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Transfer matrix solutions to axisymmetric and non-axisymmetric consolidation of multilayered soils

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Abstract Transfer matrix solutions are presented in this paper to study the axisymmetric and non-axisymmetric consolidation of a multilayered soil system under an arbitrary loading. Starting with the governing equations for consolidation problems of saturated soils, the relationship of displacements, stresses, excess pore water pressure, and flux between the points at the depth z, and on the ground surface ($z = 0$) is established in a transformed domain by introducing the displacement functions and using the integral transform technique. Then the transfer matrix method is used with the boundary conditions to obtain the analytical solutions in the transformed domain for the multilayered soil system. Numerical inversion of the integral transform of these analytical solutions results in the solutions for the actual problems. The numerical results for axisymmetric and non-axisymmetric Biot's consolidation problems of a single layer and a multi-layered soil system are obtained and compared with existing results by others.

1 Introduction

Since Biot first developed the theory of three-dimensional consolidation in 1941 [1], the displacement function method has been regarded as one of the most successful techniques for solving Biot's consolidation problem. By combining with the linear transform technique, McNamee and Gibson [2] proposed two displacement functions to solve the stresses and excess pore pressure in porous elastic media, and later they used these displacement functions to solve plane strain and axisymmetric consolidation problems of a semi-infinite clay stratum [3]. Schiffman and Fungaroli [4] extended the displacement function formulation to non-axisymmetric problems to obtain the analytical solutions for the consolidation of a semi-infinite medium subjected to tangential surface loading. Gibson et al. [5] studied plane strain and axisymmetric consolidation problems of a clay layer on a smooth impervious base using two displacement functions proposed by McNamee and Gibson [2]. Verruijt [6] pointed out that the formulation of these displacement functions was similar to Biot's original formulation [7], and he extended these displacement functions to a more general case with a compressible pore fluid. Vardoulakis and Harnpattanapanich [8] studied the layered-soil consolidation problems by means of the appropriate displacement functions and Laplace–Fourier transformation. In addition to the displacement function method mentioned above, the finite layer method [9–13], the finite element method [14], and the boundary element method [15] have been used to solve consolidation problems of poroelastic

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soils. Furthermore, other analytical techniques have been developed to solve Biot's consolidation problems, for example those by Yue and Selvadurai [16], Pan [17], Wang and Fang [18,19], Ai and Han [20], and Ai et al. [21].

The transfer matrix method is one of the most efficient methods to solve elastostatic problems in multi-layered materials because the size of the final equations does not depend on the number of multi-layered materials [22,23]. In addition, this method is accurate and convenient for computation. In this study, the transfer matrix method is used to solve the consolidation problems of the multilayered soils subjected to both axisymmetric and non-axisymmetric loads, which are prescribed either on the ground surface or within the multilayered soils. The transfer matrices in this paper are deduced using the displacement functions proposed by McNamee and Gibson [2] for the axisymmetric consolidation problem and by Schiffman and Fungaroli [4] for the non-axisymmetric consolidation problem.

2 Governing differential equations

The governing differential equations for the non-axisymmetric consolidation problem are

$$\nabla^2 u_r + (2\eta - 1)\frac{\partial e}{\partial r} - \frac{1}{r}\left(\frac{2}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + \frac{1}{G}\frac{\partial \sigma}{\partial r} = 0, \tag{1a}$$

$$\nabla^2 u_\theta + (2\eta - 1)\frac{\partial e}{r\partial \theta} - \frac{1}{r^2}\left(u_\theta - 2\frac{\partial u_r}{\partial \theta}\right) + \frac{1}{G}\frac{\partial \sigma}{r\partial \theta} = 0, \tag{1b}$$

$$\nabla^2 u_z + (2\eta - 1)\frac{\partial e}{\partial z} + \frac{1}{G}\frac{\partial \sigma}{\partial z} = 0, \tag{1c}$$

$$C\nabla^2 e = \frac{\partial e}{\partial t}, \tag{1d}$$

where u_r, u_θ and u_z are the displacements in r, θ and z directions, respectively; σ is the excess pore pressure; $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator in the cylindrical coordinate system; $e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$ is the dilatation; $\eta = \frac{1-v}{1-2v}$, C is the coefficient of consolidation, which can be expressed as $C = \frac{k}{\gamma_w}M$; k is the coefficient of permeability, γ_w is the unit weight of water, and M is the elastic modulus of the soil, i.e., $M = \lambda + 2G = \frac{E(1-v)}{(1+v)(1-2v)}$, $\lambda = \frac{vE}{(1+v)(1-2v)}$; E, G and v are Young's modulus, the shear modulus, and Poisson's ratio of the soil, respectively.

If all variables are independent of the coordinate θ, then the governing differential equations for the non-axisymmetric consolidation problem can be degenerated to the axisymmetric one as follows:

$$\nabla^2 u_r - \frac{1}{r^2}u_r + \frac{1}{1-2v}\frac{\partial e}{\partial r} + \frac{1}{G}\frac{\partial \sigma}{\partial r} = 0, \tag{2a}$$

$$\nabla^2 u_z + \frac{1}{1-2v}\frac{\partial e}{\partial z} + \frac{1}{G}\frac{\partial \sigma}{\partial z} = 0, \tag{2b}$$

$$C \cdot \nabla^2 e = \frac{\partial e}{\partial t}, \tag{2c}$$

in which $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ and $e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$.

3 Transfer matrix solutions for consolidation in a single soil layer

3.1 Transfer matrix solution for an axisymmetric problem

For the axisymmetric consolidation problem, two displacement functions E and S proposed by McNamee and Gibson [2,3] are adopted here. The stresses, displacements, excess pore pressure, and flux in the z direction can be expressed as follows by applying these two functions:

$$u_r = -\frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r}, \tag{3a}$$

$$u_z = -\frac{\partial E}{\partial z} + z\frac{\partial S}{\partial z} - S, \tag{3b}$$

$$\frac{\sigma}{2G} = \frac{\partial S}{\partial z} - \eta\nabla^2 E, \tag{3c}$$

$$\frac{\sigma_{rz}}{2G} = \frac{\partial^2 E}{\partial r\partial z} - z\frac{\partial^2 S}{\partial r\partial z}, \tag{3d}$$

$$\frac{\sigma_z}{2G} = \frac{\partial^2 E}{\partial z^2} - \nabla^2 E - z\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}, \tag{3e}$$

$$Q = \frac{k}{\gamma_w}\frac{\partial\sigma}{\partial z} = \frac{2Gk}{\gamma_w}\left[\frac{\partial^2 S}{\partial z^2} - \eta\nabla^2\frac{\partial E}{\partial z}\right], \tag{3f}$$

in which σ_{rz} is the shear stress, σ_z is the normal stress acting on the plane normal to the z axes, and Q is the flux in the z direction.

The differential equations governing E and S are

$$C\nabla^4 E = \nabla^2\left(\frac{\partial E}{\partial t}\right), \tag{4a}$$

$$\nabla^2 S = 0. \tag{4b}$$

To solve the governing equations, the Laplace–Hankel transforms of E and S [24] can be used:

$$E(r,z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty \bar{E}(\xi,z,s)K(r,\xi)e^{st}d\xi ds, \tag{5a}$$

$$S(r,z,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty \bar{S}(\xi,z,s)K(r,\xi)e^{st}d\xi ds, \tag{5b}$$

where $\bar{E}(\xi,z,s)$ and $\bar{S}(\xi,z,s)$ are the corresponding variables of $E(r,z,t)$ and $S(r,z,t)$ in the Laplace–Hankel transformed domain, respectively, and the kernel is $K(r,\xi) = \xi J_0(r\xi)$.

Using the Laplace–Hankel transform and assuming zero initial dilatation e everywhere, the partial differential equations (4) can be re-written in the following ordinary differential equations:

$$\left(\frac{d^2}{dz^2} - \xi^2\right)\left(\frac{d^2}{dz^2} - q^2\right)\bar{E} = 0, \tag{6a}$$

$$\left(\frac{d^2}{dz^2} - \xi^2\right)\bar{S} = 0, \tag{6b}$$

where $q^2 = \xi^2 + \frac{s}{C}$. The solutions for the Eqs. (6a)–(6b) can be expressed as

$$\bar{E} = A_1 sh\xi z + A_2 ch\xi z + A_3 shqz + A_4 chqz, \tag{7a}$$

$$\bar{S} = A_5 sh\xi z + A_6 ch\xi z, \tag{7b}$$

where $A_1 ... A_6$ are functions of ξ and q. The Laplace–Hankel transform of Eqs. (3a) to (3f) yields

$$\bar{u}_r = \xi \bar{E} - \xi z \bar{S}, \tag{8a}$$

$$\bar{u}_z = -\frac{\mathrm{d}\bar{E}}{\mathrm{d}z} + z\frac{\mathrm{d}\bar{S}}{\mathrm{d}z} - \bar{S}, \tag{8b}$$

$$\bar{\sigma} = 2G\left[\frac{\mathrm{d}\bar{S}}{\mathrm{d}z} - \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)\bar{E}\right], \tag{8c}$$

$$\bar{\sigma}_z = 2G\left[\xi^2\bar{E} - z\frac{\mathrm{d}^2\bar{S}}{\mathrm{d}z^2} + \frac{\mathrm{d}\bar{S}}{\mathrm{d}z}\right], \tag{8d}$$

$$\bar{\sigma}_{rz} = 2G\left[-\xi\frac{\mathrm{d}\bar{E}}{\mathrm{d}z} + \xi z\frac{\mathrm{d}\bar{S}}{\mathrm{d}z}\right], \tag{8e}$$

$$\bar{Q} = \frac{2Gk}{\gamma_w}\left[\frac{\mathrm{d}^2\bar{S}}{\mathrm{d}z^2} - \eta\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2\right)\frac{\mathrm{d}\bar{E}}{\mathrm{d}z}\right], \tag{8f}$$

where $\bar{u}_r$ is the corresponding variable of u_r in the transformed domain. Other variables in the above equation are expressed in the same manner.

Substitution of Eq. (7) into Eqs. (8a) to (8f) yields

$$\bar{u}_r = (A_1 - zA_5)\xi sh\xi z + (A_2 - zA_6)\xi ch\xi z + A_3\xi shqz + A_4\xi chqz, \tag{9a}$$
$$\bar{u}_z = (A_5\xi z - A_1\xi - A_6)ch\xi z + (A_6\xi z - A_2\xi - A_5)sh\xi z - qA_3chqz - qA_4shqz, \tag{9b}$$
$$\bar{\sigma} = 2G[-\eta(q^2 - \xi^2)A_3shqz - \eta(q^2 - \xi^2)A_4chqz + \xi A_5ch\xi z + \xi A_6sh\xi z], \tag{9c}$$
$$\bar{\sigma}_{rz} = 2G[\xi^2(zA_5 - A_1)ch\xi z + \xi^2(zA_6 - A_2)sh\xi z - \xi qA_3chqz - \xi qA_4shqz], \tag{9d}$$
$$\bar{\sigma}_z = 2G\xi[(\xi A_1 + A_6 - \xi zA_5)sh\xi z + (\xi A_2 + A_5 - \xi zA_6)ch\xi z + A_3\xi shqz + A_4\xi chqz], \tag{9e}$$
$$\bar{Q} = 2Gk'[\xi^2A_5sh\xi z + \xi^2A_6ch\xi z - \eta q(q^2 - \xi^2)A_3chqz - \eta q(q^2 - \xi^2)A_4shqz], \tag{9f}$$

where $k' = \frac{k}{\gamma_w}$.

Setting $z = 0$ in Eqs. (9a) to (9f) results in

$$\bar{u}_r(0) = A_2\xi + A_4\xi, \tag{10a}$$
$$\bar{u}_z(0) = -A_1\xi - A_3q - A_6, \tag{10b}$$
$$\bar{\sigma}(0) = 2G[A_5\xi - \eta A_4(q^2 - \xi^2)], \tag{10c}$$
$$\bar{\sigma}_{rz}(0) = -2G(\xi^2A_1 + \xi qA_3), \tag{10d}$$
$$\bar{\sigma}_z(0) = 2G(\xi^2A_2 + \xi^2A_4 + \xi A_5), \tag{10e}$$
$$\bar{Q}(0) = 2Gk'[\xi^2A_6 - \eta qA_3(q^2 - \xi^2)], \tag{10f}$$

in which $\bar{u}_r(0)$ is the value of $\bar{u}_r$ at $z = 0$. Other variables in the above equation are expressed in the same manner. Equation (10) is a linear algebraic equation system for the six variables $A_1 ... A_6$, which can be obtained by solving Eq. (10).

Substituting $A_1 ... A_6$ into Eq. (8) results in the transfer matrix solution for the axisymmetric consolidation problem of a single soil layer as follows:

$$\bar{B}(\xi, z, s) = \Phi(\xi, z, s)\bar{B}(\xi, 0, s), \tag{11}$$

where $\bar{B}(\xi, z, s) = [\bar{u}_r, \bar{u}_z, \bar{\sigma}, \bar{\sigma}_{rz}, \bar{\sigma}_z, \bar{Q}]^T$, $\Phi(\xi, z, s)$ is the transfer matrix of 6×6, which establishes the relationship of $[\bar{u}_r, \bar{u}_z, \bar{\sigma}, \bar{\sigma}_{rz}, \bar{\sigma}_z, \bar{Q}]^T$ between the ground surface ($z = 0$) and the depth z in the transformed domain, and its elements are provided in Appendix A.

Equation (11) can also be written as

$$\bar{B}(\xi, 0, s) = \Phi(\xi, -z, s)\bar{B}(\xi, z, s). \tag{12}$$

3.2 Transfer matrix solution for the non-axisymmetric problem

Three displacement functions E, S, and R proposed by Schiffman and Fungaroli [4] are required for the non-axisymmetric consolidation problem. The stresses, displacements, excess pore pressure, and flux in the z direction related to these three functions can be expressed as follows:

$$u_r = \frac{\partial^2 E}{\partial r \partial z} - \frac{2}{r}\frac{\partial R}{\partial \theta} - z\frac{\partial S}{\partial r}, \tag{13a}$$

$$u_\theta = \frac{1}{r}\frac{\partial^2 E}{\partial \theta \partial z} + 2\frac{\partial R}{\partial r} - \frac{z}{r}\frac{\partial S}{\partial \theta}, \tag{13b}$$

$$u_z = \frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z} + S, \tag{13c}$$

$$\sigma = 2G\left[\frac{\partial S}{\partial z} - \eta\frac{\partial}{\partial z}(\nabla^2 E)\right], \tag{13d}$$

$$\sigma_z = 2G\left[\frac{\partial}{\partial z}\left(\frac{\partial^2 E}{\partial z^2} - \nabla^2 E\right) - z\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}\right], \tag{13e}$$

$$\sigma_{rz} = 2G\left[\frac{\partial}{\partial r}\left(\frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z}\right) - \frac{1}{r}\frac{\partial^2 R}{\partial \theta \partial z}\right], \tag{13f}$$

$$\sigma_{\theta z} = 2G\left[\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z}\right) + \frac{\partial^2 R}{\partial r \partial z}\right], \tag{13g}$$

$$Q = 2Gk'\left[\frac{\partial^2 S}{\partial z^2} - \eta\frac{\partial^2}{\partial z^2}\left(\nabla^2 E\right)\right]. \tag{13h}$$

The differential equations governing E, S, and R are:

$$C\nabla^4 E = \nabla^2\left(\frac{\partial E}{\partial t}\right), \tag{14a}$$

$$\nabla^2 S = 0, \tag{14b}$$

$$\nabla^2 R = 0. \tag{14c}$$

To solve these governing differential equations, intermediate variables, u_v, u_h, σ_{vz}, and σ_{hz}, are introduced here:

$$u_v = \frac{1}{r}\left[\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta}\right], \tag{15a}$$

$$u_h = -\frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right], \tag{15b}$$

$$\sigma_{vz} = \frac{1}{r}\left[\frac{\partial(r\sigma_{rz})}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial \theta}\right], \tag{15c}$$

$$\sigma_{hz} = -\frac{1}{r}\left[\frac{\partial(r\sigma_{\theta z})}{\partial r} - \frac{\partial \sigma_{rz}}{\partial \theta}\right]. \tag{15d}$$

So Eqs. (13a)–(13h) can be expressed by two sets of equations as follows:

$$\begin{cases} u_v = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial E}{\partial z} - zS\right), \\ u_z = \frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z} + S, \\ \sigma = 2G\left[\frac{\partial S}{\partial z} - \eta\frac{\partial}{\partial z}(\nabla^2 E)\right], \\ \sigma_z = 2G\left[\frac{\partial}{\partial z}\left(\frac{\partial^2 E}{\partial z^2} - \nabla^2 E\right) - z\frac{\partial^2 S}{\partial z^2} + \frac{\partial S}{\partial z}\right], \\ \sigma_{vz} = 2G\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 E}{\partial z^2} - z\frac{\partial S}{\partial z}\right), \\ Q = 2Gk'\left[\frac{\partial^2 S}{\partial z^2} - \eta\frac{\partial^2}{\partial z^2}(\nabla^2 E)\right], \end{cases} \tag{16a}$$

and

$$\begin{cases} u_h = -2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)R, \\ \sigma_{hz} = -2G\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\frac{\partial R}{\partial z}. \end{cases} \tag{16b}$$

The displacements, stresses, excess pore pressure, and flux can be expressed in the Fourier expansions suggested by Muki [25] for elastostatics in the cylindrical coordinate system, i.e.,

$$\begin{aligned} u_r &= \sum_{m=0}^{\infty} u_{rm}\cos m\theta, \quad \sigma_{rz} = \sum_{m=0}^{\infty} \sigma_{rzm}\cos m\theta, \\ u_\theta &= \sum_{m=0}^{\infty} u_{\theta m}\sin m\theta, \quad \sigma_{\theta z} = \sum_{m=0}^{\infty} \sigma_{\theta zm}\sin m\theta, \\ u_z &= \sum_{m=0}^{\infty} u_{zm}\cos m\theta, \quad \sigma_z = \sum_{m=0}^{\infty} \sigma_{zm}\cos m\theta, \\ \sigma &= \sum_{m=0}^{\infty} \sigma_m\cos m\theta, \quad Q = \sum_{m=0}^{\infty} Q_m\cos m\theta. \end{aligned} \tag{17}$$

When $m = 0$, all the variables mentioned in Eq. (17) are independent of the coordinate θ, and the non-axisymmetric problem can be degenerated to an axisymmetric one.

Similarly, the three displacement functions can be expressed as follows:

$$E = \sum_{m=0}^{\infty} E_m\cos m\theta, \quad S = \sum_{m=0}^{\infty} S_m\cos m\theta, \quad R = \sum_{m=0}^{\infty} R_m\sin m\theta. \tag{18}$$

From Eq. (15), we have

$$\begin{aligned} u_v &= \sum_{m=0}^{\infty} u_{vm}\cos m\theta, \quad \sigma_{vz} = \sum_{m=0}^{\infty} \sigma_{vzm}\cos m\theta, \\ u_h &= \sum_{m=0}^{\infty} u_{hm}\sin m\theta, \quad \sigma_{hz} = \sum_{m=0}^{\infty} \sigma_{hzm}\sin m\theta. \end{aligned} \tag{19}$$

Substitution of Eqs. (17), (18) and (19) into Eqs. (16a) and (16b) yields

$$\begin{cases} u_{vm} = \nabla_m^2 \left(\frac{\partial E_m}{\partial z} - z S_m \right), \\ u_{zm} = \frac{\partial^2 E_m}{\partial z^2} - z \frac{\partial S_m}{\partial z} + S_m, \\ \sigma_m = 2G \left[\frac{\partial S_m}{\partial z} - \eta \frac{\partial}{\partial z} \left(\nabla_m^2 + \frac{\partial^2}{\partial z^2} \right) E_m) \right], \\ \sigma_{zm} = 2G \left[\frac{\partial}{\partial z} \left(-\nabla_m^2 E_m \right) - z \frac{\partial^2 S_m}{\partial z^2} + \frac{\partial S_m}{\partial z} \right], \\ \sigma_{vzm} = 2G \nabla_m^2 \left(\frac{\partial^2 E_m}{\partial z^2} - z \frac{\partial S_m}{\partial z} \right), \\ Q_m = 2Gk' \left[\frac{\partial^2 S_m}{\partial z^2} - \eta \frac{\partial^2}{\partial z^2} \left(\nabla_m^2 + \frac{\partial^2}{\partial z^2} \right) E_m \right], \end{cases} \tag{20a}$$

and

$$\begin{cases} u_{hm} = -2\nabla_m^2 R_m, \\ \sigma_{hzm} = -2G\nabla_m^2 \frac{\partial R_m}{\partial z}, \end{cases} \tag{20b}$$

where $\nabla_m^2 = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - m^2 \frac{1}{r^2})$ is an operator. To solve the governing equations, the Laplace–Hankel transform of E_m, S_m and R_m can be taken as follows [24]:

$$\begin{aligned} E_m(r, z, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \bar{E}_m(\xi, z, s) K(r, \xi) \mathrm{e}^{st} \mathrm{d}\xi \mathrm{d}s, \\ S_m(r, z, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \bar{S}_m(\xi, z, s) K(r, \xi) \mathrm{e}^{st} \mathrm{d}\xi \mathrm{d}s, \\ R_m(r, z, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \int_0^\infty \bar{R}_m(\xi, z, s) K(r, \xi) \mathrm{e}^{st} \mathrm{d}\xi \mathrm{d}s, \end{aligned} \tag{21}$$

where $\bar{E}_m(\xi, z, s)$, $\bar{S}_m(\xi, z, s)$ and $\bar{R}_m(\xi, z, s)$ are the corresponding variables of $E_m(r, z, t)$, $S_m(r, z, t)$ and $R_m(r, z, t)$ in the transformed domain, respectively, and the kernel is $K(r, \xi) = \xi J_m(r\xi)$.

Using the Laplace–Hankel transform and assuming zero initial dilatation e everywhere, Eq. (14) can be re-written into the following ordinary differential equations:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 \right) \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - q^2 \right) \bar{E}_m = 0, \tag{22a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 \right) \bar{S}_m = 0, \tag{22b}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \xi^2 \right) \bar{R}_m = 0, \tag{22c}$$

where $q^2 = \xi^2 + \frac{s}{C}$. The solutions for the above equations can be expressed as follows:

$$\bar{E}_m = A_1 sh\xi z + A_2 ch\xi z + A_3 shqz + A_4 chqz, \tag{23a}$$

$$\bar{S}_m = A_5 sh\xi z + A_6 ch\xi z, \tag{23b}$$

$$\bar{R}_m = A_7 sh\xi z + A_8 ch\xi z, \tag{23c}$$

where $A_1 \ldots A_8$ are functions of ξ and q.

Taking the Laplace–Hankel transform of Eqs. (20a) and (20b) yields

$$\begin{cases} \bar{u}_{vm} = -\xi^2[(\xi A_1 - zA_5)ch\xi z + (\xi A_2 - zA_6)sh\xi z + qA_3chqz + qA_4shqz], \\ \bar{u}_{zm} = (\xi^2 A_1 - \xi z A_6 + A_5)sh\xi z + (\xi^2 A_2 - \xi z A_5 + A_6)ch\xi z + q^2 A_3 shqz + q^2 A_4 chqz, \\ \bar{\sigma}_m = 2G[\xi A_5 ch\xi z + \xi A_6 sh\xi z - \eta(q^2 - \xi^2)(qA_3chqz + qA_4shqz)], \\ \bar{\sigma}_{zm} = 2G\xi[(A_1\xi^2 - A_6\xi z + A_5)ch\xi z + (A_2\xi^2 - A_5\xi z + A_6)sh\xi z + A_3\xi qchqz + A_4\xi qshqz], \\ \bar{\sigma}_{vzm} = -2G\xi^2[(\xi^2 A_1 - \xi z A_6)sh\xi z + (\xi^2 A_2 - \xi z A_5)ch\xi z + q^2 A_3 shqz + q^2 A_4 chqz], \\ \bar{Q}_m = 2Gk'[(\xi^2 A_5 sh\xi z + \xi^2 A_6 ch\xi z) - \eta(q^2 - \xi^2)(q^2 A_3 shqz + q^2 A_4 chqz)], \end{cases} \tag{24a}$$

and

$$\begin{cases} \bar{u}_{hm} = 2\xi^2(A_7 sh\xi z + A_8 ch\xi z), \\ \bar{\sigma}_{hzm} = 2G\xi^3(A_7 ch\xi z + A_8 sh\xi z). \end{cases} \tag{24b}$$

Substitution of $z = 0$ into Eqs. (24a) to (24b) results in

$$\begin{cases} \bar{u}_{vm}(0) = -\xi^3 A_1 - \xi^2 q A_3, \\ \bar{u}_{zm}(0) = \xi^2 A_2 + q^2 A_4 + A_6, \\ \bar{\sigma}_m(0) = -2G\eta(q^2 - \xi^2)qA_3 + 2G\xi A_5, \\ \bar{\sigma}_{zm}(0) = 2G\xi^3 A_1 + 2G\xi^2 q A_3 + 2G\xi A_5, \\ \bar{\sigma}_{vzm}(0) = -2G\xi^4 A_2 - 2G\xi^2 q^2 A_4, \\ \bar{Q}_m(0) = -2Gk'\eta(q^2 - \xi^2)q^2 A_4 + 2Gk'\xi^2 A_6, \end{cases} \tag{25a}$$

and

$$\begin{cases} \bar{u}_{hm}(0) = 2\xi^2 A_8, \\ \bar{\sigma}_{hzm}(0) = 2G\xi^3 A_7. \end{cases} \tag{25b}$$

Solving Eqs. (25a) and (25b) for $A_1 \ldots A_8$ and then substituting $A_1 \ldots A_8$ into Eqs. (24a) and (24b) yields

$$\bar{D}(\xi, z, s) = P(\xi, z, s)\bar{D}(\xi, 0, s), \tag{26a}$$
$$\bar{F}(\xi, z, s) = L(\xi, z, s)\bar{F}(\xi, 0, s), \tag{26b}$$

where $\bar{D}(\xi, z, s) = [\bar{u}_{vm}, \bar{u}_{zm}, \bar{\sigma}_m, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{Q}_m]^T$ and $\bar{F}(\xi, z, s) = [\bar{u}_{hm}, \bar{\sigma}_{hzm}]^T$. $P(\xi, z, s)$ is a transfer matrix of 6×6, which establishes the relationship of $[\bar{u}_{vm}, \bar{u}_{zm}, \bar{\sigma}_m, \bar{\sigma}_{zm}, \bar{\sigma}_{vzm}, \bar{Q}_m]^T$ between the ground surface ($z = 0$) and the depth z in the transformed domain. $L(\xi, z, s)$ is a transfer matrix of 2×2, which establishes the relationship of $[\bar{u}_{hm}, \bar{\sigma}_{hzm}]^T$ between on the ground surface ($z = 0$) and at the depth z in the transformed domain. The elements of $P(\xi, z, s)$ and $L(\xi, z, s)$ are provided in Appendix B.

Equations (26a) and (26b) can also be re-written as

$$\bar{D}(\xi, 0, s) = P(\xi, -z, s)\bar{D}(\xi, z, s), \tag{27a}$$
$$\bar{F}(\xi, 0, s) = L(\xi, -z, s)\bar{F}(\xi, z, s). \tag{27b}$$

4 Transfer matrix solutions for the consolidation of multi-layered soils

4.1 Solutions for the axisymmetric problem

Based on the transfer matrix solutions for a single soil layer, the solutions for the consolidation of multi-layered soils can also be obtained using the same transfer matrix method. As shown in Fig. 1, an axisymmetric normal load $q(r, H_{m1}, t)$ and an axisymmetric tangential load $p(r, H_{m1}, t)$ in the interior of the mth layer are considered, in which H_{m1} denotes the distance from the loading plane to the surface of the layered soils. It is assumed that the ground surface of the soil system is permeable; therefore, the boundary conditions of the surface are

$$\sigma_z(r, 0, t) = 0, \quad \sigma_{rz}(r, 0, t) = 0, \quad \sigma(r, 0, t) = 0. \tag{28}$$

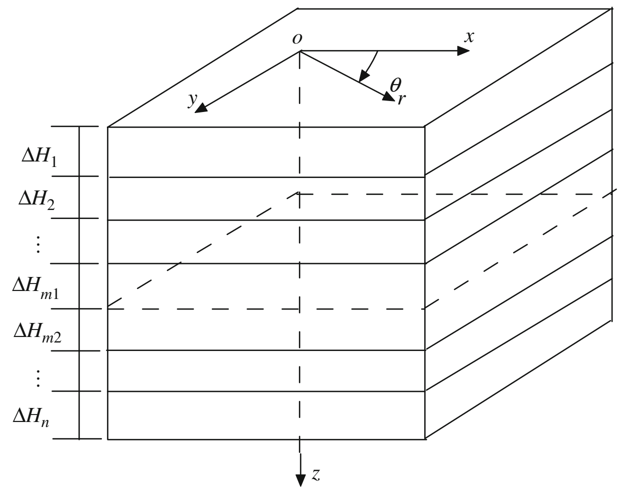

Fig. 1 A multilayered soil system in the cylindrical coordinates

Assuming a fixed and permeable bottom of the soil system, the boundary conditions of the bottom are

$$u_r(r, H_n, t) = 0, \quad u_z(r, H_n, t) = 0, \quad \sigma(r, H_n, t) = 0. \tag{29}$$

However, if a fixed and impermeable bottom of the soil system is assumed, the boundary conditions of the bottom are

$$u_r(r, H_n, t) = 0, \quad u_z(r, H_n, t) = 0, \quad Q(r, H_n, t) = 0. \tag{30}$$

Considering the continuity of the interface between two adjacent layers and at natural interfaces, we have

$$\begin{cases} u_r(r, H_i^+, t) = u_r(r, H_i^-, t), \\ u_z(r, H_i^+, t) = u_z(r, H_i^-, t), \\ \sigma(r, H_i^+, t) = \sigma_z(r, H_i^-, t), \\ \sigma_{rz}(r, H_i^+, t) = \sigma_{rz}(r, H_i^-, t), \\ \sigma_z(r, H_i^+, t) = \sigma_z(r, H_i^-, t), \\ Q(r, H_i^+, t) = Q(r, H_i^-, t), \end{cases} \tag{31}$$

At the interface of loading, we have

$$\begin{cases} u_r(r, H_{m1}^+, t) = u_r(r, H_{m1}^-, t), \\ u_z(r, H_{m1}^+, t) = u_z(r, H_{m1}^-, t), \\ \sigma(r, H_{m1}^+, t) = \sigma(r, H_{m1}^-, t), \\ \sigma_{rz}(r, H_{m1}^+, t) = \sigma_{rz}(r, H_{m1}^-, t) - p(r, H_{m1}^-, t), \\ \sigma_z(r, H_{m1}^+, t) = \sigma_z(r, H_{m1}^-, t) - q(r, H_{m1}^-, t), \\ Q(r, H_{m1}^+, t) = Q(r, H_{m1}^-, t), \end{cases} \tag{32}$$

where $u_r(r, H_i^+, t)$ stands for the variable of the $(i+1)$th layer when $z = H_i$, while $u_r(r, H_i^-, t)$ is the variable of ith layer when $z = H_i$. Other variables are expressed in the same manner.

Taking the Laplace–Hankel transform of the boundary conditions and the continuity conditions above, then applying Eq. (11) to the soil system from the last layer to the first layer sequentially, we obtain

$$\bar{B}(\xi, H_n, s) = \Pi \cdot \bar{B}(\xi, 0, s) - \Psi \cdot \Gamma, \tag{33}$$

where

$$\Pi = \Phi(\xi, \Delta H_n, s)\Phi(\xi, \Delta H_{n-1}, s) \cdots \Phi(\xi, \Delta H_1, s),$$
$$\Psi = \Phi(\xi, \Delta H_n, s)\Phi(\xi, \Delta H_{n-1}, s) \cdots \Phi(\xi, \Delta H_{m2}, s),$$
$$\Gamma = \left[0\ 0\ 0\ \bar{p}(\xi, H_{m1}, s)\ \bar{q}(\xi, H_{m1}, s)\ 0\right]^T,$$
$$\bar{q}(\xi, H_{m1}, s) = \int_0^\infty \int_0^\infty q(r, H_{m1}, t) r J_0(\xi r) \mathrm{e}^{-st} \mathrm{d}r\, \mathrm{d}t,$$
$$\bar{p}(\xi, H_{m1}, s) = \int_0^\infty \int_0^\infty p(r, H_{m1}, t) r J_1(\xi r) \mathrm{e}^{-st} \mathrm{d}r\, \mathrm{d}t,$$

in which $\Delta H_{m1} = H_{m1} - H_{m-1}$ and $\Delta H_{m2} = H_m - H_{m1}$. Obviously, $m = 1$ and $\Delta H_{m1} = 0$ represents a special case, in which the load acts on the ground surface.

From Eq. (33), $\bar{B}(\xi, 0, s)$ and $\bar{B}(\xi, H_n, s)$ can be obtained. For a given depth z in the ith layer above the horizontal plane where the load acts, the variables in the transformed domain can be obtained through the following relationship:

$$\bar{B}(\xi, z, s) = \Omega \cdot \bar{B}(\xi, 0, s), \tag{34}$$

where $\Omega = \Phi(\xi, z - H_{i-1}, s)\Phi(\xi, \Delta H_{i-1}, s) \cdots \Phi(\xi, \Delta H_1, s)$.

For a given depth z in the ith layer below the horizontal plane where the load acts, the variables in the transformed domain can be obtained through the following relationship:

$$\bar{B}(\xi, z, s) = \Theta \cdot \bar{B}(\xi, H_n, s), \tag{35}$$

where $\Theta = \Phi(\xi, z - H_i, s)\Phi(\xi, -\Delta H_{i+1}, s) \cdots \Phi(\xi, -\Delta H_n, s)$.

Equations (34) and (35) are the solutions for the consolidation problem of multi-layered soils in the transformed domain. By taking the inversion of the Laplace–Hankel transform to $\bar{B}(\xi, z, s)$, the solutions for stresses, displacements, excess pore pressure, and flux in the multi-layered soil subjected to axisymmetric loads can be obtained.

4.2 Solutions for the non-axisymmetric problem

Similarly, it is assumed that a non-axisymmetric load is applied in the interior of the mth layer of an n-layered soil. The non-axisymmetric load can be decomposed into three components $p(r, \theta, H_{m1}, t)$, $f(r, \theta, H_{m1}, t)$, and $q(r, \theta, H_{m1}, t)$ along the r, θ and z directions, respectively, and they can be rewritten in the following form in terms of the Fourier expansion suggested by Muki [25]:

$$p(r, \theta, H_{m1}, t) = \sum_{m=0}^{\infty} p_m(r, H_{m1}, t) \cos m\theta, \tag{36a}$$

$$f(r, \theta, H_{m1}, t) = \sum_{m=0}^{\infty} f_m(r, H_{m1}, t) \sin m\theta, \tag{36b}$$

$$q(r, \theta, H_{m1}, t) = \sum_{m=0}^{\infty} q_m(r, H_{m1}, t) \cos m\theta. \tag{36c}$$

Similarly, the boundary conditions of the ground surface can be obtained by assuming a permeable surface of the soil system:

$$\sigma_z(r, \theta, 0, t) = \sigma_{rz}(r, \theta, 0, t) = \sigma_{\theta z}(r, \theta, 0, t) = \sigma(r, \theta, 0, t) = 0. \tag{37}$$

For a fixed and permeable bottom of the soil system, the boundary condition is

$$u_r(r, \theta, H_n, t) = u_\theta(r, \theta, H_n, t) = u_z(r, \theta, H_n, t) = \sigma(r, \theta, H_n, t) = 0. \tag{38}$$

For a fixed but impermeable bottom, however, the boundary condition is

$$u_r(r,\theta,H_n,t)=u_\theta(r,\theta,H_n,t)=u_z(r,\theta,H_n,t)=Q(r,\theta,H_n,t)=0. \tag{39}$$

The following equations can be used to describe the fully bonded interface conditions:

$$\begin{cases} u_r(r,\theta,H_i^+,t)=u_r(r,\theta,H_i^-,t),\\ u_\theta(r,\theta,H_i^+,t)=u_\theta(r,\theta,H_i^-,t),\\ u_z(r,\theta,H_i^+,t)=u_z(r,\theta,H_i^-,t),\\ \sigma_z(r,\theta,H_i^+,t)=\sigma_z(r,\theta,H_i^-,t),\\ \sigma_{rz}(r,\theta,H_i^+,t)=\sigma_{rz}(r,\theta,H_i^-,t),\\ \sigma_{\theta z}(r,\theta,H_i^+,t)=\sigma_{\theta z}(r,\theta,H_i^-,t),\\ \sigma(r,\theta,H_i^+,t)=\sigma(r,\theta,H_i^-,t),\\ Q(r,\theta,H_i^+,t)=Q(r,\theta,H_i^-,t). \end{cases} \tag{40}$$

At the interface of loading, the interface conditions are

$$\begin{cases} u_r(r,\theta,H_{m1}^+,t)=u_r(r,\theta,H_{m1}^-,t),\\ u_\theta(r,\theta,H_{m1}^+,t)=u_\theta(r,\theta,H_{m1}^-,t),\\ u_z(r,\theta,H_{m1}^+,t)=u_z(r,\theta,H_{m1}^-,t),\\ \sigma_z(r,\theta,H_{m1}^+,t)=\sigma_z(r,\theta,H_{m1}^-,t)-q(r,\theta,H_{m1}^-,t),\\ \sigma_{rz}(r,\theta,H_{m1}^+,t)=\sigma_{rz}(r,\theta,H_{m1}^-,t)-p(r,\theta,H_{m1}^-,t),\\ \sigma_{\theta z}(r,\theta,H_{m1}^+,t)=\sigma_{\theta z}(r,\theta,H_{m1}^-,t)-f(r,\theta,H_{m1}^-,t),\\ \sigma(r,\theta,H_{m1}^+,t)=\sigma(r,\theta,H_{m1}^-,t),\\ Q(r,\theta,H_{m1}^+,t)=Q(r,\theta,H_{m1}^-,t). \end{cases} \tag{41}$$

Applying Eq. (26) to all the soil layers and utilizing the Laplace–Hankel transform of Eqs. (40) and (41), the following equations can be obtained:

$$\bar{D}(\xi,H_n,s)=\prod\nolimits_1\cdot\bar{D}(\xi,0,s)-\Psi_1\cdot\Gamma_1, \tag{42a}$$

$$\bar{F}(\xi,H_n,s)=\prod\nolimits_2\cdot\bar{F}(\xi,0,s)-\Psi_2\cdot\Gamma_2, \tag{42b}$$

where

$$\begin{gathered} \Pi_1=P(\xi,\Delta H_n,s)P(\xi,\Delta H_{n-1},s)\cdots P(\xi,\Delta H_1,s),\\ \Psi_1=P(\xi,\Delta H_n,s)P(\xi,\Delta H_{n-1},s)\cdots P(\xi,\Delta H_{m2},s),\\ \Gamma_1=\left[0\ 0\ 0\ \bar{q}_m(\xi,H_{m1},s)\ \bar{M}_m(\xi,H_{m1},s)\ 0\right]^T,\\ \Pi_2=L(\xi,\Delta H_n,s)L(\xi,\Delta H_{n-1},s)\cdots L(\xi,\Delta H_1,s),\\ \Psi_2=L(\xi,\Delta H_n,s)L(\xi,\Delta H_{n-1},s)\cdots L(\xi,\Delta H_{m2},s),\\ \Gamma_2=\left[0\ \bar{N}_m(\xi,H_{m1},s)\right]^T, \end{gathered}$$

$$\bar{M}_m(\xi, H_{m1}, s) = \int_0^\infty \int_0^\infty \left[\frac{p_m(r, H_{m1}, t)}{r} + \frac{\partial p_m(r, H_{m1}, t)}{\partial r} + \frac{m f_m(r, H_{m1}, t)}{r}\right] r J_m(\xi r) \mathrm{e}^{-st} \mathrm{d}r\, \mathrm{d}t,$$

$$\bar{N}_m(\xi, H_{m1}, s) = \int_0^\infty \int_0^\infty \left[-\frac{f_m(r, H_{m1}, t)}{r} - \frac{\partial f_m(r, H_{m1}, t)}{\partial r} - \frac{m p_m(r, H_{m1}, t)}{r}\right] r J_m(\xi r) \mathrm{e}^{-st} \mathrm{d}r\, \mathrm{d}t,$$

$$\bar{q}_m(\xi, H_{m1}, s) = \int_0^\infty \int_0^\infty [q_m(r, H_{m1}, t)] r J_m(\xi r) \mathrm{e}^{-st} \mathrm{d}r\, \mathrm{d}t.$$

From Eqs. (42a) and (42b), $\bar{D}(\xi, 0, s)$, $\bar{D}(\xi, H_n, s)$, $\bar{F}(\xi, 0, s)$, $\bar{F}(\xi, H_n, s)$ can be obtained analytically. For a given depth z in the ith layer above the horizontal plane where the load is applied, the variables in the transformed domain can be obtained through the following relationships:

$$\bar{D}(\xi, z, s) = \Omega_1 \cdot \bar{D}(\xi, 0, s), \tag{43a}$$
$$\bar{F}(\xi, z, s) = \Omega_2 \cdot \bar{F}(\xi, 0, s), \tag{43b}$$

in which

$$\Omega_1 = P(\xi, z - H_{i-1}, s) P(\xi, \Delta H_{i-1}, s) \cdots P(\xi, \Delta H_1, s),$$
$$\Omega_2 = L(\xi, z - H_{i-1}, s) L(\xi, \Delta H_{i-1}, s) \cdots L(\xi, \Delta H_1, s).$$

Similarly, for a given depth z in the ith layer below the horizontal plane where the load is applied, the variables in the transformed domain can be obtained through the following relationships:

$$\bar{D}(\xi, z, s) = \Theta_1 \cdot \bar{D}(\xi, H_n, s), \tag{44a}$$
$$\bar{F}(\xi, z, s) = \Theta_2 \cdot \bar{F}(\xi, H_n, s), \tag{44b}$$

where

$$\Theta_1 = P(\xi, z - H_i, s) P(\xi, -\Delta H_{i+1}, s) \cdots P(\xi, -\Delta H_n, s),$$
$$\Theta_2 = L(\xi, z - H_i, s) L(\xi, -\Delta H_{i+1}, s) \cdots L(\xi, -\Delta H_n, s).$$

Equations (43a), (43b), (44a) and (44b) are the solutions for the consolidation problem of multi-layered soils in the transformed domain. By taking the inversion of the Laplace–Hankel transform to $\bar{D}(\xi, z, s)$ and $\bar{F}(\xi, z, s)$, the solution for stresses, displacements, excess pore pressure, and flux in the multi-layered soil subjected to non-axisymmetric loads can be obtained.

5 Numerical results

To obtain the solutions for the actual problems, we can take the inversion of the Laplace–Hankel transform to the solutions in the transformed domain. As presented in the literature [9–11], the numerical inversion of the Laplace transform was carried out using the method proposed by Talbot [26]. The Talbot method was also used for the Laplace inversion in this study. As for the implementation of the inversion of the Hankel transform, the technique suggested by Ai et al. [23] was adopted.

The solutions in the transformed domain expressed in the Appendices A and B include the exponential functions, which increase exponentially when the depth z increases. Therefore, they can easily lead to ill-conditioned matrices and overflow of calculations. In this study, this difficulty was overcome by taking the scheme proposed by Ai et al. [23].

To verify the feasibility and the accuracy of the method proposed in this paper, the calculated results are compared against the existing results provided by Booker and Small [10] using the finite layer method. The parameters and results of the single soil layer subjected to a uniform circular vertical surface load are shown in Fig. 2. It is shown that the results from this study are in good agreement with the existing results obtained by the finite layer method.

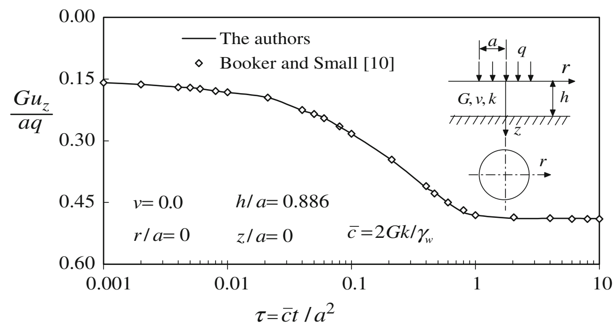

Fig. 2 Consolidation of a single soil layer under axisymmetric loading

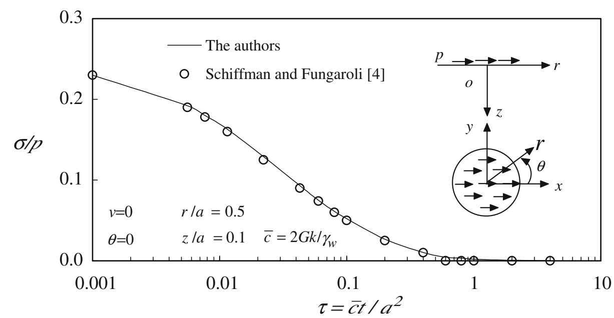

Fig. 3 Consolidation of a semi-infinite soil under non-axisymmetric loading

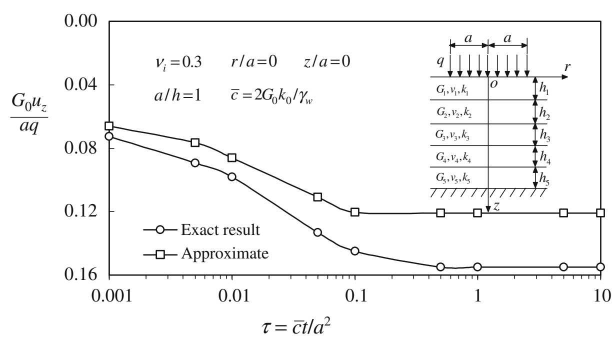

Fig. 4 Vertical displacement of the five-layered soil system under axisymmetric loading

The results for the consolidation of the semi-infinite soil obtained by Schiffman and Fungaroli [4] are compared with those obtained using the solution from this study in Fig. 3. The comparison shows that the results from this study are in good agreement with those by Schiffman and Fungaroli [4]. Therefore, the theory and the numerical method proposed in this study are proved appropriate and accurate.

In practice, the ground often comprises multi-layered soils with different types and properties. A five-layered soil profile was chosen in this study and is shown schematically in Figs. 4 and 5. It is assumed that the ground surface of the soil system is permeable and the bottom of the model is fixed and impermeable. The soil layers have the following relationships for the shear modulus, the thickness, and the permeability: $G_1 : G_2 : G_3 : G_4 : G_5 = G_0 : 2G_0 : 4G_0 : 2G_0 : G_0$; $h_1 : h_2 : h_3 : h_4 : h_5 = 1 : 2 : 4 : 1 : 2$; $k_1 : k_2 : k_3 : k_4 : k_5 = k_0 : 4k_0 : 2k_0 : k_0 : 2k_0$. The total thickness of five-layered soils is $h = h_1 + h_2 + h_3 + h_4 + h_5$. Poisson's ratio of all soil layers is taken as 0.3.

Figure 4 shows the displacement versus the time factor τ in the z direction at the point ($r = 0, z = 0$) of a five-layered soil under a surface vertical load. Figure 5 shows that the pore water pressure in the five-layered soil at the distance to the centerline of $r = a$ under a surface tangential load, changing with the depth z and the time factor, τ. The calculated vertical displacement and the excess pore water pressure for this five-layered soil system under axisymmetric and non-axisymmetric loadings are shown in Figs. 4 and 5, respectively.

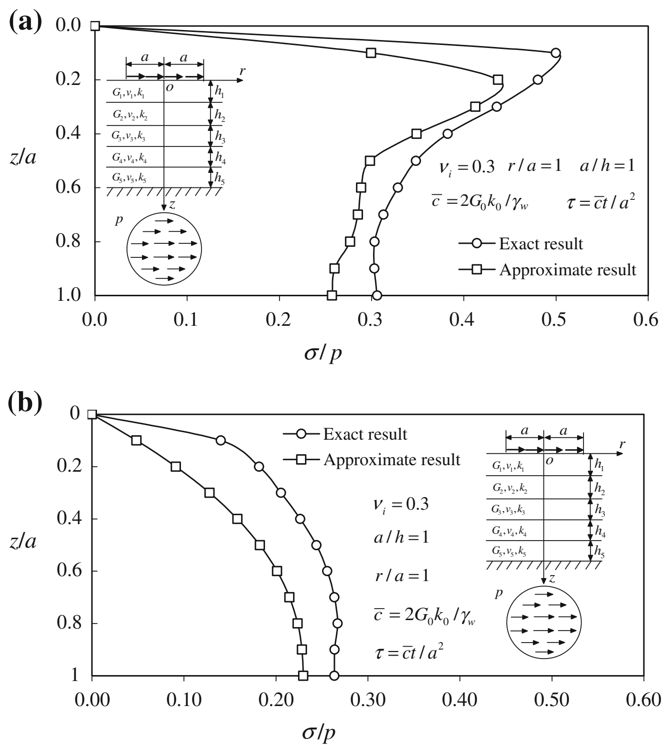

Fig. 5 Excess pore water pressure of the five-layered soil system under non-axisymmetric loading. **a** $\tau = 0.001$, **b** $\tau = 0.01$

To evaluate the influence of layered soil properties on the consolidation of multilayered soils, the weighted average shear modulus,$G' = \sum_{i=1}^{5} G_i h_i/h$ and the weighted average permeability, $k' = \sum_{i=1}^{5} k_i h_i/h$ are introduced and used to approximately calculate the vertical displacement and the excess pore water pressure shown in Figs. 4 and 5, respectively. The weighted average shear modulus and permeability for this example are $2.5G_0$ and $2.2k_0$, respectively. The exact result was obtained using the method for multi-layered soils proposed in this study with the actual properties (shear modulus and permeability) for each soil layer. However, the approximate result was obtained using the method for a single layer soil with the weighted properties within the entire depth.

Figure 4 shows that the vertical displacement at the center of the loading area on the ground surface increases with the time factor, τ. The approximation method underestimates the vertical displacement as compared with the exact solution. It is well known that the vertical displacement decreases with an increase of the shear modulus for a homogeneous soil. For the multi-layered soil system selected as an example in this study, however, the shear moduli of all the soil layers except the third layer are less than the weighted average shear modulus. In other words, the third soil layer with a high shear modulus is over-weighted for the average modulus. As a result, the approximation method underestimates the vertical displacement as compared with the exact solution.

Figure 5 shows that the excess pore water pressure at the beginning is not uniformly distributed and a peak value exists at the depth of approximately $0.15a$ based on the exact result and $0.20a$ based on the approximate result. With an increase of time, the excess pore water pressure decreases and the peak value disappears. Figure 5 also shows that the exact result has higher excess pore water pressure than the approximate result. It is well known that excess pore water pressure can dissipate faster in a homogeneous soil with higher permeability than that with lower permeability. For the multi-layered soil system selected as an example in this study, however, the permeability values of all the soil layers except the second layer are less than the weighted average permeability. In other words, the second soil layer with a high permeability value is over-weighted for the average permeability. As a result, the exact result has higher excess pore water pressure than the approximate result.

The above comparisons show that layered soil properties have a great influence on its consolidation behavior.

6 Conclusions

Based on the displacement functions, the transfer matrix solutions in the Laplace–Hankel transform domain for the consolidation problem of a single soil layer subjected to axisymmetric and non-symmetric loads are obtained. These solutions are extended for a multi-layered soil system by considering the continuity conditions between adjacent layers and the boundary conditions of the layered soil system. The actual solutions in the physical domain are acquired by the inversion of the Laplace–Hankel transform. The calculated results for the consolidation for a single soil layer using the proposed method in this study are compared well and validated against those by others. The consolidation of a multi-layered soil system (five soil layers were selected as an example in this study) under axisymmetric and non-axisymmetric loadings is investigated and discussed. The approximation method using the solution for a single soil layer with the weighted average properties (shear modulus and permeability) underestimates the vertical displacement on the ground surface and the excess pore water pressure as compared with the exact method using the solution for multi-layered soils proposed in this study with actual properties for each soil layer.

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Appendix A: Transfer matrix $\Phi(\xi, z)$ for an axisymmetric problem

$$
\begin{aligned}
\Phi_{11} &= \frac{2G\xi^2 C}{Ms}(ch\xi z - chqz) + (ch\xi z + \xi z sh\xi z) = \Phi_{44},\\
\Phi_{12} &= \frac{2G\xi^2 C}{Ms}(sh\xi z - \frac{\xi}{q} shqz) + \xi z ch\xi z = -\Phi_{54},\\
\Phi_{13} &= \frac{\xi C}{Ms}(ch\xi z - chqz),\\
\Phi_{14} &= \frac{\xi C}{Ms}\left(sh\xi z - \frac{\xi}{q} shqz\right) + \frac{1}{2G\xi}(sh\xi z + \xi z ch\xi z),\\
\Phi_{15} &= \frac{\xi C}{Ms}(ch\xi z - chqz) + \frac{1}{2G\xi}\xi z sh\xi z = -\Phi_{24},\\
\Phi_{16} &= \frac{1}{s}\left(sh\xi z - \frac{\xi}{q} shqz\right),\\
\Phi_{21} &= \frac{2G\xi C}{Ms}(qshqz - \xi sh\xi z) - \xi z ch\xi z = -\Phi_{45},\\
\Phi_{22} &= \frac{2G\xi^2 C}{Ms}(chqz - ch\xi z) + (ch\xi z - \xi z sh\xi z) = \Phi_{55},\\
\Phi_{23} &= \frac{C}{Ms}(qshqz - \xi sh\xi z),\\
\Phi_{25} &= \frac{C}{Ms}(qshqz - \xi sh\xi z) + \frac{1}{2G\xi}(sh\xi z - \xi z ch\xi z),\\
\Phi_{26} &= \frac{1}{s}(chqz - ch\xi z),\\
\Phi_{31} &= 2G\xi(chqz - ch\xi z),\\
\Phi_{32} &= 2G\xi\left(\frac{\xi}{q} shqz - sh\xi z\right),\\
\Phi_{33} &= chqz = \Phi_{66},\\
\Phi_{34} &= \frac{\xi}{q} shqz - sh\xi z,\\
\Phi_{35} &= chqz - ch\xi z,
\end{aligned}
$$

$$\Phi_{36} = \frac{M}{qC} shqz,$$

$$\Phi_{41} = 2G\left[\frac{2G\xi^2 C}{Ms}(\xi sh\xi z - qshqz) + \xi(sh\xi z + \xi zch\xi z)\right],$$

$$\Phi_{42} = 2G\left[\frac{2G\xi^3 C}{Ms}(ch\xi z - chqz) + \xi \cdot \xi zsh\xi z)\right],$$

$$\Phi_{43} = 2G\left[\frac{\xi C}{Ms}(\xi sh\xi z - qshqz)\right],$$

$$\Phi_{46} = 2G\left[\frac{\xi}{s}(ch\xi z - chqz)\right],$$

$$\Phi_{51} = 2G\left[\frac{2G\xi^3 C}{Ms}(chqz - ch\xi z) - \xi \cdot \xi zsh\xi z)\right],$$

$$\Phi_{52} = 2G\left[\frac{2G\xi^3 C}{Ms}\left(\frac{\xi}{q}shqz - sh\xi z\right) + \xi(sh\xi z - \xi zch\xi z)\right],$$

$$\Phi_{53} = 2G\frac{\xi^2 C}{Ms}(chqz - ch\xi z),$$

$$\Phi_{56} = 2G\frac{\xi}{s}\left(\frac{\xi}{q}shqz - sh\xi z\right),$$

$$\Phi_{61} = \frac{2G\xi C}{M}(qshqz - \xi sh\xi z),$$

$$\Phi_{62} = \frac{2G\xi^2 C}{M}(chqz - ch\xi z),$$

$$\Phi_{63} = \frac{qC}{M} shqz,$$

$$\Phi_{64} = \frac{\xi C}{M}(chqz - ch\xi z),$$

$$\Phi_{65} = \frac{C}{M}(qshqz - \xi sh\xi z).$$

Appendix B: Transfer matrix for a non-axisymmetric problem

$$P_{11} = ch\xi z + \xi zch\xi z + \frac{2Gk'}{s}\xi^2(ch\xi z - chqz),$$

$$P_{12} = \xi^2 zsh\xi z + \frac{2G\xi^3 k'}{s}(sh\xi z - \frac{\xi}{q}shqz),$$

$$P_{13} = -\frac{\xi^2 k'}{s}(ch\xi z - chqz),$$

$$P_{14} = \frac{1}{2G}\xi zch\xi z + \frac{\xi^2 k'}{s}(ch\xi z - chqz),$$

$$P_{15} = \frac{\xi k'}{s}\left(sh\xi z - \frac{\xi}{q}shqz\right) + \frac{1}{2G\xi}sh\xi z + \frac{z}{2G}sh\xi z,$$

$$P_{16} = -\frac{\xi}{s}\left(sh\xi z - \frac{\xi}{q}shqz\right),$$

$$P_{21} = \frac{2Gk'}{s}(qshqz - \xi sh\xi z) - zch\xi z,$$

$$P_{22} = ch\xi z + \frac{2G\xi^2 k'}{s}(chqz - ch\xi z) - \xi zsh\xi z,$$

$$P_{23} = \frac{k'}{s}(\xi sh\xi z - qshqz),$$

$$P_{24} = -\frac{k'}{s}(\xi sh\xi z - qshqz) + \frac{1}{2G\xi}sh\xi z - \frac{z}{2G}ch\xi z,$$

$$P_{25} = \frac{k'}{s}(chqz - ch\xi z) - \frac{z}{2G\xi}sh\xi z,$$

$$P_{26} = \frac{1}{s}(ch\xi z - chqz),$$

$$P_{31} = 2G(ch\xi z - chqz),$$

$$P_{32} = 2G\xi\left(sh\xi z - \frac{\xi}{q}shqz\right),$$

$$P_{33} = P_{66} = chqz,$$

$$P_{34} = ch\xi z - chqz,$$

$$P_{35} = \frac{1}{\xi}sh\xi z - \frac{1}{q}shqz,$$

$$P_{36} = \frac{1}{k'q}shqz,$$

$$P_{41} = \frac{4G^2\xi^2 k'}{s}(chqz - ch\xi z) - 2G\xi zsh\xi z,$$

$$P_{42} = 2G\xi sh\xi z - 2G\xi^2 zch\xi z + \frac{4G^2\xi^3 k'}{s}\left(\frac{\xi}{q}shqz - sh\xi z\right),$$

$$P_{43} = \frac{2Gk'\xi^2}{s}(ch\xi z - chqz),$$

$$P_{44} = ch\xi z - \xi zsh\xi z + \frac{2Gk'\xi^2}{s}(chqz - ch\xi z),$$

$$P_{45} = -zch\xi z + \frac{2Gk'\xi}{s}\left(\frac{\xi}{q}shqz - sh\xi z\right),$$

$$P_{46} = 2G\xi\left(sh\xi z - \frac{\xi}{q}shqz\right),$$

$$P_{51} = 2G\xi(sh\xi z + \xi zch\xi z) + \frac{4G^2\xi^2 k'}{s}(\xi sh\xi z - qshqz),$$

$$P_{52} = 2G\xi^3 zsh\xi z + \frac{4G^2\xi^4 k'}{s}(ch\xi z - chqz),$$

$$P_{53} = \frac{2G\xi^2 k'}{s}(qshqz - \xi sh\xi z),$$

$$P_{54} = \frac{2G\xi^2 k'}{s}(\xi sh\xi z - qshqz) + \xi^2 zch\xi z,$$

$$P_{55} = \xi zsh\xi z + ch\xi z + \frac{2G\xi^2 k'}{s}(ch\xi z - chqz),$$

$$P_{56} = \frac{2G\xi^2}{s}(chqz - ch\xi z),$$

$$
\begin{aligned}
P_{61} &= 2Gk'(\xi sh\xi z - qshqz),\\
P_{62} &= 2Gk'\xi^2(ch\xi z - chqz),\\
P_{63} &= k'qshqz,\\
P_{64} &= k'(\xi sh\xi z - qshqz),\\
P_{65} &= k'(ch\xi z - chqz),\\
L_{11} &= L_{22} = ch\xi z,\\
L_{12} &= \frac{1}{G\xi} sh\xi z,\\
L_{21} &= G\xi sh\xi z.
\end{aligned}
$$

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