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Boundary-layer equations for a power-law shear-driven flow over a plane surface of non-Newtonian fluids

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Abstract We study the boundary-layer similarity flow driven over a semi-infinite flat plate by a power-law shear with asymptotic velocity profile $u_e(y) = \beta y^\alpha$ $(y \to \infty,\ \beta > 0)$, for fluids both Newtonian and non-Newtonian. Theoretical analysis is reported to derive a range of exponents α and amplitudes β for which similarity solutions exist. The shear stress parameter $f''(0)$ is determined as a function of α and β.

1. Introduction

The purpose of this paper is to present a mathematical analysis for similarity solutions describing laminar flows past a plane surface in a class of non-Newtonian fluids corresponding to an exterior power-law velocity profile of the form

$$u_e(y) = \beta y^\alpha \ (y \to \infty,\ \beta > 0). \tag{1.1}$$

Equation (1.1) is regarded in the following sense:

$$\lim_{y\to\infty} u(x, y) y^{-\alpha} = \beta, \tag{1.2}$$

where the positive x-coordinate is measured along the plate and the positive y-coordinate is measured normal to it, with $y = 0$ is the plate, the plate origin located at $x = y = 0$ and $u(x, y)$ is the x-velocity component.

The scaling (power-type) law (1.1), where α and β depend somehow on the flow Reynolds number R_e, is proposed by Barenblatt [1] for the mean velocity in fully developed turbulent shear flow. In [2], with the help of experimental data, Barenblatt and Prostokishin confirmed the conjecture proposed in [1] that

$$\alpha = \frac{3}{2 \ln R_e}.$$

The case $\alpha = 0$ coincides with the problem considered by Blasius in his pioneering work [3] for the Newtonian case. Blasius obtained a family of numerical solutions such that the Prandtl velocity profile, $u(x, y)/u_e$, depends only on a single variable $\eta = yx^{-\frac{1}{2}}$, where u_e is assumed to be constant. This results are extended by Falkner and Skan [4] to the class where u_e is required to vary with the x coordinate only, which is the essence of similarity reduction reported in a voluminous literature.

The problem considered here is structurally quite different from the more familiar Blasius and Falkner–Skan similarity reduction. Surprisingly, in 1987 an important observation was made by Weidman et al. [5]. They noticed that assumption (1.1), under some restriction on α, may underpin the simplification from the

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boundary layer approximation for the Newtonian case which motivated the present work. First the authors proved that no similarity solution exists for $\alpha \le -1$. From numerical results, they showed for an impermeable surface that similarity solutions only exist for $\alpha > -\frac{2}{3}$ and not for $\alpha \le -\frac{2}{3}$. Exact solutions were obtained for $\alpha = -\frac{1}{2}$ and $\alpha = 1$.

Recently, the adjustment of the flow over a permeable plate with blowing or suction has been investigated for the particular cases $\alpha = -\frac{2}{3}$ and $\alpha = -\frac{1}{2}$ by Magyari et al. [6].

The aim of the present work is to provide a rigorous mathematical justification for constructing similarity solutions satisfying (1.2) for a wide range of the power-law velocity profile. The absence of similarity solutions is also considered. We shall derive a range of values of α for which no similarity solutions can exist.

2. Scaling properties

Consider a flow of a fluid past an impermeable semi-infinite solid plate-or equally. The fluid can be Newtonian or not. One of the ways to describe the flow behavior of a non-Newtonian fluid is the Ostwald-de Waele power law model, where the shear stress is related to the strain rate $\partial u/\partial y$ by the expression [7–10]

$$\tau = K \left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y},$$

where K and n are positive constants. The parameter $n > 0$ is called the power-law index. The case $n < 1$ is referred to as pseudo-plastic fluids (or shear-thinning fluids), the case $n > 1$ is known as dilatant or shear-thickening fluids. The Newtonian fluid is, of course, a special case where the power-law index n is one.

Our starting point is the boundary-value problem (the coefficient of viscosity is assumed to be equal to unity)

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\left(\left|\dfrac{\partial u}{\partial y}\right|^{n-1} \dfrac{\partial u}{\partial y}\right), \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases} \tag{2.1}$$

where u and v are the velocity components. The above system is reduced to the classical Prandtl equation [10] when the fluid index, n, is equal to unity (Newtonian fluid).

System (2.1) must be solved subject to the non–slip and the asymptotic conditions

$$u(x,0) = 0, \quad v(x,0) = 0, \quad \text{and} \quad u(x,y) \to \beta y^{\alpha} \ (y \to \infty, \beta > 0), \tag{2.2}$$

where the exponent α is a negative real number (wall jet). In terms of the stream function $\psi(x,y)$; $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, we may write Eqs. (2.1) and (2.2) as follows:

$$\begin{cases} \dfrac{\partial \psi}{\partial y}\dfrac{\partial^2 \psi}{\partial x \partial y} - \dfrac{\partial \psi}{\partial x}\dfrac{\partial^2 \psi}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\left|\dfrac{\partial^2 \psi}{\partial y^2}\right|^{n-1} \dfrac{\partial^2 \psi}{\partial y^2}\right), \\ \psi(x,0) = 0, \quad \dfrac{\partial \psi}{\partial y}(x,0) = 0 \quad \text{and} \quad \dfrac{\partial \psi}{\partial y} \to \beta y^{\alpha} \ (y \to \infty). \end{cases} \tag{2.3}$$

We postulate a similarity solution of the form

$$\psi(x,y) = x^p f(\eta), \quad \eta = yx^{-q}, \tag{2.4}$$

where the exponents p and q will be determined in order to reduce the PDE in (2.3) to an ODE and by the boundary condition at infinity ($y \to \infty$). An easy computation shows that the *shape* function f satisfies

$$x^{np-q(2n+1)}\left(|f''|^{n-1}f''\right)' + x^{2(p-q)-1}\left(pff'' - (p-q)f'^2\right) = 0 \tag{2.5}$$

and

$$y^{-\alpha}\frac{\partial \psi}{\partial y}(x,y) = x^{p-q-\alpha q}\eta^{-\alpha} f'(\eta), \tag{2.6}$$

where primes indicate differentiations with respect to η. Equation (2.5) is an ordinary differential equation if and only if (scaling relation)

$$p(2-n)+q(2n-1)=1, \tag{2.7}$$

and the right-hand side of Eq. (2.6) is a function of η only if and only if

$$p=q(1+\alpha). \tag{2.8}$$

The system (2.7), (2.8) has a unique solution (p,q) for $\alpha(n-2)\neq n+1$. In this case, we get

$$p=\frac{1+\alpha}{1+n+\alpha(2-n)},\quad q=\frac{1}{1+n+\alpha(2-n)}. \tag{2.9}$$

Hence the governing similarity equation is

$$\left(|f''|^{n-1}f''\right)'+pff''-(p-q)f'^2=0,\quad p-q=\frac{\alpha}{1+n+\alpha(2-n)}, \tag{2.10}$$

to be solved with boundary conditions

$$f(0)=0,\quad f'(0)=0, \tag{2.11}$$

and

$$f'(\eta)\to\beta\eta^{\alpha}\quad \text{as } \eta\to\infty. \tag{2.12}$$

The purpose of the present paper is to obtain a range of values of α for which solutions to (2.10)–(2.12) exist. One can understand from Eq. (2.9) that the value $\alpha=\frac{n+1}{n-2}$ is critical one if $n\neq 2$.

3. Nonexistence results for some negative values of α

The objective of this section is to show that there exists a value of α, say α_c, such that for $\alpha<\alpha_c$ solutions to (2.10)–(2.12) cannot exist. We shall assume first that $2p-q>0$ and $\alpha<-\frac{1}{2}$. This condition is satisfied if

$$\alpha<\frac{n+1}{n-2}\quad\text{and}\quad 0<n<2. \tag{3.1}$$

As in [5] the nonexistence result will be established by analyzing the large η−behavior of solutions to (2.10) or the following

$$|f''(\eta)|^{n-1}f''(\eta)+pf(\eta)f'(\eta)=|f''(0)|^{n-1}f''(0)+(2p-q)\int_0^{\eta}f'(s)^2\mathrm{d}s,\quad \forall\eta>0, \tag{3.2}$$

which is obtained from Eq. (2.10) by a simple integration. Note that assumption (3.1) ensures that $\lim_{\eta\to\infty}f(\eta)f'(\eta)=0$. Since $f'(\eta)$ goes to 0 at infinity, we deduce that there exists a sequence (η_r) such that η_r tends to infinity with r and $\lim_{r\to\infty}f''(\eta_r)=0$. Together with Eq. (3.2) one sees

$$|f''(0)|^{n-1}f''(0)+(2p-q)\int_0^{\infty}f'(s)^2\mathrm{d}s=0.$$

Consequently, $f''(0)<0$ and then $f'(\eta)<0$ for all $0<\eta<\eta_0$, η_0 small. Because $f'(\eta)\to\beta\eta^{\alpha}$ as $\eta\to\infty$, we have $f'>0$ on (η_1,∞), η_1 large. Hence, we may conclude that there exists $\eta_0<\eta_2<\eta_1$ such that $f'(\eta_2)=0$ and $f''(\eta_2)\geq 0$. Using again Eq. (3.2) we get

$$|f''(\eta)|^{n-1}f''(\eta)+pf(\eta)f'(\eta)=f''(\eta_2)^n+(2p-q)\int_{\eta_2}^{\eta}f'(s)^2\mathrm{d}s,\quad \forall\eta>\eta_2,$$

which leads to

$$f''(\eta_2)^n + (2p-q)\int\limits_{\eta_2}^{\infty} f'(s)^2 \mathrm{d}s = 0.$$

Thus $f'(\eta) = 0$ for all $\eta \geq \eta_2$. And this contradicts the physical requirement at infinity.

The above argument demonstrates, in particular, that $\alpha_c \geq \frac{n+1}{n-2}$ for $n < 2$.

Suppose on the other hand that

$$\frac{n+1}{n-2} < \alpha < -1, \quad \frac{1}{2} < n < 1. \tag{3.3}$$

In this case we write

$$f = f(\infty) + \varphi,$$

where the real number $f(\infty)$ (the volume flux) is finite and φ and its all derivatives tend to zero as η goes to infinity. Thus, (2.10) is replaced by

$$\left(|\varphi''|^{n-1}\varphi''\right)' = -p(f(\infty)+\varphi)\varphi'' + (q-p)\varphi'^2.$$

Without loss of generality we may assume that φ'' is negative on some (η_0, ∞) (see Sect. 4). Recall that $\varphi'(\eta) \to \beta\eta^\alpha,\ \alpha < -1$, for $\eta \to \infty$. Therefore,

$$\varphi''' \sim \frac{p}{n} f(\infty)(-\varphi'')^{2-n}$$

for $\eta \to \infty$. We then have

$$\varphi'(\eta) \sim \frac{1-n}{n}\left(\frac{p(1-n)}{n} f(\infty)\right)^{\frac{1}{n-1}} \eta^{\frac{n}{n-1}} \tag{3.4}$$

as $\eta \to \infty$. This algebraic decay breaks down since the exponent in (3.4) is not in accord with (3.3). According to the above results we may conclude that $\alpha_c \geq -1$ for shear-thinning fluids ($n < 1$).

The following short analysis may be used to show that $-1 < \alpha_c$. The analysis deals with explicit solutions to Eqs. (2.10)–(2.12) for $\alpha = -1$. In this situation we note that $p = 0, n \neq 1/2$ and $q = \frac{1}{2n-1}$ and consider the ODE satisfied by $g = f'$; that is

$$\left(|g'|^{n-1}g'\right)' + \frac{1}{2n-1}g^2 = 0, \quad g(\eta) \to \beta\eta^{-1} \quad \text{for } \eta \to \infty. \tag{3.5}$$

One readily verifies that

$$\frac{n}{n+1}|g'|^{n+1} + \frac{1}{3(2n-1)}g^3 = 0,$$

which has only the trivial solution $g = 0$ if $n > \frac{1}{2}$ and, for $0 < n < \frac{1}{2}$, in addition to the trivial solution,

$$g(\eta) = \left\{c + \frac{n+1}{2-n}\left[\frac{n+1}{3n(1-2n)}\right]\eta\right\}^{\frac{n+1}{n-2}}, \tag{3.6}$$

where c is an arbitrary nonnegative constant. Equation (3.6) is incompatible with the required asymptotic behavior $g(\eta) \to \beta\eta^{-1}$ for $\eta \to \infty$.

4. Similarity solutions of the problem for $-\frac{1}{2} \leq \alpha < 0$

We now turn our attention to the existence of solutions of the two-point boundary-value problem (2.10)–(2.12), where $-\frac{1}{2} \leq \alpha < 0, n > 0$ and p is defined by Eq. (2.9). We start with the case $-\frac{1}{2} < \alpha < 0$. The problem will be solved via a standard shooting method. So, we consider the initial value problem

$$\begin{cases} (|f''|^{n-1}f'')' + \dfrac{1+\alpha}{1+n+\alpha(2-n)}ff'' - \dfrac{\alpha}{1+n+\alpha(2-n)}f'^2 = 0, \quad \eta > 0, \\ f(0) = 0, \ f'(0) = 0, \ f''(0) = \tau, \end{cases} \tag{4.1}$$

where τ is the shooting parameter (shear stress parameter) which has to be determined.

By usual theorems problem (4.1) has a unique solution, f_τ, defined over the maximal interval of existence $(0, \eta_\tau)$, $\eta_\tau \leq \infty$. To obtain a solution to (2.10)–(2.12) we will investigate whether f_τ admits an entire extension and satisfies the algebraic decay. To this end we take $\tau > 0$. Recall that f_τ satisfies (3.2); that is

$$|f_\tau''(\eta)|^{n-1}f_\tau''(\eta) + \frac{1+\alpha}{1+n+\alpha(2-n)}f_\tau(\eta)f_\tau'(\eta) = \tau^n + \frac{1+2\alpha}{1+n+\alpha(2-n)}\int_0^\eta f_\tau'(s)^2 \mathrm{d}s \tag{4.2}$$

for all $0 < \eta < \eta_\tau$. Clearly f_τ' and f_τ'' cannot vanish at the same point and f_τ cannot have a local maximum. Thus, since τ is positive, f_τ is positive and monotonic strictly increasing. To show that f_τ is global ($\eta_\tau = \infty$) we observe from the ODE that

$$\frac{\mathrm{d}E}{\mathrm{d}\eta} = -\frac{1+\alpha}{1+n+\alpha(2-n)}ff''^2,$$

which is nonpositive, where

$$E = \frac{n}{n+1}|f''|^{n+1} - \frac{\alpha}{3(1+n+\alpha(2-n))}f'^3 \tag{4.3}$$

is the fundamental "energy" function for Eq. (2.10).

Hence

$$\frac{n}{n+1}|f_\tau''(\eta)|^{n+1} + \frac{|\alpha|}{3(1+n+\alpha(2-n))}f_\tau'(\eta)^3 \leq \frac{n}{n+1}\tau^{n+1}, \quad \forall\, \eta < \eta_\tau.$$

This in turn infers that $f_\tau''(\eta)$, $f_\tau'(\eta)$ and then $f_\tau(\eta)$ are bounded in any interval $(0, d)$, $d > 0$. Because of the arbitrariness of d, f_τ is global.

The task remaining is therefore to study the asymptotic behavior of f_τ'. First, we shall see that $f_\tau''(\eta)$ is negative for large η and that $f_\tau'(\eta)$ tends to zero at infinity. We assume for the sake of contradiction that f_τ'' is nonnegative on $(0, \infty)$. Because f_τ' is monotonically increasing and bounded there exists a real number $l > 0$ such that

$$\lim_{\eta\to\infty} f_\tau'(\eta) = l.$$

Moreover, there exists a sequence (η_r) tending to ∞ with r such that $\lim_{r\to\infty} f_\tau''(\eta_r) = 0$. It follows that E tends to $\dfrac{|\alpha|}{3(1+n+\alpha(2-n))}l^3$ as η approaches infinity. Thus $f_\tau''(\eta) \to 0$ as $\eta \to \infty$. Together with identity (4.2) we get, as η approaches infinity,

$$f_\tau''(\eta)^n = -\frac{1+\alpha}{1+n+\alpha(2-n)}l^2\eta + \frac{1+2\alpha}{1+n+\alpha(2-n)}l^2\eta + o(\eta),$$
$$f_\tau''(\eta)^n = \frac{\alpha}{1+n+\alpha(2-n)}l^2\eta + o(\eta)$$

as $\eta \to \infty$. This is only possible if $\alpha = 0$. This gives a contradiction. So, there hold $f_\tau''(\eta_0) = 0$, $f_\tau'' < 0$ on (η_0, η_1), for some $0 < \eta_0 < \eta_1$. Furthermore, the function f_τ satisfies

$$f_\tau''' + \frac{p}{n}f_\tau|f_\tau''|^{1-n}f_\tau'' - \frac{p-q}{n}|f_\tau''|^{1-n}f_\tau'^2 = 0$$

on (η_0, η_1). Hence

$$\left(f_\tau'' e^F\right)' = \frac{\alpha}{n(1+n+\alpha(2-n))} e^F f_\tau'^2 |f_\tau''|^{1-n}, \quad \forall\, \eta_0 < \eta < \eta_1, \tag{4.4}$$

where F is any anti-derivative of $\frac{p}{n} f_\tau |f_\tau''|^{1-n}$. So, the function $f_\tau'' e^F$ decreases and then we must have $f_\tau''(\eta) < 0$ for all $\eta > \eta_0$. Next, an argument similar to those we have previously performed reveals that $f'(\eta)$ tends to 0 as η goes to infinity. We are now in a position to give the large η behavior of f_τ'. The argument is as follows. First, observe that it may be shown, from the identity (4.2) and the ODE, that $f_\tau(\eta) f_\tau'(\eta)$ tends to infinity with η and $f_\tau'''(\eta)$ is positive for any $\eta \geq \eta_2$, η_2 large. Next, we divide the ODE by $f_\tau f_\tau'$:

$$\frac{(|f_\tau''|^{n-1} f_\tau'')'}{f_\tau f_\tau'} = (p-q)\frac{f_\tau'}{f_\tau} - p\frac{f_\tau''}{f_\tau'}.$$

Integrating over (η_3, η), η_3 large, gives

$$\int_{\eta_3}^{\eta} \frac{(|f_\tau''|^{n-1} f_\tau'')'}{f_\tau f_\tau'} \mathrm{d}s = \log\left(f_\tau^{p-q}(\eta) f_\tau'^{-p}(\eta)\right) - \log\left(f_\tau^{p-q}(\eta_3) f_\tau'^{-p}(\eta_3)\right). \tag{4.5}$$

Since $f_\tau f_\tau'$ tends to infinity with η the LHS is integrable up to infinity. Letting $\eta \to \infty$ we see that $\log\left(f_\tau^{p-q}(\eta) f_\tau'^{-p}(\eta)\right)$ tends to a finite limit. Thus, $f_\tau^{p-q}(\eta) f_\tau'^{-p}(\eta)$ has a finite limit and this limit must be positive. Therefore, there exists $A_\tau > 0$ such that $f_\tau(\eta) \to A_\tau \eta^{\frac{p}{q}}$, as $\eta \to \infty$ and this implies

$$f_\tau(\eta) \to A_\tau \eta^{1+\alpha} \quad \text{and} \quad f_\tau'(\eta) \to \beta_\tau \eta^\alpha$$

as $\eta \to \infty$, for some positive real number β_τ. Finally, since the ODE is invariant under the transformation $\eta \to a f(b\eta)$, where $a^{n-2} b^{2n-1} = 1$ we deduce that the new function f given by

$$f(\eta) = a^{-1} f_\tau\left(b^{-1}\eta\right), \quad a = \left\{\frac{\beta_\tau}{\beta}\right\}^{\frac{2n-1}{1+n+\alpha(2-n)}}, \quad b = \left\{\frac{\beta_\tau}{\beta}\right\}^{\frac{2-n}{1+n+\alpha(2-n)}} \tag{4.6}$$

is a solution to Eqs. (2.10)–(2.12).

Equation (4.6) can also be used to exhibit the wall shear stress parameter for wall-bounded flows in terms of α and β,

$$f''(0) = \tau \left\{\frac{\beta}{\beta_\tau}\right\}^{\frac{3}{1+n+\alpha(2-n)}}.$$

Herewith, the existence result of a similarity solution to (2.3) is justified for $-\frac{1}{2} < \alpha < 0$. We note that there exists a unique point η_m such that $f'(\eta_m) = f_m'$ is the maximum of f' (the jet velocity). The point η_m is the solution of equation $f''(\eta_m) = 0$. Using the energy function E it can be shown that

$$f_m'^{\,3} \leq \tau^{n+1} \frac{n(1+n+\alpha(2-n))}{(n+1)|\alpha|} \left\{\frac{\beta}{\beta_\tau}\right\}^{\frac{3(n+1)}{1+n+\alpha(2-n)}}.$$

It is also worth noticing that at the point η_m the function f''' is unbounded if $n > 1$.

The case $\alpha = -\frac{1}{2}$ is easy to analyze. In this case we infer

$$|f''|^{n-1} f'' + \frac{1}{3n} ff' = |f''(0)|^{n-1} f''(0).$$

Therefore if $f''(0)$ is positive the solution f is global unbounded and satisfies

$$f'(\eta) \to \beta \eta^{-\frac{1}{2}}$$

for all $n > 0$, where $\beta = \sqrt{\frac{3n}{2}} f''(0)^{n/2}$.

5. Conclusion

Similarity solutions of the Prandtl boundary layer equations describing a power-law shear driven flow over a plane surface with the asymptotic velocity profile $u_e(y) = \beta y^\alpha$ ($y \to \infty$, $\beta > 0$) have been investigated. The fluid can be Newtonian or not (Ostwald-de Waele). The search for solutions introduces two parameters: α and n, the power-law index. Using a shooting argument, it is shown that there are analytical solutions for any $n > 0$ and any $\alpha \in (-\frac{1}{2}, 0)$. The shear stress $f''(0)$ parameter is determined as a function of α, β and n. The analysis reveals that f' has exactly one maximum value (jet velocity) which decreases monotonically, with decreasing β and the wall shear stress parameter is a power function of β with the positive exponent $\dfrac{3}{1+n+\alpha(2-n)}$.

For $\alpha = -\frac{1}{2}$, elementary integrations show that solution exists and that the wall shear stress (skin friction) parameter satisfies

$$f''(0) = \left(\frac{2n}{3}\right)\beta^{\frac{2}{n}},$$

which coincides with the expression of $f''(0)$ found by Weidman et al. [5] for the Newtonian case.

A theoretical analysis is reported for $\alpha < \frac{n+1}{n-2}$, where $0 < n < 2$, in which case no similarity solutions exist, while for shear-thinning and Newtonian fluids similarity solutions cannot exist for $\alpha \leq -1$. The absence of similarity solutions for $\alpha = -1$ and $n > 0$ was illustrated by explicit solutions to the ODE.

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