's method are quite successful in various beams [20–23]. The significance of Cheng's method is that it opens a systematic way of developing the

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exact and approximate lower-dimensional theory from a higher-dimensional theory with the aid of the general solution of elasticity and symbolic computation.

In the present paper, from linear elastic theory of one-dimensional (1D) QCs, the exact theory of deep beams is derived by using the general solution of 1D QCs [24] and the Lur'e method [25] without ad hoc assumptions. The exact governing equations for the beams under transverse loadings are derived from the beam theory. Meanwhile, three examples are examined to illustrate the application of the exact theory of 1D QC beams.

## 2 Basic equations and the general solution

A 1D QC is defined as a 3D body of which the atom arrangement is periodic in a plane and quasi-periodic in the orthogonal direction. A theoretical description of the deformed state of QCs requires a combined consideration of interrelated phonon and phason fields. Owing to the existence of phason fields, the elasticity of QCs is more complex than that of the conventional crystals. Assume 1D QCs periodic in the $x_1 - x_2$ plane and quasi-periodic in the $x_3$-direction in a Cartesian coordinate system $(x_1, x_2, x_3)$. For a narrow straight beam, the width in the $x_2$-direction is stress free. Therefore, it is plausible to set $\sigma_{21} = \sigma_{22} = \sigma_{23} = 0$. This is a plane stress assumption. We assume that the beam length in the $x_1$-direction is denoted by $l$, the beam width in the $x_2$-direction is set to be unit, and the beam height in the $x_3$-direction is $h$.

Wang et al. [13] derived all 31 possible 1D QCs point groups, which can be further categorized into 10 Laue classes and 6 1D QCs systems: triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal systems, and obtained a generalized Hooke law for 1D QCs. To ensure the general solution of 1D QCs, the present study is concerned, in particular, with the QCs system with three orthogonal symmetry planes. Only three kinds of Laue classes fit into this symmetry condition in 1D QCs, such as Laue class 4 in orthorhombic QCs, Laue class 6 in tetragonal QCs and Laue class 10 in hexagonal QCs. For orthorhombic QCs, the point groups 2*mm*, 222, *mmm* and *mm*2 belong to Laue class 4. Following Wang et al. [13], in the absence of body forces, the general equations governing the plane stress state of 1D orthorhombic QCs can be written as:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right), \quad w_{3\beta} = \partial_\beta w_3, \tag{1}$$

$$\partial_\beta \sigma_{\alpha\beta} = 0, \quad \partial_\beta H_{3\beta} = 0, \tag{2}$$

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{13}\varepsilon_{33} + R_1 w_{33},\\
\sigma_{33} &= C_{13}\varepsilon_{11} + C_{33}\varepsilon_{33} + R_3 w_{33},\\
\sigma_{31} &= \sigma_{13} = 2C_{55}\varepsilon_{31} + R_6 w_{31},\\
H_{33} &= R_1\varepsilon_{11} + R_3\varepsilon_{33} + K_3 w_{33},\\
H_{31} &= 2R_6\varepsilon_{31} + K_1 w_{31},
\end{aligned} \tag{3}$$

where $\alpha, \beta = 1, 3$. The elastic fields of 1D QCs are divided into two parts, namely, phonon field $u_\alpha$ in physical space and phason field $w_3$ in the complementary orthogonal space. Corresponding to the phonon and phason parameters, there are two stress fields $\sigma_{\alpha\beta}$ and $H_{3\beta}$ associated with two strain fields $\varepsilon_{\alpha\beta}$ and $w_{3\beta}$, respectively, which is a new parameter in QC elasticity and is asymmetric. The former describes the change in the shape and volume of the unit cell, and the latter describes the local rearrangement of the unit cell in the QC, while the local rearrangement is indistinguishable in a crystal. For a plane stress problem, there are nine independent constants in Eq. (3):

$$\begin{aligned}
&C_{11} = \bar{C}_{11} - \frac{\bar{C}_{12}^2}{\bar{C}_{11}}, \quad C_{13} = \bar{C}_{13} - \frac{\bar{C}_{12}\bar{C}_{13}}{\bar{C}_{11}}, \quad C_{33} = \bar{C}_{33} - \frac{\bar{C}_{13}^2}{\bar{C}_{11}}, \quad C_{55} = \bar{C}_{55},\\
&R_1 = \bar{R}_1 - \frac{\bar{C}_{12}\bar{R}_1}{\bar{C}_{11}}, \quad R_3 = \bar{R}_3 - \frac{\bar{C}_{13}\bar{R}_1}{\bar{C}_{11}}, \quad R_6 = \bar{R}_6,\\
&K_1 = \bar{K}_1, \quad K_3 = \bar{K}_3 - \frac{\bar{R}_1^2}{\bar{C}_{11}},
\end{aligned}$$

which are expressed by five elastic constants $\bar{C}_{mn}$ in phonon field, 2 constants $\bar{K}_m$ in phason field and 3 constants $\bar{R}_m$ in phonon–phason coupling field. Substituting Eq. (1) into Eq. (3) and then substituting the obtained results into Eq. (2) yields the equations of equilibrium

$$\begin{aligned}
&\left(C_{11}\partial_1^2 + C_{55}\partial_3^2\right)u_1 + (C_{13}+C_{55})\,\partial_1\partial_3 u_3 + (R_1+R_6)\,\partial_1\partial_3 w_3 = 0,\\
&(C_{13}+C_{55})\,\partial_1\partial_3 u_1 + \left(C_{55}\partial_1^2 + C_{33}\partial_3^2\right)u_3 + \left(R_6\partial_1^2 + R_3\partial_3^2\right)w_3 = 0,\\
&(R_1+R_6)\,\partial_1\partial_3 u_1 + \left(R_6\partial_1^2 + R_3\partial_3^2\right)u_3 + \left(K_1\partial_1^2 + K_3\partial_3^2\right)w_3 = 0.
\end{aligned} \tag{4}$$

According to the general solution of plane elasticity of 1D orthorhombic QCs with distinct eigenvalues [24], the components of displacements take the form:

$$u_1 = \delta_{Ii}\partial_1\psi_i, \quad u_3 = m_{1i}\partial_3\psi_i, \quad w_3 = m_{2i}\partial_3\psi_i, \tag{5}$$

where $i = 1, 2, 3$; $m_{1i}$ and $m_{2i}$ are given in Appendix A; and $\delta_{ij}$ is the Kronecker delta symbol, and the following summation convention has been used throughout this paper: the Einstein summation over repeated lower case indices from 1 to 3 is applied, while upper case indices take on the same numbers as the corresponding lower case ones but are not summed. Substitution of Eq. (5) into Eq. (4) leads to

$$\begin{aligned}
&C_{11}\partial_1^2\psi_i + \left[C_{55} + (C_{13}+C_{55})\,m_{1I} + (R_1+R_6)\,m_{2I}\right]\partial_3^2\psi_i = 0,\\
&\left[C_{13} + C_{55}\,(1+m_{1I}) + R_6 m_{2I}\right]\partial_1^2\psi_i + (C_{33}m_{1I} + R_3 m_{2I})\,\partial_3^2\psi_i = 0,\\
&\left[R_1 + R_6\,(1+m_{1I}) + K_1 m_{2I}\right]\partial_1^2\psi_i + (R_3 m_{1I} + K_3 m_{2I})\,\partial_3^2\psi_i = 0.
\end{aligned} \tag{6}$$

For the sake of compactness, the differential equation (6) can be simplified as

$$\nabla_I^2\psi_i = \partial_1^2\psi_i + \frac{1}{s_I^2}\partial_3^2\psi_i = 0. \tag{7}$$

The values of $m_{1i}$, $m_{2i}$ and $s_i^2$ are related by the following expressions:

$$\begin{aligned}
\frac{C_{55} + (C_{13}+C_{55})\,m_{1i} + (R_1+R_6)\,m_{2i}}{C_{11}} &= \frac{C_{33}m_{1i} + R_3 m_{2i}}{C_{13} + C_{55}\,(1+m_{1i}) + R_6 m_{2i}}\\
&= \frac{R_3 m_{1i} + K_3 m_{2i}}{R_1 + R_6\,(1+m_{1i}) + K_1 m_{2i}} = \frac{1}{s_i^2},
\end{aligned} \tag{8}$$

where $s_i^2$ are three eigenvalues of the cubic algebraic equation of $s^2 : as^6 - bs^4 + cs^2 - d = 0$. The constants in the preceding equations can be founded in Appendix A.

For the sake of brevity and conciseness, the exact theory of 1D QC beams will be given only to the case of distinct eigenvalues $s_i^2$ in the following context. When equal eigenvalues appear, the exact theory for these cases can be obtained by using a similar analysis technique, although the general solution will take a more complicated form for these cases [24].

For the Lur'e method [25], satisfying these requirements and treating Eq. (7) as an ordinary differential equation in $x_3$ with constant coefficients, one obtains the following symbolic solution of Eq. (7):

$$\psi_i = \cos\left(s_I x_3\partial_1\right) f_i + \frac{\sin\left(s_I x_3\partial_1\right)}{s_I\partial_1} g_i, \tag{9}$$

where $f_i$ and $g_i$ are unknown functions of $x_1$ yet to be determined, and the trigonometric differential operators $\sin\left(s_i x_3\partial_1\right)/(s_i\partial_1)$ and $\cos(s_i x_3\partial_1)$ must be interpreted as representing series in powers of $(s_i x_3\partial_1)^2$, i.e.

$$\begin{aligned}
\frac{\sin\left(s_i x_3\partial_1\right)}{s_i\partial_1} &= x_3\left(1 - \frac{1}{3!}s_i^2x_3^2\partial_1^2 + \frac{1}{5!}s_i^4x_3^4\partial_1^4 - \cdots\right),\\
\cos\left(s_i x_3\partial_1\right) &= \left(1 - \frac{1}{2!}s_i^2x_3^2\partial_1^2 + \frac{1}{4!}s_i^4x_3^4\partial_1^4 - \cdots\right).
\end{aligned} \tag{10}$$

## 3 The exact theory

Based on the loadings subjected on the top and bottom surfaces of QC beams, the general deformation of beams can be decomposed into two independent parts: the symmetric part (the extension or compression of a beam) and the asymmetric part (the bending of a beam).

### 3.1 The extension or compression of a beam

In the case of extension or compression of a beam, the beam is subjected only to symmetric loadings and edge conditions, thus only even functions of $x_3$ are required for $u_1$ and odd functions of $x_3$ for $u_3$ and $w_3$. Furthermore, the first part of $\psi_i$ in Eq. (9) can be used in the extensional problem. Substituting Eq. (9) into Eq. (5), one obtains

$$\begin{aligned} u_1 &= \cos(s_i x_3 \partial_1)\,\partial_1 f_i, \quad u_3 = -m_{1i}\sin(s_I x_3 \partial_1)\, s_I \partial_1 f_i, \\ w_3 &= -\,m_{2i}\sin(s_I x_3 \partial_1)\, s_I \partial_1 f_i. \end{aligned} \tag{11}$$

Then from Eq. (11), the displacement $U$, transverse normal strains $\phi$ and $\chi$ of the mid-plane can be found to be

$$U = u_1|_{x_3=0} = \delta_{Ii}\partial_1 f_i, \quad \phi = \partial_3 u_3|_{x_3=0} = -m_{1i}s_I^2\partial_1^2 f_i, \quad \chi = \partial_3 w_3|_{x_3=0} = -m_{2i}s_I^2\partial_1^2 f_i. \tag{12}$$

From Eq. (12), one obtains

$$f_i = \frac{1}{A}\left(n_{i1}\frac{U}{\partial_1} + n_{i2}\frac{\phi}{\partial_1^2} + n_{i3}\frac{\chi}{\partial_1^2}\right), \tag{13}$$

where the parameters $n_{ij}$ and $A$ can be found in Appendix A. From Eqs. (11) and (13), the final expressions for the displacements are

$$\begin{aligned} Au_1 &= \cos(s_i x_3 \partial_1)\left(n_{i1}U + n_{i2}\frac{\phi}{\partial_1} + n_{i3}\frac{\chi}{\partial_1}\right), \\ Au_3 &= -\sin(s_I x_3 \partial_1)\, s_I m_{1i}\left(n_{i1}U + n_{i2}\frac{\phi}{\partial_1} + n_{i3}\frac{\chi}{\partial_1}\right), \\ Aw_3 &= -\sin(s_I x_3 \partial_1)\, s_I m_{2i}\left(n_{i1}U + n_{i2}\frac{\phi}{\partial_1} + n_{i3}\frac{\chi}{\partial_1}\right). \end{aligned} \tag{14}$$

By using the generalized Hooke's law in Eq. (3), expressions (14) can be used to determine the stress components as

$$\begin{aligned} A\sigma_{31} &= -\sin(s_I x_3 \partial_1)\, s_I l_{1i}\,(n_{i1}\partial_1 U + n_{i2}\phi + n_{i3}\chi), \\ A\sigma_{11} &= -\cos(s_I x_3 \partial_1)\, s_I^2 l_{2i}\,(n_{i1}\partial_1 U + n_{i2}\phi + n_{i3}\chi), \\ A\sigma_{33} &= \cos(s_I x_3 \partial_1)\, l_{2i}\,(n_{i1}\partial_1 U + n_{i2}\phi + n_{i3}\chi), \\ AH_{31} &= \sin(s_I x_3 \partial_1)\, s_I l_{3i}\,(n_{i1}\partial_1 U + n_{i2}\phi + n_{i3}\chi), \\ AH_{33} &= \cos(s_I x_3 \partial_1)\, l_{3i}\,(n_{i1}\partial_1 U + n_{i2}\phi + n_{i3}\chi), \end{aligned} \tag{15}$$

where the parameters $l_{ij}$ are available in Appendix A.

Now let us consider the case that the beam is subjected only to the transverse surface loadings, i.e.

$$\sigma_{31} = 0, \quad \sigma_{33} = q_1(x_1)/2, \quad H_{31} = 0 \quad (x_3 = \pm h/2). \tag{16}$$

Substituting the stress expressions in Eq. (15) into the boundary conditions (16) of beams, we get the following nonhomogeneous matrix equation:

$$\begin{bmatrix} l_{1i}n_{i1}s_I SN_I\partial_1 & l_{1i}n_{i2}s_I SN_I & l_{1i}n_{i3}s_I SN_I \\ l_{2i}n_{i1}CS_I\partial_1 & l_{2i}n_{i2}CS_I & l_{2i}n_{i3}CS_I \\ l_{3i}n_{i1}CS_I\partial_1 & l_{3i}n_{i2}CS_I & l_{3i}n_{i3}CS_I \end{bmatrix} \begin{bmatrix} U \\ \phi \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ Aq_1/2 \\ 0 \end{bmatrix}, \tag{17}$$

where the differential operators $SN_i$ and $CS_i$ are defined by $SN_i = \sin(s_i h \partial_1/2)$ and $CS_i = \cos(s_i h \partial_1/2)$. Let $L_0$ be the determinant of the $3 \times 3$ matrix of the preceding equation, then there is

$$L_0 = A^2 \left(D_1 s_1 SN_1 CS_2 CS_3 + D_2 s_2 CS_1 SN_2 CS_3 + D_3 s_3 CS_1 CS_2 SN_3\right) \partial_1, \tag{18}$$

where the parameters $D_i$ can be seen in Appendix A, and satisfy $D_1 + D_2 + D_3 = 0$. In virtue of the Lur'e method [25], we get the exact governing equations in terms of $\phi$ from Eq. (17),

$$\begin{aligned} \frac{L_0}{A^2}\phi = (&E_1 CS_2 SN_3 s_3 + E_2 SN_2 CS_3 s_2 + E_3 CS_3 SN_1 s_1 \\ &+ E_4 SN_3 CS_1 s_3 + E_5 CS_1 SN_2 s_2 + E_6 SN_1 CS_2 s_1)\, \partial_1 q_1/2, \end{aligned} \tag{19}$$

where the parameters $E_m (m = 1, 2, \ldots, 6)$ can be seen in Appendix A.

## 3.2 The bending of a beam

In the case of bending of a beam, the beam is subjected only to asymmetrical loadings and edge conditions, thus only odd functions of $x_3$ are required for $u_1$ and even functions of $x_3$ for $u_3$ and $w_3$. In this case, the last part of $\psi_i$ in Eq. (9) can be used. Following the same manipulation as the extensional problem, we can obtain

$$u_1 = \frac{\sin(s_i x_3 \partial_1)}{s_I} g_i, \quad u_3 = m_{1i} \cos(s_I x_3 \partial_1)\, g_i, \quad w_3 = m_{2i} \cos(s_I x_3 \partial_1)\, g_i. \tag{20}$$

The angle of rotation $\psi$, the deflections $w$ and $s$ of the mid-plane can be found to be

$$\psi = -\partial_3 u_1 \big|_{x_3=0} = -\delta_{Ii} \partial_1 g_i, \quad w = u_3 \big|_{x_3=0} = m_{1i} g_i, \quad s = w_3 \big|_{x_3=0} = m_{2i} g_i. \tag{21}$$

From Eq. (21), one obtains

$$g_i = \frac{1}{B}\left(k_{i1}\frac{\psi}{\partial_1} + k_{i2} w + k_{i3} s\right), \tag{22}$$

in which the parameters $k_{ij}$ and $B$ can be seen in Appendix A. From Eqs. (20) and (22), final expressions for the displacements and stresses can be indicated as

$$\begin{aligned} Bu_1 &= \frac{\sin(s_i x_3 \partial_1)}{s_I}\left(k_{i1}\frac{\psi}{\partial_1} + k_{i2} w + k_{i3} s\right), \\ Bu_3 &= \cos(s_I x_3 \partial_1)\, m_{1i}\left(k_{i1}\frac{\psi}{\partial_1} + k_{i2} w + k_{i3} s\right), \\ Bw_3 &= \cos(s_I x_3 \partial_1)\, m_{2i}\left(k_{i1}\frac{\psi}{\partial_1} + k_{i2} w + k_{i3} s\right), \end{aligned} \tag{23}$$

$$\begin{aligned} B\sigma_{31} &= \cos(s_I x_3 \partial_1)\, l_{1i}\, (k_{i1}\psi + k_{i2}\partial_1 w + k_{i3}\partial_1 s), \\ B\sigma_{11} &= -\frac{\sin(s_I x_3 \partial_1)}{s_I} s_I^2 l_{2i}\, (k_{i1}\psi + k_{i2}\partial_1 w + k_{i3}\partial_1 s), \\ B\sigma_{33} &= \frac{\sin(s_I x_3 \partial_1)}{s_I} l_{2i}\, (k_{i1}\psi + k_{i2}\partial_1 w + k_{i3}\partial_1 s), \\ BH_{31} &= -\cos(s_I x_3 \partial_1)\, l_{3i}\, (k_{i1}\psi + k_{i2}\partial_1 w + k_{i3}\partial_1 s), \\ BH_{33} &= \frac{\sin(s_I x_3 \partial_1)}{s_I} l_{3i}\, (k_{i1}\psi + k_{i2}\partial_1 w + k_{i3}\partial_1 s). \end{aligned} \tag{24}$$

The bending problem considers the following boundary conditions:

$$\sigma_{31} = 0, \quad \sigma_{33} = \pm q_2(x_1)/2, \quad H_{31} = 0 \quad \left(x_3 = \pm h/2\right). \tag{25}$$

Substituting the stress expressions in Eq. (24) into the boundary conditions (25) of beams, we get the following matrix equation:

$$\begin{bmatrix} l_{1i}k_{i1}CS_I & l_{1i}k_{i2}CS_I\partial_1 & l_{1i}k_{i3}CS_I\partial_1 \\ l_{2i}k_{i1}\frac{SN_I}{s_I} & l_{2i}k_{i2}\frac{SN_I}{s_I}\partial_1 & l_{2i}k_{i3}\frac{SN_I}{s_I}\partial_1 \\ l_{3i}k_{i1}\frac{SN_I}{s_I} & l_{3i}k_{i2}\frac{SN_I}{s_I}\partial_1 & l_{3i}k_{i3}\frac{SN_I}{s_I}\partial_1 \end{bmatrix} \begin{bmatrix} \psi \\ w \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ Bq_2/2 \\ 0 \end{bmatrix}. \tag{26}$$

Let $M_0$ be the determinant of the $3 \times 3$ matrix of the preceding equation, then there is

$$M_0 = B^2\left(D_1\frac{CS_1SN_2SN_3}{s_2s_3\partial_1^4} + D_2\frac{SN_1CS_2SN_3}{s_3s_1\partial_1^4} + D_3\frac{SN_1SN_2CS_3}{s_1s_2\partial_1^4}\right)\partial_1^6. \tag{27}$$

Eliminating $\psi$ and $s$ from Eq. (26), we obtain

$$\begin{aligned} \frac{M_0}{B^2}w = \Bigg(& F_1\frac{SN_2CS_3}{s_2\partial_1} + F_2\frac{CS_2SN_3}{s_3\partial_1} + F_3\frac{SN_3CS_1}{s_3\partial_1} \\ & +F_4\frac{CS_3SN_1}{s_1\partial_1} + F_5\frac{SN_1CS_2}{s_1\partial_1} + F_6\frac{CS_1SN_2}{s_2\partial_1}\Bigg)\frac{\partial_1^2 q_2}{2}, \end{aligned} \tag{28}$$

where the parameters $F_m$ can be seen in Appendix A.

Up to here, Eqs. (19) and (28) are the exact governing equations for transverse normal strain $\phi$ and the deflection $w$ at the neutral surface of beams subject to the transverse surface loadings. In a similar way, the corresponding exact governing equations for $U$, $\chi$, $\psi$ and $s$ can be obtained. Therefore, all the expressions of displacements and stresses for QC beams can be acquired in terms of the mid-plane displacement functions.

## 4 Examples

To illustrate the applications of the exact theory developed in the previous Sections, we present the following three examples of beams: a simply supported stretching beam with an exponentially distributed load, a cantilever bending beam with a transverse concentrated load applied at the free end and a simply supported bending beam with a constant transverse distributed load. Note that the same examples for transversely isotropic elastic beams have been discussed by Gao et al. [22].

### 4.1 The exponentially loaded and simply supported stretching beam

Considering a simple stretching beam of uniform cross-section, which is simply supported at $x_1 = \pm l$ and is subjected to an exponentially distributed load along $x_1$-direction, i.e., $q_1 = q_0 \exp(\rho x_1)$, where $\rho$ and $q_0$ are constants. Since the load $q_1$ is exponentially distributed along the length of the beam, from the exact governing differential equation (28) and the Taylor series of the trigonometric differential operators in Eq. (10), the transverse normal strain of the mid-plane has the form

$$\begin{aligned} \phi = \big(& E_1CS_2'SN_3's_3 + E_2SN_2'CS_3's_2 + E_3CS_3'SN_1's_1 \\ & +E_4SN_3'CS_1's_3 + E_5CS_1'SN_2's_2 + E_6SN_1'CS_2's_1\big)\frac{A^2\rho}{2L_0'}q_1, \end{aligned} \tag{29}$$

where the expressions of $L_0'$, $SN_i'$ and $CS_i'$ are similar to those of $L_0$, $SN_i$ and $CS_i$, respectively, only except that the differential symbol $\partial_1$ is replaced by the constant $\rho$.

### 4.2 The end loaded cantilever bending beam

When the bending beam carries a transverse concentrated load or a uniformly distributed load, obviously the right part of Eq. (28) is zero, so Eq. (28) is a homogeneous differential equation in these cases. Substituting the Taylor series of the trigonometric differential operators in Eq. (10) into Eq. (28), by dropping all the terms associated with $h^2$ and the higher orders, the result turns out to be

$$D\partial_1^4 w = q_2. \tag{30}$$

After tedious manipulation, from Eq. (24) $\psi$ and $s$ can be expressed as

$$\psi = \left(1 + \frac{N + N'}{24} h^2 \partial_1^2\right) \partial_1 w, \quad s = -\frac{N''}{24} h^2 \partial_1^2 w, \tag{31}$$

where $D$ is the flexural rigidity of the QC beam,

$$D = \frac{h^3}{12} \frac{D_i s_i^2}{B\left(R_6^2 - C_{44} K_1\right)},$$

and $N$, $N'$ and $N''$ can be seen in Appendix A.

The bending moment $M(x_1)$ and the shear force $Q(x_1)$ of beams can be found to be

$$M = \int_{-h/2}^{h/2} x_3 \cdot \sigma_{33} \mathrm{d}x_3, \quad Q = \int_{-h/2}^{h/2} \sigma_{13} \mathrm{d}x_3. \tag{32}$$

Substitution of $\sigma_{33}$ and $\sigma_{13}$ from Eq. (15) into Eq. (32) with $w$, $\psi$ and $s$ given by Eqs. (30) and (31) leads to

$$M = \left(\xi + \eta h^2 \partial_1^2\right) \partial_1^2 w, \quad Q = \left(\delta + \lambda h^2 \partial_1^2\right) \partial_1 w, \tag{33}$$

where the parameters $\xi$, $\eta$, $\delta$ and $\lambda$ are available in Appendix A.

Considering a cantilever beam of uniform cross-section loaded by a transverse shear force of magnitude $Q_0$ at $x_1 = 0$ and clamped at $x_1 = l$, and the boundary conditions are

$$M(0) = 0, \quad Q(0) = Q_0, \quad w(l) = \psi(l) = 0. \tag{34}$$

From Eqs. (30), (31) and (34), one obtains the solution for the mid-plane deflection

$$w = \frac{4Q_0 l^3}{12l^2\delta + (N + N') h^2\delta - 24h^2\lambda} \left[1 - \frac{x_1^3}{l^3} - \frac{6h^2\lambda}{l^2\delta}\left(1 - \frac{x_1}{l}\right)\right] - \frac{Q_0 l}{\delta}\left(1 - \frac{x_1}{l}\right). \tag{35}$$

### 4.3 The uniformly loaded and simply supported bending beam

The third example is a bending beam of uniform cross-section which is simply supported at $x_1 = \pm l$ and which carries a uniformly distributed load of intensity $q_2 = q_0$, and the boundary conditions are

$$M(\pm l) = 0, \quad w(\pm l) = 0. \tag{36}$$

The solution for the mid-plane deflection is

$$w = \frac{q_0 l^4}{24D}\left(\frac{x_1^4}{l^4} - 6\frac{x_1^2}{l^2} + 5\right) + \frac{q_0 l^2 h^2 \eta}{2D\xi}\left(1 - \frac{x_1^2}{l^2}\right). \tag{37}$$

Up to here, these examples show that the exact or accurate solutions may be obtained by applying the exact theory deduced herein. To verify the correctness of the above solutions, we will discuss a special case to investigate its validity, i.e., a 1D QC beam reduces to a transversely isotropic elastic beam.

In this case, no phonon–phason field coupling effect is taken into account, i.e., $R_m = 0$. $s_1^2$ and $s_2^2$ relate only to elastic constants in the phonon field, while $s_3^2$ associates only with elastic constants in the phason field. The constants $m_{1i}$ and $m_{2i}$ degenerated from expressions (8) reduce to

$$m_{1k} = \frac{C_{11} - C_{44}s_k^2}{(C_{13} + C_{44})\, s_k^2} = \frac{C_{13} + C_{44}}{C_{33}s_k^2 - C_{44}}, \quad m_{2k} = 0,$$

where $k = 1, 2$. On the other hand, $m_{13} = 0$ and $m_{23} \neq 0$, which associates with $s_3^2$. For the transversely isotropic elastic beam, the flexural rigidity $D$ and some parameters in Appendix A have the forms

$$\begin{aligned} D &= \frac{C_{44}h^3\,(1 + m_{11})\,(1 + m_{12})\left(s_2^2 - s_1^2\right)}{m_{11} - m_{12}}, \quad N + N' = \frac{3\,(1 + m_{11})\,(1 + m_{12})\left(s_2^2 - s_1^2\right)}{m_{11} - m_{12}}, \\ \delta &= \frac{2C_{44}h}{3}, \quad \lambda = \frac{C_{44}l^2}{3h}, \quad \frac{\eta}{\xi} = \frac{(4m_{11} + m_{12} + 5)\, s_2^2 - (m_{11} + 4m_{12} + 5)\, s_1^2}{40\,(m_{11} + m_{12})}. \end{aligned} \tag{38}$$

Noticeably, the solutions of the mid-plane deflection in Eqs. (35) and (37) described by Eq. (38) are the same as the corresponding solutions deduced by Gao et al. [22]. Therefore, the exact theory of 1D QC beams can be degenerated into that of transversely isotropic elastic beams by omitting the phonon–phason field coupling effect. In comparison with the exact theory of transversely isotropic elastic beams, the existence of phason field influences strongly the deformation and mechanical behavior of QC materials. A theoretical description of the deformed state of QC beams requires a combined consideration of interrelated phonon and phason fields, so the beam theory of QCs is more complex than that of the conventional crystals.

## 5 Conclusion

Based on the elastic theory of QCs, an exact theory for the extension or compression and bending of a 1D QC beam has been deduced systematically and directly by using the general solution and the Lur'e method. Because the present theory is derived without requirement of any ad hoc assumptions concerning the deformation or the stress state, results based on them are of high accuracy, appeal to application and help to describe problems in an incisive way. Meanwhile, as three illustrative examples, explicit expressions of the transverse normal strain and deflection of the mid-plane are obtained for QC beams subjected to an exponentially distributed load, a transverse concentrated load and a constant distributed load, respectively. Results show that the theory developed in this paper is reliable and can serve as a basis for further applications.

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## Appendix A

$$m_{1i} = \frac{[R_3\,(R_1 + R_6) - K_3\,(C_{13} + C_{55})]\, s_i^2 + K_1\,(C_{13} + C_{55}) - R_6\,(R_1 + R_6)}{\left(R_3^2 - C_{33}K_3\right) s_i^4 + (C_{33}K_1 + C_{55}K_3 - 2R_3R_6)\, s_i^2 + R_6^2 - C_{55}K_1},$$

$$m_{2i} = \frac{[R_3\,(C_{13} + C_{44}) - C_{33}\,(R_1 + R_6)]\, s_i^2 + C_{13}R_6 - C_{55}R_1}{\left(R_3^2 - C_{33}K_3\right) s_i^4 + (C_{33}K_1 + C_{55}K_3 - 2R_3R_6)\, s_i^2 + R_6^2 - C_{55}K_1},$$

$$a = C_{55}\left(R_3^2 - C_{33}K_3\right),$$

$$\begin{aligned} b &= \left(C_{13}^2 + 2C_{13}C_{55} - C_{11}C_{33}\right) K_3 - C_{33}C_{55}K_1 - 2\,(C_{13} + C_{55})\, R_1R_3 \\ &\quad + 2C_{33}R_1\,(R_1 + R_6) - 2C_{13}R_3R_6 + C_{11}R_3^2, \end{aligned}$$

$$c = \left(C_{13}^2 + 2C_{13}C_{55} - C_{11}C_{33}\right) K_1 - C_{11}C_{55}K_3 - 2C_{13}R_6\,(R_1 + R_3) + 2C_{11}R_3R_6 + C_{55}R_1^2,$$

$$d = C_{11}\left(R_6^2 - C_{55}K_1\right),$$

$$n_{11} = (m_{12}m_{23} - m_{13}m_{22})\, s_2^2 s_3^2, \quad n_{12} = m_{23}s_3^2 - m_{22}s_2^2, \quad n_{13} = m_{12}s_2^2 - m_{13}s_3^2,$$

$$n_{21} = (m_{13}m_{21} - m_{11}m_{23})\, s_3^2 s_1^2, \quad n_{22} = m_{21}s_1^2 - m_{23}s_3^2, \quad n_{23} = m_{13}s_3^2 - m_{11}s_1^2,$$

$$n_{31} = (m_{11}m_{22} - m_{12}m_{21})\, s_1^2 s_2^2, \quad n_{32} = m_{22}s_2^2 - m_{21}s_1^2, \quad n_{33} = m_{11}s_1^2 - m_{12}s_2^2,$$

$$A = \left(m_{11}s_1^2 - m_{12}s_2^2\right)\left(m_{21}s_1^2 - m_{23}s_3^2\right) + \left(m_{11}s_1^2 - m_{13}s_3^2\right)\left(m_{22}s_2^2 - m_{21}s_1^2\right),$$

$$l_{1i} = -l_{2i} = C_{55}\,(1 + m_{1i}) + R_6 m_{2i}, \quad l_{3i} = -R_6\,(1 + m_{1i}) - K_1 m_{2i},$$

$$D_1 = l_{11}\,(l_{22}l_{33} - l_{23}l_{32}), \quad D_2 = l_{12}\,(l_{23}l_{31} - l_{21}l_{33}), \quad D_3 = l_{13}\,(l_{21}l_{32} - l_{22}l_{31}),$$

$$E_1 = l_{13}l_{32}m_{11}s_1^2, \quad E_2 = -l_{12}l_{33}m_{11}s_1^2, \quad E_3 = l_{11}l_{33}m_{12}s_2^2,$$

$$E_4 = -l_{13}l_{31}m_{12}s_2^2, \quad E_5 = l_{12}l_{31}m_{13}s_3^2, \quad E_6 = -l_{11}l_{32}m_{13}s_3^2,$$

$$k_{11} = m_{12}m_{23} - m_{13}m_{22}, \quad k_{12} = m_{23} - m_{22}, \quad k_{13} = m_{12} - m_{13},$$

$$k_{21} = m_{13}m_{21} - m_{11}m_{23}, \quad k_{22} = m_{21} - m_{23}, \quad k_{23} = m_{13} - m_{11},$$

$$k_{31} = m_{11}m_{22} - m_{12}m_{21}, \quad k_{32} = m_{22} - m_{21}, \quad k_{33} = m_{11} - m_{12},$$

$$B = (m_{11} - m_{12})\,(m_{23} - m_{21}) + (m_{11} - m_{13})\,(m_{21} - m_{22}),$$

$$F_1 = l_{13}l_{32}m_{11}, \quad F_2 = -l_{12}l_{33}m_{11}, \quad F_3 = l_{11}l_{33}m_{12},$$

$$F_4 = -l_{13}l_{31}m_{12}, \quad F_5 = l_{12}l_{31}m_{13}, \quad F_6 = -l_{11}l_{32}m_{13},$$

$$N = -\frac{[3F_3 + F_4 + F_5 + 3F_6]\, s_1^2 + [F_1 + 3F_2 + 3F_5 + F_6]\, s_2^2 + [3F_1 + F_2 + F_3 + 3F_4]\, s_3^2}{B\left(R_6^2 - C_{44}K_1\right)},$$

$$N' = -\frac{\left[3F_3' + F_4' + F_5' + 3F_6'\right] s_1^2 + \left[F_1' + 3F_2' + 3F_5' + F_6'\right] s_2^2 + \left[3F_1' + F_2' + F_3' + 3F_4'\right] s_3^2}{B\left(R_6^2 - C_{44}K_1\right)},$$

$$F_1' = l_{13}l_{32}, \quad F_2' = -l_{12}l_{33}, \quad F_3' = l_{11}l_{33},$$

$$F_4' = -l_{13}l_{31}, \quad F_5' = l_{12}l_{31}, \quad F_6' = -l_{11}l_{32},$$

$$N'' = -\frac{\left[3F_3'' + F_4'' + F_5'' + 3F_6''\right] s_1^2 + \left[F_1'' + 3F_2'' + 3F_5'' + F_6''\right] s_2^2 + \left[3F_1'' + F_2'' + F_3'' + 3F_4''\right] s_3^2}{B\left(R_6^2 - C_{44}K_1\right)},$$

$$F_1'' = l_{13}l_{32}m_{21}, \quad F_2'' = -l_{12}l_{33}m_{21}, \quad F_3'' = l_{11}l_{33}m_{22},$$

$$F_4'' = -l_{13}l_{31}m_{22}, \quad F_5'' = l_{12}l_{31}m_{23}, \quad F_6'' = -l_{11}l_{32}m_{23},$$

$$\xi = \frac{h^3 s_I^2}{12B} l_{1i}\,(k_{i1} + k_{i2}), \eta = \frac{h^3 s_I^2}{12B} l_{1i}\left[\frac{k_{i1}\left(N + N'\right) - k_{i3}N''}{24} - \frac{1}{40}s_i^2\right],$$

$$\delta = \frac{h}{B} l_{1i}\,(k_{i1} + k_{i2}), \quad \lambda = \frac{h}{24B} l_{1i}\left[k_{i1}\left(N + N'\right) - k_{i3}N'' - s_i^2\right].$$

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