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# Bending response of inhomogeneous fiber-reinforced viscoelastic sandwich plates

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**Abstract** The static response of an inhomogeneous fiber-reinforced viscoelastic sandwich plate is investigated by using the first-order shear deformation theory. Several types of sandwich plates are considered taking into account the symmetry of the plate and the thickness of each layer. In addition, two cases are considered depending on the viscoelastic material which are included in the core or the faces of the sandwich plates. The method of effective moduli and Illyushin's approximation method are used to solve the equations governing the bending of simply supported inhomogeneous fiber-reinforced viscoelastic sandwich plates. Numerical computations were carried out to study the effect of the time parameter on deflections and stresses at different values of the aspect ratio, side-to-thickness ratio and constitutive parameter.

## 1 Introduction

The composite structure is commonly defined as a combination of two or more distinct materials, each of which retains its own distinctive properties, to create a new material with properties that cannot be achieved by any of the components acting alone. Using this definition, it can be determined that a wide range of engineering materials fall into this category. For example, concrete is a composite because it is a mixture of Portland cement and aggregate. Fiberglass sheet is a composite since it is made of glass fibers imbedded in a polymer.

A sandwich structure consists of three distinct layers (i.e., the top face, the core and the bottom face), which are bonded together to form an efficient load carrying assembly. The greatest advantage with sandwich construction compared to solid laminates is that the strength and stiffness are increased without a corresponding increase in the weight. Sandwich plates are widely used in modern engineering applications, especially in aviation, marine, civil, and mechanical industries. This is because they have a combination of features like lightweight, high stiffness, high structural efficiency and durability.

Sandwich plates have been the subject of many investigations; an extensive list of references up to 1965 can be found in the monograph by Plantema [1]. Originally, most authors dealt with sandwiches in which the

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facings were thin, stiff and heavy as compared with the core, henceforth, this configuration will be referred to as “classical”. Reissner [2] has suggested a simple and useful model to describe such plates. He only took into account the transverse shear stiffness of the core and the in-plane or membrane stiffness of the facings. Hoff [3] later added the flexural rigidity of the face-plates.

The first-order shear deformation theory proposed by Reissner [4] and Mindlin [5] is extended by Yang et al. [6] to laminated plates, followed by many variants of the first-order theory. Reissner [7], Noor and Burton [8], and Reddy [9] have reviewed these developments. Extension of the first-order theory to laminated anisotropic plates has not been as successful as it has been for an isotropic plate, particularly for the recovery of the interlaminar stress state without integrating the equilibrium equations. It is also difficult to determine properly the shear correction factor of laminates, upon which the accuracy of the prediction of the first-order theory is strongly dependent. Many theories have been developed to overcome the deficiency of the first-order theory of a constant or uniform transverse shear strain distribution through the thickness (see, e.g., [10–12]). So the transient behavior of composite plates has long been a main subject of many studies. But, these studies are limited to the response of homogeneous composite plates. Even the few studies accounting for the structural response of non-homogeneous composite plates deal with special cases of non-homogeneity and anisotropy, and the reported results in open literature are rare [13–17]. Fares and Zenkour [18] have investigated the free vibration and buckling problems of inhomogeneous composite plates with various plate theories.

For the viscoelastic heterogeneous media of several discrete linear viscoelastic phases with known stress–strain relations it has been shown that the effective relaxation and creep functions can be obtained by the corresponding principle of the theory of linear viscoelasticity. In some cases explicit results in terms of general linear viscoelastic matrix properties have been given, thus permitting direct use of experimental information [19]. Some adopted their models to study the damping mechanism of the viscoelastic layer, see, e.g., Douglas and Yang [20]. The stability of rectangular, viscoelastic, orthotropic plates subjected to biaxial compression was analyzed by Wilson and Vinson [21]. In their analysis, the equations governing the stability were obtained by using the quasi-elastic approximation, which overlooks the hereditary material behavior. Kim and Hong [22] have examined the viscoelastic-buckling load of sandwich plates with cross-ply faces. Huang [23] has studied the viscoelastic buckling and post-buckling of circular cylindrical laminated shells. Pan [24] has analyzed the dynamic response problem of isotropic viscoelastic plates by extending, for this case, Mindlin’s shear-deformation plate theory. Librescu and Chandiramani [25] have presented a paper that deals with the dynamic stability analysis of transversely isotropic viscoelastic plates subjected to in-plane biaxial edge-load systems. Zenkour [26] has performed quasi-static stability analysis of fiber-reinforced viscoelastic rectangular plates subjected to in-plane edge-load systems. Zenkour [27] has investigated the static thermo-viscoelastic responses of fiber-reinforced composite plates by the use of a refined shear deformation theory.

In this paper, the first-order shear deformation plate theory (FSDT) is used to study the static response of inhomogeneous fiber-reinforced viscoelastic sandwich plates. Two types of sandwich plates are considered. In the first one, the core is made from an isotropic viscoelastic material and the faces are made from an isotropic elastic material with the same elastic properties. In the other case, the core is an isotropic elastic material while the faces are isotropic viscoelastic material with the same viscoelastic modulus properties. With the help of the effective moduli method [28] as well as Illyushin’s approximation method [29], a wide variety of results is presented for the symmetric analysis of inhomogeneous fiber-reinforced viscoelastic rectangular sandwich plates.

## 2 Problem formulation

Let us consider the case of a flat sandwich plate composed of three inhomogeneous layers as shown in Fig. 1. Rectangular Cartesian coordinates $(x, y, z)$ are used to describe infinitesimal deformations of a three-layer sandwich elastic plate occupying the region $x \in [0, a]$, $y \in [0, b]$, and $z \in [-h/2, +h/2]$, in the unstressed reference configuration. The mid-plane of the composite sandwich plate is defined by $z = 0$ and its external bounding planes are defined by $z = \pm h/2$. The layers of the sandwich plate are made of an isotropic inhomogeneous material with material properties varying smoothly in the $z$ (thickness) direction only. The effective material properties for each layer, like Young’s modulus, can be expressed as

$$E_k(z) = E_k e^{-kz/h}, \quad k = 1, 2, 3. \tag{1}$$

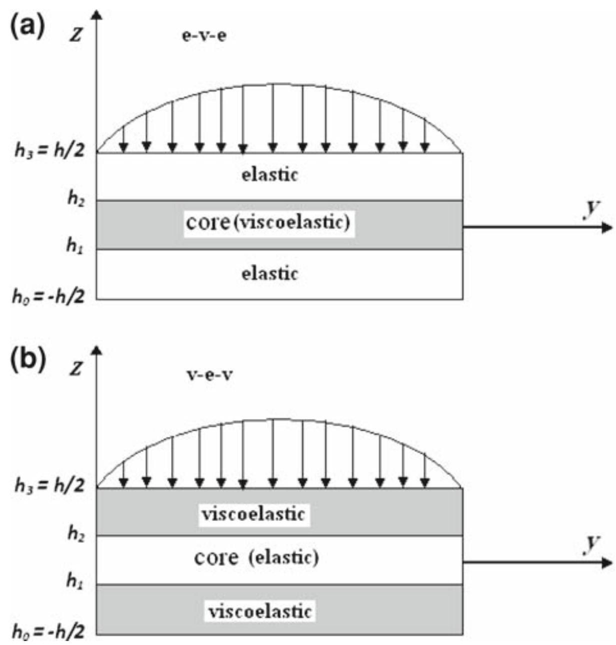



**Fig. 1** Geometry and coordinates of the two types of in sandwich plate

A normal traction $\sigma_z = q(x, y)$ is applied on the upper surface, while the lower surface is traction free. The displacements of a material point located at $(x, y, z)$ in the plate may be written as:

$$u_1 = u + z\left(\varphi_1 - \frac{\partial w}{\partial x}\right), \quad u_2 = v + z\left(\varphi_2 - \frac{\partial w}{\partial y}\right), \quad u_3 = w, \tag{2}$$

where $(u_1, u_2, u_3)$ are the displacements corresponding to the co-ordinate system and are functions of the spatial co-ordinate; $(u, v, w)$ are the displacements along the axes $x$, $y$ and $z$, respectively, and $\varphi_1$ and $\varphi_2$ are the rotations about the $y$- and $x$-axes. All of the generalized displacements $(u, v, w, \varphi_1, \varphi_2)$ are functions of $x$ and $y$.

The six strain components compatible with the displacement field (2) are given by

$$\begin{aligned}
\varepsilon_1 &= \frac{\partial u}{\partial x} + z\frac{\partial}{\partial x}\left(\varphi_1 - \frac{\partial w}{\partial x}\right), \quad \varepsilon_3 = 0,\\
\varepsilon_2 &= \frac{\partial v}{\partial y} + z\frac{\partial}{\partial y}\left(\varphi_2 - \frac{\partial w}{\partial y}\right), \quad \varepsilon_4 = \varphi_2,\\
\varepsilon_5 &= \varphi_1, \quad \varepsilon_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + z\left(\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y} - 2\frac{\partial^2 w}{\partial x \partial y}\right).
\end{aligned} \tag{3}$$

By treating each layer as an individual non-homogeneous plate, the stress–strain relationships, accounting for the transverse shear deformation in the plate coordinates for the $k$th layer, can be expressed as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix}^{(k)} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ & c_{22} & 0 & 0 & 0 \\ & & c_{66} & 0 & 0 \\ & & & c_{44} & 0 \\ \text{symm.} & & & & c_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}, \tag{4}$$

where $c_{ij}^{(k)}$ are the transformation elastic coefficients, which depend on the material properties of each layer,

$$c_{11}^{(k)} = c_{22}^{(k)} = \frac{E_k(z)}{1-\nu_k^2}, \qquad c_{12}^{(k)} = \frac{\nu_k E_k(z)}{1-\nu_k^2}, \qquad c_{44}^{(k)} = c_{55}^{(k)} = c_{66}^{(k)} = \frac{E_k(z)}{2(1-\nu_k)}, \tag{5}$$

in which $E_k$ and $\nu_k$ are Young's modulus and Poisson's ratio of layer $k$, respectively.

## 3 Governing equations

The principle of virtual displacement for the present problem may be expressed as follows:

$$\int_{\Omega} \left\{ \int_{-h/2}^{+h/2} \left[\sigma_1^{(k)}\delta\varepsilon_1 + \sigma_2^{(k)}\delta\varepsilon_2 + \cdots\right]\mathrm{d}z - q\delta w \right\} \mathrm{d}\Omega = 0. \tag{6}$$

The governing equilibrium equations can be derived from the above equation by integrating the displacement gradient in $\varepsilon_i$ by parts and setting the coefficients of $\delta u, \delta v, \delta w, \delta\varphi_1$ and $\delta\varphi_2$ to zero separately. Thus one obtains

$$\begin{aligned}
&\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = 0, \quad \frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = 0,\\
&\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 M_6}{\partial x \partial y} + \frac{\partial^2 M_2}{\partial y^2} + q = 0,\\
&\frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} - Q_5 = 0, \quad \frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} - Q_4 = 0,
\end{aligned} \tag{7}$$

where $N_i$ and $M_i$ are the basic components of stress resultants and stress couples, and $Q_j$ are transverse shear stress resultants. They can be expressed as

$$\begin{aligned}
\{N_i, M_i\} &= \sum_{k=1}^{3} \int_{h_{k-1}}^{h_k} \sigma_i^{(k)} \{1, z\}\mathrm{d}z,\\
Q_j &= \sum_{k=1}^{3} \int_{h_{k-1}}^{h_k} \sigma_j^{(k)}\mathrm{d}z \quad (i = 1, 2, 6;\ j = 4, 5)
\end{aligned} \tag{8}$$

where $h_k$ and $h_{k-1}$ are the top and bottom $z$-coordinates of the $k$th layer. Substituting Eq. (4) into Eq. (8), the force and moment resultants can be related to the total strains.

## 4 Exact solutions for sandwich plates

Rectangular plates are generally classified in accordance with the type support used. We are here concerned with the exact solution of Eqs. (7) for a simply supported sandwich plate. The following boundary conditions are imposed at the side edges:

$$\begin{aligned}
v = w = N_1 = M_1 = 0, &\quad \text{at } x = 0, a,\\
u = w = N_2 = M_2 = 0, &\quad \text{at } y = 0, b.
\end{aligned} \tag{9}$$

To solve this problem, Navier presented the external force for the case of a sinusoidally distributed load,

$$q(x, y) = q_0 \sin(\lambda x) \sin(\mu y), \tag{10}$$

where $\lambda = \pi/a$, $\mu = \pi/b$ and $q_0$ represents the intensity of the load at the plate center. Following the Navier solution procedure, we assume the following solution form for $(u, v, w, \varphi_1, \varphi_2)$ that satisfies the boundary conditions,

$$\begin{Bmatrix} u \\ v \\ w \\ \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} U\cos(\lambda x) & \sin(\mu y) \\ V\sin(\lambda x) & \cos(\mu y) \\ W\sin(\lambda x) & \sin(\mu y) \\ X\cos(\lambda x) & \sin(\mu y) \\ Y\sin(\lambda x) & \cos(\mu y) \end{Bmatrix}, \tag{11}$$

where $U$, $V$, $W$, $X$ and $Y$ are arbitrary parameters to be determined. Substituting Eq. (11) into Eq. (7), we obtain

$$[C]\{\Delta\} = \{F\}, \tag{12}$$

where $\{\Delta\}$ and $\{F\}$ denote the columns

$$\begin{aligned} \{\Delta\}^t &= \{U, V, W, X, Y\}, \\ \{F\}^t &= \{0, 0, -q_0, 0, 0\}. \end{aligned} \tag{13}$$

The elements $C_{ij} = C_{ji}$ of the coefficient matrix $[C]$ are given by

$$\begin{aligned}
C_{11} &= -\lambda^2 A_{11} - \mu^2 A_{66}, \\
C_{12} &= -\lambda\mu(A_{12} + A_{66}), \\
C_{13} &= \lambda[\lambda^2 B_{11} + (B_{12} + 2B_{66})\mu^2], \\
C_{14} &= -\lambda^2 B_{11} - \mu^2 B_{66}, \\
C_{15} &= -\lambda\mu(B_{12} + B_{66}), \\
C_{22} &= -\lambda^2 A_{66} - \mu^2 A_{22}, \\
C_{23} &= \mu[(B_{12} + 2B_{66})\lambda^2 + B_{22}\mu^2], \\
C_{24} &= C_{15}, \\
C_{25} &= -\lambda^2 B_{66} - \mu^2 B_{22}, \\
C_{33} &= -\lambda^4 D_{11} - 2(D_{12} + 2D_{66})\lambda^2\mu^2 - \mu^4 D_{22}, \\
C_{34} &= -\lambda[\lambda^2 D_{11} + (D_{12} + 2D_{66})\mu^2], \\
C_{35} &= -\mu[(D_{12} + 2D_{66})\lambda^2 + D_{22}\mu^2], \\
C_{44} &= -\lambda^2 D_{11} - \mu^2 D_{66} - A_{55}, \\
C_{45} &= -\lambda\mu(D_{12} + D_{66}), \\
C_{55} &= -\lambda^2 D_{66} - \mu^2 D_{22} - A_{44},
\end{aligned} \tag{14}$$

where $A_{ij}$, $B_{ij}$ and $D_{ij}$ are the stiffness coefficients defined by

$$\begin{aligned}
\left\{A_{ij}, B_{ij}, D_{ij}\right\} &= \sum_{k=1}^{3} \int_{h_{k-1}}^{h_k} c_{ij}^{(k)}(z)\left\{1, z, z^2\right\}\mathrm{d}z, \\
A_{ll} &= \sum_{k=1}^{3} \int_{h_{k-1}}^{h_k} K_l c_{ll}^{(k)}(z)\mathrm{d}z, \qquad (i, j = 1, 2, 6; l = 4, 5),
\end{aligned} \tag{15}$$

in which $K_l$ are the shear correction factors, $K_4 = K_5 = 5/6$.

Moreover, substituting Eq. (11) into Eq. (4), one can obtain the stress components in terms of Young's modulus and the arbitrary parameters $U$, $V$, $W$, $X$ and $Y$ as follows:

$$\begin{aligned}
\sigma_1^{(k)} &= -\frac{E_k(z)}{1 - \nu_k^2}\left\{\lambda U + \nu_k\mu V - z\left[\left(\lambda^2 + \nu_k\mu^2\right)W - \lambda X - \nu_k\mu Y\right]\right\}\sin(\lambda x)\sin(\mu y), \\
\sigma_2^{(k)} &= -\frac{E_k(z)}{1 - \nu_k^2}\left\{\nu_k\lambda U + \mu V - z\left[\left(\nu_k\lambda^2 + \mu^2\right)W - \nu_k\lambda X - \mu Y\right]\right\}\sin(\lambda x)\sin(\mu y), \\
\sigma_4^{(k)} &= \frac{E_k(z)}{2(1 - \nu_k)} Y \sin(\lambda x)\cos(\mu y), \\
\sigma_5^{(k)} &= \frac{E_k(z)}{2(1 - \nu_k)} X \cos(\lambda x)\sin(\mu y), \\
\sigma_6^{(k)} &= \frac{E_k(z)}{2(1 - \nu_k)}\left\{\mu U + \lambda V - z\left[2\lambda\mu W - \mu X - \lambda Y\right]\right\}\cos(\lambda x)\cos(\mu y).
\end{aligned} \tag{16}$$

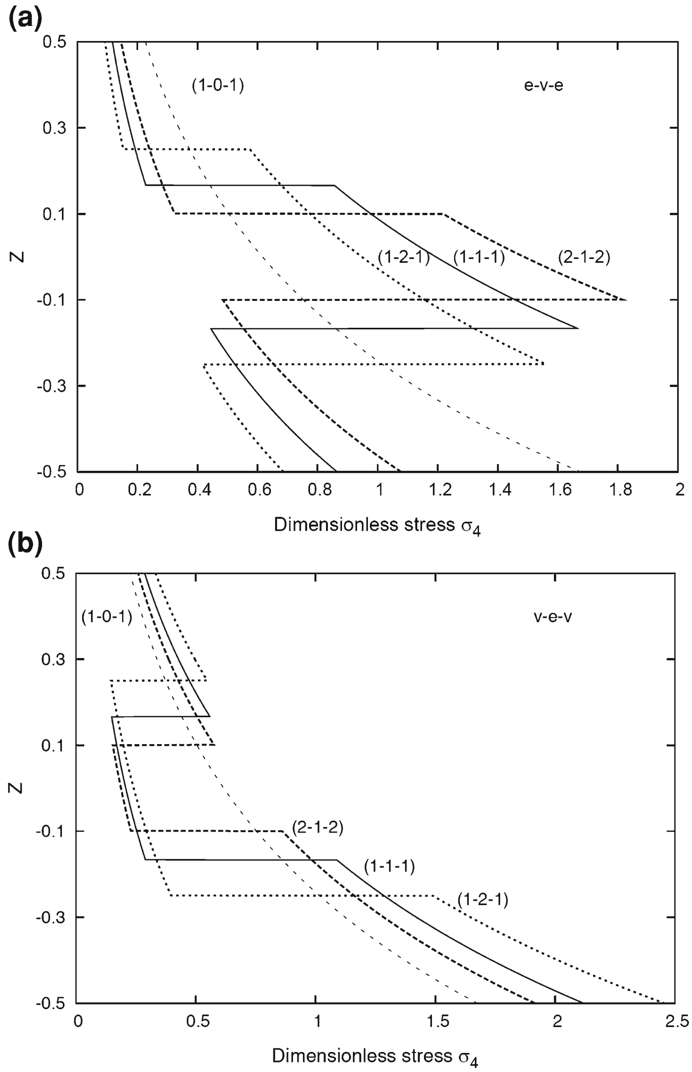



**Fig. 2** Variation of the transverse shear stress $\sigma_4$ through the plate thickness for different types of sandwich plate **a** e–v–e and **b** v–e–v

## 5 Viscoelastic solution

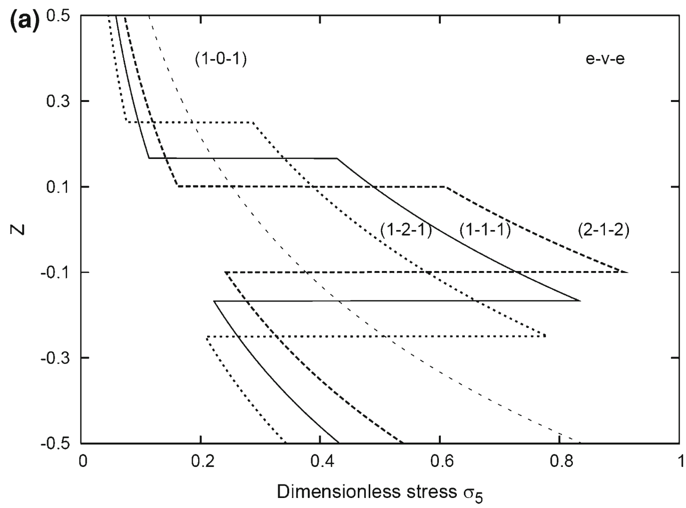
### 5.1 Elastic–viscoelastic–elastic sandwich plate (e–v–e)

In this problem, the core of the sandwich plate is considered to be made of an isotropic viscoelastic and the faces are made of an isotropic elastic material with the same elastic properties, i.e., $E_1 = E_3 = E$ and $\nu_1 = \nu_3 = \nu$. Note that the viscoelastic modulus of the core layer is given by

$$E_2 = \frac{9K\bar{\omega}}{2+\bar{\omega}}, \tag{17}$$

where $K$ is the coefficient of volume compression (the bulk modulus) and it is assumed to be not relaxed, i.e., $K$ = constant, and $\bar{\omega}$ is the dimensionless kernel of the relaxation function which is related to the corresponding Poisson's ratio of the core layer by the formula

$$\nu_2 = \frac{1-\bar{\omega}}{2+\bar{\omega}}. \tag{18}$$



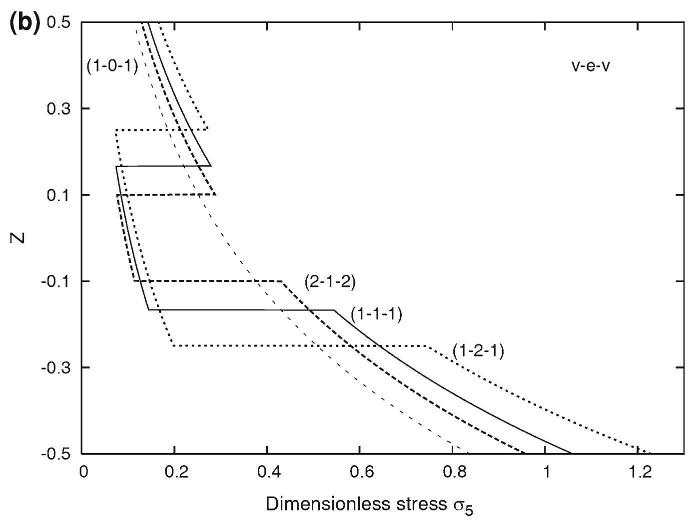




**Fig. 3** Variation of the transverse shear stress $\sigma_5$ through the plate thickness for different types of sandwich plate **a** e–v–e and **b** v–e–v

5.2 Viscoelastic–elastic–viscoelastic sandwich plate (v–e–v)

Here, we take the core of the sandwich plate as an isotropic elastic material while the faces are made of a viscoelastic material with the same viscoelastic properties, i.e., $E_2 = E$ and $\nu_2 = \nu$. The viscoelastic properties of the two faces are given by

$$E_1 = E_3 = \frac{9K\bar{\omega}}{2+\bar{\omega}}, \quad \nu_1 = \nu_3 = \frac{1-\bar{\omega}}{2+\bar{\omega}}. \tag{19}$$

To solve the quasi-static problem of the linear theory for a viscoelastic composite material, we can use the method of reducing the non-homogeneous isotropic viscoelastic problem to a sequence of successive homogeneous anisotropic ones, as is done in the elastic case (see [28,29]).

The substitution of Eqs. (17)–(19) into Eq. (16) gives for the two problems:

$$\sigma_{i,j}^{(k)} = F_{ij}^{(k)}(\bar{\omega})q_0(t), \quad i = 1, 2, 4, 5, 6, \quad j = 1, 2, \quad k = 1, 2, 3, \tag{20}$$

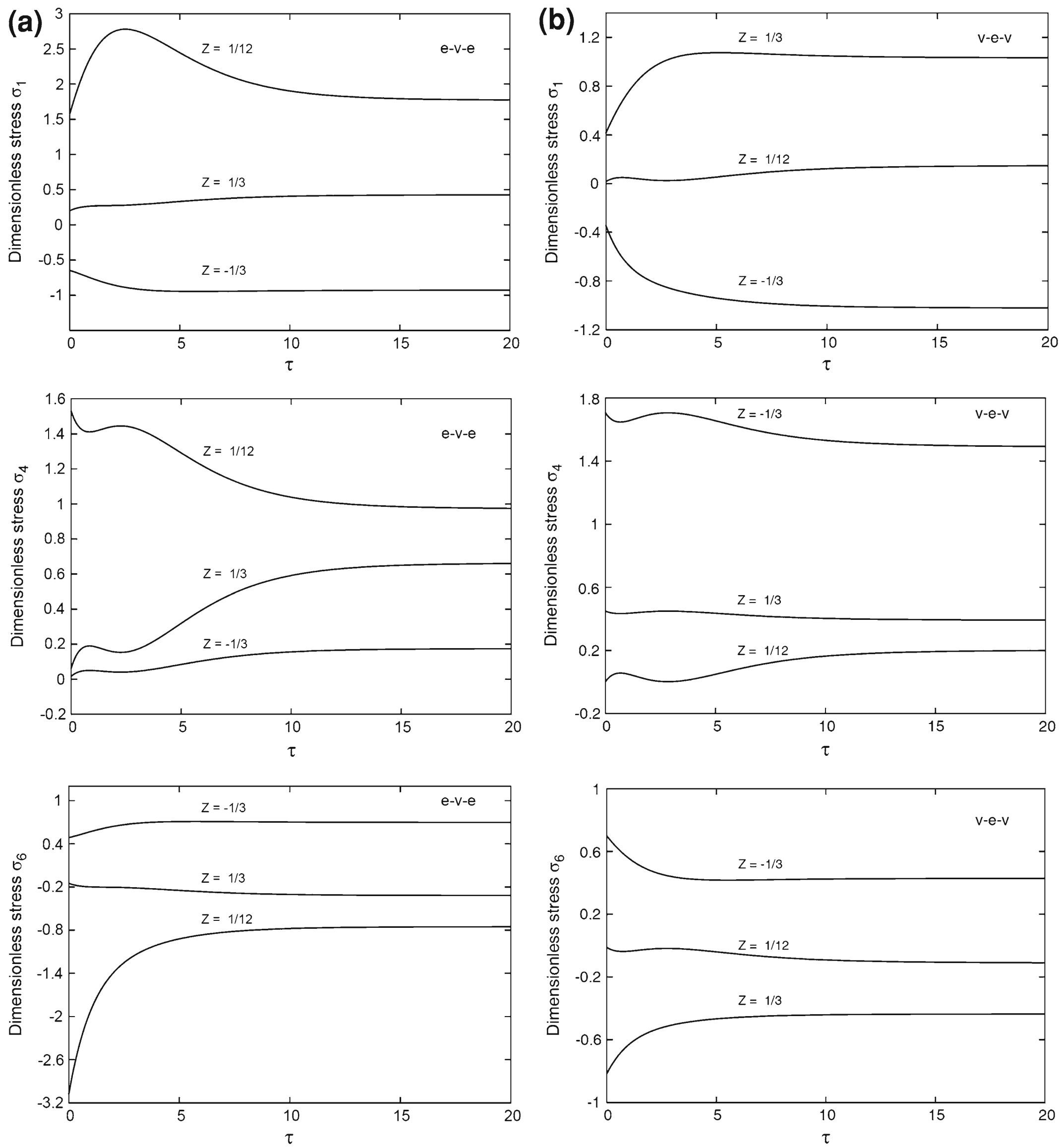



**Fig. 4** Variation of the dimensionless stresses versus the time parameter $\tau$ at different values of the thickness $Z$ for the two cases of the (1–1–1) sandwich plates

where $q_0(t)$ is a transient function accounting for the viscoelastic response of the bending problem. In an elastic composite $F_{ij}^{(k)}$ is a function of $\bar{\omega}$, while in an viscoelastic composite it is an operator function of the time $t$. According to Illyushin's approximation method [29], the function $F_{ij}^{(k)}$ can be represented in the form

$$F_{ij}^{(k)} = \sum_{l=1}^{5} A_{ijl}^{(k)} \Phi(\bar{\omega}), \tag{21}$$

where $\Phi(\bar{\omega})$ are some known kernels, constructed on the basis of the kernel $\bar{\omega}$ and may be chosen in the form

$$\Phi_1 = 1, \quad \Phi_2 = \bar{\omega}, \quad \Phi_3 = \frac{1}{\bar{\omega}} = \bar{\Pi}, \quad \Phi_4 = \bar{g}_{\beta_1}, \quad \Phi_5 = \bar{g}_{\beta_2}, \tag{22}$$

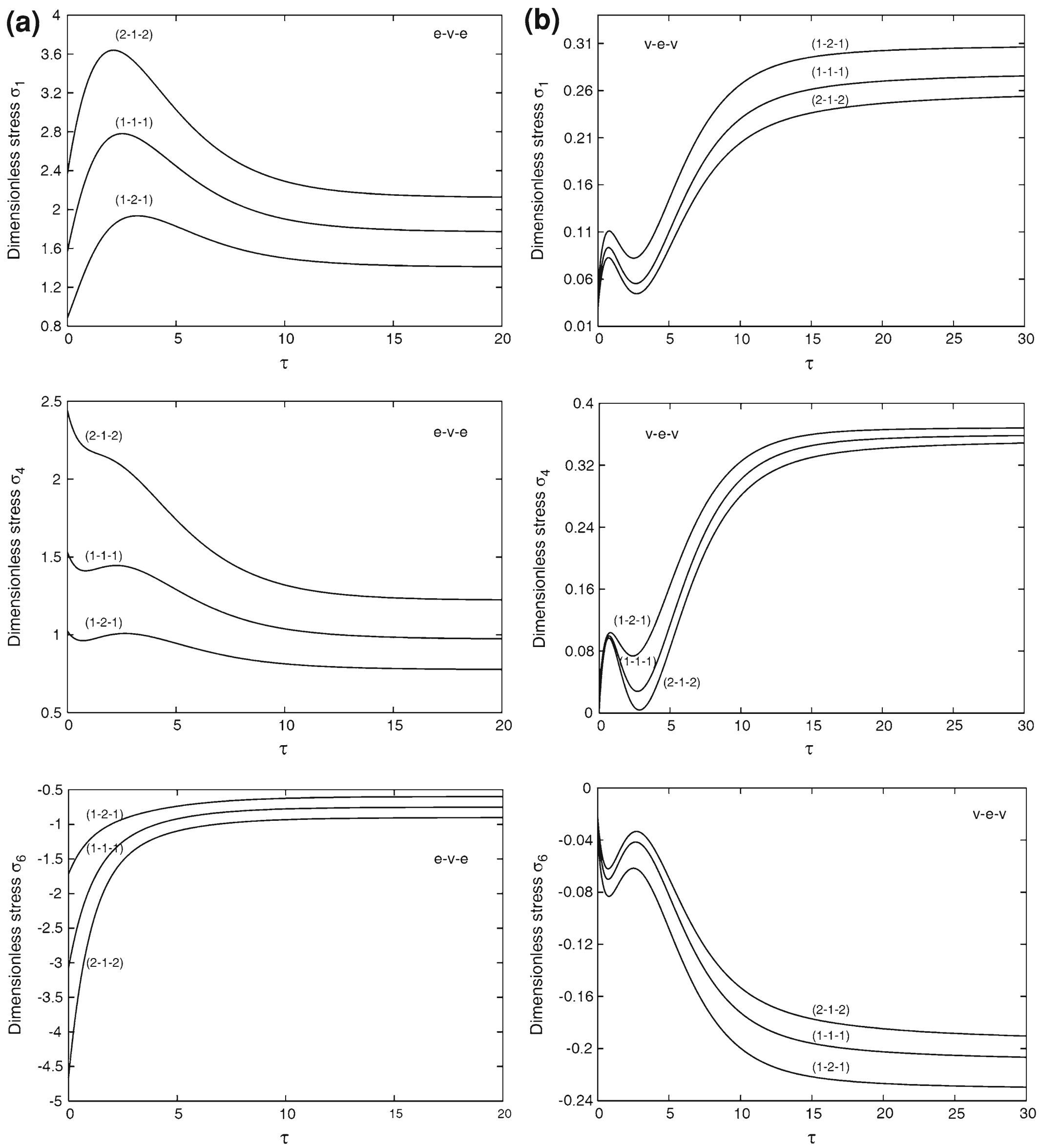



**Fig. 5** Variation of the dimensionless stresses versus the time parameter $\tau$ for different types of the two cases of the sandwich plates ($Z = 1/12$)

where $\bar{g}_{\beta_m} = \frac{1}{1+\beta_m \bar{\omega}}$, $m = 1, 2$. The coefficients $A_{ijl}^{(k)}$ are determined from the following system of algebraic equations:

$$\sum_{l=1}^{5} L_{ij} A_{ijl}^{(k)} = B_{ijl}^{(k)}, \tag{23}$$

where

$$L_{ij} = \int_0^1 \Phi_i(\bar{\omega})\Phi_j(\bar{\omega})\mathrm{d}\bar{\omega}, \quad B_{ijl}^{(k)} = \int_0^1 \Phi_i(\bar{\omega}) F_{ij}^{(k)}(\bar{\omega})\mathrm{d}\bar{\omega}. \tag{24}$$

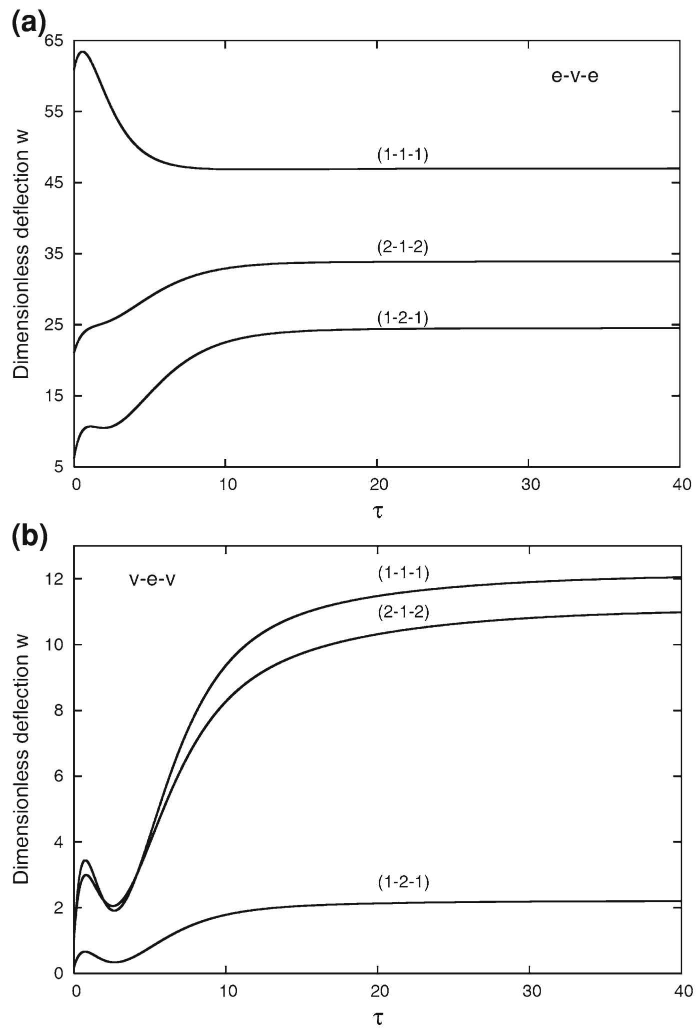



**Fig. 6** Variation of the dimensionless deflection versus the time parameter $\tau$ for different types of the two cases of sandwich plates ($Z = 1/12$)

The viscoelastic solution may now record to obtain explicit formulae for stresses $\sigma_{i,j}^{(k)}$ as functions of the time $t$. Then,

$$\sigma_{i,j}^{(k)} = A_{ij1}^{(k)} q_0(t) + A_{ij2}^{(k)} \int_0^1 \bar{\omega}(t-\tau)\mathrm{d}q_0(t) + A_{ij3}^{(k)} \int_0^1 \bar{\Pi}(t-\tau)\mathrm{d}q_0(t)$$
$$+A_{ij4}^{(k)} \int_0^1 \bar{g}_{\beta_1}(t-\tau)\mathrm{d}q_0(t) + A_{ij5}^{(k)} \int_0^1 \bar{g}_{\beta_2}(t-\tau)(t-\tau)\mathrm{d}q_0(t). \tag{25}$$

Taking $q_0(t) = \bar{q}_0 H(t)$ where $H(t)$ is Heaviside's unit step function,

$$H(t) = \begin{cases} 1 & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0. \end{cases} \tag{26}$$

So, Eq. (25) for the two problems takes the form

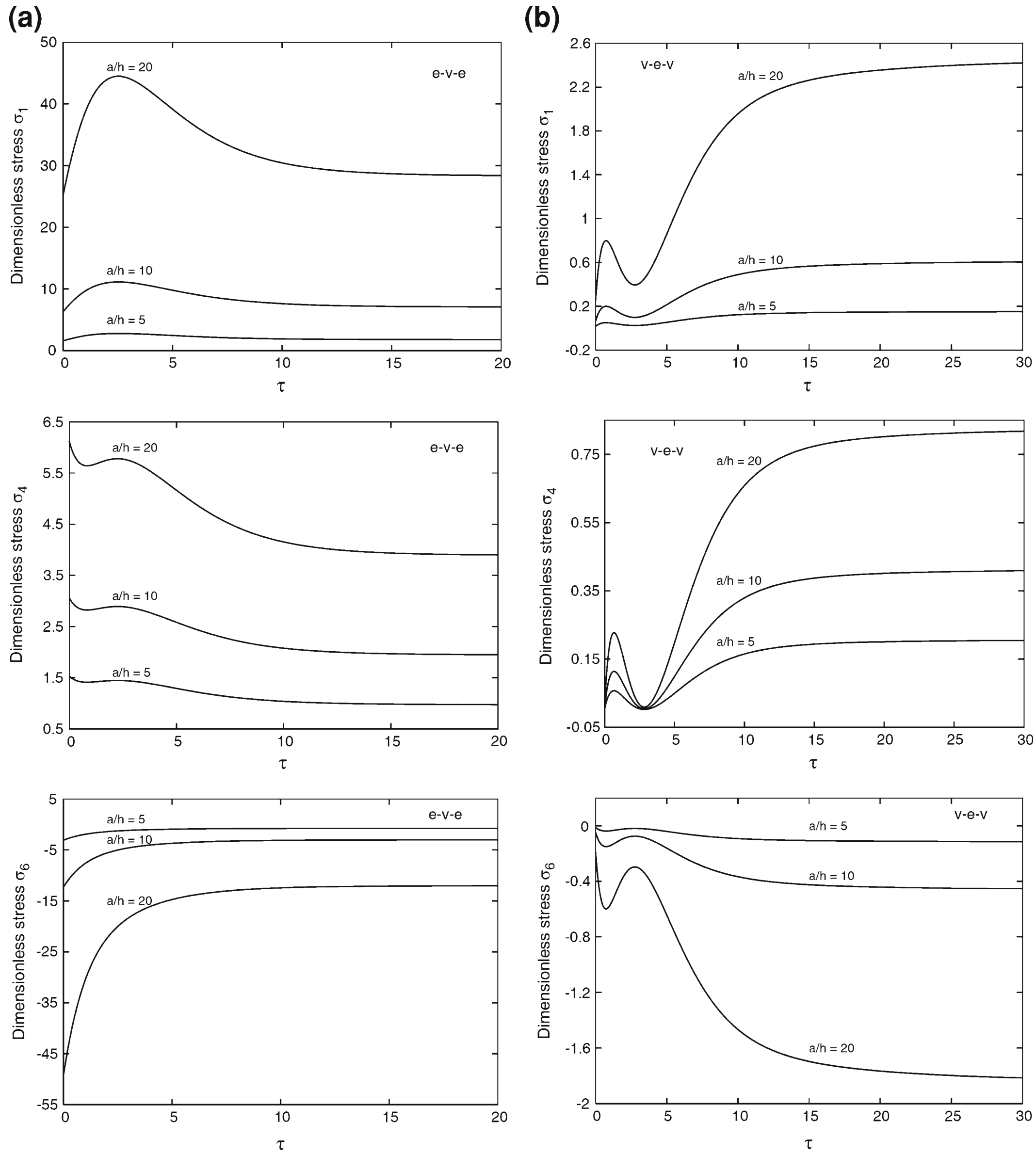



**Fig. 7** Variation of the dimensionless stresses versus the time parameter $\tau$ with different values of side-to-thickness $a/h$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

$$\sigma_{i,j}^{(k)} = \bar{q}_0 \left\{ A_{ij1}^{(k)} H(t) + A_{ij2}^{(k)} \omega(t) + A_{ij3}^{(k)} \Pi(t) + A_{ij4}^{(k)} g_{\beta_1}(t) + A_{ij5}^{(k)} g_{\beta_2}(t) \right\}, \tag{27}$$

where $\omega(t) \equiv \bar{\omega}$, $\Pi(t) \equiv \bar{\Pi}$ and $g_{\beta_m}(t) \equiv \bar{g}_{\beta_m}$.

Assume an exponential relaxation function

$$\omega(t) = c_1 + c_2 e^{-\alpha t}, \tag{28}$$

where $c_1$, $c_2$ are constants that are to be determined, $\alpha = 1/t_s$ in which $t_s$ is the relaxation time. The function $\Pi(t)$ and $g_{\beta_m}(t)$ can be determined by deducing the Laplace–Carson transform of these functions from the known Laplace–Carson transform of the function $\omega(t)$, which are given in detail in Appendix A. They take the following form:

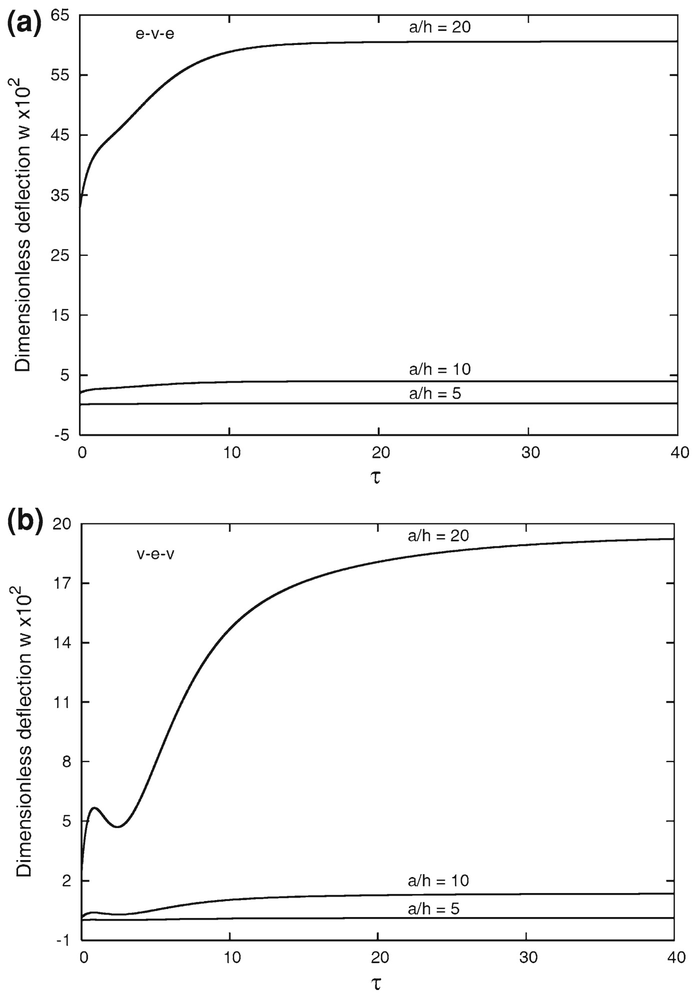



**Fig. 8** Variation of the dimensionless deflection versus the time parameter $\tau$ with different values of side-to-thickness $a/h$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

$$\Pi(t) = \frac{1}{c_1}\left(1 - \frac{c_2}{c_1 + c_2} e^{-\frac{c_1 \tau}{c_1 + c_2}}\right), \quad \tau = \alpha t, \tag{29}$$

$$g_{\beta_m}(t) = \frac{1}{1 + c_1 \beta_m}\left(1 - \frac{c_2 \beta_m}{1 + (c_1 + c_2)\beta_m} e^{-\frac{(1 + c_1 \beta_m)\tau}{1 + (c_1 + c_2)\beta_m}}\right). \tag{30}$$

So, the final forms of the stress load in terms of the time parameter $\tau$ are

$$\begin{aligned}\Sigma_{i,j}^{(k)} = A_{ij1}^{(k)} H(t) + A_{ij2}^{(k)}\left(c_1 + c_2 e^{-\alpha t}\right) + \frac{A_{ij3}^{(k)}}{c_1}\left(1 - \frac{c_2}{c_1 + c_2} e^{-\frac{c_1 \tau}{c_1 + c_2}}\right) \\ + \frac{A_{ij4}^{(k)}}{1 + c_1 \beta_1}\left(1 - \frac{c_2 \beta_1}{1 + (c_1 + c_2)\beta_1} e^{-\frac{(1 + c_1 \beta_1)\tau}{1 + (c_1 + c_2)\beta_1}}\right) \\ + \frac{A_{ij5}^{(k)}}{1 + c_1 \beta_2}\left(1 - \frac{c_2 \beta_2}{1 + (c_1 + c_2)\beta_2} e^{-\frac{(1 + c_1 \beta_2)\tau}{1 + (c_1 + c_2)\beta_2}}\right),\end{aligned} \tag{31}$$

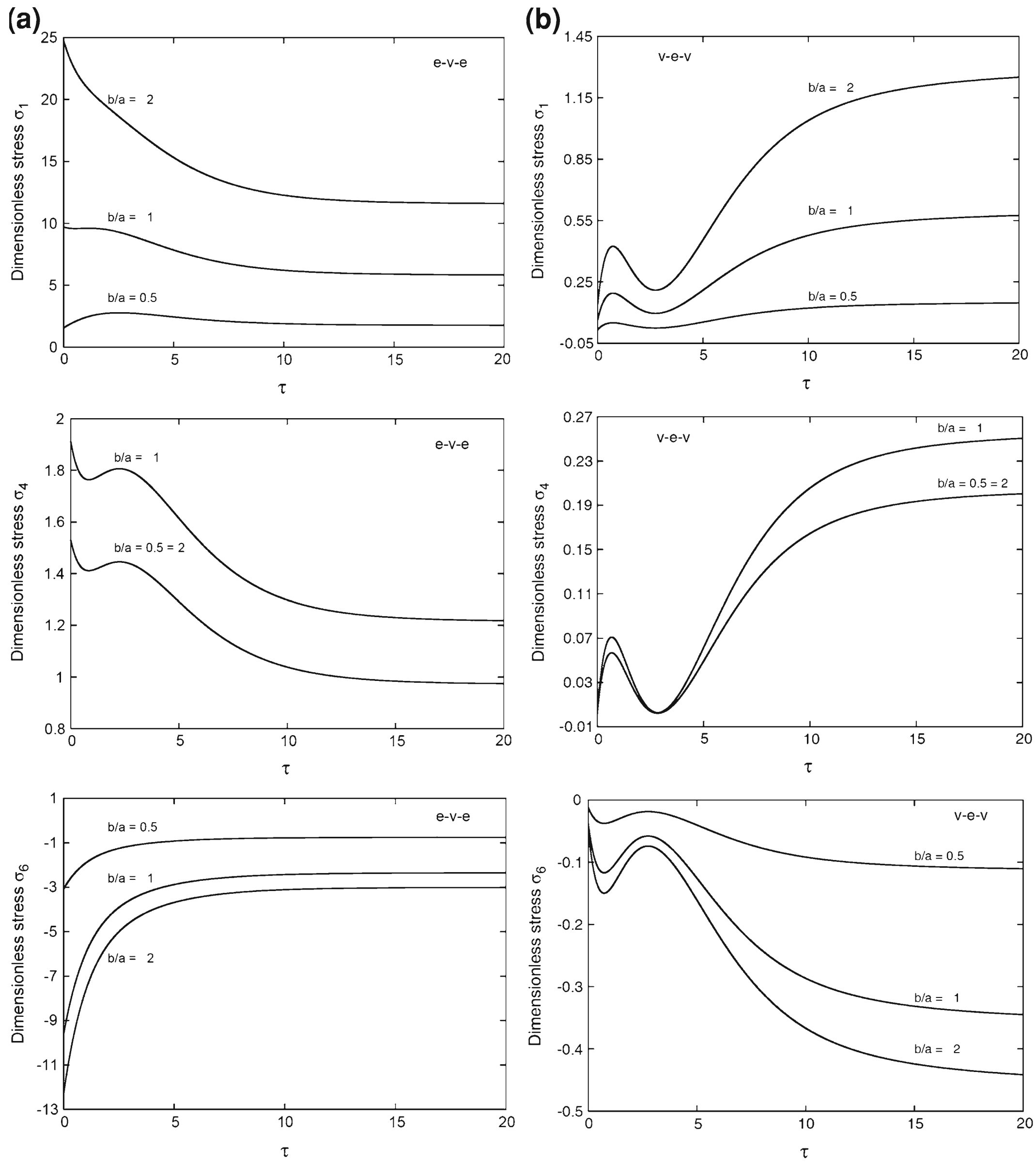



**Fig. 9** Variation of the dimensionless stresses versus the time parameter $\tau$ with different values of aspect ratio $b/a$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

where $\Sigma_{i,j}^{(k)} = \sigma_{i,j}^{(k)}/\bar{q}_0$.

## 6 Several kinds of sandwich plates

### 6.1 (1–2–1) sandwich plate

Here the plate is symmetric, in which the core thickness equals the sum of faces thicknesses. In this case, we have

$$h_1 = -\frac{1}{4}, \quad h_2 = \frac{1}{4}.$$

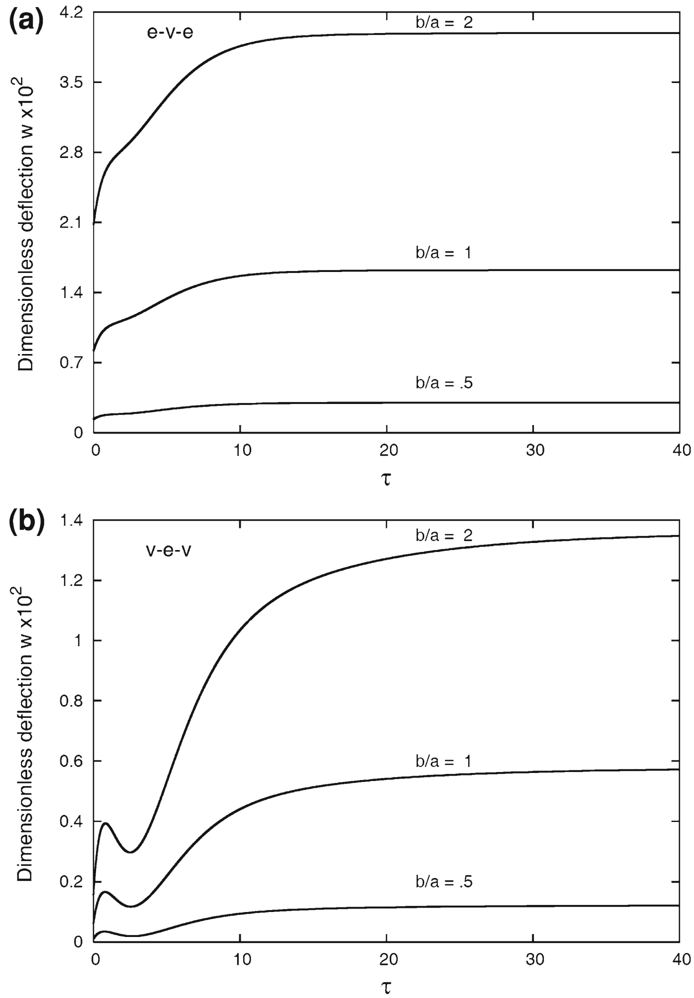



**Fig. 10** Variation of the dimensionless deflection versus the time parameter $\tau$ with different values of aspect ratio $b/a$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

### 6.2 (1–1–1) sandwich plate

The plate is symmetric and made of three equal-thickness layers. So, one has

$$h_1 = -\frac{1}{6}, \quad h_2 = \frac{1}{6}.$$

### 6.3 (2–1–2) sandwich plate

In this case the plate is also symmetric and the thickness of the core is half the face thickness. Then

$$h_1 = -\frac{1}{10}, \quad h_2 = \frac{1}{10}.$$

### 6.4 (1–0–1) sandwich plate

In this case the plate is symmetric and made of only two equal-thickness layers, i.e., there is no core layer. Thus,

$$h_1 = h_2 = 0.$$

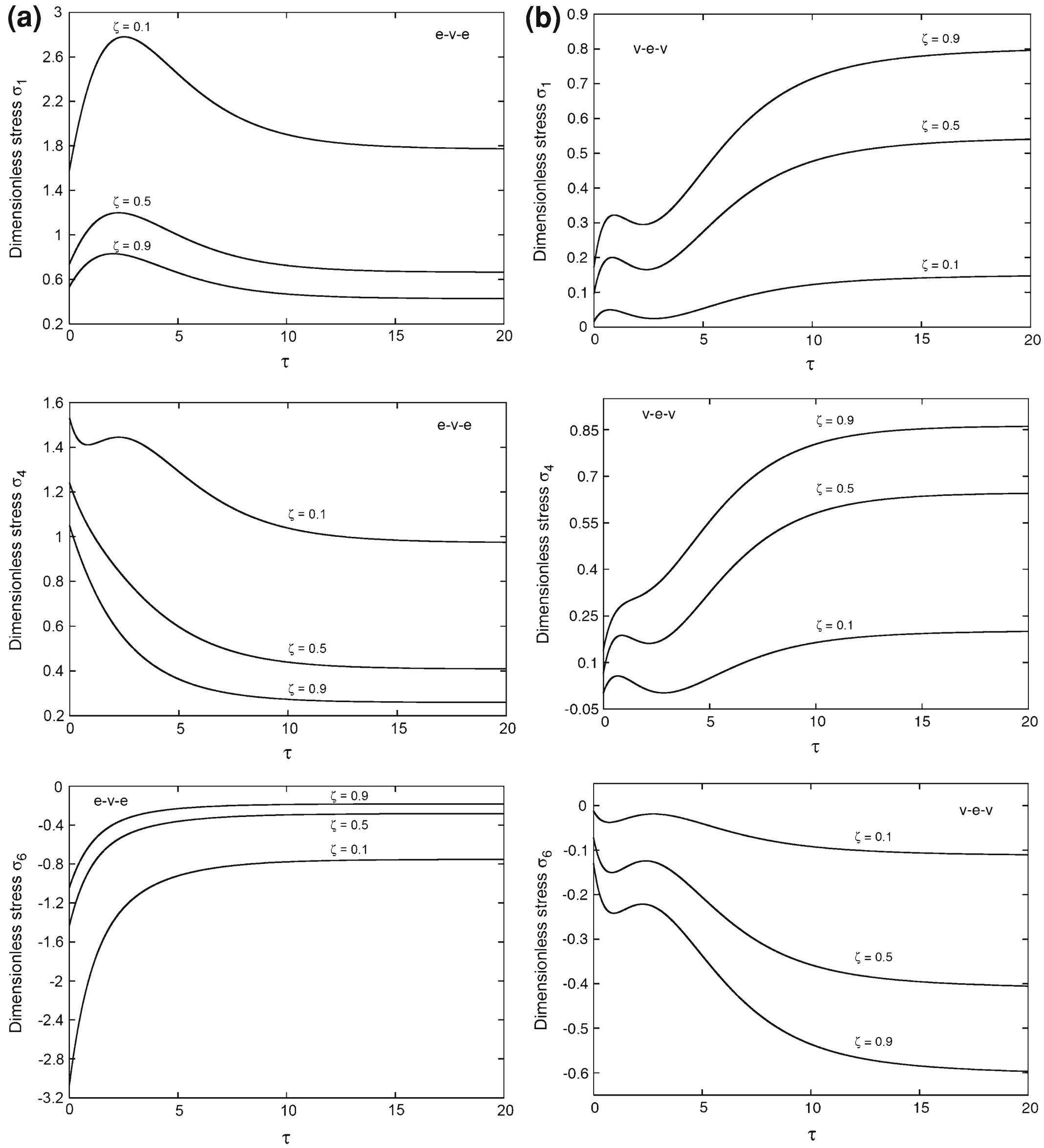



**Fig. 11** Variation of the dimensionless stresses versus the time parameter $\tau$ with different values of the constitutive parameter $\zeta$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

## 7 Numerical results and discussion

Numerical results for the stresses of simply supported sandwich plates are obtained. The relaxation time $\alpha$ is still unknown and the time parameter $\tau = \alpha t$ is given in terms of it. Poisson's ratio for the elastic plate is given the value 0.25. In addition, unless otherwise stated, it is assumed that

$$b/a = 0.5, \quad \zeta = 0.1, \quad a/h = 5, \quad c_1 = 0.1, \quad c_2 = 0.9.$$

The following non-dimensional response characteristics determined are used throughout the figures:

$$\sigma_1 = \Sigma_1\left(\frac{a}{2}, \frac{b}{2}, Z\right), \qquad \sigma_2 = \Sigma_2\left(\frac{a}{2}, \frac{b}{2}, Z\right), \qquad \sigma_6 = \Sigma_6\left(0, 0, Z\right),$$
$$\sigma_4 = \Sigma_4\left(\frac{a}{2}, 0, Z\right), \qquad \sigma_5 = \Sigma_5\left(0, \frac{b}{2}, Z\right), \qquad w = \frac{K}{h\bar{q}_0} w\left(\frac{a}{2}, \frac{b}{2}\right),$$

in which $Z = z/h$.

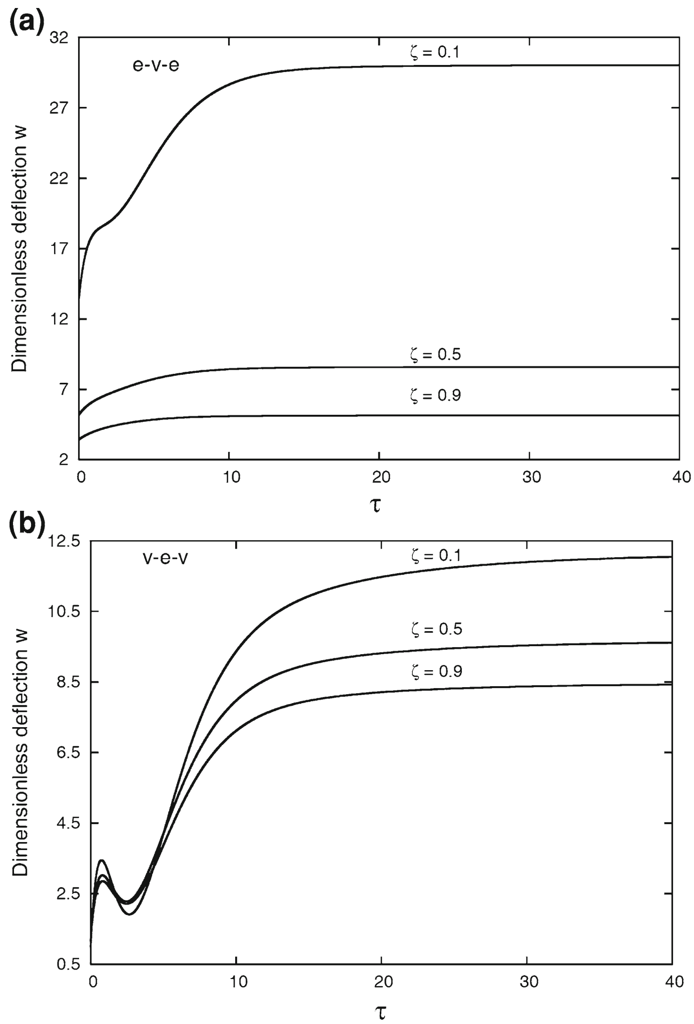



**Fig. 12** Variation of the dimensionless deflection versus the time parameter $\tau$ with different values of the constitutive parameter $\zeta$ for the two cases of the (1–1–1) sandwich plates ($Z = 1/12$)

The variation of dimensionless stresses and deflection through the plate thickness and with the time parameter $\tau$ for different types of inhomogeneous viscoelastic sandwich plates are shown graphically in Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Results are obtained for different values of side-to-thickness ratio $a/h$, aspect ratio $b/a$ and constitutive parameter $\zeta$ for two cases of uniformly loaded sandwich plates: (a) e–v–e and (b) v–e–v. Figures 2 and 3 illustrate the transversal shear stresses $\sigma_4$ and $\sigma_5$ through-the-thickness of the two cases with different types of inhomogeneous fiber-reinforced viscoelastic sandwich plates. It can be seen that the dimensionless stresses take larger values at the core (viscoelastic) for the first case (e–v–e) and vice versa for the second case (v–e–v). Also note that in the first case the stresses increase with the decrease of the thickness of the core compared with the thickness of the other faces, while for the other case the stresses decrease. If there is no core, this means that the plate becomes fully elastic for the first case and it is fully viscoelastic for the second case. The stresses take the same curve-related shape.

Figure 4 illustrates the variation of dimensionless stresses $\sigma_1$, $\sigma_4$ and $\sigma_6$ versus the time parameter $\tau$ at the different values of thickness $Z$ for the e–v–e and v–e–v sandwich plates. The variation of stresses for the two cases at different layers appears clearly with the variation of the time parameter $\tau$ and becomes constant for $\tau > 12$. Figure 5 illustrates the variation of the dimensionless stresses $\sigma_1$, $\sigma_4$ and $\sigma_6$ versus the time parameter $\tau$ at the core layer ($Z = 1/12$) for the two cases of the sandwich plates. We can see that for the e–v–e plate the dimensionless stresses and deflections increase with the decrease in the thickness of the core compared with the thickness of the other layer (top and bottom) and vice versa for the other case. However, Figure 6 shows that the behavior of the deflection $w$ of a plate with thin core, i.e., (2–1–2) plate, intermediates the behavior of other plates with equal or double thickness compared with their faces and this irrespective of the case studied.

Figures 7 and 8 illustrate the variation of dimensionless stresses and the deflection versus the time parameter $\tau$ at the core layer ($Z = 1/12$) and at different values of side-to-thickness ratio $a/h$. It is to be noted that the dimensionless stresses and deflections increase with the increase of the side-to-thickness ratio $a/h$. However, the e–v–e plates give largest stresses and deflections compared with the v–e–v plates.

Figures 9 and 10 illustrate the variation of dimensionless stresses and deflection versus the time parameter $\tau$ at the core layer ($Z = 1/12$) and at the different values of aspect ratio $b/a$ for the two cases. Results increase as the aspect ratio $b/a$ increases. Also, the e–v–e plates give largest stresses and deflections compared with the v–e–v plates.

Finally, Figs. 11 and 12 illustrate the variation of dimensionless stresses and deflection versus the time parameter $\tau$ at the core layer ($Z = 1/12$) and at the different values of the constitutive parameter $\zeta$. The dimensionless stresses increase with the decrease of the constitutive parameter $\zeta$ for the e–v–e plate and vice versa for the other plate. However, the dimensionless deflection increases as $\zeta$ decreases for the two cases.

## 8 Concluding remarks

A consistent FSDT for inhomogeneous fiber-reinforced viscoelastic sandwich plates is presented. Two cases of e–v–e and v–e–v sandwich plates are considered. In addition, different types of the various-thickness sandwich plates are also considered. Numerical computations were carried out to study the effect of the time parameter $\tau$ on deflections and stresses for different values of aspect ratio $b/a$, side-to-thickness $a/h$ and constitutive parameter $\zeta$. The obtained results show how the dimensionless stresses and deflection depend on the elastic properties of the layers and time parameter.

## Appendix A

The functions $\Pi(t)$ and $g_{\beta_m}(t)$ can be determined by deducing the Laplace–Carson transform of these functions from the known Laplace–Carson transform of the function $\omega(t)$, which can be written in the form

$$\omega^*(s) = s\int_0^\infty \omega(t)e^{-st}\mathrm{d}t. \tag{A.1}$$

Using (28), one obtains

$$\omega^*(s) = s\int_0^\infty \left(c_1 + c_2 e^{-\alpha t}\right)e^{-st}\mathrm{d}t. \tag{A.2}$$

Then by integration of the above function we get

$$\omega^*(s) = s\left\{-\frac{c_1}{s}e^{-st}\Big|_0^\infty - \frac{c_2}{\alpha+s}e^{-(\alpha+s)t}\Big|_0^\infty\right\} = c_1 + c_2\frac{s}{\alpha+s}. \tag{A.3}$$

But we have

$$\Pi^*(s) = \frac{1}{\omega^*(s)} = \frac{1}{c_1 + c_2 s(\alpha+s)^{-1}}, \tag{A.4}$$

then

$$\Pi^*(s) = \frac{1}{c_1}\left[1 - \frac{c_2 s}{c_1\alpha + (c_1+c_2)s}\right] = \frac{1}{c_1}\left[1 - c_3\frac{s}{c_4+s}\right], \tag{A.5}$$

where

$$c_3 = \frac{c_2}{c_1+c_2}, \quad c_4 = \frac{c_1\alpha}{c_1+c_2}. \tag{A.6}$$

So, we can find the function $\Pi(t)$ by using the inverse Laplace–Carson transform of (A.5) in the form

$$\Pi(t) = \frac{1}{c_1}\left[1 - c_3 e^{-c_4\tau}\right] = \frac{1}{c_1}\left[1 - \frac{c_2}{c_1 + c_2} e^{-\frac{c_1\tau}{c_1+c_2}}\right]. \tag{A.7}$$

Similarly (for more details, one can refer to [26]),

$$g_{\beta_m}(t) = \frac{1}{1 + c_1\beta_m}\left[1 - \frac{c_2\beta_m}{1 + (c_1 + c_2)\beta_m} e^{-\frac{(1+c_1\beta_m)\tau}{1+(c_1+c_2)\beta_m}}\right], \tag{A.8}$$

where $\beta_1 = \frac{1}{2}$ and $\beta_2 = 2$.

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