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Nonlinear analysis of non-uniform beams on nonlinear elastic foundation

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Abstract In this paper a boundary integral equation solution to the nonlinear problem of non-uniform beams resting on a nonlinear triparametric elastic foundation is presented, which permits also the treatment of nonlinear boundary conditions. The nonlinear subgrade model which describes the foundation includes the linear and nonlinear Winkler (normal) parameters and the linear Pasternak (shear) foundation parameter. The governing equations are derived in terms of the displacements for nonlinear analysis in the deformed configuration and for linear analysis in the undeformed one. Moreover, as the cross-sectional properties of the beam vary along its axis, the resulting coupled nonlinear differential equations have variable coefficients which complicate the mathematical problem even more. Their solution is achieved using the analog equation method of Katsikadelis. Several beams are analyzed under various boundary conditions and load distributions, which illustrate the method and demonstrate its efficiency and accuracy. Finally, useful conclusions are drawn from the investigation of the nonlinear response of non-uniform beams resting on nonlinear elastic foundation.

1 Introduction

Beams resting on an elastic foundation are very often come across in engineering practice [1] and therefore an accurate and reliable method of analysis is required, especially when the properties of their cross-section are variable. The complexity of the problem highly increases with the increase of the applied external load since the beam's transverse deflection influences the axial force and the resulting equations become coupled and nonlinear [2]. In this case, the linear elastic subgrade model is inadequate to describe the real behavior of the foundation and the use of a more sophisticated nonlinear one becomes inevitable.

The work has been done on the subject is limited only to the *linear* response of uniform beams on linear elastic foundation (for an analytic presentation see [3]), uniform beams on nonlinear elastic foundation [4–9] and non-uniform beams on nonlinear elastic foundation [10]. Distefano and Todeschini [4] presented the solution of a beam on nonlinear elastic foundation by means of the theory of quasilinearization. Sharma and DasGupta [5] studied the bending of axially constrained beams on nonlinear Winkler-type elastic foundation by an iteration method using Green's functions. Beaufait and Hoadley [6] used the midpoint difference method for the solution of elastic deformation of a beam supported on an elastic nonlinear foundation on rigid or elastic supports. A numerically integrated Finite Element Method developed in [7] for the analysis of a three-dimensional, nonlinear Winkler foundation while Yankelevsky et al. presented an iterative procedure based on the exact stiffness matrix for beams on a nonlinear Winkler foundation [8] and on a nonlinear Winkler foundation with gaps [9]. Finally, Kuo and Lee [10] examined the static deflection of a general elastically end

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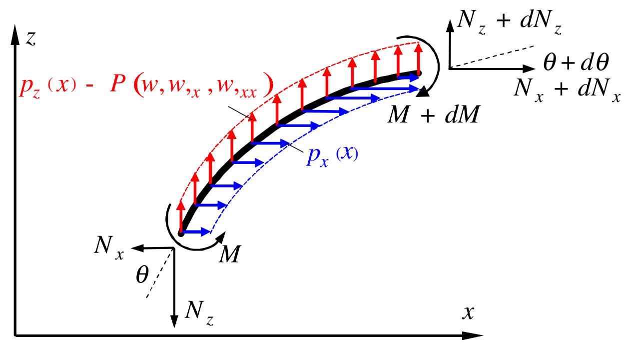

Fig. 1 Forces and moments acting on the deformed element

restrained non-uniform beam resting on a nonlinear elastic foundation by using the method of perturbation. To the author's knowledge publications on the solution to the general problem of the *nonlinear* analysis of non-uniform beams resting on a nonlinear elastic foundation do not exist.

In this paper the nonlinear response of beams with variable properties resting on a nonlinear elastic foundation is investigated. The governing equations are derived in terms of the displacements for nonlinear analysis in the deformed configuration and for linear analysis in the undeformed one. The linear analysis is performed in order to reveal the difference of the nonlinear response. The nonlinear model which describes the foundation includes the linear and nonlinear Winkler (normal) parameters and the linear Pasternak (shear) foundation parameter [11,12]. Moreover, the variable cross-sectional properties of the beam result in governing differential equations with variable coefficients which complicate the mathematical problem even more. The solution of the problems was achieved using the analog equation method (AEM) of Katsikadelis. The method was developed for the nonlinear analysis of beams by Katsikadelis and Tsiatas [2] and has been used for the solution of complicated nonlinear beam problems (e.g. [13]). Using the principle of the analog equation, the two coupled nonlinear differential equations with variable coefficients are replaced by two uncoupled linear ones pertaining to the axial and transverse deformation of a substitute beam with unit axial and bending stiffness, respectively, under fictitious load distributions. Several beams are analyzed under various boundary conditions and load distributions, which illustrate the method and demonstrate its efficiency and accuracy. Moreover, useful conclusions are drawn from the investigation of the nonlinear response of non-uniform beams resting on a nonlinear foundation.

2 Governing equations

Consider an initially straight beam of length l having variable axial stiffness EA and bending stiffness EI, which may result from variable cross-section, $A = A(x)$, and/or from inhomogeneous linearly elastic material, $E = E(x)$; $I = I(x)$ is the moment of inertia of the cross-section. The x-axis coincides with the neutral axis of the beam, which is bent in its plane of symmetry xz under the combined action of the distributed loads $p_x = p_x(x)$ and $p_z = p_z(x)$ in the x- and z-direction, respectively. It is assumed that there is no abrupt variation in cross-section of the beam so that the Euler–Bernoulli theory is valid [14,15]. The beam is resting on a nonlinear elastic foundation. In the following the equilibrium equations in terms of the displacements are derived (a) for nonlinear response and (b) for linear response.

2.1 Nonlinear theory

When the magnitude of the external applied loads is large, the nonlinear theory of finite strains must be employed [16]. However, in this work the external applied loads produce small strains as compared with unity and the theory of moderately large deflections is considered. In this case the nonlinear kinematic relation retains the square of the slope of the deflection. The equilibrium equations are derived by considering the equilibrium of the deformed element [2] taking into account the subgrade reaction which is opposite to the distributed load $p_z = p_z(x)$ acting in the z-direction (see Fig. 1). Thus, we obtain

$$N_{x,x} = -p_x(x) \tag{1}$$

$$N_z{,_x} = -p_z(x) + P(w, w{,_x}, w{,_{xx}}), \tag{2}$$
$$M{,_x} = Q, \tag{3}$$

where

$$N_x = N - Qw{,_x}, \tag{4}$$
$$N_z = Nw{,_x} + Q, \tag{5}$$

are the stress resultants in the x- and z- direction, respectively, and $P(w, w{,_x}, w{,_{xx}})$ is a nonlinear function of the deflection and its derivatives up to the second order representing the foundation reaction. It should be noted, however, that usual realistic foundation models depend mostly on the deflection and its second derivative [17], and without restricting the generality, the model used in this investigation is [11,12]

$$P(w, w{,_x}, w{,_{xx}}) = k_1 w + k_2 w^3 - k_3 w{,_{xx}}, \tag{6}$$

where k_1 and k_2 are the linear and nonlinear Winkler (normal) foundation parameters, respectively, and k_3 is the linear Pasternak (shear) foundation parameter.

The axial force and the bending moment are evaluated by integrating appropriately the normal stress, that is

$$N = EA\left(u{,_x} + \frac{1}{2}w{,_x^2}\right), \tag{7}$$
$$M = -EIw{,_{xx}}. \tag{8}$$

Substituting Eqs. (4), (5) and (6) into Eqs. (1) and (2) and using Eq. (3) to eliminate Q yields

$$N{,_x} - (M{,_x}\, w{,_x}){,_x} = -p_x(x), \tag{9}$$
$$M{,_{xx}} + (Nw{,_x}){,_x} - (k_1 w + k_2 w^3 - k_3 w{,_{xx}}) = -p_z(x), \tag{10}$$

which by virtue of Eqs. (7) and (8) become

$$\left[EA\left(u{,_x} + \frac{1}{2}w{,_x^2}\right)\right]{,_x} + (EIw{,_{xx}}\, w{,_x}){,_x} = -p_x(x), \tag{11}$$
$$-(EIw{,_{xx}}){,_{xx}} + \left[EA\left(u{,_x} + \frac{1}{2}w{,_x^2}\right) w{,_x}\right]{,_x} - (k_1 w + k_2 w^3 - k_3 w{,_{xx}}) = -p_z(x). \tag{12}$$

The pertinent boundary conditions are [2]

$$a_1 u(0) + a_2 N_x(0) = a_3 \tag{13}$$

and

$$\bar{a}_1 u(l) + \bar{a}_2 N_x(l) = \bar{a}_3, \tag{14}$$
$$\beta_1 w(0) + \beta_2 N_z(0) = \beta_3 \tag{15}$$

and

$$\bar{\beta}_1 w(l) + \bar{\beta}_2 N_z(l) = \bar{\beta}_3, \tag{16}$$
$$\gamma_1 w{,_x}(0) + \gamma_2 M(0) = \gamma_3 \tag{17}$$

and

$$\bar{\gamma}_1 w{,_x}(l) + \bar{\gamma}_2 M(l) = \bar{\gamma}_3, \tag{18}$$

where $a_k, \bar{a}_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k$ $(k = 1, 2, 3)$ are given constants. Equations (13)–(18) describe the most general boundary conditions associated with the problem and can include elastic support or restrain.

2.2 Linear theory

Equation (9) after dropping the nonlinear term $M,_x\, w,_x$ can be readily integrated independently [15]:

$$N(x) = \int_x^l p(x)\mathrm{d}x \tag{19}$$

Subsequently, Eq. (10) using Eq. (8) yields the linear counterpart of Eq. (12), that is, the linear bending equation of a non-uniform beam resting on nonlinear foundation

$$-\left(EIw,_{xx}\right),_{xx} + \left[N(x)w,_x\right],_x - \left(k_1 w + k_2 w^3 - k_3 w,_{xx}\right) = -p_z(x) \tag{20}$$

together with the pertinent boundary conditions

$$\beta_1 w(0) + \beta_2 Q(0) = \beta_3 \tag{21}$$

and

$$\bar{\beta}_1 w(l) + \bar{\beta}_2 Q(l) = \bar{\beta}_3, \tag{22}$$

$$\gamma_1 w,_x(0) + \gamma_2 M(0) = \gamma_3 \tag{23}$$

and

$$\bar{\gamma}_1 w,_x(l) + \bar{\gamma}_2 M(l) = \bar{\gamma}_3. \tag{24}$$

3 The AEM solution for the nonlinear analysis of non-uniform beams

Equations (11) and (12) are solved using the AEM, as developed for the large deflection analysis of beams with variable stiffness by Katsikadelis and Tsiatas [2]. This method is applied to the problem at hand as follows. Let $u = u(x)$ and $w = w(x)$ be the sought solutions, which are two and four times differentiable, respectively, in $(0, l)$. Noting that Eqs. (11) and (12) are of the second order with respect to u and of fourth order with respect to w, one obtains by differentiating

$$u,_{xx} = b_1(x), \tag{25}$$

$$w,_{xxxx} = b_2(x). \tag{26}$$

Equations (25) and (26) describe the axial and bending linear response of a beam with constant unit axial and flexural stiffness subjected to the fictitious axial b_1 and transverse b_2 loads, respectively. They indicate that the solution of Eqs. (11) and (12) can be established by solving Eqs. (25) and (26) under the boundary conditions (13)–(18), provided that the fictitious load distributions b_1, b_2 are first determined. Note that Eqs. (25) and (26) are referred to as the analog equations to Eqs. (11) and (12). The fictitious loads are established by developing a procedure based on the integral equation method for one-dimensional problems. Thus, the integral representations of the solutions of Eqs. (25) and 26) are written as

$$u(x) = c_1 x + c_2 + \int_0^l G_1(x,\xi) b_1(\xi)\mathrm{d}\xi, \tag{27}$$

$$w(x) = c_3 x^3 + c_4 x^2 + c_5 x + c_6 + \int_0^l G_2(x,\xi) b_2(\xi)\mathrm{d}\xi, \tag{28}$$

where $c_i \quad (i = 1, 2, \ldots, 6)$ are arbitrary integration constants to be determined from the boundary conditions and

$$G_1 = \frac{1}{2}\left|x - \xi\right|, \tag{29}$$

$$G_2 = \frac{1}{12}\left|x - \xi\right|(x - \xi)^2 \tag{30}$$

are the fundamental solutions (free space Green's functions) of Eqs. (25) and (26), respectively.

The derivatives of u and w are obtained by direct differentiation of Eqs. (27) and (28). Thus, we have

$$u_{,x}(x) = c_1 + \int_0^l G_{1,x}(x,\xi) b_1(\xi)\mathrm{d}\xi, \tag{31}$$

$$u_{,xx}(x) = b_1(x), \tag{32}$$

$$w_{,x}(x) = 3c_3x^2 + 2c_4x + c_5 + \int_0^l G_{2,x}(x,\xi) b_2(\xi)\mathrm{d}\xi, \tag{33}$$

$$w_{,xx}(x) = 6c_3x + 2c_4 + \int_0^l G_{2,xx}(x,\xi) b_2(\xi)\mathrm{d}\xi, \tag{34}$$

$$w_{,xxx}(x) = 6c_3 + \int_0^l G_{2,xxx}(x,\xi) b_2(\xi)\mathrm{d}\xi, \tag{35}$$

$$w_{,xxxx}(x) = b_2(x). \tag{36}$$

Substituting Eqs. (27), (28) and the above derivatives into Eqs. (11) and (12) yields the equations from which the fictitious sources b_1 and b_2 can be determined. This can be implemented only numerically as follows.

The interval $(0, l)$ is divided into N equal elements on which b_1 and b_2 are assumed to vary according to a certain law (constant, linear, parabolic etc). The constant element assumption is employed here, because the numerical implementation becomes very simple and the obtained numerical results are very good.

After discretization of Eqs. (27) and (28) we obtain

$$u(x) = \sum_{j=1}^{2} x^{2-j} c_j + b_1(\xi_j) \sum_{j=1}^{N} \int_j G_1(x,\xi)\mathrm{d}\xi, \tag{37}$$

$$w(x) = \sum_{j=1}^{4} x^{4-j} c_{j+2} + b_2(\xi_j) \sum_{j=1}^{N} \int_j G_2(x,\xi)\mathrm{d}\xi, \tag{38}$$

or using matrix notation

$$u(x) = \mathbf{H}_1(x)\mathbf{c}_1 + \mathbf{G}_1(x)\mathbf{b}_1, \tag{39}$$

$$w(x) = \mathbf{H}_2(x)\mathbf{c}_2 + \mathbf{G}_2(x)\mathbf{b}_2, \tag{40}$$

where $\mathbf{G}_1(x)$ and $\mathbf{G}_2(x)$ are $1 \times N$ known matrices originating from the integration of the kernels $G_1(x,\xi)$ and $G_2(x,\xi)$ on the elements, respectively; $\mathbf{H}_1(x) = \begin{bmatrix} x & 1 \end{bmatrix}$ and $\mathbf{H}_2(x) = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix}$; $\mathbf{c}_1 = \{c_1, c_2\}^T$; $\mathbf{c}_2 = \{c_3, c_4, c_5, c_6\}^T$; $\mathbf{b}_1$, $\mathbf{b}_2$ are the vectors containing the values of the fictitious loads at the nodal points, respectively. Similarly, we obtain for Eqs. (31)–(36)

$$u_{,x}(x) = \mathbf{H}_{1x}(x)\mathbf{c}_1 + \mathbf{G}_{1x}(x)\mathbf{b}_1, \tag{41}$$

$$u_{,xx}(x) = \mathbf{b}_1, \tag{42}$$

$$w_{,x}(x) = \mathbf{H}_{2x}(x)\mathbf{c}_2 + \mathbf{G}_{2x}(x)\mathbf{b}_2, \tag{43}$$

$$w_{,xx}(x) = \mathbf{H}_{2xx}(x)\mathbf{c}_2 + \mathbf{G}_{2xx}(x)\mathbf{b}_2, \tag{44}$$

$$w_{,xxx}(x) = \mathbf{H}_{2xxx}(x)\mathbf{c}_2 + \mathbf{G}_{2xxx}(x)\mathbf{b}_2, \tag{45}$$

$$w_{,xxxx}(x) = \mathbf{b}_2, \tag{46}$$

where $\mathbf{G}_{1x}(x), \mathbf{G}_{2xx}(x), \ldots, \mathbf{G}_{2xxx}(x)$ are $1 \times N$ known matrices, originating from the integration of the derivatives of the kernels $G_1(x,\xi)$, $G_2(x,\xi)$ on the elements; $\mathbf{H}_{1x}(x)$ is a 1×2 known matrix resulting from the differentiation of $\mathbf{H}_1(x)$, whereas $\mathbf{H}_{2x}(x)$, $\mathbf{H}_{2xx}(x)$, $\mathbf{H}_{2xxx}(x)$ are 1×4 known matrices resulting from the differentiation of $\mathbf{H}_2(x)$.

Table 1 Deflection (m) and bending moment (kNm) in Example 4.1, for various values of N

	Present study			Kuo and Lee [10]
	N = 11	N = 21	N = 31	
$w(0.5)$	0.022	0.022	0.022	0.022
$w(1.0)$	0.082	0.082	0.081	0.082
$M(0.0)$	77.057	76.278	76.116	76.236
$M(0.5)$	36.505	36.319	36.279	36.373

Linear response ($p_{z0} = 500$ kN/m)

Finally, collocating Eqs. (11) and (12) at the N nodal points and substituting the discretized Eqs. (39)–(46) yields the following equations:

$$\mathbf{F}_1 \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) = -\mathbf{p}_x, \tag{47}$$

$$\mathbf{F}_2 \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) - \mathbf{P} \left(\mathbf{b}_2, \mathbf{c}\right) = -\mathbf{p}_z, \tag{48}$$

where $\mathbf{F}_i(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c})$ are generalized stiffness vectors, $\mathbf{P}(\mathbf{b}_2, \mathbf{c})$ is the foundation's reaction vector and $\mathbf{c} = \{c_1, c_2, \ldots, c_6\}^T$. Equations (47) and (48) constitute a system of $2N$ nonlinear algebraic equations with $2N + 6$ unknowns. The required six additional equations result from the boundary conditions. Thus, after substituting the relevant derivatives into Eqs. (13)–(18), we obtain

$$\mathbf{f}_i \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) = \mathbf{0} \quad (i = 1, 2, \ldots, 6). \tag{49}$$

The nonlinear equations (47)–(49) are solved numerically to yield $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{c}$ by minimizing the function

$$S \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) = \sum_{i=1}^{N} \left\{ \left[F_1^i \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) + p_x^i \right]^2 + \left[F_2^i \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) - P^i \left(\mathbf{b}_2, \mathbf{c}\right) + p_z^i \right]^2 \right\} + \sum_{j=1}^{6} \left[f_j \left(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}\right) \right]^2. \tag{50}$$

4 Numerical examples

On the base of the procedure described in previous section a FORTRAN program has been written for establishing the nonlinear response of non-uniform beams resting on nonlinear foundation. In all examples the results have been obtained using $N = 21$ elements.

4.1 Cantilever beam with variable cross-section

A non-uniform cantilever beam with length $l = 1.0$ m has been studied solving both sets of equations (linear and nonlinear) in order to compare the results with those obtained by Kuo and Lee [10] using the small nonlinear foundation reaction $P = 500(1.6 - 2x + x^2)w + 250w^2$. The employed data are: $EI = 500(1 - 0.5x)^3$ kNm^2 and $p_z = p_{z0}(x^4 - x^5)$ kN/m. The linear analysis is first performed with $N(x) = -2{,}500(1 - x)$ kN and $p_{z0} = 500$ kN/m. In Table 1 results for the deflection $w(x)$ and the bending moment $M(x)$ are presented at certain cross-sections for various values of N. The results are in very good agreement while the convergence and accuracy of the solution method is ensured for only $N = 21$ elements.

Subsequently the nonlinear response of the same beam was investigated with $EA = 266{,}667(1 - 0.5x)$ kN and $p_x = -2{,}500$ kN/m. In Fig. 2 the end deflection of the cantilever $w(l)$ versus the load p_{z0} for linear and nonlinear analysis is depicted. It is obvious that even for small nonlinear foundation reaction linear analysis is inadequate to predict the real response of the beam and the use of the nonlinear one is essential. In Table 2 results for the deflection $w(x)$, the axial force $N(x)$, the shear force $Q(x)$ and the bending moment $M(x)$ are presented as compared with those obtained from the linear response ($p_{z0} = 500$ kN/m). Moreover, in Fig. 3 the profiles of the shear force and the bending moment are shown for $p_{z0} = 1{,}500$ kN/m.

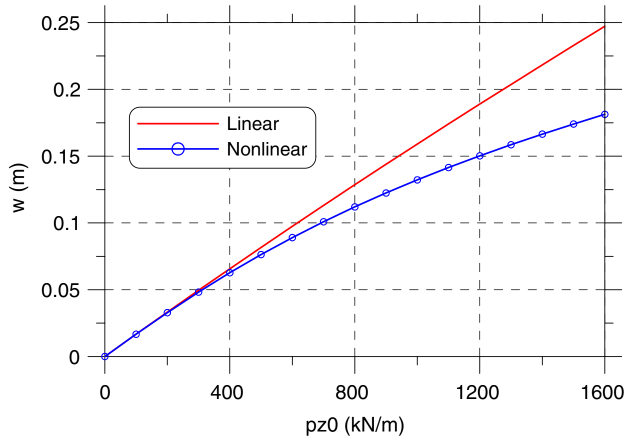

Fig. 2 End deflection versus load in Example 4.1

Table 2 Deflection (m), axial force (kN), shear force (kN) and bending moment (kNm) in Example 4.1

	$w(0.5)$	$w(1.0)$	$N(0.0)$	$N(0.5)$	$Q(0.0)$	$Q(0.5)$	$M(0.0)$	$M(0.5)$
Linear	0.022	0.082	−2,500	−1,250	6.213	11.950	76.278	36.319
Nonlinear	0.021	0.076	−2,499	−1,241	6.864	11.202	71.670	33.808

Linear and nonlinear response ($p_{z0} = 500\,\mathrm{kN/m}$)

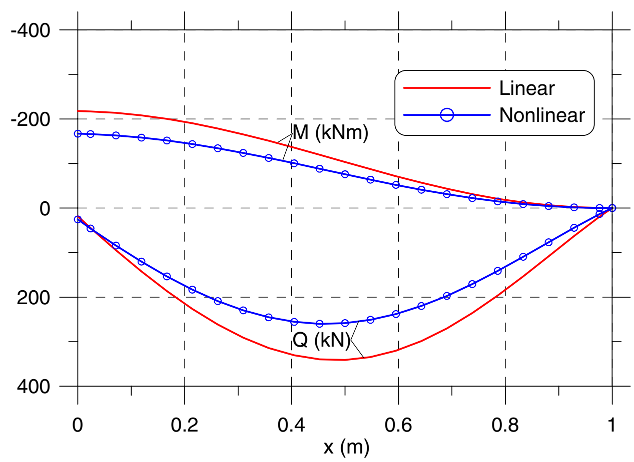

Fig. 3 Profile of the shear force and the bending moment in Example 4.1 ($p_{z0} = 1{,}500\,\mathrm{kN/m}$)

4.2 Rectangular beam with linearly varying height

The nonlinear response of a non-uniform rectangular beam with constant width b, variable height $h(x)$ and length $l = 3.0\,\mathrm{m}$ resting on a nonlinear foundation ($P = k_1 w + k_2 w^3 - k_3 w,_{xx}$) has been studied. The height of the beam varies according to the linear law $h(x) = h_0 + a(x - l/2)$ with $a = 2\tan\phi$ being the taper ratio and h_0 the height at the half length. The employed data are: $E = 2.9 \times 10^6\,\mathrm{kN/m^2}$, $b = 0.1\,\mathrm{m}$, $h_0 = 0.20\,\mathrm{m}$, $p_x = 0\,\mathrm{kN/m}$ and $p_z = 500\,\mathrm{kN/m}$. In order to examine the influence of the taper ratio on the nonlinear response, the volume of the beam, i.e. $V = bh_0 l$, was kept constant. The resulting beam should have no abrupt change of the cross-section so that the *Euler–Bernoulli* theory remains valid. Boley [14] has shown that, for a beam with unit constant width, a rate of change of the cross-section $|a| \simeq 0.35$ yields an error of 7.5%, while for $|a| \simeq 0.17$ the error is 1.8%. This was also verified by the author who treated the beam as a 2D elasticity problem and used the BEM to obtain the solution [18]. Three types of boundary conditions with immovable ends are considered: (i) hinged–hinged, (ii) fixed–hinged and (iii) fixed–fixed.

Table 3 Central deflection of the hinged–hinged beam of Example 4.2 for various values of the three foundation parameters and the taper ratio

k_1	k_2	k_3	$a = -0.05$	$a = 0.00$	$a = 0.05$
0	0	0	0.36040	0.31272	0.28336
1,000	0	0	0.28491	0.25546	0.23408
0	1,000	0	0.35251	0.30825	0.28020
0	0	1,000	0.28259	0.25176	0.23063
1,000	1,000	0	0.28015	0.25262	0.23206
1,000	0	1,000	0.21869	0.20154	0.18748
0	1,000	1,000	0.27822	0.24909	0.22871
1,000	1,000	1,000	0.21653	0.20009	0.18640

Table 4 Central deflection of the fixed–hinged beam of Example 4.2 for various values of the three foundation parameters and the taper ratio

k_1	k_2	k_3	$a = -0.05$	$a = 0.00$	$a = 0.05$
0	0	0	0.31527	0.28207	0.26025
1,000	0	0	0.24975	0.23038	0.21441
0	1,000	0	0.30972	0.27871	0.25779
0	0	1,000	0.24234	0.22242	0.20754
1,000	1,000	0	0.24653	0.22832	0.21289
1,000	0	1,000	0.18975	0.17910	0.16923
0	1,000	1,000	0.23956	0.22058	0.20616
1,000	1,000	1,000	0.18839	0.17812	0.16848

Table 5 Central deflection of the fixed–fixed beam of Example 4.2 for various values of the three foundation parameters and the taper ratio

k_1	k_2	k_3	$a = -0.05$	$a = 0.00$	$a = 0.05$
0	0	0	0.31627	0.25324	0.19432
1,000	0	0	0.24994	0.20675	0.16068
0	1,000	0	0.31067	0.25086	0.19323
0	0	1,000	0.23992	0.19390	0.15100
1,000	1,000	0	0.24680	0.20535	0.16006
1,000	0	1,000	0.18727	0.15757	0.12602
0	1,000	1,000	0.23720	0.19274	0.15049
1,000	1,000	1,000	0.18598	0.15696	0.12574

In Tables 3, 4 and 5 results for the central deflections for various values of the three foundation parameters and the taper ratio are presented. It is observed that the deflections are most sensitive in case of the two linear parameter Pasternak foundation with cubic Winkler nonlinearity ($k_1 = k_2 = k_3 = 1{,}000$) and the least sensitive in case of cubic Winkler foundation ($k_2 = 1{,}000$, $k_1 = k_3 = 0$). It is also observed that the deflections of the beam decrease with increasing the taper ratio, which means that the material must be shifted toward the end $x = l$ in order to obtain the minimum deflections. However, the differences of the deflections between the taper values $a = -0.05$ and $a = 0.05$ reduce as the foundation parameters k_1, k_2, k_3 increase. Moreover, the percentage of the deflection reduction as compared with the case $k_1 = k_2 = k_3 = 0$ is greater for taper ratio $a = -0.05$ and reduces for $a = 0.00$ and $a = 0.05$. Finally, in Figs. 4, 5 and 6 the profiles of the bending moment are shown for the three cases of boundary conditions and for various values of the three foundation parameters and the taper ratio.

4.3 Elastically restrained rectangular beam with parabolically varied modulus of elasticity

To demonstrate the computational efficiency of the method the nonlinear response of an elastically restrained non-uniform rectangular beam with constant cross-section and parabolically varied modulus of elasticity $E = 2.9 \times 10^6(1 + x^2)\,\mathrm{kN/m^2}$ resting on a nonlinear foundation ($P = 10{,}000w + 10{,}000w^3 - 1{,}000w_{,xx}$) has been studied. The employed data are: $b = 0.1\,\mathrm{m}$, $h = 0.20\,\mathrm{m}$, $l = 1.0\,\mathrm{m}$, $p_z = 5{,}000(1 + x)\,\mathrm{kN/m}$, $p_x = -1{,}500\,\mathrm{kN/m}$. Two types of classical boundary conditions, namely, hinged–hinged (H-H) and

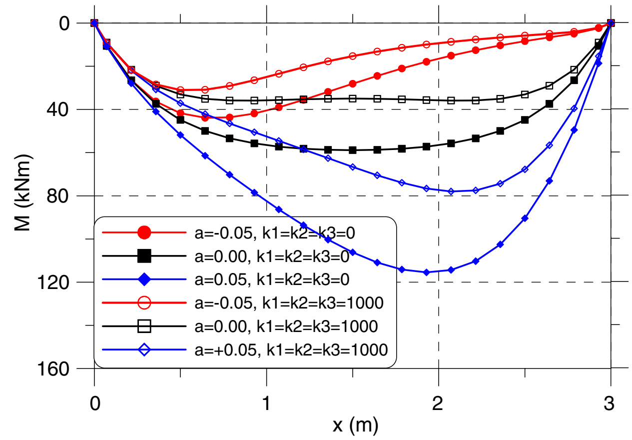

Fig. 4 Profile of the bending moment of the hinged–hinged beam in Example 4.2

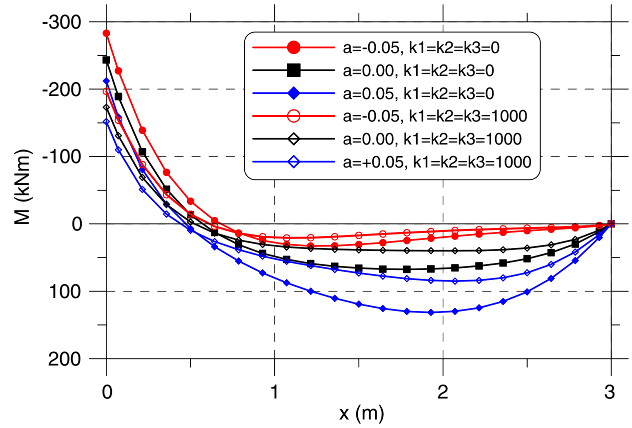

Fig. 5 Profile of the bending moment of the fixed–hinged beam in Example 4.2

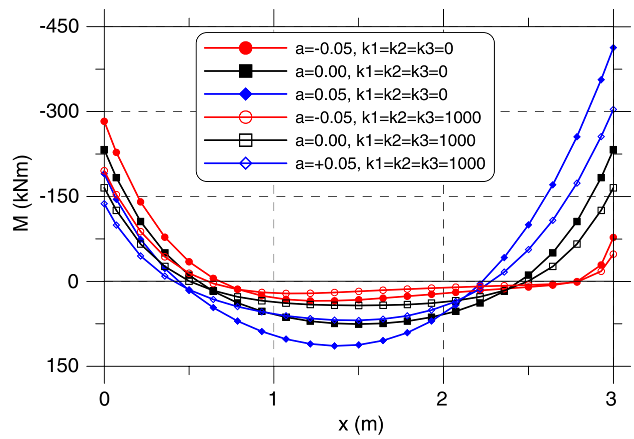

Fig. 6 Profile of the bending moment of the fixed–fixed beam in Example 4.2

fixed–hinged (F-H) are considered as well as a hinged–hinged one with a rotational elastic support (HRS-H) at $x = 0$ (all coefficients in the boundary conditions are set to zero except for $a_1 = \beta_1 = \bar{a}_1 = \bar{\beta}_1 = \gamma_2 = \bar{\gamma}_2 = 1$ and $\gamma_1 = K_r$). In Figs. 7, 8, 9 and 10 the profiles of the deflection, the axial force, the shear force and the bending moment are presented, respectively, for various values of the rotational stiffness coefficient K_r. As it was expected, with the increase of K_r the results tend to coincide with those obtained for fixed–hinged boundary conditions.

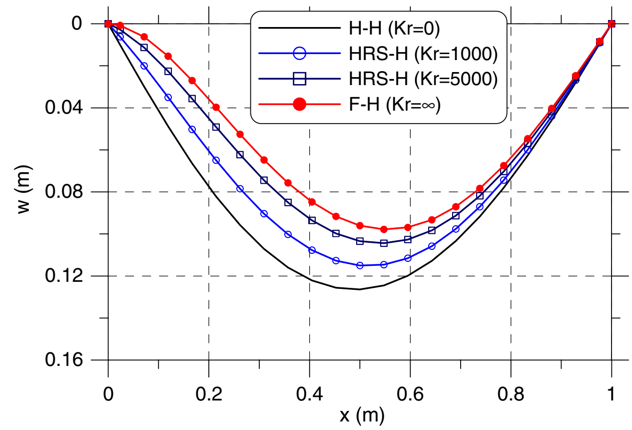

Fig. 7 Profile of the deflection in Example 4.3

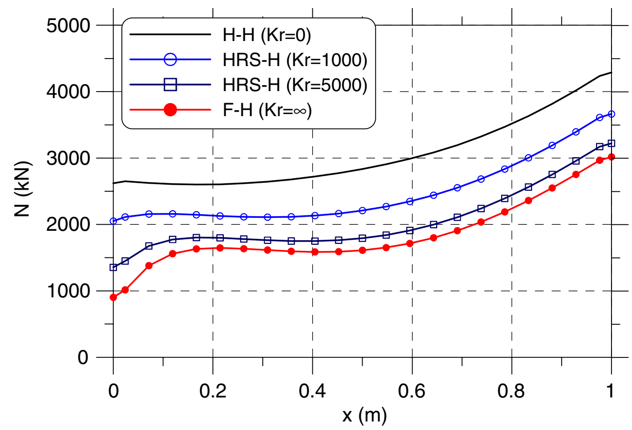

Fig. 8 Profile of the axial force in Example 4.3

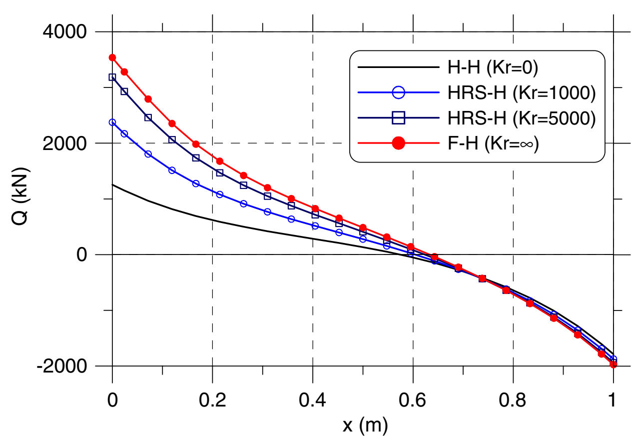

Fig. 9 Profile of the shear force in Example 4.3

5 Conclusions

In this paper the nonlinear response of beams with variable properties resting on a nonlinear elastic foundation is investigated. The solution of the derived coupled nonlinear governing equations with variable coefficients was achieved effectively using the AEM. This investigation has yielded several interesting findings concerning

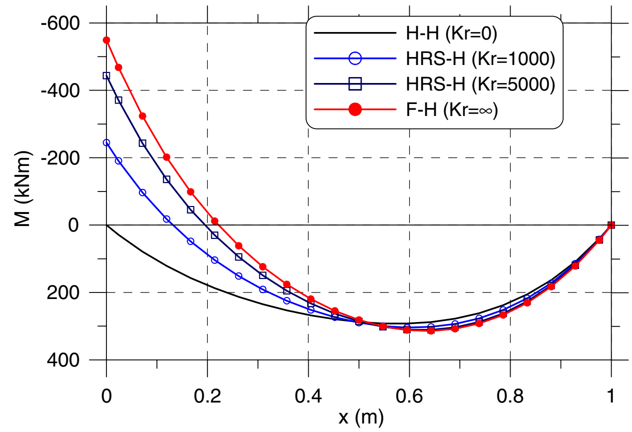

Fig. 10 Profile of the bending moment in Example 4.3

the employed solution method, as well as the nonlinear response of non-uniform beams resting on a nonlinear triparametric elastic foundation. These findings can be summarized as:

- The employed solution method exhibits stability and a small number of constant elements are adequate to obtain accurate results for the displacements and the stress resultants.
- Even for small nonlinearity in the foundation reaction the linear analysis is inadequate to predict the real response of the beam and the use of the nonlinear analysis is essential.
- The deflections are most sensitive in case of the two linear parameter Pasternak foundation with cubic Winkler nonlinearity and least sensitive on a cubic Winkler foundation.
- For tapered beams with immovable ends (hinged–hinged, fixed–hinged and fixed–fixed) the deflections decrease with increasing the taper ratio. This suggests shifting of the material towards the end $x = l$ in order to obtain minimum deflections.
- However, the differences of the deflections between the taper values $a = -0.05$ and $a = 0.05$ decrease as the foundation parameters k_1, k_2, k_3 increase.
- Moreover, the percentage of the deflection reduction as compared with the case $k_1 = k_2 = k_3 = 0$ is greater for taper ratio $a = -0.05$ and reduces for $a = 0.00$ and $a = 0.05$.

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