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Bending analysis of FG viscoelastic sandwich beams with elastic cores resting on Pasternak's elastic foundations

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Abstract The investigation of bending response of a simply supported functionally graded (FG) viscoelastic sandwich beam with elastic core resting on Pasternak's elastic foundations is presented. The faces of the sandwich beam are made of FG viscoelastic material while the core is still elastic. Material properties are graded from the elastic interfaces through the viscoelastic faces of the beam. The elastic parameters of the faces are considered to be varying according to a power-law distribution in terms of the volume fraction of the constituent. The interaction between the beam and the foundations is included in the formulation. Numerical results for deflections and stresses obtained using the refined sinusoidal shear deformation beam theory are compared with those obtained using the simple sinusoidal shear deformation beam theory, higher- and first-order shear deformation beam theories. The effects due to material distribution, span-to-thickness ratio, foundation stiffness and time parameter on the deflection and stresses are investigated.

1 Introduction

Sandwich structures are often found in aerospace applications such as in the skin of wings, vertical fin torque box, aileron, spoilers, etc. The advantages of these structures are that they provide high specific stiffness and strength-to-weight ratios, good fatigue properties, good thermal and acoustical insulation and ease of mass production [1]. Recently, sandwich construction becomes even more attractive due to the introduction of advanced composite materials for the faces and the core. Considerable effort has been devoted to study viscoelastic sandwich beam problems, with relatively little work directed toward developing numerical models that might be applicable to more general beams and support conditions.

The damping behavior of a 0° laminated sandwich composite beam inserted with a viscoelastic layer is investigated by Yim et al. [2]. Barbosa and Farage [3] studied a sandwich viscoelastic beam based on a finite element model. In this work, an assessment of a time-domain formulation for numerical modelling of viscoelastic materials was made. This formulation is based on a second-order time-domain realization of Laplace-domain

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motion equations. Bekuit et al. [4] presented a quasi-two-dimensional finite element formulation for the static and dynamic analysis of sandwich viscoelastic beams that are composed of three layers. Nayfeh [5] developed a model to study the vibration parallel to the plane of lamination of a symmetric five-layer elastic–viscoelastic sandwich beam. The dynamic stability of a rotating composite beam with a viscoelastic core subjected to axial periodic loads was investigated by Lin and Chen [6] using the finite element method. In Yan et al. [7], the two-dimensional problem of a simply supported laminated orthotropic strip with viscoelastic interfaces under static loading was investigated. Barkanov et al. [8] employed the finite element method to analyze the transient response of sandwich beams, plates and shells with viscoelastic layers under impulse loading. Galucio et al. [9] presented a finite element formulation for transient dynamic analysis of sandwich beams with embedded viscoelastic material using fractional derivative constitutive equations. In their analysis, the sandwich configuration is composed of a viscoelastic core sandwiched between elastic faces. Analysis and calculating approaches as to sandwich viscoelastic beams may also be found in references [10–13].

In the above conventional sandwich structures, homogeneous elastic and viscoelastic layers are bonded together to obtain enhanced mechanical properties. However, the sudden change in material properties across the interfaces among different materials can result in large interlaminar stresses. To overcome these adverse effects, a new class of advanced inhomogeneous composite materials, composed of two or more phases with different material properties and continuously varying composition distribution, has been developed which is referred to as functionally graded materials (FGMs). Such materials were introduced as to take advantage of the desired material properties of each constituent material without interface problems.

The mechanical properties of FGMs are often being represented in the exponentially graded form [14–16] and power-law variations one [17–26]. Reddy [22] analyzed the static behavior of FG rectangular plates based on his third-order shear deformation plate theory. Reddy and Chin [23] studied the dynamic thermoelastic response of FG cylinders and plates. In Reddy and Cheng [24], three-dimensional thermomechanical deformations of simply supported, FG rectangular plates were studied by using an asymptotic method. A new beam element has been developed to study the thermoelastic behavior of FG beam structures by Chakraborty et al. [27] using the first-order shear deformation theory. A crack in a viscoelastic strip of an FGM under tensile loading conditions was studied by Jin and Paulino [28]. More reports on FG structures may also be found in the literature, such as Refs. [29–31].

Although there are research works reported on general FG structures, studies related to FGM sandwich structures are few in number. Zenkour [17,18] employed the sinusoidal shear deformation plate theory to study the bending response, buckling and free vibration of a simply supported functionally graded sandwich plate. Three-dimensional finite element simulations for analyzing low velocity impact behavior of sandwich beams with a functionally graded core were conducted by Etemadi et al. [32]. Anderson [33] presented an analytical three-dimensional elasticity solution method for a sandwich composite with a functionally graded core subjected to transverse loading by a rigid spherical indentor. Ávila [34] proposed a failure mode criterion for piece-wise functionally graded sandwich beams and compared it against experimental data. In Bhangale and Ganesan [35], the buckling and vibration of a FGM sandwich beam having viscoelastic layer was studied in thermal environment by using a finite element formulation.

Beams resting on elastic foundations have wide applications in modern engineering and pose great technical problems in structural design. This motivated many researchers to analyze the behavior of beam structures on various types of elastic foundations. Thermo-mechanical vibration analysis of FG sandwich beams resting on variable elastic foundations was investigated by Pradhan and Murmu [36]. Ying et al. [37] presented exact solutions for bending and free vibration of functionally graded beams resting on Winkler-Pasternak elastic foundations. Aköz and Kadioğlu [38] studied circular beams with variable cross-sections on elastic foundations under arbitrary loading using the finite element method and using the Winkler's hypothesis for the foundation. Chen et al. [39] proposed a mixed method, which combined the state space method and the differential quadrature method, for bending and free vibration of arbitrarily thick beams resting on Pasternak's elastic foundations. Natural frequencies and buckling stresses of a deep beam-column on two-parameter elastic foundations were analyzed by Matsunaga [40] using a one-dimensional higher order theory and taking into account the effect of shear deformation depth change and rotatory inertia. Sato et al. [41] presented the mathematical hypothesis that a beam on equidistant elastic supports can be considered as a beam on an elastic foundation in static and free vibration problems. Tsiatas [42] has presented the solution of the nonlinear problem of non-uniform beams on nonlinear triparametric elastic foundations.

In this article, we restrict our attention on the FG viscoelastic sandwich beam with elastic core, which has found application in functionally graded materials. The present beam is symmetric about its mid-plane and resting on Pasternak's elastic foundations. The faces are considered as FG viscoelastic material whereas the core is assumed to be elastic. The elastic properties of the faces are considered to vary according to power-law variations. The refined sinusoidal shear deformation beam theory (RSBT) (see [14,20]) is used for the bending response of the present beam resting on Pasternak's elastic foundations. One of the advantages of this theory is that the effects of normal and shear deformation are both included. To investigate this effect, the convergence of the deflection and stresses are compared with those obtained using the simple sinusoidal shear deformation beam theory (SSBT), the higher-order shear deformation beam theory (HOBT) and the first-order shear deformation beam theory (FOBT). The effective moduli methods are used to solve the governing equations. The effects of the foundations on the FG viscoelastic sandwich beams are included in the formulation with a two-parameter Pasternak model.

2 Problem formulation

Figure 1 shows an elastic–viscoelastic FG beam of length a, width b and thickness h. Rectangular Cartesian coordinates x_i are used to describe infinitesimal deformations of a three-layer viscoelastic sandwich beam. The mid-plane of the composite sandwich beam is defined by $x_3 = 0$. The vertical positions of the bottom and top, and of the two interfaces between the layers are denoted by $h_0 = -h/2,\ h_1,\ h_2,\ h_3 = h/2$. The faces are made of FG viscoelastic material while the core is still elastic. The material properties vary smoothly in the x_3 direction from the interfaces ($x_3 = h_1$ and $x_3 = h_2$) to the outer surfaces of the faces ($x_3 = \pm h/2$).

It is assumed that the top surface of the FG viscoelastic sandwich beam is subjected to sinusoidal distribution load $q(x)$ and the lower one is supported by two-parameter elastic foundations. In this model, the boundary conditions on the beam surfaces are:

$$\begin{aligned} &\sigma_3 = K_1 w - K_2 w_{,11}, \quad \sigma_5 = 0 \quad \text{at} \quad x_3 = h_0; \\ &\sigma_3 = -q(x), \quad \sigma_5 = 0 \quad \text{at} \quad x_3 = h_3, \end{aligned} \tag{1}$$

where K_1 is Winkler's foundation stiffness while K_2 is the shear stiffness of the elastic foundation and the comma followed by index j, for example, denotes the differentiation with respect to the position x_j ($j = 1, 3$) of material particle. Here, we replace the indices xx, zz and zx of the stresses and strains by 1, 3 and 5, respectively.

The stress–strain relationships accounting for transverse shear deformation in the beam coordinates can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_5 \end{Bmatrix}^{(n)} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{13} & 0 \\ \bar{c}_{13} & \bar{c}_{33} & 0 \\ 0 & 0 & \bar{c}_{55} \end{bmatrix}^{(n)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_5 \end{Bmatrix}, \tag{2}$$

or

$$\{\sigma\}^{(n)} = [\bar{c}(x_3)]^{(n)} \{\varepsilon\}, \quad (n = 1,\ 2,\ 3). \tag{3}$$

The elastic stiffness coefficients are assumed to vary in the x_3 direction. Using the power-law index distribution, the elasticity matrix for each layer can be written as

$$[\bar{c}(x_3)]^{(n)} = [c_v] + ([c_e] - [c_v])\, V^{(n)}, \tag{4}$$

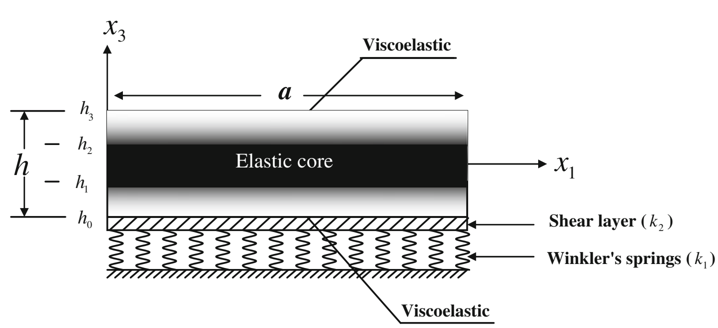

Fig. 1 Geometry of the FG viscoelastic sandwich beam resting on Pasternak's elastic foundations

where $[c_v]$ and $[c_e]$ represent the matrix of viscoelastic and elastic parameters, respectively. These matrices are defined as

$$[c_i] = \frac{E_i}{1-\nu_i^2}\begin{bmatrix} 1 & \nu_i & 0 \\ \nu_i & 1 & 0 \\ 0 & 0 & \frac{1-\nu_i}{2} \end{bmatrix}, \quad (i = e,\ v), \tag{5}$$

where E_e and ν_e are Young's modulus and Poisson's ratio of the elastic material and they take constant values. The viscoelastic modulus E_v and the corresponding Poisson's ratio ν_v may be given in terms of the coefficient of volume compression K and the kernel of the relaxation function $\omega(t) = \hat{\omega}$ by

$$E_v(t) = \frac{9K\hat{\omega}}{2+\hat{\omega}}, \qquad \nu_v(t) = \frac{1-\hat{\omega}}{2+\hat{\omega}}. \tag{6}$$

The volume fraction $V^{(n)}$ follows a simple power-law through the thickness of the sandwich beam faces while it equals unity in the core layer. It reads [17,18]

$$V^{(1)} = \left(\frac{x_3 - h_0}{h_1 - h_0}\right)^m, \qquad h_0 \le x_3 \le h_1, \tag{7}$$

$$V^{(2)} = 1, \qquad h_1 \le x_3 \le h_2, \tag{8}$$

$$V^{(3)} = \left(\frac{x_3 - h_3}{h_2 - h_3}\right)^m, \qquad h_2 \le x_3 \le h_3, \tag{9}$$

where m denotes the power-law index ($0 \le m \le \infty$). The core layer is independent of the value of m, which makes it fully elastic layer, whereas the value of m equal to zero represents a fully elastic beam. The above power-law assumption given in Eqs. (7) and (9) reflects a simple rule of mixtures used to obtain the effective properties of the elastic–viscoelastic beam faces (see Fig. 1). Note that the volume fraction of the elastic material is high near the interfaces, and that of the viscoelastic one is high near the bottom and the top surfaces of the beam. In addition, Eqs. (7)–(9) indicate that the top and bottom surfaces of the beam are viscoelastic-rich while the bottom ($x_3 = h_1$) and top ($x_3 = h_2$) surfaces of the core are elastic-rich.

The displacements of a material point located at $(x_1,\ x_3)$ in the beam may be written as follows:

$$\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} u \\ w \end{Bmatrix} + \alpha\, x_3 \begin{Bmatrix} w_{,1} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \psi\, \varphi_1 \\ \beta \psi_{,3}\, \varphi_3 \end{Bmatrix}, \tag{10}$$

where u and w denote the in-plane and transverse displacements of points on the neutral axis of the beam, φ_1 represents the rotation of a transverse normal about the x_2-axis, φ_3 is an additional undetermined function of x_1. All of the generalized displacements $(u,\ w,\ \varphi_1,\ \varphi_3)$ are functions of x_1.

The above displacement field gives the following theories:

- Refined sinusoidal beam theory (RSBT): $\alpha = -1,\ \beta = 1,\ \psi(x_3) = h\sin(\pi x_3/h)/\pi$.
- Simple sinusoidal beam theory (SSBT): $\alpha = -1,\ \beta = 0,\ \psi(x_3) = h\sin(\pi x_3/h)/\pi$.
- Higher-order beam theory (HOBT): $\alpha = -1,\ \beta = 0,\ \psi(x_3) = x_3\left[1 - 4x_3^2/3h^2\right]$.
- First-order beam theory (FOBT): $\alpha = \beta = 0,\ \psi(x_3) = x_3$.

The corresponding strain components associated with the displacement field in Eq. (10) are

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_5 \end{Bmatrix} = \begin{Bmatrix} e_1 \\ 0 \\ e_5 \end{Bmatrix} + x_3 \begin{Bmatrix} \zeta \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \psi\, \eta_1 \\ \psi_{,33}\, \eta_3 \\ \psi_{,3}\, \eta_5 \end{Bmatrix}, \tag{11}$$

where

$$\begin{aligned} &e_1 = u_{,1}, \qquad \zeta = \alpha w_{,11}, \qquad \eta_1 = \varphi_{1,1}, \\ &\eta_3 = \beta\, \varphi_3, \quad e_5 = (\alpha + 1) w_{,1}, \quad \eta_5 = \varphi_1 + \beta\, \varphi_{3,1}. \end{aligned} \tag{12}$$

The principle of virtual work in the present study yields

$$\int_{\Omega}\left[\int_{-h/2}^{+h/2}\left(\sigma_1^{(n)}\delta\varepsilon_1+\sigma_3^{(n)}\delta\varepsilon_3+\sigma_5^{(n)}\delta\varepsilon_5\right)\mathbf{d}x_3+\left(K_1w-K_2w_{,11}-q\right)\delta w\right]\mathbf{d}\Omega=0, \tag{13}$$

or

$$\int_0^L\left[N\delta e_1+Q\delta e_5+S\delta\eta_3+R\delta\eta_5+M\delta\zeta+P\delta\eta_1+\left(K_1w-K_2w_{,11}-q\right)\delta w\right]b\mathbf{d}x_1=0, \tag{14}$$

where N, Q, S, R, M and P are the stress resultants which can be expressed as

$$\begin{aligned}&(N,\ M,\ P)=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}(1,\ x_3,\ \psi)\sigma_1^{(n)}\mathbf{d}x_3,\qquad S=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\psi_{,33}\sigma_3^{(n)}\mathbf{d}x_3,\\&(Q,\ R)=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\bar{K}(1,\ \psi_{,3})\sigma_5^{(n)}\mathbf{d}x_3,\end{aligned} \tag{15}$$

where $\bar{K}$ is the shear correction factor of FOBT and it is fixed to be 5/6; h_n and h_{n-1} are the top and bottom x_3-coordinates of the nth layers.

3 Governing equations

The governing equations of the equilibrium associated with the displacement field in Eq. (10) are

$$\begin{aligned}&N_{,1}=0,\\&\alpha\,M_{,11}+(\alpha+1)Q_{,1}+K_1w-K_2w_{,11}-q=0,\\&P_{,1}-R=0,\\&\beta\,(P_{,11}-S)=0.\end{aligned} \tag{16}$$

Using Eqs. (2) and (15), the force and moment resultants can be related to the total strains in the following equations:

$$\begin{aligned}&\begin{Bmatrix}N\\M\\P\\S\end{Bmatrix}=\begin{bmatrix}F_{11}&F_{12}&F_{13}&F_{14}\\&F_{22}&F_{23}&F_{24}\\&&F_{33}&F_{34}\\\text{symm.}&&&F_{44}\end{bmatrix}\begin{Bmatrix}e_1\\\zeta\\\eta_1\\\eta_3\end{Bmatrix},\\&\begin{Bmatrix}Q\\R\end{Bmatrix}=\begin{bmatrix}H_{11}&0\\0&H_{22}\end{bmatrix}\begin{Bmatrix}e_5\\\eta_5\end{Bmatrix}.\end{aligned} \tag{17}$$

The elements of matrices $[F]$ and $[H]$ are given as follows:

$$\begin{aligned}&\begin{bmatrix}F_{11}&F_{12}&F_{13}\\&F_{22}&F_{23}\\\text{symm.}&&F_{33}\end{bmatrix}=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\bar{c}_{11}^{(n)}\begin{bmatrix}1\\x_3\\\psi\end{bmatrix}\left[1\ x_3\ \psi\right]\mathbf{d}x_3,\\&\{F_{14},\ F_{24},\ F_{34}\}=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\bar{c}_{13}^{(n)}\{1,\ x_3,\ \psi\}\,\psi_{,33}\,\mathbf{d}x_3,\quad F_{44}=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\bar{c}_{33}^{(n)}(\psi_{,33})^2\mathbf{d}x_3,\\&\{H_{11},\ H_{22}\}=\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}\bar{c}_{55}^{(n)}\psi_{,3}\{1,\ \psi_{,3}\}\mathbf{d}x_3.\end{aligned} \tag{18}$$

Substituting Eq. (17) into Eq. (16), we obtain the following equations for all theories:

$$\begin{gathered}F_{11}\nabla^2 u + F_{12}\alpha\nabla^2 w_{,1} + F_{13}\nabla^2\varphi_1 + F_{14}\beta\varphi_{3,1} = 0,\\ \alpha(F_{12}\nabla^2 u_{,1} + F_{22}\alpha\nabla^2 w_{,11} + F_{23}\nabla^2\varphi_{1,1} + F_{24}\beta\nabla^2\varphi_3) + [(\alpha+1)^2 H_{11} + K_2]\nabla^2 w - K_1 w + q = 0\\ F_{13}\nabla^2 u + F_{23}\alpha\nabla^2 w_{,1} + F_{33}\nabla^2\varphi_1 + F_{34}\beta\varphi_{3,1} - H_{22}(\varphi_1 + \beta\varphi_{3,1}) = 0,\\ \beta(F_{14}u_{,1} + F_{24}\alpha\nabla^2 w + F_{34}\varphi_{1,1} + F_{44}\beta\varphi_3) - H_{22}\beta(\varphi_{1,1} + \beta\nabla^2\varphi_3) = 0.\end{gathered} \tag{19}$$

For further computational reasons, the converted expressions of the stresses as functions of the volume fraction V^n are also recorded. They read

$$\sigma_1^{(n)} = (\phi^{(n)} + \hat{\phi}^{(n)})(u_{,1} + \alpha\, x_3 w_{,11} + \psi\ \varphi_{1,1}) + (\phi^{(n)}\nu_v + \hat{\phi}^{(n)}\ \nu_e)(\beta\psi_{,33}\ \varphi_3), \tag{20}$$

$$\sigma_3^{(n)} = (\phi^{(n)}\nu_v + \hat{\phi}^{(n)}\ \nu_e)\left(u_{,1} + \alpha\, x_3 w_{,11} + \psi\ \varphi_{1,1}\right) + (\phi^{(n)} + \hat{\phi}^{(n)})\beta\ \psi_{,33}\ \varphi_3, \tag{21}$$

$$\sigma_5^{(n)} = \frac{1}{2}\left[(1-\nu_v)\phi^{(n)} + (1-\nu_e)\hat{\phi}^{(n)}\right]\left[(1+\alpha)w_{,1} + (\varphi_1 + \beta\,\varphi_{3,1})\psi_{,3}\right], \tag{22}$$

where

$$\phi^{(n)} = \frac{(1-V^{(n)})E_v}{1-\nu_v^2}, \qquad \hat{\phi}^{(n)} = \frac{V^{(n)}E_e}{1-\nu_e^2}. \tag{23}$$

4 Exact solutions for FGMs sandwich beams

In order to show the applicability and reliability of the present refined sinusoidal shear deformation beam theory for the analysis of transverse normal and shear deformations in beams, a simply supported FGMs sandwich viscoelastic beam resting on two-parameter elastic foundations is analyzed. The following boundary conditions for all theories are assumed:

$$w = \beta\varphi_3 = N = \alpha M = P = 0 \quad \text{at} \quad x_1 = 0,\ a. \tag{24}$$

Following the Navier solution procedure, displacement components and the external force for the case of sinusoidally distributed load may be expressed as

$$\begin{Bmatrix} u\\ w\\ \varphi_1\\ \varphi_3\end{Bmatrix} = \begin{Bmatrix} U\cos(\lambda x_1)\\ W\sin(\lambda x_1)\\ \Phi_1\cos(\lambda x_1)\\ \Phi_3\sin(\lambda x_1)\end{Bmatrix}, \tag{25}$$

$$q = q_0(t)\sin(\lambda x_1), \tag{26}$$

where $\lambda = \pi/a$ and $U,\ W,\ \Phi_1$ and Φ_3 are arbitrary parameters. Note that $q_0(t)$ is a transient function according to the viscoelastic response of the bending problem.

With the help of Eqs. (25) and (26), Eq. (19) becomes

$$[A]\{\Delta\} = \{f\}, \tag{27}$$

where $\{\Delta\}$ and $\{f\}$ denote the columns

$$\begin{aligned}\{\Delta\}^{\mathrm{T}} &= \{U,\ W,\ \Phi_1,\ \Phi_3\},\\ \{f\}^{\mathrm{T}} &= \{0,\ q_0(t),\ 0,\ 0\},\end{aligned} \tag{28}$$

and the elements $A_{ij} = A_{ji}$ of matrix $[A]$ are given by:

$$\begin{aligned}&A_{11} = \lambda^2 F_{11},\quad A_{12} = \alpha\lambda^3 F_{12},\quad A_{13} = \lambda^2 F_{13},\quad A_{14} = -\beta\lambda F_{14},\\ &A_{22} = \alpha^2\lambda^4 F_{22} - \left[(1+\alpha)^2 H_{11} - K_2\right]\lambda^2 + K_1,\\ &A_{23} = \alpha\lambda^3 F_{23},\qquad A_{24} = -\alpha\,\beta\lambda^2\,F_{24},\\ &A_{33} = \lambda^2 F_{33} + H_{22},\qquad A_{34} = -\beta\lambda\,(F_{34} - H_{22}),\\ &A_{44} = \beta^2(F_{44} + \lambda^2 H_{22}).\end{aligned} \tag{29}$$

5 Method of time-dependent solution

For a given set of values of $\hat{\omega}$, the corresponding elastic solution is obtained numerically with $q_0(t) = \hat{q}_0$ where $\hat{q}_0$ is a constant. In the elastic composites, the stresses are functions of $x_1,\ x_3$ and $\hat{\omega}$, while in the viscoelastic composites they are operator functions of $x_1,\ x_3$ and time t. According to Illyushin's approximation method [43] (see also [44,45]), the stresses $\sigma_i^{(n)}$, $i = 1,\ 3,\ 5$ can be represented in the form

$$\sigma_i^{(n)}(x_1,\ x_3,\ \hat{\omega}) = \sum_{j=1}^{5} \Re_{ij}^{(n)}(x_1,\ x_3)\Psi_j(\hat{\omega}), \tag{30}$$

where $\Psi_j(\hat{\omega})$ are some known kernels, constructed on the base of the kernel $\hat{\omega}$ and may be chosen in the form

$$\Psi_1 = 1, \quad \Psi_2 = \hat{\omega}, \quad \Psi_3 = \Pi(t) = \frac{1}{\hat{\omega}}, \quad \Psi_4 = g_{\frac{1}{2}}(t) = \frac{1}{1 + \frac{1}{2}\hat{\omega}}, \quad \Psi_5 = g_2(t) = \frac{1}{1 + 2\hat{\omega}}. \tag{31}$$

The functions $\Re_{ij}^{(n)}(x_1,\ x_3)$ can be determined, for example, so as to achieve a minimum squared error fit to $\sigma_i^{(n)}(x_1,\ x_3,\ \hat{\omega})$. For this reason, we have the following expression:

$$J = \int_{\hat{\omega}_0}^{\hat{\omega}_1} \left[\sigma_i^{(n)}(x_1,\ x_3,\ \hat{\omega}) - \sum_{j=1}^{5} \Re_{ij}^{(n)}(x_1,\ x_3)\Psi_j(\hat{\omega}) \right]^2 \mathbf{d}\hat{\omega}, \tag{32}$$

where $\hat{\omega}_0$ and $\hat{\omega}_1$ are the limits of kernel $\hat{\omega}$ and $0 \le \hat{\omega}_0 \le \hat{\omega}_1 \le 1$. Taking $\hat{\omega}_0 = 0$ and $\hat{\omega}_1 = 1$ and equating $\partial J / \partial \Re_{ij}^{(n)}$ to zero, we have the following system of algebraic equations:

$$\int_0^1 \sum_{j=1}^{5} \Re_{ij}^{(n)}(x_1,\ x_3)\Psi_k(\hat{\omega})\Psi_j(\hat{\omega})\mathbf{d}\hat{\omega} = \int_0^1 \Psi_k(\hat{\omega})\sigma_i^{(n)}(x_1,\ x_3,\ \hat{\omega})\mathbf{d}\hat{\omega}. \tag{33}$$

The viscoelastic solution may be recorded to determine explicit formulae for $\sigma_i^{(n)}$ as functions of the coordinates x_1 and x_3 and time t. Then

$$\begin{aligned} \sigma_i^{(n)}(x_1,\ x_3,\ t) = {} & \Re_{i1}^{(n)} q_0(t) + \Re_{i2}^{(n)} \int_0^t \omega(t-\tau)\mathbf{d}q_0(\tau) + \Re_{i3}^{(n)} \int_0^t \Pi(t-\tau)\mathbf{d}q_0(\tau) \\ & + \Re_{i4}^{(n)} \int_0^t g_{\frac{1}{2}}(t-\tau)\mathbf{d}q_0(\tau) + \Re_{i5}^{(n)} \int_0^t g_2(t-\tau)\mathbf{d}q_0(\tau). \end{aligned} \tag{34}$$

Taking $q_0(t) = \hat{q}_0 H(t)$, where $H(t)$ is Heaviside's unit step function,

$$H(t) = \begin{cases} 1 & \text{if} \quad t \ge 0, \\ 0 & \text{if} \quad t < 0. \end{cases} \tag{35}$$

Then Eq. (34) takes the form

$$\sigma_i^{(n)}(x_1,\ x_3,\ t) = \hat{q}_0 \left[\Re_{i1}^{(n)} H(t) + \Re_{i2}^{(n)} \omega(t) + \Re_{i3}^{(n)} \Pi(t) + \Re_{i4}^{(n)} g_{\frac{1}{2}}(t) + \Re_{i5}^{(n)} g_2(t) \right]. \tag{36}$$

Assuming an exponential relaxation function [45]

$$\omega(t) = c_1 + c_2 \mathrm{e}^{-t/t_s}, \tag{37}$$

where c_1 and c_2 are constants to be experimentally determined, and t_s is the relaxation time, the Laplace-Carson transform can be used to determine the functions $\Pi(t)$, $g_{\frac{1}{2}}(t)$ and $g_2(t)$. Since the transform of $\omega(t)$ is

$$\omega^*(s) = c_1 + c_2 \frac{s}{s + 1/t_s}, \tag{38}$$

we can obtain the following three equations (see Appendix A):

$$\Pi(t) = \frac{1}{c_1}\left[1 - \frac{c_2}{c_1 + c_2} \mathrm{e}^{-\frac{c_1 \tau}{(c_1+c_2)}}\right], \tag{39}$$

$$g_{\frac{1}{2}}(t) = \frac{2}{2 + c_1}\left[1 - \frac{c_2}{2 + c_1 + c_2} \mathrm{e}^{-\frac{(2+c_1)\tau}{2+c_1+c_2}}\right], \tag{40}$$

$$g_2(t) = \frac{1}{1 + 2c_1}\left[1 - \frac{2c_2}{1 + 2(c_1 + c_2)} \mathrm{e}^{-\frac{(1+2c_1)\tau}{1+2(c_1+c_2)}}\right]. \tag{41}$$

Using Eqs. (37), (39), (40) and (41), the final form of the stresses in terms of the time parameter τ is given by

$$\begin{aligned} \sigma_i^{(n)}(x_1,\ x_3,\ t) = \hat{q}_0 \Bigg\{ & \Re_{i1}^{(n)} H(t) + \Re_{i2}^{(n)}[c_1 + c_2 \mathrm{e}^{-\tau}] + \frac{\Re_{i3}^{(n)}}{c_1}\left[1 - \frac{c_2}{c_1 + c_2} \mathrm{e}^{-\frac{c_1 \tau}{(c_1+c_2)}}\right] \\ & + \frac{2\Re_{i4}^{(n)}}{2 + c_1}\left[1 - \frac{c_2}{2 + c_1 + c_2} \mathrm{e}^{-\frac{(2+c_1)\tau}{2+c_1+c_2}}\right] \\ & + \frac{\Re_{i5}^{(n)}}{1 + 2c_1}\left[1 - \frac{2c_2}{1 + 2(c_1 + c_2)} \mathrm{e}^{-\frac{(1+2c_1)\tau}{1+2(c_1+c_2)}}\right] \Bigg\}. \end{aligned} \tag{42}$$

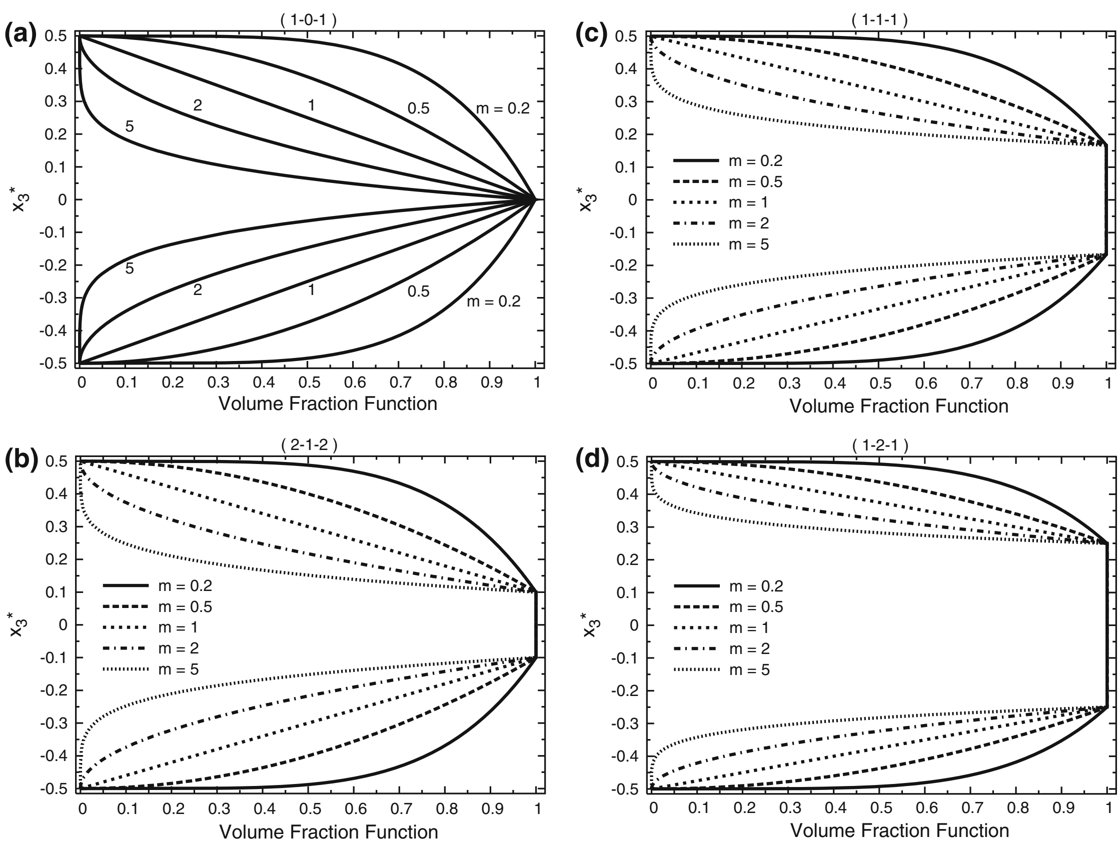

Fig. 2 Through-the-thickness distributions of volume fraction function for various values of the power-law index m and different types of viscoelastic sandwich beams. **a** The (1-0-1) FGM sandwich beam, **b** the (2-1-2) FGM sandwich beam, **c** the (1-1-1) FGM sandwich beam, **d** the (1-2-1) FGM sandwich beam

Table 1 Effects of the stress resultants on the results of an FG viscoelastic sandwich beam (1-2-1) resting on elastic foundations

Theory	w^*	σ_1^*	σ_5^*
FOBT	0.25455	0.16004	0.40132
HOBT	0.24879	0.15680	0.31629
SSBT	0.24879	0.15684	0.29897
RSBT	0.24896	0.14837	0.28136

Table 2 Effects of the power-law index and the relaxation function on dimensionless deflection of an FG viscoelastic sandwich beam resting on or without elastic foundations

Scheme	k_1	k_2	$\hat{\omega}$	w^*				
				$m = 0$	$m = 1$	$m = 2$	$m = 5$	$m = 10$
1-0-1	0	0	0.1	12.55080	43.65980	87.94342	201.58951	263.68312
			0.5	12.55080	32.27011	47.32595	63.72490	67.99140
			0.9	12.55080	26.86554	35.15822	42.58908	44.32917
	0.02	0	0.1	3.57556	4.48623	4.73102	4.87899	4.90695
			0.5	3.57556	4.32922	4.52222	4.63623	4.65749
			0.9	3.57556	4.21545	4.37746	4.47467	4.49320
	0.02	0.01	0.1	3.45368	4.29601	4.51997	4.65484	4.68029
			0.5	3.45368	4.15182	4.32901	4.43337	4.45281
			0.9	3.45368	4.04708	4.19617	4.28541	4.30241
2-1-2	0	0	0.1	12.55080	31.10842	49.39640	87.38209	115.08457
			0.5	12.55080	25.70429	34.82109	47.16011	53.01452
			0.9	12.55080	22.63956	28.40463	35.06975	37.87535
	0.02	0	0.1	3.57556	4.30764	4.54041	4.72938	4.79181
			0.5	3.57556	4.18578	4.37219	4.52071	4.56907
			0.9	3.57556	4.09550	4.25160	4.37609	4.41691
	0.02	0.01	0.1	3.45368	4.13197	4.34567	4.51848	4.57543
			0.5	3.45368	4.01972	4.19133	4.32762	4.37192
			0.9	3.45368	3.93639	4.08038	4.19491	4.23241
1-1-1	0	0	0.1	12.55080	31.10842	49.39640	87.38209	115.08387
			0.5	12.55080	25.70428	34.82109	47.16011	53.01449
			0.9	12.55080	22.63956	28.40463	35.06975	37.87535
	0.02	0	0.1	3.57556	4.30764	4.54041	4.72938	4.79181
			0.5	3.57556	4.18578	4.37219	4.52071	4.56907
			0.9	3.57556	4.09550	4.25160	4.37609	4.41691
	0.02	0.01	0.1	3.45368	4.13197	4.34567	4.51848	4.57543
			0.5	3.45368	4.01972	4.19133	4.32762	4.37192
			0.9	3.45368	3.93639	4.08038	4.19491	4.23241
1-2-1	0	0	0.1	12.55080	25.22554	34.85778	50.76771	60.84807
			0.5	12.55080	22.06429	27.82573	35.40348	39.27493
			0.9	12.55080	20.09147	24.06573	28.75462	30.94833
	0.02	0	0.1	3.57556	4.17288	4.37277	4.55171	4.62034
			0.5	3.57556	4.07627	4.23840	4.38124	4.43535
			0.9	3.57556	4.00365	4.13988	4.25936	4.30456
	0.02	0.01	0.1	3.45368	4.00782	4.19186	4.35602	4.41884
			0.5	3.45368	3.91862	4.06822	4.19964	4.24933
			0.9	3.45368	3.85146	3.97737	4.08753	4.12913

6 Several types of viscoelastic sandwich beams

Through-the-thickness variations of the volume fraction function of the elastic properties for $m = 0.2,\ 0.5,\ 1,\ 2$ and 5 are plotted in Fig. 2. Note that the core of the beam is a fully elastic material while the bottom and top surfaces of the beam are viscoelastic-rich. Also note that the beam is symmetric about the mid-plane.

6.1 The (1-0-1) FGM viscoelastic sandwich beam

Here the beam is made of only two equal-thickness layers, i.e. there is no core layer as shown in Fig. 2a. Thus,

$$h_1 = h_2 = 0. \tag{43}$$

Table 3 Effects of the power-law index and the relaxation function on σ_1^* for an FG viscoelastic sandwich beam resting on or without elastic foundations ($x_3 = h/4$)

Scheme	k_1	k_2	$\hat{\omega}$	σ_1^*				
				$m = 0$	$m = 1$	$m = 2$	$m = 5$	$m = 10$
1-0-1	0	0	0.1	29.96630	55.40711	60.60551	35.92500	27.99675
			0.5	29.96630	46.36033	44.53615	31.90629	29.80320
			0.9	29.96630	42.42696	40.08022	31.40443	30.10742
	0.2	0	0.1	1.14806	0.62735	0.34262	0.08888	0.05299
			0.5	1.14806	0.70736	0.46561	0.24840	0.21757
			0.9	1.14806	0.77519	0.56201	0.36441	0.33580
	0.2	1	0.1	0.77857	0.42164	0.22984	0.05956	0.03550
			0.5	0.77857	0.47603	0.31284	0.16675	0.14603
			0.9	0.77857	0.52220	0.37806	0.24494	0.22568
2-1-2	0	0	0.1	29.96630	56.36297	65.10285	43.39451	23.62487
			0.5	29.96630	48.06738	49.38573	35.18787	28.29363
			0.9	29.96630	44.05965	43.93772	33.56259	29.14396
	0.2	0	0.1	1.14806	0.76932	0.50550	0.16474	0.06470
			0.5	1.14806	0.82853	0.60471	0.31311	0.22683
			0.9	1.14806	0.88128	0.68611	0.42377	0.34460
	0.2	1	0.1	0.77857	0.51745	0.33934	0.11045	0.04336
			0.5	0.77857	0.55794	0.40655	0.21027	0.15229
			0.9	0.77857	0.59401	0.46178	0.28494	0.23164
1-1-1	0	0	0.1	29.96630	57.52296	70.13072	57.97754	27.89126
			0.5	29.96630	49.71426	54.62257	43.31838	29.31396
			0.9	29.96630	45.63644	48.28950	39.23804	29.74121
	0.2	0	0.1	1.14806	0.90993	0.70276	0.32986	0.12065
			0.5	1.14806	0.94859	0.77323	0.45445	0.27389
			0.9	1.14806	0.98611	0.83532	0.55156	0.38750
	0.2	1	0.1	0.77857	0.61247	0.47212	0.22128	0.08090
			0.5	0.77857	0.63918	0.52017	0.30535	0.18396
			0.9	0.77857	0.66503	0.56253	0.37104	0.26059
1-2-1	0	0	0.1	29.96630	61.15344	84.85233	123.89513	148.52759
			0.5	29.96630	53.56673	67.92522	86.85145	96.52851
			0.9	29.96630	49.18251	59.49758	71.83887	77.68345
	0.2	0	0.1	1.14806	1.18857	1.19991	1.20832	1.21053
			0.5	1.14806	1.18698	1.19900	1.20951	1.21345
			0.9	1.14806	1.19424	1.21099	1.22782	1.23511
	0.2	1	0.1	0.77857	0.80099	0.80720	0.81168	0.81273
			0.5	0.77857	0.80064	0.80753	0.81361	0.81589
			0.9	0.77857	0.80611	0.81634	0.82679	0.83137

6.2 The (2-1-2) FGM viscoelastic sandwich beam

In this state the core of the beam is half the face thickness. Figure 2b shows that

$$h_1 = -\frac{h}{10}, \quad h_2 = \frac{h}{10}. \tag{44}$$

6.3 The (1-1-1) FGM viscoelastic sandwich beam

As shown in Fig. 2c the beam is made of three equal-thickness layers. So, one gets

$$h_1 = -\frac{h}{6}, \quad h_2 = \frac{h}{6}. \tag{45}$$

6.4 The (1-2-1) FGM viscoelastic sandwich beam

In this state the core of the beam is twice the face thickness (see Fig. 2d). Thus, we have

$$h_1 = -\frac{h}{4}, \quad h_2 = \frac{h}{4}. \tag{46}$$

Table 4 Effects of the power-law index and the relaxation function on σ_5^* for an FG viscoelastic sandwich beam resting on or without elastic foundations ($x_3 = 0$)

Scheme	k_1	k_2	$\hat{\omega}$	σ_5^*				
				$m = 0$	$m = 1$	$m = 2$	$m = 5$	$m = 10$
1-0-1	0	0	0.1	49.21399	74.31893	97.74550	156.51355	241.29382
			0.5	49.21399	68.14954	82.71728	114.60840	149.47881
			0.9	49.21399	63.62495	73.37899	92.04598	108.71093
	0.2	0	0.1	1.88548	0.84148	0.55259	0.38724	0.45668
			0.5	1.88548	1.03981	0.86477	0.89224	1.09122
			0.9	1.88548	1.16250	1.02892	1.06809	1.21250
	0.2	1	0.1	1.27866	0.56555	0.37069	0.25950	0.30597
			0.5	1.27866	0.69976	0.58104	0.59897	0.73242
			0.9	1.27866	0.78311	0.69215	0.71792	0.81487
2-1-2	0	0	0.1	49.21399	63.63327	73.84076	90.23121	101.36005
			0.5	49.21399	60.29625	66.87662	76.82730	84.16457
			0.9	49.21399	57.87601	62.55255	69.40293	74.28460
	0.2	0	0.1	1.88548	0.86855	0.57335	0.34255	0.27758
			0.5	1.88548	1.03931	0.81888	0.68364	0.67476
			0.9	1.88548	1.15763	0.97678	0.87630	0.87835
	0.2	1	0.1	1.27866	0.58420	0.38489	0.22965	0.18603
			0.5	1.27866	0.69989	0.55053	0.45910	0.45301
			0.9	1.27866	0.78028	0.65742	0.58921	0.59043
1-1-1	0	0	0.1	49.21399	59.17414	65.41666	74.31618	79.49252
			0.5	49.21399	56.92465	60.97747	66.19808	69.36168
			0.9	49.21399	55.32791	58.25298	61.93375	64.19022
	0.2	0	0.1	1.88548	0.93605	0.65552	0.42282	0.34387
			0.5	1.88548	1.08617	0.86319	0.69448	0.64806
			0.9	1.88548	1.19553	1.00768	0.87060	0.83635
	0.2	1	0.1	1.27866	0.63005	0.44038	0.28364	0.23058
			0.5	1.27866	0.73189	0.58069	0.46663	0.43527
			0.9	1.27866	0.80625	0.67860	0.58565	0.56242
1-2-1	0	0	0.1	49.21399	55.46584	58.89103	63.28368	65.58467
			0.5	49.21399	54.11565	56.40965	59.03203	60.35796
			0.9	49.21399	53.18123	54.87069	56.71751	57.65533
	0.2	0	0.1	1.88548	1.07803	0.83279	0.61719	0.53452
			0.5	1.88548	1.19914	0.99573	0.82209	0.75875
			0.9	1.88548	1.29134	1.11681	0.96938	0.91667
	0.2	1	0.1	1.27866	0.72649	0.56023	0.41459	0.35887
			0.5	1.27866	0.80884	0.67063	0.55300	0.51016
			0.9	1.27866	0.87165	0.75286	0.65276	0.61702

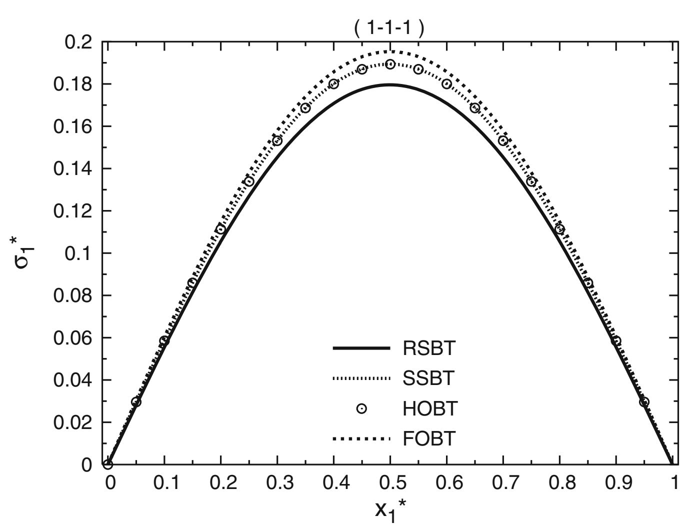

Fig. 3 Variation of axial normal stress σ_1^* along-the-length of the FG viscoelastic sandwich beam ($a/h = 8$)

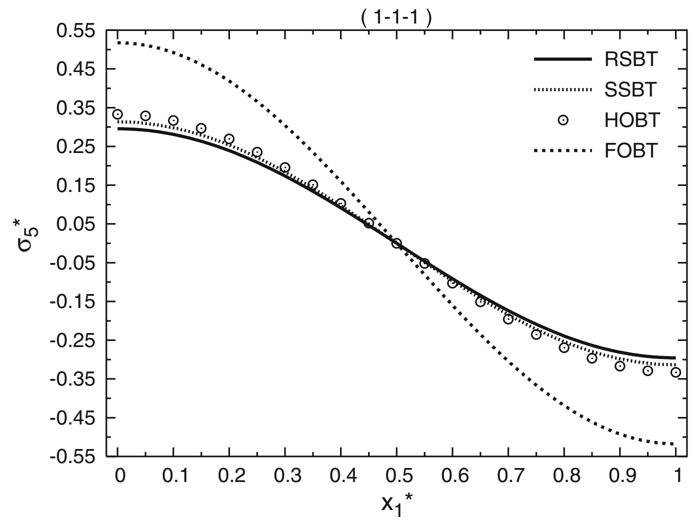

Fig. 4 Variation of transverse shear stress σ_5^* along-the-length of the FG viscoelastic sandwich beam $(a/h = 8)$

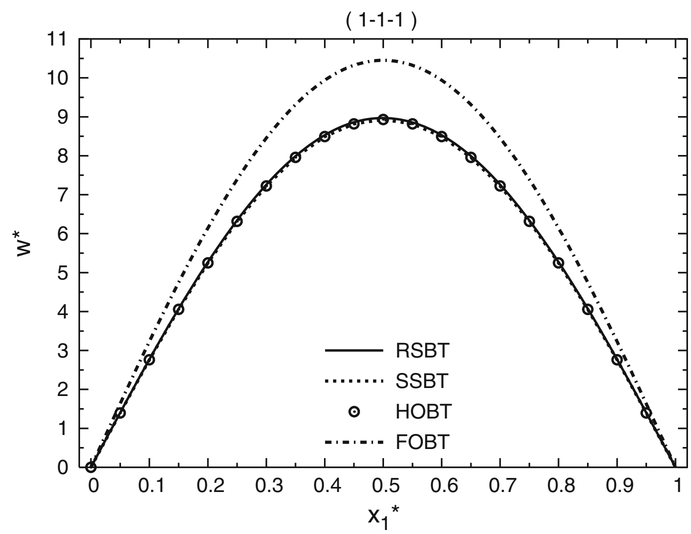

Fig. 5 Variation of the central deflection w^* along-the-length of the FG viscoelastic sandwich beam $(a/h = 15,\ k_1 = k_2 = 0.01)$

7 Numerical results and discussions

In this paper, several examples are performed concerning bending response of FG viscoelastic sandwich beams with rectangular cross section resting on elastic foundations. The results are evaluated for loading case involving sinusoidal load on the top face whereas the bottom face is resting on two-parameter elastic foundations and hence the effect on stress distribution and deflection is investigated thoroughly. Based on the analysis of the previous sections, the stresses and transverse displacements of simply supported beams are obtained by the refined sinusoidal shear deformation beam theory and compared with those obtained by the simple sinusoidal, higher-order and first-order shear theories.

Nondimensionalized central deflection and stresses given here are presented according to the following definitions:

$$w^* = \frac{E_e}{100h\hat{q}_0} w_0(x_1), \quad \sigma_1^* = \frac{1}{\hat{q}_0}\sigma_1(x_1,\ x_3), \quad \sigma_5^* = \frac{10}{\hat{q}_0}\sigma_5(x_1,\ x_3),$$
$$\xi = \frac{K}{E_e}, \quad k_1 = \frac{K_1 h}{K}, \quad k_2 = \frac{K_2}{Kh}, \quad x_1^* = \frac{x_1}{a}, \quad x_3^* = \frac{x_3}{h}.$$

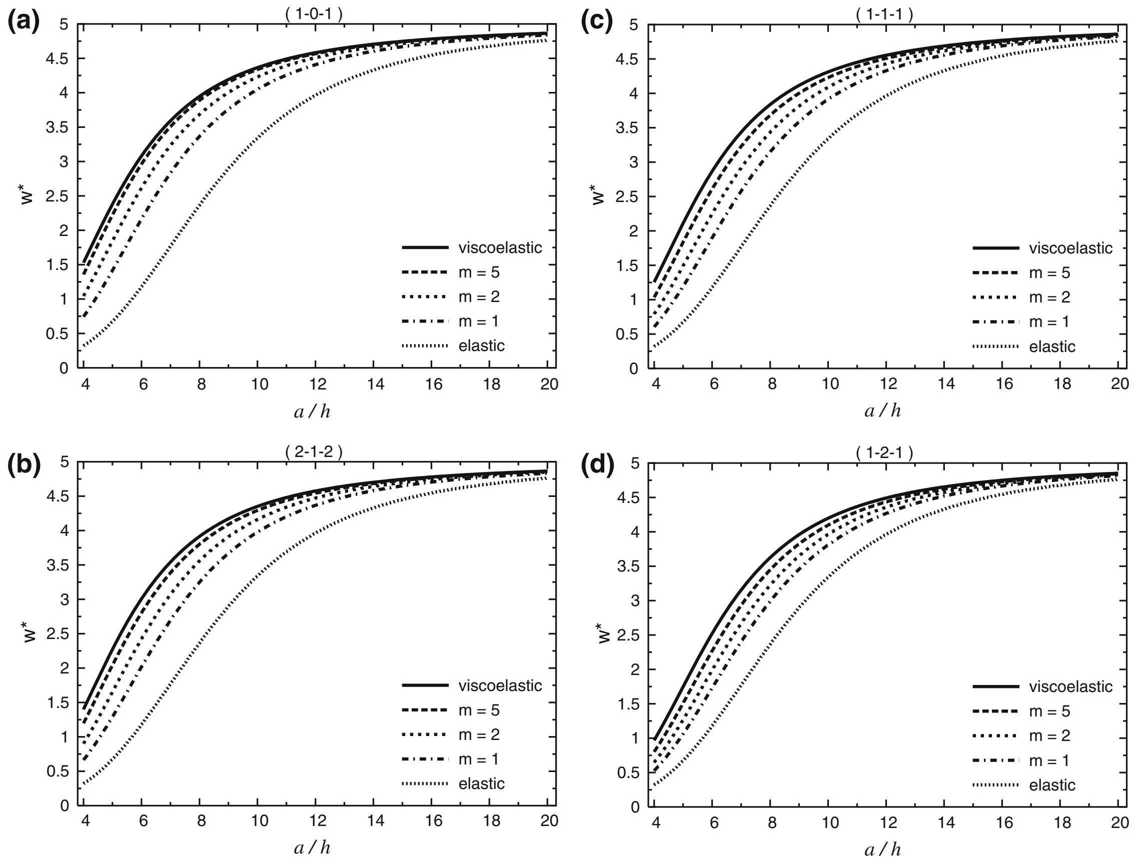

Fig. 6 The maximum central deflection w^* versus a/h for various values of the power-law index m and different types of viscoelastic sandwich beams ($k_1 = k_2 = 0.02$). **a** The (1-0-1) FGM sandwich beam, **b** the (2-1-2) FGM sandwich beam, **c** the (1-1-1) FGM sandwich beam, **d** the (1-2-1) FGM sandwich beam

Numerical results are tabulated in Tables 1–4 and are plotted in Figs. 3–14. Plots of the volume fraction function through the thickness of the FG viscoelastic sandwich beam are given in Fig. 2 for different values of the power-law index m. The longitudinal stress σ_1^* is computed at $x_1 = a/2,\ x_3 = h/2$ and the transverse shear stress σ_5^* is computed at $x_1 = 0,\ x_3 = 3h/10$ while the transverse deflection w^* is computed at $x_1 = a/2$. The value of Poisson's ratio of the elastic material is taken to be $\nu_e = 0.25$. It is assumed (unless otherwise stated) for the FG sandwich beam that

$$\xi = 0.1, \quad m = 1, \quad \hat{\omega} = 0.3, \quad c_1 = 0.1, \quad c_2 = 0.9, \quad a/h = 10, \quad k_1 = 0.2, \quad k_2 = 2.$$

Note that, for the time-dependence solution, the stresses and the deflection may be given in terms of the time parameter in the following forms:

$$\sigma_1^* = 0.05884073 + 0.16010276\ \mathrm{e}^{-\tau} + 0.00004406\ \mathrm{e}^{-0.1\tau} + 0.04895176\ \mathrm{e}^{-0.7\tau} + 0.09640154\ \mathrm{e}^{-0.4\tau},$$
$$\sigma_5^* = 0.17163142 + 0.09687815\ \mathrm{e}^{-\tau} + 0.00000534\ \mathrm{e}^{-0.1\tau} - 0.00696426\ \mathrm{e}^{-0.7\tau} + 0.01198476\ \mathrm{e}^{-0.4\tau},$$
$$w^* = 13.91219899 - 0.57564870\ \mathrm{e}^{-\tau} - 0.00043891\ \mathrm{e}^{-0.1\tau} - 0.87743037\ \mathrm{e}^{-0.7\tau} - 0.70937452\ \mathrm{e}^{-0.4\tau}.$$

For the sake of completeness and comparison, the influence of the higher-order stress resultants P and R on the numerical results of FG viscoelastic sandwich beam (1-2-1) resting on elastic foundations is explained in Table 1. It is to be noticed that the results obtained by the higher-order shear deformation beam theories are lower than those obtained by the first-order one, indicating the influence of the higher-order stress resultants P and R. Note that for the FOBT, the stress couple M and the stress couple associated with the transverse shear effect P are equal. In addition, the transverse stress resultants Q and R are also equal.

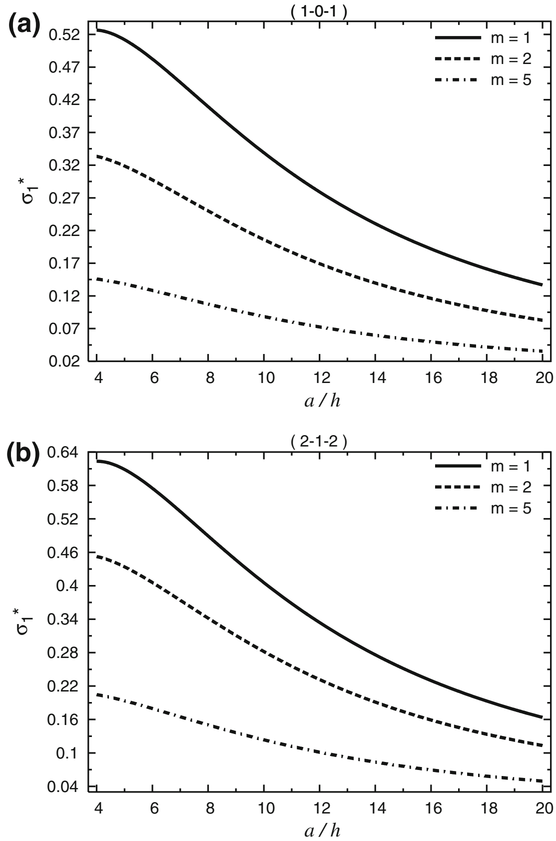

Fig. 7 The stress σ_1^* versus a/h for various values of the power-law index m and different types of viscoelastic sandwich beams ($x_3 = h/4$). **a** The (1-0-1) FGM sandwich beam, **b** the (2-1-2) FGM sandwich beam, **c** the (1-1-1) FGM sandwich beam, **d** the (1-2-1) FGM sandwich beam

The stresses σ_1^* and σ_5^* and deflection w^* of the mid-plane of FG viscoelastic sandwich beams for different values of the power-law index m and the relaxation function $\hat{\omega}$ and different states of the sandwich beam with various types of elastic foundations are listed in Tables 2–4. Since the homogeneous elastic beam is independent of the relaxation function, the results do not change with the variation of $\hat{\omega}$. Table 2 reveals that, irrespective of the elastic foundations, the dimensionless deflection w^* of the FG sandwich beams increases with the increase of m (i.e. with the increase of viscoelastic constituents), whereas its change is reversed with the increase of the relaxation function $\hat{\omega}$. For the sandwich beams without elastic foundation, regardless of the relaxation function $\hat{\omega}$, the axial normal stress σ_1^* increases to its maximum values and then decreases as m increases, except σ_1^* for the (1-2-1) sandwich beam, which increases directly (see Table 3). It is also observed from this table that, with the presence of Winkler's or Pasternak's elastic foundations, the normal stress σ_1^* decreases directly as the power-law index m increases for the all values of $\hat{\omega}$, except σ_1^* for the (1-2-1) sandwich beam, its change is reversed because it is estimated at the top interface. It can be seen that the stress σ_1^* for the FG foundation beams is proportional to the relaxation function $\hat{\omega}$ excluding σ_1^* for the (1-2-1) FG sandwich beam, while the normal stress for the foundationless beams decreases as $\hat{\omega}$ increases for $m = 1, 2, 5$. Table 4 shows that the transverse shear stress σ_5^* increases directly as the power-law index m increases for foundationless sandwich beams, whereas, for foundation beams, its change is reversed excluding σ_5^* for (1-0-1) beams at the all values of $\hat{\omega}$ and (2-1-2) beams at $\hat{\omega} = 0.9$. In general, the results show that the presence of Winkler's or Pasternak's elastic foundations leads to a significant reduction in the variation of the deflections and stresses.

Figures 3–5 demonstrate the stresses σ_1^* and σ_5^* and central deflection w^* plots, respectively, using the RSBT, SSBT, HOBT and FOBT. It can be seen that the stress σ_1^* and deflection w^* have their maximum values at the center of the beam ($x_1 = a/2$), while σ_5^* has its maximums at the tip of the beam. From Fig. 3, significant differences between the results obtained by the refined theory (RSBT) and those obtained by sinusoidal, higher-order and first-order theories are noted, indicating the effect of transverse normal strain which is included in the stress-strain relation of RSBT. It is clear from Figs. 4 and 5 that the disagreement

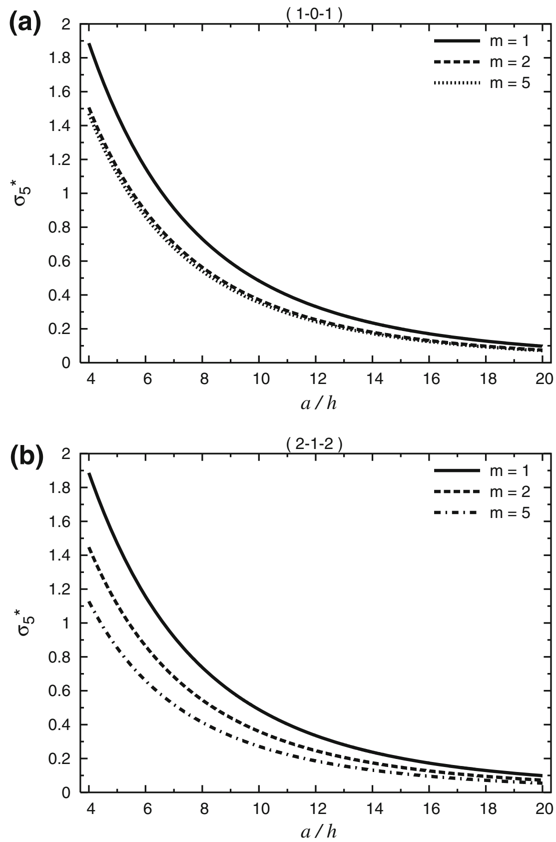

Fig. 8 The stress σ_5^* versus a/h for various values of the power-law index m and different types of viscoelastic sandwich beams ($x_3 = 0$). **a** The (1-0-1) FGM sandwich beam, **b** the (2-1-2) FGM sandwich beam, **c** the (1-1-1) FGM sandwich beam, **d** the (1-2-1) FGM sandwich beam

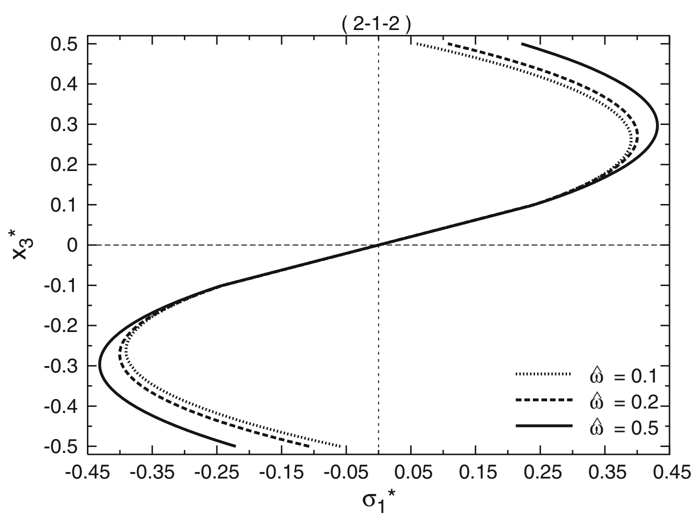

Fig. 9 Through-the-thickness distributions of axial normal stress σ_1^* for different values of the relaxation function $\hat{\omega}$

between RSBT, SSBT and HOBT is much less than the disagreement between any of them and FOBT. Also, it is to be noticed that the sinusoidal shear deformation beam theory suggested by Zenkour [14,17–19,45] is in extremely good agreement with the higher-order shear deformation beam theory.

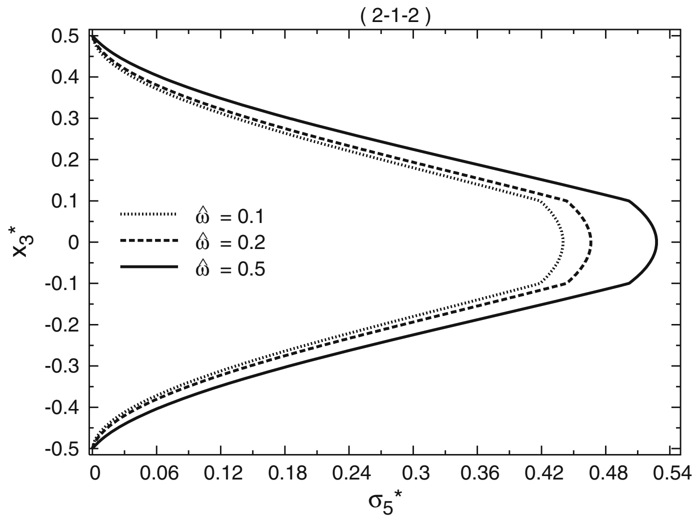

Fig. 10 Through-the-thickness distributions of transverse shear stress σ_5^* for different values of the relaxation function $\hat{\omega}$

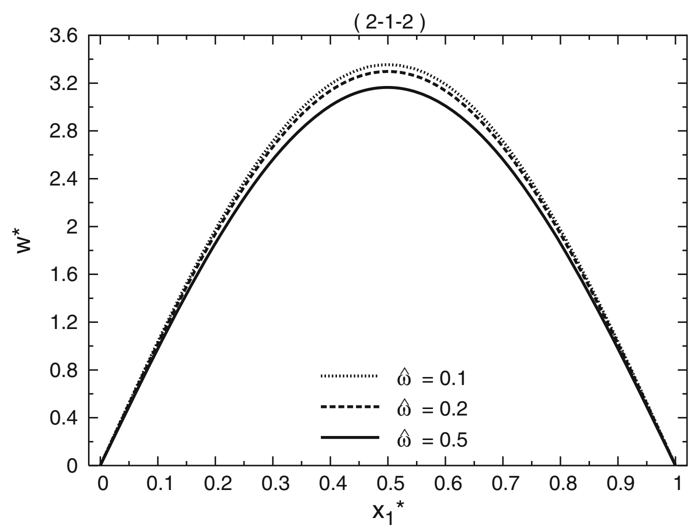

Fig. 11 Along-the-length distributions of the central deflection w^* for various values of the relaxation function $\hat{\omega}$ ($a/h = 8,\ k_1 = k_2 = 0.02$)

Figure 6 illustrates the variation of central deflection w^* as a function of the span-to-thickness ratio a/h for various values of the power-law index m and different types of viscoelastic sandwich beams. It is to be noted that the deflection w^* of the homogeneous sandwich beams and FG ones increases gradually as the span-to-thickness ratio a/h increases. Note that since the stiffness of the elastic materials is higher than that of the viscoelastic materials, the central deflection of the fully elastic beam ($m = 0$) is less than that of the viscoelastic sandwich beam, especially for the thick beam. All the FG sandwich beams with intermediate properties undergo corresponding intermediate values of the deflection.

Figures 7 and 8 exhibit the variations of axial normal stress σ_1^* and transverse shear stress σ_5^* against the span-to-thickness ratio a/h for various values of the power-law index m and different types of viscoelastic sandwich beams. It is to be seen that the stress σ_1^* decreases gradually as the span-to-thickness ratio a/h increases just as σ_5^*. Note that for the (1-2-1) FG sandwich beam, the axial normal stress increases with the increase of the power-law index m, while for the other types of beams it is monotonically decreasing as the power-law index increases just as σ_5^* for the all types of sandwich beams.

Variations of the stresses σ_1^* and σ_5^* through the thickness of the FG sandwich beam are shown graphically in Figs. 9 and 10. The results for the relaxation function $\hat{\omega} = 0.1,\ 0.2,\ 0.5$ are presented here. The maximum

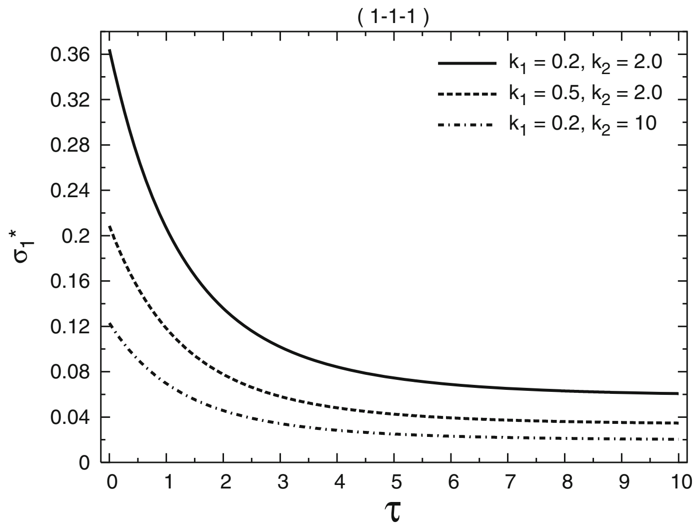

Fig. 12 Dimensionless stress σ_1^* as a function of the time parameter τ for different values of the elastic foundation parameters k_1 and k_2

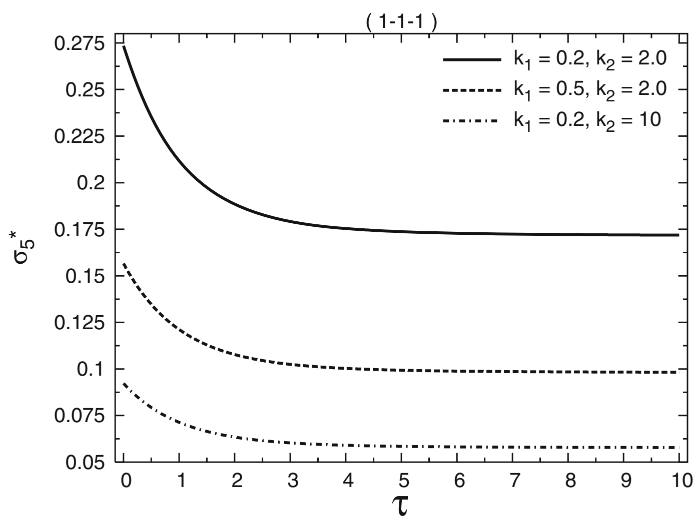

Fig. 13 Dimensionless stress σ_5^* as a function of the time parameter τ for different values of the elastic foundation parameters k_1 and k_2

compressive {tensile} stress σ_1^* occurs in the top {bottom} layer of the beam as shown in Fig. 9. Note that the plots read negative sign for tensile stresses and positive sign for compressive stresses. Figure 10 shows the sensitivity of the transverse shear stress σ_5^* through the thickness of the sandwich beams. It is to be seen from Figs. 9 and 10 that the stresses are proportional to the relaxation function $\hat{\omega}$.

Variations of the central deflection w^* along the length of FG sandwich beams for different values of the relaxation function $\hat{\omega}$ are shown graphically in Fig. 11. It is noticed that the deflection decreases as $\hat{\omega}$ increases.

The plots of the axial stress σ_1^*, transverse shear stress σ_5^* and central deflection w^* as functions of the time parameter τ for different values of the elastic foundation parameters k_1 and k_2 are presented in Figs. 12–14. As expected the results decrease as the elastic foundation stiffnesses increase. The longitudinal stress σ_1^* and transverse shear stress σ_5^* decrease directly to reach its constant values with the increase of the time parameter τ. Figure 14 shows that, for $k_1 = 0.002$ and $k_2 = 0.02$, the time parameter τ effect is more pronounced on the deflection of the beam, and it is less pronounced on w^* for $k_1 = 0.002$ and $k_2 = 0.1$. But for $k_1 = 0.005$ and $k_2 = 0.02$, the deflection takes intermediate values between the two above states.

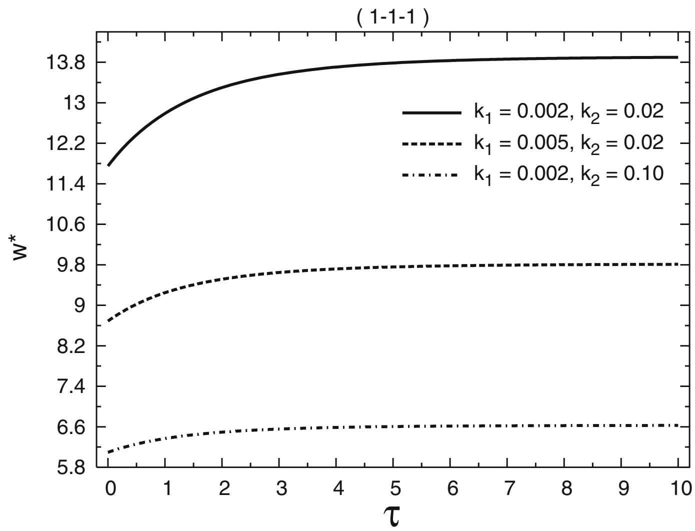

Fig. 14 Dimensionless deflection w^* as a function of the time parameter τ for different values of the elastic foundation parameters k_1 and k_2

8 Conclusions

Refined sinusoidal shear deformation beam theory is effectively used to extensively study the stresses and central deflection of the bending of FG viscoelastic sandwich beams resting on two-parameter (Pasternak's) elastic foundations. The effects of transverse normal strain as well as the shear deformation are included in the present theory. In order to show that effect, the three shear deformation theories SSBT, HOBT and FOBT are used in our study. Each face of the present sandwich beam is considered to be made of an FG viscoelastic material while the core is considered to be made of an elastic material. The variation of properties through the thickness of the faces follows a power-law type. The deflection for the given boundary conditions is high for a viscoelastic rich beam when compared to an elastic rich beam and it increases as the power-law index increases. Investigations on the effects of the relaxation function, span-to-thickness ratio, elastic foundation parameters and the time parameter on the FG viscoelastic sandwich beams with elastic cores are presented.

Appendix A

Denoting the transforms of $\Pi(t)$ and $g_{\chi_i}(t)$ by $\Pi^*(t)$ and $g^*_{\chi_i}(t)$, respectively, where $i = 1, 2$ and $\chi_1 = 1/2,\ \chi_2 = 2$, we may deduce the Laplace–Carson transform of them as follows:

$$\Pi^*(s) = \frac{1}{\omega^*(s)} = \frac{1}{c_1 + c_2 s/(s + 1/t_s)}, \tag{47}$$

or

$$\Pi^*(s) = \frac{1}{c_1}\left(1 - c_3 \frac{s}{s + c_4}\right), \tag{48}$$

where

$$c_3 = \frac{c_2}{c_1 + c_2}, \qquad c_4 = \frac{c_1}{(c_1 + c_2)t_s}. \tag{49}$$

Then we can find the function $\Pi(t)$ by using the inverse Laplace–Carson transform of Eq. (48) in the form

$$\Pi(t) = \frac{1}{c_1}\left[1 - \frac{c_2}{c_1 + c_2} \mathrm{e}^{-\frac{c_1 \tau}{(c_1 + c_2)}}\right], \qquad \tau = \frac{t}{t_s}. \tag{50}$$

Similarly

$$g^*_{\chi_i}(s) = \frac{1}{1 + \chi_i \omega^*(s)} = \frac{1}{1 + \chi_i \{c_1 + c_2 [s/(s + 1/t_s)]\}}, \tag{51}$$

or

$$g^*_{\chi_i}(s) = \frac{1}{1 + c_1 \chi_i} \left(1 - c_5 \frac{s}{s + c_6}\right), \tag{52}$$

where

$$c_5 = \frac{c_2 \chi_i}{1 + (c_1 + c_2)\chi_i}, \qquad c_6 = \frac{1 + c_1 \chi_i}{[1 + (c_1 + c_2)\chi_i] t_s}. \tag{53}$$

Using the Laplace–Carson transform of Eq. (52), one obtains

$$g_{\chi_i}(t) = \frac{1}{1 + c_1 \chi_i} \left[1 - c_5 \mathrm{e}^{-c_6 t}\right], \tag{54}$$

or

$$g_{\chi_i}(t) = \frac{1}{1 + c_1 \chi_i} \left[1 - \frac{c_2 \chi_i}{1 + (c_1 + c_2)\chi_i} \mathrm{e}^{-(1 + c_1 \chi_i)\tau / [1 + (c_1 + c_2)\chi_i]}\right]. \tag{55}$$

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