Acta Mech 207, 163–181 (2009)
DOI 10.1007/s00707-008-0109-4

X.-L. Gao · H. M. Ma

Green's function and Eshelby's tensor based on a simplified strain gradient elasticity theory

Received: 4 June 2008 / Revised: 11 September 2008 / Published online: 13 November 2008

Abstract The Eshelby problem of an infinite homogeneous isotropic elastic material containing an inclusion is analytically solved using a simplified strain gradient elasticity theory that involves one material length scale parameter in addition to two classical elastic constants. The Green's function in the simplified strain gradient elasticity theory is first obtained in terms of elementary functions by applying Fourier transforms, which reduce to the Green's function in classical elasticity when the strain gradient effect is not considered. The Eshelby tensor is then derived in a general form for an inclusion of arbitrary shape, which consists of a classical part and a gradient part. The former contains Poisson's ratio only, while the latter includes the length scale parameter additionally, thereby enabling the interpretation of the size effect. By applying the general form of the Eshelby tensor derived, the explicit expressions of the Eshelby tensor for the special case of a spherical inclusion are obtained. The numerical results quantitatively show that the components of the new Eshelby tensor for the spherical inclusion vary with both the position and the inclusion size, unlike their counterparts based on classical elasticity. It is found that when the inclusion radius is small, the contribution of the gradient part is significantly large and thus should not be ignored. For homogenization applications, the volume average of this newly obtained Eshelby tensor over the spherical inclusion is derived in a closed form. It is observed that the components of the averaged Eshelby tensor change with the inclusion size: the smaller the inclusion radius, the smaller the components. Also, these components are seen to approach from below the values of their counterparts based on classical elasticity when the inclusion size becomes sufficiently large.

1 Introduction

Eshelby's eigenstrain method and inclusion problem solutions [1,2] are monumental in the development of micromechanics. The Eshelby tensor provides a direct link between the actual (induced) strain in an infinite homogeneous isotropic elastic material and the stress-free uniform transformation strain (eigenstrain) in an inclusion embedded in the infinite material [1]. This fourth-order tensor plays a key role in homogenization methods by Mori and Tanaka [3] and others [4–9]. However, Eshelby's tensor in its original form is based on classical elasticity and depends only on the elastic constants and the inclusion shape (e.g., the aspect ratio for an ellipsoidal inclusion [9]). As a result, Eshelby's tensor and the subsequent homogenization methods cannot capture the size effect exhibited by inclusion-matrix composites [10–12]. This motivated the studies on the Eshelby inclusion problem using higher-order elasticity theories, which, unlike classical elasticity, contain microstructure-dependent material length scale parameters and are therefore capable of explaining the size effect.

X.-L. Gao (✉) · H. M. Ma
Department of Mechanical Engineering, Texas A&M University,
3123 TAMU, College Station, TX 77843-3123, USA
E-mail: xlgao@tamu.edu
Tel.: +1-979-845-4835

The Eshelby tensors for a spherical inclusion and a cylindrical inclusion were respectively derived by Cheng and He [13] and Cheng and He [14] using a micropolar elasticity theory. The same micropolar theory was employed by Ma and Hu [15] to obtain Eshelby's tensor for an ellipsoidal inclusion. The analytical expressions of Eshelby's tensor for a spherical inclusion embedded in an infinite microstretch material were derived by Liu and Hu [16] and Kiris and Inan [17]. Zheng and Zhao [18] studied the Eshelby inclusion problem and obtained explicit expressions of Eshelby's tensor for a spherical inclusion using a couple stress elasticity theory. The Eshelby tensor for a spherical inclusion in a microelongated medium was derived by Kiris and Inan [19]. Zhang and Sharma [20] obtained Eshelby's tensor for an inclusion of arbitrary shape based on a strain gradient elasticity theory, with the spherical inclusion problem solved explicitly. More recently, Ma and Hu [21] provided an analytical form of the Eshelby tensor for an ellipsoidal inclusion using a microstretch elasticity theory. However, the higher-order elasticity theories used in the afore-mentioned studies other than that in [18] contain at least four elastic constants, with two or more being material length scale parameters. Due to the difficulties in determining these microstructure-dependent length scale parameters [22–24] and in dealing with the fourth-order Eshelby tensor, it is very desirable to study the Eshelby inclusion problem using a higher-order elasticity theory containing only one material length scale parameter in addition to the two classical elastic constants. The work reported in [18] appears to be the only study that involves just one additional length scale parameter, which is based on a couple stress theory modified from the classical couple stress theory [25] that contains four elastic constants in the constitutive equations but three in the displacement-equations of equilibrium. There is still a lack of studies on the Eshelby inclusion problem based on strain gradient elasticity theories involving only one additional elastic constant.

The objective of this paper is to provide such a study. A simplified strain gradient elasticity theory involving only one additional material length scale parameter [26,27] is used in the current study to analytically solve the Eshelby problem of an infinite homogeneous isotropic elastic medium containing an inclusion of arbitrary shape. The procedure to be followed here is similar to that used in [18], where an illuminating Green's function based approach is also used. The rest of this paper is organized as follows. In Sect. 2, Green's function in the simplified strain gradient elasticity theory is obtained from directly solving the governing equations using Fourier transforms, which reduce to the Green's function in classical elasticity when the strain gradient effect is ignored. Based on the Green's function obtained, the Eshelby tensor is derived in Sect. 3 in a general form for an inclusion of arbitrary shape, which consists of a classical part and a gradient part. The former contains only one classical elastic constant (Poisson's ratio), while the latter includes the length scale parameter additionally. In Sect. 4, the explicit expressions for the Eshelby tensor are obtained for the special case of a spherical inclusion by directly applying the general form of the newly derived Eshelby tensor. This specific Eshelby tensor is position-dependent even inside the inclusion, unlike its counterpart based on classical elasticity. For homogenization applications, the volume average of this Eshelby tensor over the spherical inclusion is also analytically determined in Sect. 4. Sample numerical results are provided in Sect. 5 to illustrate the Eshelby tensor for the spherical inclusion. The paper concludes with a summary in Sect. 6.

2 Green's function

The Navier-like basic governing equations in the simplified strain gradient elasticity theory are given by [27]

$$(\lambda + \mu)u_{i,ij} + \mu u_{j,kk} - L^2 \left[(\lambda + \mu)u_{i,ij} + \mu u_{j,kk}\right]_{,mm} + f_j = 0, \tag{1}$$

which are the displacement-equations of *equilibrium* that have incorporated the *geometrical* equations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.1}$$

$$\kappa_{ijk} = \varepsilon_{ij,k} = \frac{1}{2}\left(u_{i,jk} + u_{j,ik}\right), \tag{2.2}$$

and the *constitutive* equations:

$$\sigma_{ij} \equiv \tau_{ij} - \mu_{ijk,k} \tag{3.1}$$

$$\tau_{ij} = \lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{3.2}$$

$$\mu_{ijk} = L^2\tau_{ij,k} = L^2(\lambda\kappa_{llk}\delta_{ij} + 2\mu\kappa_{ijk}). \tag{3.3}$$

In Eqs. (1)–(3.1), u_i $(i \in \{1, 2, 3\})$ are the Cartesian components of the displacement vector, L is a material length scale parameter(with $L^2 = c$, c being the strain gradient coefficient used in [27]), λ and μ are the Lamé constants, f_j are the Cartesian components of the body force vector (force per unit volume), ε_{ij} are the components of the classical (infinitesimal) strain, κ_{ijk} are the components of the strain gradient, σ_{ij} are the components of the total stress, τ_{ij} are the components of the Cauchy stress conjugated to ε_{ij}, μ_{ijk} are the components of the double stress conjugated to κ_{ijk}, and δ_{ij} is the Kronecker delta. Note that the standard index notation, together with the Einstein summation convention, is used in Eqs. (1)–(3.1–3.3) and throughout this paper, with each Latin index (subscript) ranging from 1 to 3 unless otherwise stated.

The solution of Eq. (1), which is the final form of the governing equations that has incorporated Eqs. (2.1, 2.2) and (3.1–3.3), subject to the boundary conditions of u_i and their derivatives vanishing at infinity, provides the fundamental solution and Green's function in the simplified strain gradient elasticity theory, as will be shown next.

The three-dimensional (3D) Fourier transform of a sufficiently smooth function $F(\mathbf{x})$ and its inverse can be defined as

$$\hat{F}(\boldsymbol{\xi}) = \int\int\int_{-\infty}^{+\infty} F(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}, \tag{4.1}$$

$$F(\mathbf{x}) = \frac{1}{(2\pi)^3} \int\int\int_{-\infty}^{+\infty} \hat{F}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}\boldsymbol{\xi}, \tag{4.2}$$

where $\mathbf{x}$ is the position vector of a point in the 3D physical space, $\boldsymbol{\xi}$ is the position vector of the same point in the Fourier (transformed) space, i is the usual imaginary number with $\mathrm{i}^2 = -1$, and $\hat{F}(\boldsymbol{\xi})$ is the Fourier transform of $F(\mathbf{x})$.

Suppose that u_i are sufficiently differentiable and that u_i and their derivatives vanish at $|\ \mathbf{x}\ | \to \infty$. Then, applying Eq. (4.1), the product rule and the divergence theorem gives

$$\hat{u}_i(\boldsymbol{\xi}) = \int\int\int_{-\infty}^{+\infty} u_i(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}\mathbf{x}, \quad \hat{u}_{k,ij}(\boldsymbol{\xi}) = -\xi_i \xi_j \hat{u}_k(\boldsymbol{\xi}), \quad \hat{u}_{k,ijll}(\boldsymbol{\xi}) = \xi_i \xi_j \xi_l \xi_l \hat{u}_k(\boldsymbol{\xi}). \tag{5}$$

Taking Fourier transforms on Eq. (1) and using Eqs. (4.1) and (5) will lead to

$$\xi^2(1 + L^2\xi^2) \left[(\lambda + 2\mu)\xi_i^0 \xi_j^0 + \mu(\delta_{ij} - \xi_i^0 \xi_j^0) \right] \hat{u}_i = \hat{f}_j, \tag{6}$$

where $\xi \equiv |\ \boldsymbol{\xi}\ | = (\xi_k \xi_k)^{1/2}$, and $\xi_i^0 = \xi_i/\xi$ are the components of the unit vector $\boldsymbol{\xi}^0 = \boldsymbol{\xi}/\xi$. Equation (6) gives a system of three algebraic equations to solve for the three unknowns $\hat{u}_i$. This equation system can be readily solved to obtain

$$\hat{u}_i(\boldsymbol{\xi}) = \hat{G}_{ij}(\boldsymbol{\xi}) \hat{f}_j(\boldsymbol{\xi}), \tag{7.1}$$

where $\hat{G}_{ij}(\boldsymbol{\xi})$ is the inverse of the coefficient matrix of $\hat{u}_i(\boldsymbol{\xi})$ in Eq. (6) given by (see Appendix A)

$$\hat{G}_{ij}(\boldsymbol{\xi}) = \frac{1}{\xi^2(1 + L^2\xi^2)} \left[\frac{1}{\mu} \left(\delta_{ij} - \xi_i^0 \xi_j^0 \right) + \frac{1}{\lambda + 2\mu} \xi_i^0 \xi_j^0 \right]. \tag{7.2}$$

Taking inverse Fourier transforms on both sides of Eq. (7.1) then yields, with the help of the convolution theorem, the solution of Eq. (1) as

$$u_i(\mathbf{x}) = \int\int\int_{-\infty}^{+\infty} G_{ij}(\mathbf{x} - \mathbf{y}) f_j(\mathbf{y}) \mathrm{d}\mathbf{y}, \tag{8}$$

where $G_{ij}(\mathbf{x})$, as the inverse Fourier transform of $\hat{G}_{ij}(\boldsymbol{\xi})$ listed in Eq. (7.2), is (see Eq. (4.2))

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi^3} \int\int\int_{-\infty}^{+\infty} \hat{G}_{ij}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}} \mathrm{d}\boldsymbol{\xi}. \tag{9}$$

Equation (8) gives the fundamental solution in the simplified strain gradient elasticity theory in terms of the Green's function $G_{ij}(\mathbf{x})$ defined in Eq. (9).

To evaluate the definite integral in Eq. (9), a convenient spherical coordinate system (ξ, θ, ϕ) in the transformed space is chosen such that the angle between $\mathbf{x}$ and $\boldsymbol{\xi}$ is θ, with the direction of $\mathbf{x}$ being the axis where $\theta = 0$. Then, it follows that $\boldsymbol{\xi} \cdot \mathbf{x} = \xi_k x_k = \xi x \cos\theta$, with $x = \mid \mathbf{x} \mid = (x_k x_k)^{1/2}$, and the volume element $\mathrm{d}\boldsymbol{\xi} = \xi^2 \sin\theta \mathrm{d}\xi \mathrm{d}\theta \mathrm{d}\phi$. Substituting Eq. (7.2) into Eq. (9) yields

$$
\begin{aligned}
G_{ij}(\mathbf{x}) &= \frac{1}{8\pi^3} \int_0^{2\pi} \left\langle \int_0^{\pi} \left\{ \int_0^{\infty} \frac{1}{1+L^2\xi^2} \left[\frac{1}{\mu}\left(\delta_{ij} - \xi_i^0 \xi_j^0\right) + \frac{1}{\lambda + 2\mu} \xi_i^0 \xi_j^0 \right] \mathrm{e}^{\mathrm{i}\xi x \cos\theta} \mathrm{d}\xi \right\} \sin\theta \mathrm{d}\theta \right\rangle \mathrm{d}\phi \\
&= \frac{1}{8\pi^3} \int_0^{\pi} \left\langle \left\{ \int_0^{2\pi} \left[\frac{1}{\mu}\left(\delta_{ij} - \xi_i^0 \xi_j^0\right) + \frac{1}{\lambda + 2\mu} \xi_i^0 \xi_j^0 \right] \mathrm{d}\phi \right\} \left(\int_0^{\infty} \frac{1}{1+L^2\xi^2} \mathrm{e}^{\mathrm{i}\xi x \cos\theta} \mathrm{d}\xi \right) \right\rangle \sin\theta \mathrm{d}\theta.
\end{aligned}
\tag{10}
$$

From Eq. (7.2) it is seen that $\hat{G}_{ij}(\boldsymbol{\xi})$ is an even function of $\boldsymbol{\xi}$ with $\hat{G}_{ij}(-\boldsymbol{\xi}) = \hat{G}_{ij}(\boldsymbol{\xi})$, and from Eq. (9) it then follows that $G_{ij}(\mathbf{x})$ is also an even function of $\mathbf{x}$ with $G_{ij}(-\mathbf{x}) = G_{ij}(\mathbf{x})$. Using this fact and the expression of $G_{ij}(\mathbf{x})$ in Eq. (10) gives

$$
\int_0^{\infty} \frac{1}{1+L^2\xi^2} \mathrm{e}^{\mathrm{i}\xi x \cos\theta} \mathrm{d}\xi = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+L^2\xi^2} \mathrm{e}^{\mathrm{i}\xi x \cos\theta} \mathrm{d}\xi = \frac{\pi}{2L} \mathrm{e}^{-|x\cos\theta|/L}, \tag{11}
$$

where the second equality follows from the Euler formula, integration properties of even and odd functions, and a known integration result in calculus. Also, it can be shown that (see Appendix B)

$$
\int_0^{2\pi} \xi_i^0 \xi_j^0 \mathrm{d}\phi = \pi \left[\delta_{ij} \sin^2\theta - x_i^0 x_j^0 (1 - 3\cos^2\theta) \right], \tag{12}
$$

where $x_i^0 = x_i / x$ are the components of the unit vector $\mathbf{x}^0 = \mathbf{x}/x$. Substituting Eqs. (11) and (12) into Eq. (10) then yields

$$
G_{ij}(\mathbf{x}) = \frac{1}{16\pi L} \int_{-1}^{1} \left\{ \left[\frac{2}{\mu} + \left(\frac{1}{\lambda+2\mu} - \frac{1}{\mu} \right)(1-t^2) \right] \delta_{ij} - \left(\frac{1}{\lambda+2\mu} - \frac{1}{\mu} \right) x_i^0 x_j^0 (1-3t^2) \right\} \mathrm{e}^{-|xt|/L} \mathrm{d}t, \tag{13}
$$

where use has been made of $t = -\cos\theta$ to facilitate the integration.

Evaluating the integral in Eq. (13) finally gives the Green's function as

$$
G_{ij}(\mathbf{x}) = \frac{1}{32\pi\mu(1-\nu)} \left[\Psi(x)\delta_{ij} + \mathrm{X}(x) x_i^0 x_j^0 \right], \tag{14}
$$

where ν is Poisson's ratio, and

$$
\Psi(x) = \frac{2}{x} \left\{ (3-4\nu)\left(1 - \mathrm{e}^{-\frac{x}{L}}\right) + \frac{1}{x^2} \left[2L^2 - (x^2 + 2Lx + 2L^2)\mathrm{e}^{-\frac{x}{L}} \right] \right\}, \tag{15.1}
$$

$$
\mathrm{X}(x) = \frac{2}{x} \left[\left(1 - \frac{6L^2}{x^2} \right) + \left(2 + \frac{6L}{x} + \frac{6L^2}{x^2} \right) \mathrm{e}^{-\frac{x}{L}} \right] \tag{15.2}
$$

are two convenient functions. Note that in reaching Eq. (14) use has also been made of the identities [28]:

$$
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \tag{16}
$$

where E is Young's modulus.

The Green's function derived here in Eqs. (14) and (15.1, 15.2) can be shown to be the same as that obtained by Polyzos et al. [29] using a different approach based on the use of the Helmholtz decomposition and potential functions. This Green's function can also be reduced to the Green's function in classical elasticity when the strain gradient effect is ignored. That is, by setting $L = 0$, Eqs. (14) and (15.1, 15.2) become

$$G_{ij}(x) = \frac{1}{16\pi\mu(1-\nu)x}\left[(3-4\nu)\delta_{ij} + x_i^0 x_j^0\right], \tag{17}$$

which is the Green's function for 3D problems in classical elasticity [5,9].

To facilitate the differentiation of the Green's function needed for determining Eshelby's tensor, the expressions given in Eqs. (14) and (15.1, 15.2) can be rewritten as follows. Note that $x_{,i} = x_i/x = x_i^0$ and $x_{i,j} = \partial x_i/\partial x_j = \delta_{ij}$. It then follows that

$$x_{,ij} = \frac{1}{x}\left(\delta_{ij} - x_i^0 x_j^0\right) \Rightarrow x_i^0 x_j^0 = \delta_{ij} - x x_{,ij}. \tag{18}$$

Inserting Eq. (18) into Eq. (14) then gives

$$G_{ij}(\mathbf{x}) = \frac{1}{32\pi\mu(1-\nu)}\left[(\Psi(x) + \mathrm{X}(x))\,\delta_{ij} - \mathrm{X}(x) x x_{,ij}\right]. \tag{19}$$

Next, using Eq. (15.2) and the following two identities:

$$\frac{1}{x^2}x_{,ij} = \frac{2}{3x^3}\delta_{ij} - \frac{1}{3}\left(\frac{1}{x}\right)_{,ij}, \tag{20.1}$$

$$\left(1 + \frac{3L^2}{x^2} + \frac{3L}{x}\right)\mathrm{e}^{-\frac{x}{L}} x_{,ij} = \left(\frac{1}{x} + \frac{2L}{x^2} + \frac{2L^2}{x^3}\right)\mathrm{e}^{-\frac{x}{L}}\delta_{ij} - L^2\left(\frac{1}{x}\mathrm{e}^{-\frac{x}{L}}\right)_{,ij} \tag{20.2}$$

leads to

$$\mathrm{X}(x) x x_{,ij} = 2\left\{\left[-\frac{4L^2}{x^3} + 2\left(\frac{1}{x} + \frac{2L}{x^2} + \frac{2L^2}{x^3}\right)\mathrm{e}^{-\frac{x}{L}}\right]\delta_{ij} + \left(x + \frac{2L^2}{x} - \frac{2L^2}{x}\mathrm{e}^{-\frac{x}{L}}\right)_{,ij}\right\}. \tag{21}$$

Substituting Eqs. (15.1, 15.2) and (21) into Eq. (19) finally yields

$$G_{ij}(\mathbf{x}) = \frac{1}{32\pi\mu(1-\nu)}\left[A(x)\delta_{ij} - B(x)_{,ij}\right], \tag{22}$$

where

$$A(x) \equiv 8(1-\nu)\frac{1}{x}\left(1 - \mathrm{e}^{-\frac{x}{L}}\right), \quad B(x) \equiv 2\left(x + \frac{2L^2}{x} - \frac{2L^2}{x}\mathrm{e}^{-\frac{x}{L}}\right). \tag{23}$$

It can be readily shown that when $L = 0$, Eqs. (22) and (23) reduce to Eq. (17), the Green's function in classical elasticity.

Equations (22) and (23) give the final form of the strain gradient Green's function for 3D elastic deformations in terms of elementary functions, which is different from the form obtained in Eqs. (14) and (15.1, 15.2) that involves $x_i^0 (= x_i/x)$ and $x_j^0 (= x_j/x)$ and is not convenient for differentiation. Equations (22) and (23) will be directly used in the next section to derive the general expressions of the Eshelby's tensor based on the simplified strain gradient elasticity theory.

3 Eshelby tensor

Consider an infinite homogenous isotropic elastic body containing an inclusion. An eigenstrain $\boldsymbol{\varepsilon}^*$ and an eigenstrain gradient $\boldsymbol{\kappa}^*$ are prescribed in the inclusion, while no body force or any other external force is present in the elastic body. $\boldsymbol{\varepsilon}^*$ and $\boldsymbol{\kappa}^*$ may have been induced by inelastic deformations such as thermal expansion, phase transformation, residual stress, and plastic flow [8].

According to the simplified strain gradient elasticity theory [27], the stress-equations of equilibrium in the absence of body forces are

$$\tau_{ij,j} - \mu_{ijp,pj} = 0, \tag{24}$$

where the Cauchy stress τ_{ij} is related to the elastic strain $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^*$ through the generalized Hooke's law:

$$\tau_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^*), \tag{25.1}$$

and the double stress μ_{ijk} is obtained from Eqs. (3.3) and (25.1) as

$$\mu_{ijp} = L^2 C_{ijkl}(\kappa_{klp} - \kappa_{klp}^*), \tag{25.2}$$

with C_{ijkl} being the components of the stiffness tensor of the isotropic elastic body given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{26}$$

Substituting Eqs. (25.1, 25.2) into Eq. (24) then yields the displacement-equations of equilibrium as

$$C_{ijkl}(\varepsilon_{kl} - L^2\kappa_{klp,p})_{,j} - C_{ijkl}(\varepsilon_{kl,j}^* - L^2\kappa_{klp,pj}^*) = 0, \tag{27}$$

where C_{ijkl} are given in Eq. (26). A comparison of Eq. (27) with Eq. (1) shows that Eq. (27) will be the same as that of Eq. (1) if the body force components f_j there are now replaced by $-C_{ijkl}(\varepsilon_{kl,j}^* - L^2\kappa_{klp,pj}^*)$ and Eq. (26) is used. As a result, the solution of Eq. (27) can be readily obtained from Eq. (8) as

$$u_i(\mathbf{x}) = -\int\int\int_{-\infty}^{+\infty} G_{ij}(\mathbf{x}-\mathbf{y})\, C_{jklm}\varepsilon_{lm,k}^* \mathrm{d}\mathbf{y} + \int\int\int_{-\infty}^{+\infty} G_{ij}(\mathbf{x}-\mathbf{y}) \left[L^2(C_{jklm}\kappa_{lmp,pk}^*)\right] \mathrm{d}\mathbf{y}. \tag{28}$$

The use of the product rule, the divergence theorem and the fact that $\varepsilon_{lm}^* = 0, \kappa_{lmp}^* = 0$ outside the inclusion (and thus at infinity) in Eq. (28), together with $C_{ijkl} =$ constants, gives

$$u_i(\mathbf{x}) = \int\int\int_{-\infty}^{+\infty} G_{ij,k}(\mathbf{x}-\mathbf{y})\, C_{jklm}\varepsilon_{lm}^* \mathrm{d}\mathbf{y} + \int\int\int_{-\infty}^{+\infty} G_{ij,kp}(\mathbf{x}-\mathbf{y})[L^2(C_{jklm}\kappa_{lmp}^*)]\mathrm{d}\mathbf{y}. \tag{29}$$

Equation (29) is valid for any (uniform or non-uniform) ε_{lm}^* and κ_{lmp}^*. For the Eshelby problem with ε_{lm}^* and κ_{lmp}^* being uniform in the inclusion and vanishing outside the inclusion and the elastic body being homogeneous (with C_{ijkl}= constants), Eq. (29) can be rewritten as

$$u_i(\mathbf{x}) = C_{jklm}\varepsilon_{lm}^* \int\int\int_{\Omega} G_{ij,k}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} + L^2 C_{jklm}\kappa_{lmp}^* \int\int\int_{\Omega} G_{ij,kp}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}, \tag{30}$$

where Ω denotes the region occupied by the inclusion.

It should be mentioned that all the derivatives in the integrals introduced so far are with respect to $\mathbf{y}$, which is the integration variable. However, it can be easily proved that

$$\frac{\partial G_{ij}(\mathbf{x}-\mathbf{y})}{\partial y_k} = -\frac{\partial G_{ij}(\mathbf{x}-\mathbf{y})}{\partial x_k}. \tag{31}$$

Using Eq. (31) in Eq. (30) then gives the displacement as

$$u_i(\mathbf{x}) = -C_{jklm}\varepsilon^*_{lm}\frac{\partial}{\partial x_k}\left[\iiint_\Omega G_{ij}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}\right] + L^2 C_{jklm}\kappa^*_{lmp}\frac{\partial}{\partial x_k \partial x_p}\left[\iiint_\Omega G_{ij}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}\right]. \quad (32)$$

Let

$$\langle F\rangle \equiv \iiint_\Omega F(\mathbf{y})\mathrm{d}\mathbf{y} \quad (33)$$

be the volume integral of a sufficiently smooth function $F(\mathbf{y})$ over the inclusion occupying region Ω. Then, Eq. (32) can be written as

$$u_i(\mathbf{x}) = -C_{jklm}\varepsilon^*_{lm}\left\langle G_{ij}\right\rangle_{,k} + L^2 C_{jklm}\kappa^*_{lmp}\left\langle G_{ij}\right\rangle_{,kp}, \quad (34)$$

where $\langle G_{ij}\rangle$ is the volume integral of the Green's function $G_{ij}(\mathbf{x}-\mathbf{y})$ defined according to Eq. (33), and the derivatives indicated are now with respect to $\mathbf{x}$. Inserting Eq. (34) into Eq. (2.1) then yields

$$\begin{aligned}\varepsilon_{ij} &= -\frac{1}{2}\left(\left\langle G_{iq}\right\rangle_{,kj} + \left\langle G_{jq}\right\rangle_{,ki}\right) C_{qklm}\varepsilon^*_{lm} + \frac{L^2}{2}\left(\left\langle G_{iq}\right\rangle_{,kpj} + \left\langle G_{jq}\right\rangle_{,kpi}\right) C_{qklm}\kappa^*_{lmp} \\ &\equiv S_{ijlm}\varepsilon^*_{lm} + T_{ijlmp}\kappa^*_{lmp}\end{aligned} \quad (35)$$

as the actual (disturbance) strain, ε_{ij}, induced by the presence of the eigenstrain, ε^*_{lm}, and the eigenstrain gradient, κ^*_{lmp}, where

$$S_{ijlm} \equiv -\frac{1}{2}\left(\left\langle G_{iq}\right\rangle_{,kj} + \left\langle G_{jq}\right\rangle_{,ki}\right) C_{qklm}, \quad (36.1)$$

$$T_{ijlmp} \equiv \frac{L^2}{2}\left(\left\langle G_{iq}\right\rangle_{,kpj} + \left\langle G_{jq}\right\rangle_{,kpi}\right) C_{qklm}. \quad (36.2)$$

Clearly, Eq. (35) shows that ε_{ij} is solely related to ε^*_{lm} in the absence of κ^*_{lmp}, and ε_{ij} is linked to only κ^*_{lmp} if $\varepsilon^*_{lm} = 0$.

The fourth-order tensor S_{ijlm} defined in Eqs. (35) and (36.1) is known as the Eshelby tensor. Since ε_{ij} and ε^*_{ij} are both symmetric, S_{ijlm} satisfies $S_{ijlm} = S_{ijml} = S_{jilm}$ (a minor symmetry rather than the major symmetry that requires $S_{ijmn} = S_{mnij}$ additionally) and therefore has 36 independent components. From Eqs. (22), (23), (33) and (36.1) it then follows that

$$\begin{aligned}S_{ijlm} = &-\frac{1}{8\pi\mu}\left[\Lambda_{,kj}\delta_{iq} + \Lambda_{,ki}\delta_{jq} - \frac{1}{2(1-\nu)}\Phi_{,ijkq}\right] C_{qklm} \\ &+ \frac{1}{8\pi\mu(1-\nu)}\left[(1-\nu)(\Gamma_{,kj}\delta_{iq} + \Gamma_{,ki}\delta_{jq}) + L^2(\Lambda-\Gamma)_{,ijkq}\right] C_{qklm},\end{aligned} \quad (37)$$

where

$$\Phi(x) \equiv \langle|\mathbf{x}-\mathbf{y}|\rangle, \quad \Lambda(x) \equiv \left\langle\frac{1}{|\mathbf{x}-\mathbf{y}|}\right\rangle, \quad \Gamma(x) \equiv \left\langle\frac{\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{y}|}{L}}}{|\mathbf{x}-\mathbf{y}|}\right\rangle \quad (38)$$

are three scalar-valued functions that can be obtained analytically or numerically by evaluating the volume integrals. Clearly, among these three functions only $\Gamma(x)$ depends on the length scale parameter L. As a result, the Eshelby tensor given in Eq. (37) can be separated into the classical part, S^C_{ijlm}, which is independent of the material length scale parameter L, and the gradient part, S^G_{ijlm}, which depends on L, thereby being microstructure-dependent. Accordingly, the general form of the Eshelby tensor in the simplified strain gradient elasticity theory derived in Eq. (37) for an inclusion of arbitrary shape can be rewritten as

$$S_{ijlm} = S^C_{ijlm} + S^G_{ijlm}, \tag{39.1}$$

$$S^C_{ijlm} = -\frac{1}{8\pi\mu}\left[\Lambda_{,kj}\delta_{iq} + \Lambda_{,ki}\delta_{jq} - \frac{1}{2(1-\nu)}\Phi_{,ijkq}\right] C_{qklm}, \tag{39.2}$$

$$S^G_{ijlm} = \frac{1}{8\pi\mu(1-\nu)}\left[(1-\nu)(\Gamma_{,kj}\delta_{iq} + \Gamma_{,ki}\delta_{jq}) + L^2(\Lambda - \Gamma)_{,ijkq}\right] C_{qklm}, \tag{39.3}$$

where the scalar-valued functions $\Lambda(x)$, $\Phi(x)$ and $\Gamma(x)$ are defined in Eq. (38) along with Eq. (33). Clearly, when $L = 0$ (i.e., when the strain gradient effect is ignored), Eqs. (38) and (39.1–39.3) show that $S^G_{ijlm} = 0$ and $S_{ijlm} = S^C_{ijlm}$. That is, the Eshelby tensor obtained in Eqs. (39.1–39.3) using the simplified strain gradient elasticity theory reduces to that based on classical elasticity.

The Eshelby-like fifth-order tensor T_{ijlmp} defined in Eqs. (35) and (36.2) links the eigenstrain gradient, κ^*_{lmp}, to the actual (induced) strain, ε_{ij}. Since ε_{ij} is symmetric and $\kappa^*_{lmp} = \kappa^*_{mlp}$, T_{ijlmp} satisfies $T_{ijlmp} = T_{ijmlp} = T_{jilmp}$ and therefore has 108 independent components (as opposed to $3^5 = 243$ such components). From Eqs. (22), (23), (33) and (36.2) it follows that

$$T_{ijlmp} = \frac{L^2}{32\pi\mu(1-\nu)}\left\{4(1-\nu)\left[(\Lambda-\Gamma)_{,kpj}\,\delta_{iq} + (\Lambda-\Gamma)_{,kpi}\,\delta_{jq}\right] - 2\left[\Phi + 2L^2(\Lambda-\Gamma)\right]_{,ijkpq}\right\} C_{qklm} \tag{40}$$

as the expression of the fifth-order tensor, with the scalar-valued functions $\Lambda(x)$, $\Phi(x)$ and $\Gamma(x)$ defined in Eq. (38) along with Eq. (33). Clearly, T_{ijlmp} has only the gradient part and vanishes when $L = 0$ (i.e., when the strain gradient effect is not considered). In fact, in this special case without the microstructural effect (i.e., $L = 0$), both S^G_{ijlm} and T_{ijlmp} vanish, and Eq. (35) simply becomes $\varepsilon_{ij} = S^C_{ijlm}\varepsilon^*_{ij}$, the defining relation for the Eshelby tensor based on classical elasticity [1], as expected.

It can be readily demonstrated that for a sufficiently smooth function $F(x)$ the following differential relations hold:

$$\begin{aligned}
F_{,i} &= x_i D_1 F,\\
F_{,ij} &= x_i x_j D_2 F + \delta_{ij} D_1 F,\\
F_{,ijk} &= x_i x_j x_k D_3 F + \langle\delta_{ij}x_k\rangle_3\, D_2 F,\\
F_{,ijkl} &= x_i x_j x_k x_l D_4 F + \langle\delta_{ij}x_k x_l\rangle\, D_3 F + \langle\delta_{ij}\delta_{kl}\rangle\, D_2 F,\\
F_{,ijmm} &= x^2 x_i x_j D_4 F + \left(x^2\delta_{ij} + 7x_i x_j\right) D_3 F + 5\delta_{ij} D_2 F,\\
F_{,ijklm} &= x_i x_j x_k x_l x_m D_5 F + \langle\delta_{ij}x_k x_l x_m\rangle_{10}\, D_4 F + \langle\delta_{ij}\delta_{kl}x_m\rangle_{15}\, D_3 F,\\
F_{,ijkkm} &= x_i x_j x_m x^2 D_5 F + \left(x^2\langle\delta_{ij}x_m\rangle_3 + 9x_i x_j x_m\right) D_4 F + 7\langle\delta_{ij}x_m\rangle_3\, D_3 F,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
&D_1 F = \tfrac{F'}{x},\; D_2 F = \tfrac{1}{x^2}\left(F'' - \tfrac{F'}{x}\right),\; D_3 F = \tfrac{1}{x^3}\left(F''' - \tfrac{3F''}{x} + \tfrac{3F'}{x^2}\right),\\
&D_4 F = \tfrac{1}{x^4}\left[F^{(4)} - \tfrac{6F'''}{x} + \tfrac{15F''}{x^2} - \tfrac{15F'}{x^3}\right],\\
&D_5 F = \tfrac{1}{x^5}\left[F^{(5)} - \tfrac{10F^{(4)}}{x} + \tfrac{45F^{(3)}}{x^2} - \tfrac{105F''}{x^3} + \tfrac{105F'}{x^4}\right],\\
&\langle\delta_{ij}x_k\rangle_3 = \delta_{ij}x_k + \delta_{kj}x_i + \delta_{ik}x_j,\\
&\langle\delta_{ij}x_k x_l\rangle \equiv \delta_{ij}x_k x_l + \delta_{kl}x_i x_j + \delta_{jl}x_i x_k + \delta_{jk}x_i x_l + \delta_{il}x_j x_k + \delta_{ik}x_j x_l = \delta_{ij}x_k x_l + \delta_{kl}x_i x_j + \langle\delta_{ij}x_k x_l\rangle_4,\\
&\qquad\langle\delta_{ij}x_k x_l\rangle_4 \equiv \delta_{jl}x_i x_k + \delta_{jk}x_i x_l + \delta_{il}x_j x_k + \delta_{ik}x_j x_l,\\
&\langle\delta_{ij}x_k x_l x_m\rangle_{10} = \delta_{ij}x_k x_l x_m + \delta_{ik}x_j x_l x_m + \delta_{il}x_k x_j x_m + \delta_{im}x_k x_l x_j + \delta_{jk}x_i x_l x_m + \delta_{jl}x_k x_i x_m\\
&\qquad + \delta_{jm}x_k x_l x_i + \delta_{kl}x_i x_j x_m + \delta_{km}x_i x_l x_j + \delta_{lm}x_k x_i x_j,\\
&\langle\delta_{ij}\delta_{kl}x_m\rangle_{15} = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})x_m + (\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl} + \delta_{il}\delta_{jm})x_k + (\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})x_l\\
&\qquad + (\delta_{jm}\delta_{kl} + \delta_{jk}\delta_{ml} + \delta_{jl}\delta_{mk})x_i + (\delta_{im}\delta_{kl} + \delta_{ik}\delta_{ml} + \delta_{il}\delta_{mk})x_j.\\
&\langle\delta_{ij}\delta_{kl}\rangle \equiv \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}.
\end{aligned} \tag{42}$$

In Eq. (42), $F' = \mathrm{d}F/\mathrm{d}x$, $F'' = \mathrm{d}^2F/\mathrm{d}x^2$, $F''' = \mathrm{d}^3F/\mathrm{d}x^3$, $F^{(4)} = \mathrm{d}^4F/\mathrm{d}x^4$, and $F^{(5)} = \mathrm{d}^5F/\mathrm{d}x^5$, as usual. Also, in Eqs. (41) and (42) F can be $\Phi(x)$, $\Lambda(x)$ or $\Gamma(x)$ involved in Eqs. (39.1–39.3) and Eq. (40).

Using Eqs. (26), (41) and (42) in Eq. (39.2) leads to

$$S^C_{ijlm} = K^C_1(x)\,\delta_{ij}\delta_{lm} + K^C_2(x)\,(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + K^C_3(x)\delta_{lm}x^0_ix^0_j + K^C_4(x)\delta_{ij}x^0_lx^0_m + K^C_5(x)(\delta_{il}x^0_jx^0_m + \delta_{im}x^0_jx^0_l + \delta_{jl}x^0_ix^0_m + \delta_{jm}x^0_ix^0_l) + K^C_6(x)(x^0_ix^0_jx^0_lx^0_m), \tag{43}$$

where

$$K^C_1(x) = \frac{1}{8\pi(1-\nu)(1-2\nu)}\left[-4\nu(1-\nu)D_1\Lambda + \nu x^2D_3\Phi + (1+3\nu)D_2\Phi\right], \tag{44.1}$$

$$K^C_2(x) = \frac{1}{8\pi(1-\nu)}\left[-2(1-\upsilon)\,D_1\Lambda + D_2\Phi\right], \tag{44.2}$$

$$K^C_3 = \frac{x^2}{8\pi(1-\nu)(1-2\nu)}\left[-4\nu(1-\nu)D_2\Lambda + \nu x^2D_4\Phi + (1+5\nu)D_3\Phi\right], \tag{44.3}$$

$$K^C_4 = \frac{x^2}{8\pi(1-\nu)}D_3\Phi, \tag{44.4}$$

$$K^C_5 = \frac{x^2}{8\pi(1-\nu)}\left[-(1-\nu)D_2\Lambda + D_3\Phi\right], \tag{44.5}$$

$$K^C_6 = \frac{x^4}{8\pi(1-\nu)}D_4\Phi. \tag{44.6}$$

It is seen from Eqs. (43) and (44.1–44.6) that S^C_{ijlm} depends only on one material constant (i.e., Poisson's ratio ν) even for an inclusion of arbitrary shape. Similarly, applying Eqs. (26), (41) and (42) to Eq. (39.3) results in

$$S^G_{ijlm} = K^G_1(x)\,\delta_{ij}\delta_{lm} + K^G_2(x)(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) + K^G_3(x)\delta_{lm}x^0_ix^0_j + K^G_4(x)\delta_{ij}x^0_lx^0_m + K^G_5(x)(\delta_{il}x^0_jx^0_m + \delta_{im}x^0_jx^0_l + \delta_{jl}x^0_ix^0_m + \delta_{jm}x^0_ix^0_l) + K^G_6(x)(x^0_ix^0_jx^0_lx^0_m), \tag{45}$$

where

$$K^G_1 = \frac{1}{4\pi(1-\nu)}\left[\frac{2\nu(1-\nu)}{1-2\nu}D_1\Gamma - \frac{\nu}{1-2\nu}L^2x^2D_3(\Gamma-\Lambda) - \frac{1+3\nu}{1-2\nu}\,L^2D_2(\Gamma-\Lambda)\right], \tag{46.1}$$

$$K^G_2 = \frac{1}{4\pi(1-\nu)}\left[(1-\nu)\,D_1\Gamma - L^2D_2(\Gamma-\Lambda)\right], \tag{46.2}$$

$$K^G_3 = \frac{x^2}{4\pi(1-\nu)}\left[\frac{2\nu(1-\nu)}{1-2\nu}D_2\Gamma - \frac{\nu}{1-2\nu}L^2x^2D_4(\Gamma-\Lambda) - \frac{1+5\nu}{1-2\nu}\,L^2D_3(\Gamma-\Lambda)\right], \tag{46.3}$$

$$K^G_4 = -\frac{L^2x^2}{4\pi(1-\nu)}D_3(\Gamma-\Lambda), \tag{46.4}$$

$$K^G_5 = \frac{x^2}{8\pi(1-\nu)}\left[(1-\nu)D_2\Gamma - 2\,L^2D_3(\Gamma-\Lambda)\right], \tag{46.5}$$

$$K^G_6 = -\frac{L^2x^4}{4\pi(1-\nu)}D_4(\Gamma-\Lambda). \tag{46.6}$$

Clearly, Eqs. (45) and (46.1–46.6) show that S^G_{ijlm} depends not only on Poisson's ratio ν but also on the material length scale parameter L, unlike S^C_{ijlm} given in Eqs. (43) and (44.1–44.6). Finally, the use of Eqs. (26), (41) and (42) in Eq. (40) yields

$$\begin{aligned} T_{ijlmp} = \frac{L^2}{8\pi(1-\nu)} \Bigg\{ & -\frac{4\nu(1-\nu)}{1-2\nu}\delta_{lm}\left[x_i x_j x_p D_3(\Gamma-\Lambda) + \langle x_p \delta_{ij}\rangle_3 D_2(\Gamma-\Lambda)\right] \\ & -(1-\nu)(x_i x_m x_p \delta_{jl} + x_j x_m x_p \delta_{il} + x_i x_l x_p \delta_{jm} + x_j x_l x_p \delta_{im}) D_3(\Gamma-\Lambda) \\ & -(1-\nu)(\langle x_i \delta_{mp}\rangle_3 \delta_{jl} + \langle x_j \delta_{mp}\rangle_3 \delta_{il} + \langle x_i \delta_{lp}\rangle_3 \delta_{jm} + \langle x_j \delta_{lp}\rangle_3 \delta_{im}) D_2(\Gamma-\Lambda) \\ & + \frac{2\nu L^2}{1-2\nu}\delta_{lm}\left[x_i x_j x_p x^2 D_5\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right) + (x^2\langle x_p\delta_{ij}\rangle_3 + 9x_i x_j x_p) D_4\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right)\right. \\ & \left. +7\langle x_p\delta_{ij}\rangle_3 D_3\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right)\right] + 2L^2\left[x_i x_j x_l x_m x_p D_5\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right)\right. \\ & \left. + \langle \delta_{ij} x_l x_m x_p\rangle_{10} D_4\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right) + \langle \delta_{ij}\delta_{lm} x_p\rangle_{15} D_3\left(\Gamma-\Lambda-\frac{\Phi}{2L^2}\right)\right]\Bigg\}. \end{aligned} \tag{47}$$

Equations (39.1) and (43)–(47) provide the general formulas for determining $S_{ijlm}(= S^C_{ijlm} + S^G_{ijlm})$ and T_{ijlmp} for an inclusion of arbitrary shape in terms of the scalar-valued functions $\Lambda(x)$, $\Phi(x)$ and $\Gamma(x)$ defined in Eq. (38) along with Eq. (33). As volume integrals over the inclusion region Ω, these three functions cannot be obtained in closed forms in general for an inclusion of arbitrary shape. However, for simple shapes such as a spherical inclusion, explicit expressions can be derived for $\Lambda(x)$, $\Phi(x)$ and $\Gamma(x)$ and thus for the Eshelby tensor given in Eqs. (39.1) and (43)–(46.1–46.6), as will be shown in the next section.

4 Spherical inclusion

The problem of a spherical inclusion embedded in an infinite elastic body and prescribed with a uniform eigenstrain is directly related to composites filled by spherical particles [4,12]. The closed-form expressions for the components of the Eshelby tensor for a spherical inclusion will be derived here by directly applying the general formulas obtained in the preceding section.

Consider a spherical inclusion of radius R and centered at the origin of the coordinate system (x_1, x_2, x_3) in the physical space. In this case, the three volume integrals defined in Eq. (38) along with Eq. (33) can be exactly evaluated to obtain the following closed-form expressions:

$$\Phi(x) = \begin{cases} -\frac{\pi}{15}x^4 + \frac{2\pi}{3}R^2x^2 + \pi R^4, & \mathbf{x}\in\Omega, \\ \frac{4\pi}{15}\frac{R^5}{x} + \frac{4\pi}{3}R^3 x, & \mathbf{x}\notin\Omega; \end{cases} \tag{48.1, 48.2}$$

$$\Lambda(x) = \begin{cases} -\frac{2\pi}{3}x^2 + 2\pi R^2, & \mathbf{x}\in\Omega, \\ \frac{4\pi}{3}\frac{R^3}{x}, & \mathbf{x}\notin\Omega; \end{cases} \tag{48.3, 48.4}$$

$$\Gamma(x) = \begin{cases} 4\pi L^2 - 4\pi L^2(L+R)\mathrm{e}^{-\frac{R}{L}}\frac{1}{x}\sinh\left(\frac{x}{L}\right), & \mathbf{x}\in\Omega, \\ \frac{-4\pi L^3}{x}\left[\sinh\left(\frac{R}{L}\right) - \frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]\mathrm{e}^{-\frac{x}{L}}, & \mathbf{x}\notin\Omega. \end{cases} \tag{48.5, 48.6}$$

Note that in Eqs. (48.1–48.6), $x \equiv |\mathbf{x}| = (x_k x_k)^{1/2}$, as defined earlier in Sect. 2. These expressions can be readily shown to be equivalent to those provided in [13] and [18], where different definitions and notation were used for the three scalar-valued functions. Clearly, $\Phi(x)$, $\Lambda(x)$ and $\Gamma(x)$ given in Eqs. (48.1–48.6) are infinitely differentiable at any $x \neq 0$.

Then, it follows from Eqs. (48.1, 48.3, 48.5), (42) and (43) and (44.1–44.6) that the classical part of the Eshelby tensor for the interior case with $\mathbf{x}$ locating inside the spherical inclusion (i.e., $\mathbf{x} \in \Omega$ or $x < R$) is

$$S^C_{ijlm} = \frac{5\nu - 1}{15(1-\nu)}\delta_{ij}\delta_{lm} + \frac{4-5\nu}{15(1-\nu)}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}). \tag{49}$$

Next, using Eqs. (48.1, 48.3, 48.5) and (42) leads to

$$\begin{aligned}
D_1\Lambda &= -\frac{4}{3}\pi, \quad D_2\Lambda = D_3\Lambda = D_4\Lambda = 0,\\
D_1\Phi &= \frac{4}{15}\pi(-x^2 + 5R^2), \quad D_2\Phi = -\frac{8}{15}\pi, \quad D_3\Phi = D_4\Phi = 0,\\
D_1\Gamma &= -\frac{4\pi L(L+R)e^{-\frac{R}{L}}}{x^3}\left[x\cosh\left(\frac{x}{L}\right) - L\sinh\left(\frac{x}{L}\right)\right],\\
D_2\Gamma &= -\frac{4\pi(L+R)e^{-\frac{R}{L}}}{x^5}\left[-3Lx\cosh\left(\frac{x}{L}\right) + \left(x^2+3L^2\right)\sinh\left(\frac{x}{L}\right)\right],\\
D_3\Gamma &= -\frac{4\pi(L+R)e^{-\frac{R}{L}}}{Lx^7}\left[x\left(x^2+15L^2\right)\cosh\left(\frac{x}{L}\right) - 3L\left(2x^2+5L^2\right)\sinh\left(\frac{x}{L}\right)\right],\\
D_4\Gamma &= -\frac{4\pi(L+R)e^{-\frac{R}{L}}}{L^2x^9}\left[-5Lx\left(2x^2+21L^2\right)\cosh\left(\frac{x}{L}\right) + \left(x^4+45L^2x^2+105L^4\right)\sinh\left(\frac{x}{L}\right)\right],\\
D_5\Gamma &= -\frac{4\pi(L+R)e^{-\frac{R}{L}}}{L^3x^{11}}\left[\left(x^5+105L^2x^3+945L^4x\right)\cosh\left(\frac{x}{L}\right) - 15L\left(x^4+28L^2x^2+63L^4\right)\sinh\left(\frac{x}{L}\right)\right]
\end{aligned} \tag{50}$$

for any interior point $\mathbf{x} \in \Omega$ (or $x < R$). Substituting Eq. (50) into Eqs. (45) and (46.1–46.6) will then give the closed-form expression of the gradient part of the Eshelby tensor for the interior case with $\mathbf{x}$ locating inside the spherical inclusion. Similarly, the use of Eq. (50) in Eq. (47) will yield the explicit formula for determining T_{ijlmp} at any $\mathbf{x}$ inside the spherical inclusion (i.e., $\mathbf{x} \in \Omega$ or $x < R$).

Note that Eq. (49) clearly shows that for the spherical inclusion considered here the classical part of the Eshelby tensor, S^C_{ijlm}, is uniform inside the inclusion, independent of L, R and x. In fact, S^C_{ijlm} listed in Eq. (49) is identical to that based on classical elasticity (see, e.g., Equation (3.123) in Li and Wang [9]). In contrast, the gradient part, S^G_{ijlm}, given in Eqs. (45), (46.1–46.6) and (50) depends on L, R and x in a complicated manner, and is therefore non-uniform inside the spherical inclusion and differs for different materials (with distinct values of L) and inclusion sizes (with distinct values of R). However, if the strain gradient effect is ignored, then $L = 0$ and Eqs. (45), (46.1–46.6) and (50) give $S^G_{ijlm} = 0$. It thus follows from Eq. (39.1) that $S_{ijlm} = S^C_{ijlm}$. That is, the Eshelby tensor for the spherical inclusion derived here using the simplified strain gradient elasticity theory reduces to that based on classical elasticity when $L = 0$.

Considering that S^G_{ijlm} is position-dependent inside the spherical inclusion, its volume average over the spherical region occupied by the inclusion is examined next. This averaged Eshelby tensor is needed for predicting the effective elastic properties of a heterogeneous composite containing spherical inclusions. The volume average of a sufficiently smooth function $F(\mathbf{x})$ over the spherical inclusion occupying region Ω is defined by

$$\langle F\rangle_v = \frac{1}{\mathrm{Vol}(\Omega)}\iiint_\Omega F\,\mathrm{d}V = \frac{1}{\frac{4}{3}\pi R^3}\int_0^R\int_0^{2\pi}\int_0^\pi F x^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}x, \tag{51}$$

where use has been made of the volume element $\mathrm{d}V = x^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}x$ in a spherical coordinate system. Letting S^G_{ijlm} given in Eqs. (45) and (46.1–46.6) be $F(\mathbf{x})$ in Eq. (51) will lead to $\left\langle S^G_{ijlm}\right\rangle_v$.

Note that in the spherical coordinate system adopted here,

$$x^0_1 = \sin\theta\cos\phi, \quad x^0_2 = \sin\theta\sin\phi, \quad x^0_3 = \cos\theta. \tag{52}$$

It then follows from Eq. (52) that

$$\int_0^{2\pi}\int_0^{\pi} x_i^0 x_j^0 \ \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \frac{4}{3}\pi \ \delta_{ij},$$
$$\int_0^{2\pi}\int_0^{\pi} \left(\delta_{il}x_j^0 x_m^0 + \delta_{im}x_j^0 x_l^0 + \delta_{jl}x_i^0 x_m^0 + \delta_{jm}x_i^0 x_l^0\right) \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \frac{8}{3}\pi \ \left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right), \tag{53}$$
$$\int_0^{2\pi}\int_0^{\pi} \left(x_i^0 x_j^0 x_l^0 x_m^0\right) \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \frac{4}{15}\pi \ \left(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right).$$

Using Eqs. (53) and (45) in Eq. (51) then gives

$$\left\langle S_{ijlm}^G \right\rangle_v = \frac{1}{R^3}\left[\left(3\overline{K_1^G} + \overline{K_3^G} + \overline{K_4^G} + \frac{1}{5}\overline{K_6^G}\right)\delta_{ij}\delta_{lm} + \left(3\overline{K_2^G} + 2\overline{K_5^G} + \frac{1}{5}\overline{K_6^G}\right)(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\right], \tag{54}$$

where

$$\overline{K_n^G} \equiv \int_0^R x^2 K_n^G(x)\ \mathrm{d}x, \tag{55}$$

with K_n^G ($n = 1, 2, \ldots, 6$) to be substituted from Eqs. (46.1–46.6) and (50). The six integrals in Eq. (55) can be exactly evaluated, and Eq. (54) becomes

$$\left\langle S_{ijlm}^G \right\rangle_v = \frac{1}{10(1-\nu)}\left(\frac{L}{R}\right)^3\left[1-\left(\frac{R}{L}\right)^2-\left(1+\frac{R}{L}\right)^2 \mathrm{e}^{-\frac{2R}{L}}\right]\left[(5\nu-1)\,\delta_{ij}\delta_{lm}+(4-5\nu)\,(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\right], \tag{56}$$

which gives

$$\begin{aligned}
\langle S_{1111}^G\rangle_v &= \frac{7-5\nu}{10(1-\nu)}\left(\frac{L}{R}\right)^3\left[1-\left(\frac{R}{L}\right)^2-\left(1+\frac{R}{L}\right)^2\mathrm{e}^{-\frac{2R}{L}}\right] = \left\langle S_{2222}^G\right\rangle_v = \left\langle S_{3333}^G\right\rangle_v, \\
\langle S_{1122}^G\rangle_v &= \frac{5\nu-1}{10(1-\nu)}\left(\frac{L}{R}\right)^3\left[1-\left(\frac{R}{L}\right)^2-\left(1+\frac{R}{L}\right)^2\mathrm{e}^{-\frac{2R}{L}}\right] \\
&= \left\langle S_{1133}^G\right\rangle_v = \left\langle S_{2233}^G\right\rangle_v = \left\langle S_{2211}^G\right\rangle_v = \left\langle S_{3311}^G\right\rangle_v = \left\langle S_{3322}^G\right\rangle_v, \\
\langle S_{1212}^G\rangle_v &= \frac{4-5\nu}{10(1-\nu)}\left(\frac{L}{R}\right)^3\left[1-\left(\frac{R}{L}\right)^2-\left(1+\frac{R}{L}\right)^2\mathrm{e}^{-\frac{2R}{L}}\right] = \left\langle S_{2323}^G\right\rangle_v = \left\langle S_{3131}^G\right\rangle_v
\end{aligned} \tag{57}$$

as the 12 non-vanishing, volume-averaged components of the gradient part of the Eshelby tensor inside the inclusion. Clearly, these components are constants, but they depend on the inclusion size (R), the length scale parameter (L) and Poisson's ratio (ν) of the material. This differs from the components of the classical part of the Eshelby tensor inside the inclusion, which, as given in Eq. (49), are constants depending only on ν. However, when $L = 0$ (or $R/L \to \infty$), Eq. (57) shows that all non-zero components of $\left\langle S_{ijlm}^G \right\rangle_v$ will vanish, as will be further illustrated in the next section.

By following the same procedure, the volume average of the classical part of the Eshelby tensor inside the inclusion, $\left\langle S^C_{ijlm}\right\rangle_v$, can also be obtained using Eqs. (49) and (51). Since S^C_{ijlm} is uniform inside the inclusion, there will be $\left\langle S^C_{ijlm}\right\rangle_v = S^C_{ijlm}$. It then follows from Eqs. (39.1), (51), (49) and (56) that

$$\langle S_{ijlm}\rangle_v = \frac{1}{15(1-\nu)}\left\{1+\frac{3}{2}\left(\frac{L}{R}\right)^3\left[1-\left(\frac{R}{L}\right)^2-\left(1+\frac{R}{L}\right)^2 \mathrm{e}^{-\frac{2R}{L}}\right]\right\}$$
$$\times\left[(5\nu-1)\,\delta_{ij}\delta_{lm}+(4-5\nu)\,(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\right] \tag{58}$$

as the volume average of the Eshelby tensor inside the spherical inclusion based on the simplified strain gradient elasticity theory. Clearly, when $L=0$ (or $R/L\to\infty$), Eq. (58) reduces to $\left\langle S^C_{ijlm}\right\rangle_v = S^C_{ijlm}$ given in Eq. (49).

The volume average of T_{ijlmp} for $\mathbf{x}$ locating inside the spherical inclusion (i.e., $\mathbf{x}\in\Omega$ or $x<R$) can be readily shown to vanish, i.e.,

$$\langle T_{ijlmp}\rangle_v = \frac{1}{\mathrm{Vol}(\Omega)}\iiint_\Omega T_{ijlmp}\mathrm{d}V = \frac{1}{\frac{4}{3}\pi R^3}\int_0^R\int_0^{2\pi}\int_0^{\pi} T_{ijlmp}x^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}x \equiv 0. \tag{59}$$

The reason for this is that T_{ijlmp} involved in Eq. (59) and to be substituted from Eqs. (47) and (50) is odd in x^0_i, which makes the integration of T_{ijlmp} over any spherical surface vanish [30].

Similarly, the Eshelby tensor for the exterior case with $\mathbf{x}$ locating outside the spherical inclusion (i.e., $\mathbf{x}\notin\Omega$ or $x>R$) can be determined by using Eqs. (48.2, 48.4, 48.6) in the general formulas derived in Sect. 3 for an inclusion of arbitrary shape. Specifically, from Eqs. (42) and (48.2, 48.4, 48.6) it follows that

$$\begin{aligned}
&D_1\Lambda = -\frac{4\pi}{3}\frac{R^3}{x^3},\quad D_2\Lambda = \frac{4\pi R^3}{x^5},\quad D_3\Lambda = -\frac{20\pi R^3}{x^7},\quad D_4\Lambda = \frac{140\pi R^3}{x^9},\\
&D_1\Phi = -\frac{4\pi}{15}\frac{R^3}{x^3}(R^2-5x^2),\quad D_2\Phi = -\frac{4\pi}{15}\frac{R^3}{x^5}(-3R^2+5x^2),\\
&D_3\Phi = -\frac{4\pi R^3}{x^7}(R^2-x^2),\quad D_4\Phi = -\frac{4\pi R^3}{x^9}(-7R^2+5x^2),\\
&D_1\Gamma = \frac{4\pi L^2\left[\sinh\left(\frac{R}{L}\right)-\frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]}{x^3}(x+L)\,\mathrm{e}^{-\frac{x}{L}},\\
&D_2\Gamma = -\frac{4\pi L\left[\sinh\left(\frac{R}{L}\right)-\frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]}{x^5}\left(x^2+3Lx+3L^2\right)\mathrm{e}^{-\frac{x}{L}},\\
&D_3\Gamma = \frac{4\pi\left[\sinh\left(\frac{R}{L}\right)-\frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]}{x^7}\left(x^3+6Lx^2+15L^2x+15L^3\right)\mathrm{e}^{-\frac{x}{L}},\\
&D_4\Gamma = -\frac{4\pi\left[\sinh\left(\frac{R}{L}\right)-\frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]}{Lx^9}\left(x^4+10Lx^3+45L^2x^2+105L^3x+105L^4\right)\mathrm{e}^{-\frac{x}{L}},\\
&D_5\Gamma = \frac{4\pi\left[\sinh\left(\frac{R}{L}\right)-\frac{R}{L}\cosh\left(\frac{R}{L}\right)\right]}{L^2x^{11}}\left(x^5+15Lx^4+105L^2x^3+420L^3x^2+945L^4x+945L^5\right)\mathrm{e}^{-\frac{x}{L}}
\end{aligned} \tag{60}$$

for any exterior point $\mathbf{x}\notin\Omega$ (or $x>R$). Note that the functions listed in Eq. (60) for the exterior case with $\mathbf{x}\notin\Omega$ (or $x>R$) are clearly different from those defined in Eq. (50) for the interior case with $\mathbf{x}\in\Omega$ (or $x<R$). From Eqs. (60), (43) and (44.1–44.6) the classical part of the Eshelby tensor for any $\mathbf{x}$ outside the spherical inclusion (i.e., $\mathbf{x}\notin\Omega$ or $x>R$) is then obtained as

$$\begin{aligned}
S^C_{ijlm} = \frac{R^3}{x^5(1-\nu)}\Bigg\{&\frac{1}{30}\left[-5(1-2\nu)x^2+3R^2\right]\delta_{ij}\delta_{lm}+\frac{1}{30}\left[5(1-2\nu)x^2+3R^2\right](\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\\
&+\frac{1}{2}\left[(1-2\nu)x^2-R^2\right]\delta_{lm}x^0_ix^0_j-\frac{1}{2}\left(R^2-x^2\right)\delta_{ij}x^0_lx^0_m\\
&-\frac{1}{2}\left(R^2-\nu x^2\right)\left(\delta_{il}x^0_jx^0_m+\delta_{im}x^0_jx^0_l+\delta_{jl}x^0_ix^0_m+\delta_{jm}x^0_ix^0_l\right)-\frac{1}{2}\left(5x^2-7R^2\right)x^0_ix^0_jx^0_lx^0_m\Bigg\}.
\end{aligned} \tag{61}$$

It can be readily shown that the expression given in Eq. (61) is the same as that based on classical elasticity [13]. Clearly, a comparison of Eq. (61) with Eq. (49) shows that S^C_{ijlm} is not uniform outside the spherical inclusion, although it is uniform inside the same spherical inclusion.

Finally, using Eq. (60) in Eqs. (45) and (46.1–46.6) will result in the explicit formula for determining S^G_{ijlm} at any exterior point $\mathbf{x} \notin \Omega$ (or $x > R$), and the substitution of Eq. (60) into Eq. (47) will lead to the closed-form expression for T_{ijlmp} at any point $\mathbf{x}$ locating outside the spherical inclusion.

5 Numerical results

By using the closed-form expressions of the Eshelby tensor for the spherical inclusion derived in the preceding section, some numerical results are obtained and presented here to quantitatively illustrate how the components of the newly obtained Eshelby tensor vary with position and inclusion size.

From Eqs. (45), (46.1–46.6) and (50), the components of the gradient part of the Eshelby tensor at any $\mathbf{x}$ inside the spherical inclusion along the x_1 axis (with $x_2 = 0 = x_3$) can be obtained as

$$S^G_{1111} = \frac{L+R}{x^5(1-\nu)} e^{-\frac{R}{L}} \left\{ \left[-(1-\nu)x^4 + 2(\nu+4)x^2L^2 + 24L^4 \right] \sinh\left(\frac{x}{L}\right) - 2xL(\nu x^2 + 12L^2)\cosh\left(\frac{x}{L}\right) \right\}, \tag{62.1}$$

$$S^G_{1122} = S^G_{1133} = \frac{L+R}{x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ -\left[\nu x^4 + (2\nu+5)x^2L^2 + 12L^4 \right] \sinh\left(\frac{x}{L}\right) + xL(x^2 + 2\nu x^2 + 12L^2)\cosh\left(\frac{x}{L}\right) \right\}, \tag{62.2}$$

$$S^G_{1212} = S^G_{1313} = \frac{L+R}{2x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ -\left[(1-\nu)x^4 + (11-\nu)x^2L^2 + 24L^4 \right] \sinh\left(\frac{x}{L}\right) + xL\left[(3-\nu)x^2 + 24L^2 \right]\cosh\left(\frac{x}{L}\right) \right\}, \tag{62.3}$$

$$S^G_{2211} = S^G_{3311} = \frac{L(L+R)}{x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ -L\left[(5-\nu)x^2 + 12L^2 \right] \sinh\left(\frac{x}{L}\right) + x\left[(1-\nu)x^2 + 12L^2 \right]\cosh\left(\frac{x}{L}\right) \right\}, \tag{62.4}$$

$$S^G_{2222} = S^G_{3333} = -\frac{L(L+R)}{x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ -L\left[(5-\nu)x^2 + 9L^2 \right] \sinh\left(\frac{x}{L}\right) + x\left[(2-\nu)x^2 + 9L^2 \right]\cosh\left(\frac{x}{L}\right) \right\}, \tag{62.5}$$

$$S^G_{2233} = S^G_{3322} = \frac{L(L+R)}{x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ L\left[(1+\nu)x^2 + 3L^2 \right] \sinh\left(\frac{x}{L}\right) - x\ (\nu x^2 + 3L^2)\cosh\left(\frac{x}{L}\right) \right]\right\}, \tag{62.6}$$

$$S^G_{2323} = \frac{L(L+R)}{x^5(1-\nu)} \mathrm{e}^{-\frac{R}{L}} \left\{ L\left[(2-\nu)x^2 + 3L^2 \right] \sinh\left(\frac{x}{L}\right) - x\left[(1-\nu)x^2 + 3L^2 \right]\cosh\left(\frac{x}{L}\right) \right\}. \tag{62.7}$$

Note that in this special case (with $x = x_1, x_2 = 0 = x_3$) there are only 12 non-zero components among the 36 independent components of S^G_{ijlm}.

In the numerical analysis, the Poisson's ratio v is taken to be 0.3, and the material length scale parameter L to be 17.6 µm. Figure 1 shows the distribution of $S_{1111} = S^C_{1111} + S^G_{1111}$ along the x_1 axis (or a radial direction of the inclusion due to the spherical symmetry) for five different values of the inclusion radius, where the values of S^C_{1111} and S^G_{1111} are, respectively, obtained from Eqs. (49) and (62.1).

It is seen from Fig. 1 that S_{1111} varies with x (the position) and depends on R (the inclusion size), unlike the classical part S^C_{1111} which is a constant (i.e.,$S^C_{1111} = 0.5238$ from Eq. (49), as shown) independent of both x and R. When R is small (comparable to the length scale parameter $L = 17.6$ µm here), S_{1111} is much smaller than S^C_{1111}, which indicates that the magnitude of $S^G_{1111} (= S_{1111} - S^C_{1111})$ is very large and the strain gradient effect is significant. However, when R is much larger than L (e.g., $R = 6L = 105.6$ µm shown here), S_{1111} is seen to be quite uniform and its value approaches from below S^C_{1111} (= 0.5238), indicating that the magnitude of S^G_{1111} is very small and the strain gradient effect become insignificant and can therefore be ignored.

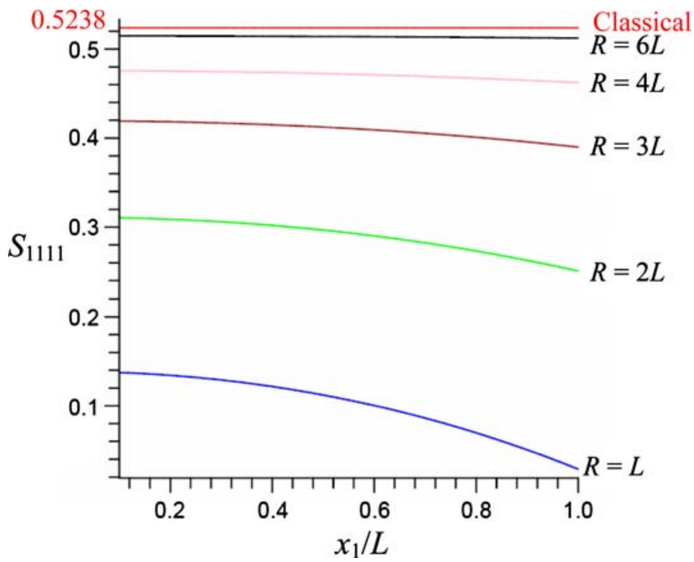

Fig. 1 S_{1111} along a radial direction of the spherical inclusion

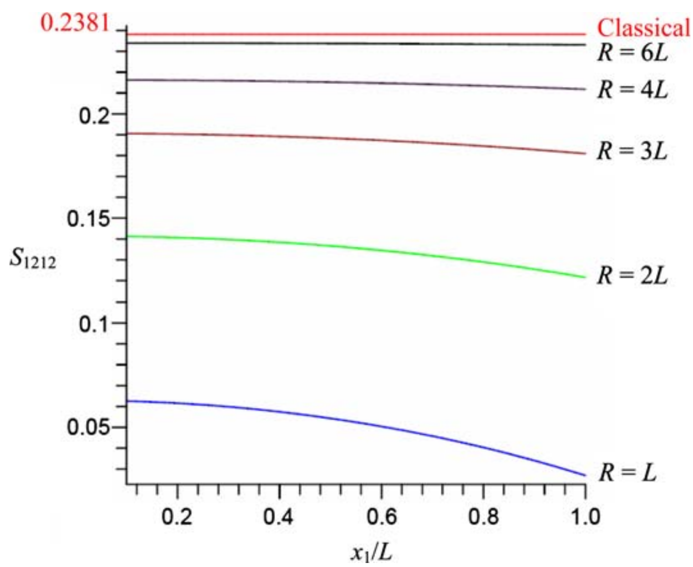

Fig. 2 S_{1212} along a radial direction of the spherical inclusion

Similar trends are observed from Figs. 2 and 3, where the values of S_{1212} and S_{2222} varying with x and R are displayed together with those of their classical parts that are horizontal lines independent of both x and R. The values of S^G_{1212} and S^G_{2222} included in S_{1212} $(= S^C_{1212} + S^G_{1212})$ and S_{2222} $(= S^C_{2222} + S^G_{2222})$ that are illustrated in Figs. 2 and 3 are, respectively, obtained from Eqs. (62.3) and (62.5), while those of S^C_{1212} and S^C_{2222} are both calculated using Eq. (49).

The variation of the component of the averaged Eshelby tensor inside the spherical inclusion, $\langle S_{1111}\rangle_v$, with the inclusion size (i.e., radius R) is shown in Fig. 4, where its counterpart in classical elasticity, $\langle S^C_{1111}\rangle_v$, is also displayed for comparison. Note that $\langle S_{1111}\rangle_v$ is obtained from Eq. (58), while $\langle S^C_{1111}\rangle_v$ $(= S^C_{1111} = 0.5238)$ is from Eq. (49). The material properties used here are $\upsilon = 0.3$ and $L = 17.6\ \mu$m, which are the same as those used in generating the results shown in Figs. 1, 2, and 3. It is observed from Fig. 4 that $\langle S_{1111}\rangle_v$ is indeed varying with R: the smaller R, the smaller $\langle S_{1111}\rangle_v$, while $\langle S^C_{1111}\rangle_v$ is a constant independent of R. Moreover, the difference between $\langle S_{1111}\rangle_v$ and $\langle S^C_{1111}\rangle_v$, which is $\langle S^G_{1111}\rangle_v$ $(=\langle S_{1111}\rangle_v - \langle S^C_{1111}\rangle_v)$, is seen to be significantly large only when the inclusion is small (with $R/L < 25$ or $R < 440\ \mu$m here). As the inclusion size increases,

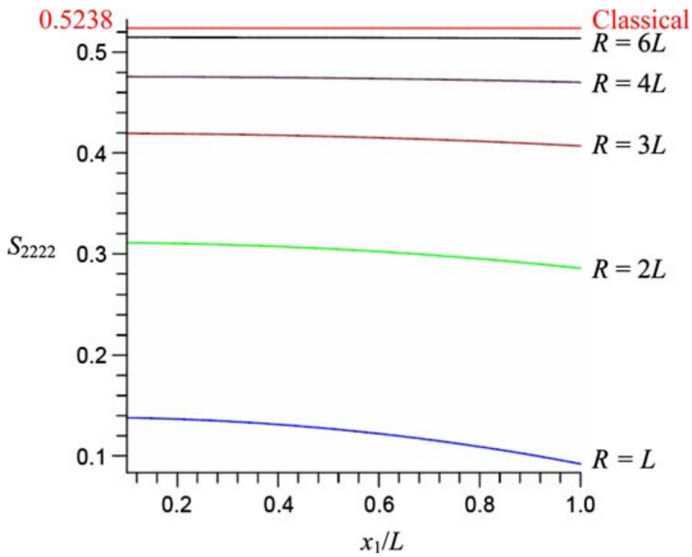

Fig. 3 S_{2222} along a radial direction of the spherical inclusion

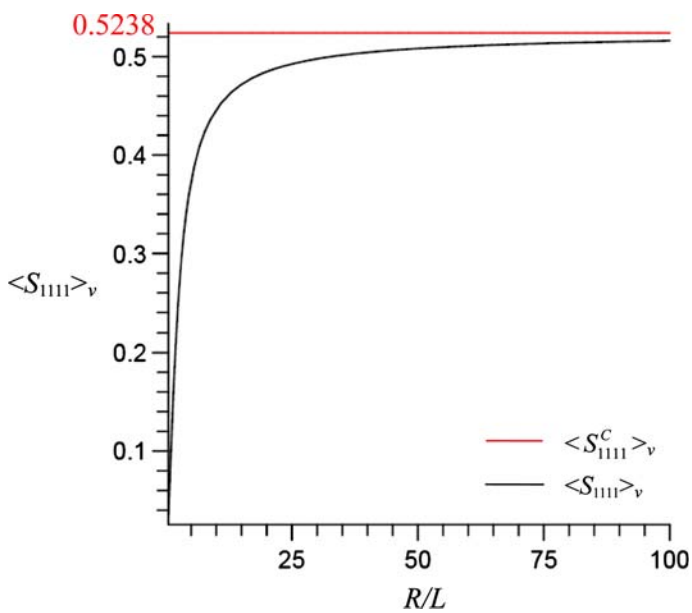

Fig. 4 $\langle S_{1111}\rangle_v$ varying with the inclusion radius

$\langle S_{1111}\rangle_v$ approaches from below the corresponding value of S^C_{1111} ($= 0.5238$) based on classical elasticity. The same is true for all the other non-vanishing components of $\left\langle S_{ijkm}\right\rangle_v$, as seen from Eqs. (58) and (49). These observations, once again, indicate that the strain gradient effect is insignificant for large inclusions and may be neglected.

Clearly, the numerical results presented above quantitatively show that the newly obtained Eshelby tensor captures the size effect at the micron scale, unlike that based on classical elasticity.

6 Conclusion

The Eshelby inclusion problem is solved analytically by using a simplified strain gradient elasticity theory. This is accomplished by first deriving the Green's function in the strain gradient elasticity theory in terms of elementary functions using Fourier transforms. The resulting Green's function reduces to that in classical elasticity when the strain gradient effect is ignored. The Eshelby tensor is then obtained in a general form for an inclusion of arbitrary shape using the Green's function method. The newly derived Eshelby tensor consists of two parts: a classical part depending only on Poisson's ratio, and a gradient part depending on the length scale parameter additionally.

The Eshelby tensor for the special case of a spherical inclusion is explicitly obtained by employing the general form of the newly derived Eshelby tensor. To further illustrate this Eshelby tensor, sample numerical results are provided, which reveal that the components of the new Eshelby tensor vary with both the position and the inclusion size, thereby capturing the size effect at the micron scale. In addition, the volume average of this new Eshelby tensor over the spherical inclusion is derived in a closed form, which is needed in homogenization analyses. The components of the averaged Eshelby tensor are found to decrease as the inclusion radius decreases, and these components are observed to approach from below the values of the corresponding components of the Eshelby tensor based on classical elasticity when the inclusion size is large enough.

Acknowledgments The work reported in this paper is supported by a grant from the U.S. National Science Foundation, with Dr. Clark V. Cooper as the program manager. This support is gratefully acknowledged. The authors also wish to thank Professor George J. Weng and one anonymous reviewer for their encouragement and helpful comments on an earlier version of the paper.

Appendix A

Note that in reaching Eq. (7.2) use has been made of the following identity:

$$\left(\alpha I_{ij}^P + \beta I_{ij}^S\right)^{-1} = \left(\frac{1}{\alpha} I_{ij}^P + \frac{1}{\beta} I_{ij}^S\right), \tag{A.1}$$

where α, β are two arbitrary non-zero scalars, $\mathrm{I}_{ij}^S = \xi_i^0 \xi_j^0$ are the components of a second-order spin tensor $\mathbf{I}^S = \boldsymbol{\xi}^0 \otimes \boldsymbol{\xi}^0$ (with $\boldsymbol{\xi}^0$ being a unit vector introduced in Eq. (6)), $I_{ij}^P = \delta_{ij} - \xi_i^0 \xi_j^0$ are the components of the associated projection tensor $\mathbf{I}^P = \mathbf{I} - \mathbf{I}^S$, with $\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ being the second-order identity tensor. Equation (A.1) can be easily proved by using the definition of an inverse tensor.

Appendix B

In this appendix, it is shown that the integration result given in Eq. (12) is true. That is,

$$\int_0^{2\pi} \xi_i^0 \xi_j^0 \mathrm{d}\phi = \pi \left[\delta_{ij} \sin^2\theta - x_i^0 x_j^0 (1 - 3\cos^2\theta)\right]. \tag{B.1}$$

Proof For the chosen spherical coordinate system (ξ, θ, ϕ) in the transformed space where the position vector $\boldsymbol{\xi} = \xi \boldsymbol{\xi}^0$ makes the angle θ with the position vector $\mathbf{x}$ (with the direction of $\mathbf{x}$ being where $\theta = 0$) in the physical space, one can write the unit vector in the $\boldsymbol{\xi}$ direction as

$$\boldsymbol{\xi}^0 = \mathbf{x}^0 \cos\theta + \left(\mathbf{y}^0 \cos\phi + \mathbf{z}^0 \sin\phi\right) \sin\theta, \tag{B.2}$$

where $\mathbf{x}^0$ is the unit vector along the $\mathbf{x}$ direction, and $\mathbf{y}^0$ and $\mathbf{z}^0$ are the unit vectors perpendicular to $\mathbf{x}^0$. In component form, Eq. (B.2) reads

$$\xi_i^0 = x_i^0 \cos\theta + \left(y_i^0 \cos\phi + z_i^0 \sin\phi\right) \sin\theta. \tag{B.3}$$

Then, it follows from Eq. (B.3) that

$$\begin{aligned} \xi_i^0 \xi_j^0 = {} & x_i^0 x_j^0 \cos^2\theta + x_i^0 y_j^0 \sin\theta \cos\theta \cos\phi + x_i^0 z_j^0 \sin\theta \cos\theta \sin\phi \\ & + y_i^0 x_j^0 \sin\theta \cos\theta \cos\phi + y_i^0 y_j^0 \sin^2\theta \cos^2\phi + y_i^0 z_j^0 \sin^2\theta \sin\phi \cos\phi \\ & + z_i^0 x_j^0 \sin\theta \cos\theta \sin\phi + z_i^0 y_j^0 \sin^2\theta \sin\phi \cos\phi + z_i^0 z_j^0 \sin^2\theta \sin^2\phi. \end{aligned} \tag{B.4}$$

Note that

$$\int_0^{2\pi} \cos\phi \mathrm{d}\phi = 0, \quad \int_0^{2\pi} \sin\phi \mathrm{d}\phi = 0, \quad \int_0^{2\pi} \sin\phi\cos\phi \mathrm{d}\phi = 0, \quad \int_0^{2\pi} \cos^2\phi \mathrm{d}\phi = \pi, \quad \int_0^{2\pi} \sin^2\phi \mathrm{d}\phi = \pi. \tag{B.5}$$

Integrating on both sides of Eq. (B.4), together with the use of Eq. (B.5), results in

$$\int_0^{2\pi} \xi_i^0 \xi_j^0 \mathrm{d}\phi = 2\pi x_i^0 x_j^0 \cos^2\theta + \pi y_i^0 y_j^0 \sin^2\theta + \pi z_i^0 z_j^0 \sin^2\theta. \tag{B.6}$$

Notice that

$$\begin{aligned} x_i^0 x_j^0 + y_i^0 y_j^0 + z_i^0 z_j^0 &= (\mathbf{x}^0 \cdot \mathbf{e}_i)(\mathbf{x}^0 \cdot \mathbf{e}_j) + (\mathbf{y}^0 \cdot \mathbf{e}_i)(\mathbf{y}^0 \cdot \mathbf{e}_j) + (\mathbf{z}^0 \cdot \mathbf{e}_i)(\mathbf{z}^0 \cdot \mathbf{e}_j) \\ &= \mathbf{e}_i \cdot (\mathbf{x}^0 \otimes \mathbf{x}^0)\mathbf{e}_j + \mathbf{e}_i \cdot (\mathbf{y}^0 \otimes \mathbf{y}^0)\mathbf{e}_j + \mathbf{e}_i \cdot (\mathbf{z}^0 \otimes \mathbf{z}^0)\mathbf{e}_j \\ &= \mathbf{e}_i \cdot \left[(\mathbf{x}^0 \otimes \mathbf{x}^0) + (\mathbf{y}^0 \otimes \mathbf{y}^0) + (\mathbf{z}^0 \otimes \mathbf{z}^0)\right]\mathbf{e}_j \\ &= \mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j = \delta_{ij}, \end{aligned} \tag{B.7}$$

where the fourth equality is based on the fact that the three orthogonal unit vectors $\mathbf{x}^0$, $\mathbf{y}^0$ and $\mathbf{z}^0$ form a set of base vectors in the 3D physical space. Using Eq. (B.7) and the identity $\sin^2\theta = 1 - \cos^2\theta$ in Eq. (B.6) will immediately give Eq. (B.1). □

References

1. Eshelby, J.D.: The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. Lond. A **241**, 376–396 (1957)
2. Eshelby, J.D.: The elastic field outside an ellipsoidal inclusion. Proc. R. Soc. Lond. A **252**, 561–569 (1959)
3. Mori, T., Tanaka, K.: Average stress in matrix and average elastic energy of materials with misfitting inclusions. Acta Metall. **21**, 571–574 (1973)
4. Weng, G.J.: Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions. Int. J. Eng. Sci. **22**, 845–856 (1984)
5. Mura, T.: Micromechanics of Defects in Solids, 2nd edn. Martinus Nijhoff, Dordrecht (1987)
6. Zhao, Y.H., Tandon, G.P., Weng, G.J.: Elastic moduli for a class of porous materials. Acta Mech. **76**, 105–130 (1989)
7. Nemat-Nasser, S., Hori, M.: Micromechanics: Overall Properties of Heterogeneous Materials, 2nd edn. Elsevier Science, Amsterdam, The Netherlands (1999)
8. Qu, J., Cherkaoui, M.: Fundamentals of Micromechanics of Solids. Wiley, Hoboken, NJ (2006)
9. Li, S., Wang, G.: Introduction to Micromechanics and Nanomechanics. World Scientific, Singapore (2008)
10. Lloyd, D.J.: Particle reinforced aluminum and magnesium matrix composites. Int. Mater. Rev. **39**, 1–23 (1994)
11. Kouzeli, M., Mortensen, A.: Size dependent strengthening in particle reinforced aluminum. Acta Mater. **50**, 39–51 (2002)
12. Gao, X.-L.: Analytical solution for the stress field around a hard spherical particle in a metal matrix composite incorporating size and finite volume effects. Math. Mech. Solids **13**, 357–372 (2008)
13. Cheng, Z.-Q., He, L.-H.: Micropolar elastic fields due to a spherical inclusion. Int. J. Eng. Sci. **33**, 389–397 (1995)
14. Cheng, Z.-Q., He, L.-H.: Micropolar elastic fields due to a circular cylindrical inclusion. Int. J. Eng. Sci. **35**, 659–668 (1997)
15. Ma, H.S., Hu, G.K.: Eshelby tensors for an ellipsoidal inclusion in a micropolar material. Int. J. Eng. Sci. **44**, 595–605 (2006)
16. Liu, X.N., Hu, G.K.: Inclusion problem of microstretch continuum. Int. J. Eng. Sci. **42**, 849–860 (2004)
17. Kiris, A., Inan, E.: Eshelby tensors for a spherical inclusion in microstretch elastic fields. Int. J. Solids Struct. **43**, 4720–4738 (2006)
18. Zheng, Q.-S., Zhao, Z.-H.: Green's function and Eshelby's fields in couple-stress elasticity. Int. J. Multiscale Comput. Eng. **2**, 15–27 (2004)
19. Kiris, A., Inan, E.: Eshelby tensors for a spherical inclusion in microelongated elastic fields. Int. J. Eng. Sci. **43**, 48–58 (2005)
20. Zhang, X., Sharma, P.: Inclusions and inhomogeneities in strain gradient elasticity with couple stresses and related problems. Int. J. Solids Struct. **42**, 3833–3851 (2005)
21. Ma, H.S., Hu, G.K.: Eshelby tensors for an ellipsoidal inclusion in a microstretch material. Int. J. Solids Struct. **44**, 3049–3061 (2007)
22. Lakes, R.: Experimental methods for study of Cosserat elastic solids and other generalized elastic continua. In: Muhlhaus, H. (ed.) Continuum Models for Materials with Micro-Structure, pp. 1–22. Wiley, New York (1995)
23. Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J., Tong, P.: Experiments and theory in strain gradient elasticity. J. Mech. Phys. Solids **51**, 1477–1508 (2003)

24. Maranganti, R., Sharma, P.: A novel atomistic approach to determine strain-gradient elasticity constants: Tabulation and comparison for various metals, semiconductors, silica, polymers and the (ir) relevance for nanotechnologies. J. Mech. Phys. Solids **55**, 1823–1852 (2007)
25. Koiter, W.T.: Couple-stresses in the theory of elasticity: I and II. Proc. K. Ned. Akad. Wet. B **67**, 17–44 (1964)
26. Altan, B.S., Aifantis, E.C.: On some aspects in the special theory of gradient elasticity. J. Mech. Behav. Mater. **8**, 231–282 (1997)
27. Gao, X.-L., Park, S.K.: Variational formulation of a simplified strain gradient elasticity theory and its application to a pressurized thick-walled cylinder problem. Int. J. Solids Struct. **44**, 7486–7499 (2007)
28. Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity, 3rd edn. McGraw-Hill, New York (1970)
29. Polyzos, D., Tsepoura, K.G., Tsinopoulos, S.V., Beskos, D.E.: A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I. Integral formulation. Comput. Methods Appl. Mech. Eng. **192**, 2845–2873 (2003)
30. Li, S., Sauer, R.A., Wang, G.: The Eshelby tensors in a finite spherical domain. Part I. Theoretical formulations. ASME J. Appl. Mech. **74**, 770–783 (2007)