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R. Bustamante

Transversely isotropic non-linear electro-active elastomers

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Abstract Electro-active or electro-sensitive elastomers are 'smart materials', which are composed of a rubber-like basis material filled with electro-active particles, and as a result, their properties are able to change significantly by the application of electric fields. In this paper, we provide the theoretical basis of the non-linear properties for a special class of these materials, namely, the transversely isotropic electro-active elastomers, whose characteristic is that during the curing process, due to the presence of an external applied field, the electro-active particles are aligned in a preferred direction. The theory is applied to some boundary value problems. As well as this, a linear approximation is obtained from the general non-linear formulation, which is compared with the results of the classical theory for piezoelectric materials.

1 Introduction

There has been recently a growing interest in a class of rubber-like materials, called electro-active and magneto-active elastomers.[1] These materials are composed of a rubber-like matrix filled with electro-active or magneto-active particles. They are capable of developing large non-linear elastic deformations by applying an electric or a magnetic field. Due to these characteristics, they have been used in applications where we need a quick response in the properties of the material (see, for example, [1,2]).

It has been shown [3–5] that if an external field is applied during the curing process, then the particles align in a preferred direction, and as a result the capability of these materials to deform, in the presence of an external electric or magnetic field, is enhanced significantly in comparison with the same kind of material but with a random distribution of particles.

The first theoretical study of large elastic deformations due to the presence of an electric field was done by Toupin [6], who used the virtual work principle to find the general equations that relate the stress, deformation and the electric field inside a highly non-linear elastic dielectric body. There is not a unique way to write the fundamental equations in the theory of deformable media and electromagnetic fields, a complete review and comparison of the different theories can be found, for example, in [7,8]. Modern treatises on the interaction of electromagnetism and mechanics have been written, for example, by Eringen and Maugin [9] and Kovetz [10].

Dorfmann and Ogden [11] developed a theory for non-linear electro-active solids, which does not take the polarization as the main independent electric variable to characterize the behaviour of the material; instead, they developed the constitutive equations assuming either the electric field or the electric displacement as the independent electric variable. With this assumption and the assumption of existence of an energy function,

[1] These materials are denoted also as ES and MS elastomers. These are abbreviations for the phrases 'electro-sensitive' and 'magneto-sensitive' elastomers respectively.

R. Bustamante (✉)
Departamento de Ingeniería Mecánica, Universidad de Chile, Santiago, Chile
E-mail: rogbusta@ing.uchile.cl
Tel.: +56-2-9784597

Dorfmann and Ogden developed simple expressions for the stress and the electric displacement (or electric field), which are obtained as simple derivatives of this energy potential.

Dorfmann and Ogden have been working with isotropic electro- or magneto-active elastomers, which basically implies the assumption of a random distribution of particles inside the rubber-like matrix material [11–14]. For this particular case, several boundary value problems have been solved. The complete set of controllable or universal solutions is also available [15].

The situation is not the same for the transversely isotropic electro-active elastomers. The general theory developed, for example, by Eringen and Maugin [9], or by Dorfmann and Ogden [11,14] are good starting points to study this problem.

Using as basis the work by Dorfmann and Ogden [11], we develop the constitutive equations for transversely isotropic ES elastomers; a similar theory for transversely isotropic MS elastomer has been developed recently by Bustamante and Ogden [16].

In Sect. 2, we have a short review of the theory for ES elastomers of Dorfmann and Ogden [11,14].

In Sect. 3.1, we study the form of the constitutive equations using the electric field as the independent electric variable. In Sect. 3.2 an equivalent set of equations is found assuming the electric-displacement as the independent electric variable.

Most of the researches on electro-elasticity have been focused on the linear theory, which means the assumption of small deformations, displacements and electric field. Derivations of linear theories from the non-linear formulation can be found, for example, in [9,17]. This process of linear approximation has been formulated starting from the general expressions for the stress and the independent electric variable as derivatives of the energy function. In Sect. 4, we obtain a linear approximation in a rather different way. First, for the full non-linear formulation, we compute the stress and the independent electric variable as functions of the invariants, then we approximate the expressions assuming small deformation and fields. We obtain the same kind of linear constitutive equation as, for example, for some well-known piezoelectric materials such as certain polarized ceramics (see, for example, [17]).

In Sect. 5, we study some boundary value problems: the simple shear of a slab, the uniform extension of a bar, the extension and inflation of a tube, the extension and torsion of a tube and helical shear. For some of these boundary value problems, we study the effect of assuming different alignments for the particles in the reference configuration on the controllability of the solutions. In the particular case of helical shear [18] we are interested in finding a non-linear universal relation, similar to the one found for the isotropic case by Bustamante and Ogden [19].

In Sect. 6, we give some final remarks.

This paper is based on Chapter 8 of the Ph.D. thesis by Bustamante [20].

2 Basic equations

2.1 Kinematics

The reference configuration is denoted as $\mathcal{B}_0$ and a material particle is labelled by its position vector $\mathbf{X}$. Let $\mathcal{B}$ denote the deformed configuration in which the particle $\mathbf{X}$ has position vector $\mathbf{x}$ defined by the mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. The boundary of the body in the reference configuration is denoted as $\partial\mathcal{B}_0$, and in the deformed configuration is denoted as $\partial\mathcal{B}$. The deformation gradient $\mathbf{F}$ is defined as

$$\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}, \tag{1}$$

where the operator Grad is the gradient in $\mathcal{B}_0$. Let us denote $J = \det \mathbf{F}$ and by convention we take $J > 0$.

The right and left Cauchy Green deformation tensors are given as

$$\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{2}$$

Finally, let us define the displacement field $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{3}$$

2.2 Electric balance equations

In the current configuration $\mathcal{B}$ we denote by $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$ the electric field, the electric induction and the polarization density, respectively. For condensed matter, these vectors are related by the standard equation [10]

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}, \tag{4}$$

where ε_o is the electric permittivity for vacuum (or free space).

For free space $\mathbf{P} = 0$ and we have

$$\mathbf{D} = \varepsilon_o \mathbf{E}. \tag{5}$$

For the quasi-static case and in the absence of magnetic fields, free currents and electric charges, the electric field and the electric displacement satisfy the simplified form of Maxwell's equations

$$\operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0, \tag{6.1,2}$$

where the operators curl and div are defined in $\mathcal{B}$.

From the global forms of (6.1,2) we can find the Lagrangian counterparts of the electric field and the electric induction, which we denote $\mathbf{E}_l$ and $\mathbf{D}_l$, respectively, and are given by [11]

$$\mathbf{E}_l = \mathbf{F}^{\mathrm{T}} \mathbf{E}, \quad \mathbf{D}_l = J \mathbf{F}^{-1} \mathbf{D}. \tag{7.1,2}$$

Standard identities ensure that (6.1,2) are equivalent to

$$\operatorname{Curl} \mathbf{E}_l = \mathbf{0}, \quad \operatorname{Div} \mathbf{D}_l = 0, \tag{8}$$

provided that $\boldsymbol{\chi}$ is suitably regular, where Curl and Div are the curl and divergence operators in $\mathcal{B}_0$.

For the polarization density $\mathbf{P}$ there is no natural way to define a corresponding Lagrangian counterpart. For convenience we define a Lagrangian form of $\mathbf{P}$, which we denote $\mathbf{P}_l$ as

$$\mathbf{P}_l = J \mathbf{F}^{-1} \mathbf{P}. \tag{9}$$

Therefore, from (4) and (7.1,2), we obtain

$$\mathbf{D}_l = \varepsilon_o J \mathbf{c}^{-1} \mathbf{E}_l + \mathbf{P}_l. \tag{10}$$

More details about the above expressions can be found in [11].

2.3 Mechanical balance laws

If we denote by ρ_0 and ρ the mass densities in the reference and current configurations, respectively, then the conservation of mass equation is given by

$$J\rho = \rho_0. \tag{11}$$

We can write the electric body forces as the divergence of a second-order tensor (see, for example, [8]), and add this tensor to the Cauchy stress tensor to define a 'total (Cauchy) stress tensor' (see, for example, [11,21]), which we denote $\boldsymbol{\tau}$. In such a case, the equilibrium equation in the absence of mechanical body forces can be written in the form

$$\operatorname{div} \boldsymbol{\tau} = \mathbf{0}. \tag{12}$$

Since the electric body forces and couples are included in the definition of the stress, then from the balance of angular momentum we have $\boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}$.

The counterpart of the nominal stress tensor is denoted here by $\mathbf{T}$, and is defined by

$$\mathbf{T} = J \mathbf{F}^{-1} \boldsymbol{\tau}, \tag{13}$$

and the equilibrium equation (12) can be written in the alternative form

$$\operatorname{Div} \mathbf{T} = \mathbf{0}. \tag{14}$$

2.4 Boundary conditions

We assume the body $\mathcal{B}$ completely surrounded by an infinite free (vacuum) space.

In the deformed configuration, in the absence of surface charge, the standard continuity conditions for the electric field and the electric displacement through $\partial\mathcal{B}$ are

$$\mathbf{n}\cdot[\![\mathbf{D}]\!]=0, \quad \mathbf{n}\times[\![\mathbf{E}]\!]=\mathbf{0}, \tag{15}$$

where the double square brackets indicates discontinuity across the surface and $\mathbf{n}$ is the unit normal outward to the surface. The Lagrangian form of these equations is

$$\mathbf{N}\cdot[\![\mathbf{D}_l]\!]=0, \quad \mathbf{N}\times[\![\mathbf{E}_l]\!]=\mathbf{0}, \tag{16}$$

where $\mathbf{N}$ is the normal vector in the reference configuration.

The boundary condition for the total stress in the current configuration is given by

$$[\![\boldsymbol{\tau}]\!]\mathbf{n}=\mathbf{0}, \tag{17}$$

where the Maxwell stress, denoted by $\boldsymbol{\tau}_m$, must be accounted as an external surface load [21]. In vacuum ($\mathbf{D}=\varepsilon_o\mathbf{E}$), this Maxwell stress is given by [22]

$$\boldsymbol{\tau}_m=\mathbf{D}\otimes\mathbf{E}-\varepsilon_o\frac{1}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{I}, \tag{18}$$

where $\mathbf{I}$ is the identity tensor.

2.5 Constitutive equations

Following Dorfmann and Ogden [11], we assume the existence of a free energy function Φ, which depends on the deformation gradient $\mathbf{F}$ and the electric field or the electric displacement. In the case, we choose the electric field as the independent electric variable and have

$$\Phi=\Phi(\mathbf{F},\mathbf{E}_l). \tag{19}$$

It can be proved that the total stress and the polarization are given by (see, for example, [11])

$$\boldsymbol{\tau}=\rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{F}}+\boldsymbol{\tau}_m, \quad \mathbf{P}=-\rho\mathbf{F}\frac{\partial\Phi}{\partial\mathbf{E}_l}, \tag{20.1,2}$$

where $\boldsymbol{\tau}_m$ is given by Eq. (18).

Dorfmann and Ogden defined the amended (or total) free energy function $\Omega=\Omega(\mathbf{F},\mathbf{E}_l)$ by [11,13,14]

$$\Omega=\rho_0\Phi-\frac{1}{2}\varepsilon_o J\mathbf{E}_l\cdot(\mathbf{c}^{-1}\mathbf{E}_l). \tag{21}$$

From Eqs. (20.1), (13) and (21) we can obtain the simple forms for the stresses

$$\mathbf{T}=\frac{\partial\Omega}{\partial\mathbf{F}} \tag{22}$$

and

$$\boldsymbol{\tau}=J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{F}}. \tag{23}$$

From Eq. (20.2), with Eqs. (7.2), (9) and (10) we have

$$\mathbf{D}_l=-\frac{\partial\Omega}{\partial\mathbf{E}_l} \tag{24}$$

and

$$\mathbf{D}=-J^{-1}\mathbf{F}\frac{\partial\Omega}{\partial\mathbf{E}_l}. \tag{25}$$

The expressions listed above require some modifications in the case of incompressible materials, which are subjected to the constraint

$$J = \det \mathbf{F} = 1. \tag{26}$$

In this case, we have

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \quad \mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{D} = -\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{E}_l}, \tag{27.1–3}$$

where p is a Lagrange multiplier associated with the constraint.

3 Constitutive equations for transversely isotropic ES materials

3.1 The electric field as the independent electric variable

Consider the case of a transversely isotropic electro-elastic solid. The energy function is given as

$$\Omega = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0), \quad |\mathbf{a}_0| = 1, \tag{28}$$

where $\mathbf{a}_0$ is a field that represents the particular alignment of the electro-active particles in the reference configuration. In the current configuration, we have

$$\mathbf{a} = \mathbf{F}\mathbf{a}_0. \tag{29}$$

For the energy function $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0)$ we have that Ω depends on the following set of invariants[2] [23,24]

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det \mathbf{c} = J^2, \tag{30.1,2}$$

$$I_4 = \mathbf{E}_l \cdot \mathbf{E}_l, \quad I_5 = \mathbf{E}_l \cdot \mathbf{c}\mathbf{E}_l, \quad I_6 = \mathbf{E}_l \cdot \mathbf{c}^2\mathbf{E}_l, \tag{31}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \quad I_9 = \mathbf{a}_0 \cdot \mathbf{E}_l, \quad I_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{E}_l. \tag{32}$$

Then $\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10})$. Consider the derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{F}^{\mathrm{T}}), \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{33}$$

$$\frac{\partial I_5}{\partial \mathbf{F}} = 2\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{F}} = 2(\mathbf{E}_l \otimes \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{E}_l) + \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{E}_l \otimes \mathbf{F}\mathbf{E}_l, \tag{34}$$

$$\frac{\partial I_7}{\partial \mathbf{F}} = 2\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{F}} = 2(\mathbf{a}_0 \otimes \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{a}_0 + \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{a}_0 \otimes \mathbf{F}\mathbf{a}_0), \tag{35}$$

$$\frac{\partial I_{10}}{\partial \mathbf{F}} = \mathbf{a}_0 \otimes \mathbf{F}\mathbf{E}_l + \mathbf{E}_l \otimes \mathbf{F}\mathbf{a}_0. \tag{36}$$

Using the chain rule and these derivatives in (23) the total stress tensor is given as[3]

$$\begin{aligned}\boldsymbol{\tau} = {} & J^{-1}[2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 + 2I_3\mathbf{I}\Omega_3 + 2\mathbf{b}\mathbf{E} \otimes \mathbf{b}\mathbf{E}\Omega_5 \\ & + 2(\mathbf{b}\mathbf{E} \otimes \mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E} \otimes \mathbf{b}\mathbf{E})\Omega_6 + 2\mathbf{a} \otimes \mathbf{a}\Omega_7 + 2(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a})\Omega_8 \\ & + (\mathbf{a} \otimes \mathbf{b}\mathbf{E} + \mathbf{b}\mathbf{E} \otimes \mathbf{a})\Omega_{10}].\end{aligned} \tag{37}$$

[2] There is an error in Zheng's paper on the theory of invariants [24]. According to Zheng for this problem, where we have one tensor field and two vectors fields as arguments in the energy function, we would need to work with 11 invariants, where the extra invariant would be $\mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{E}_l$. However, it can be proved (see, for example, the Appendix B of the thesis by Bustamante [20]) that this invariant is not independent of the rest of the invariants of the list (30.1,2)–(32).

[3] We use the notation $\Omega_i = \frac{\partial \Omega}{\partial I_i}$ for $i = 1, \ldots, 10$.

From Eq. (27.1) for an incompressible material we have

$$\boldsymbol{\tau} = 2\mathbf{b}\Omega_1 + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2 - p\mathbf{I} + 2\mathbf{bE}\otimes\mathbf{bE}\Omega_5 + 2(\mathbf{bE}\otimes\mathbf{b}^2\mathbf{E} + \mathbf{b}^2\mathbf{E}\otimes\mathbf{bE})\Omega_6$$
$$+2\mathbf{a}\otimes\mathbf{a}\Omega_7 + 2(\mathbf{a}\otimes\mathbf{ba} + \mathbf{ba}\otimes\mathbf{a})\Omega_8 + (\mathbf{a}\otimes\mathbf{bE} + \mathbf{bE}\otimes\mathbf{a})\Omega_{10}. \quad (38)$$

Consider the derivatives

$$\frac{\partial I_4}{\partial \mathbf{E}_l} = 2\mathbf{E}_l, \quad \frac{\partial I_5}{\partial \mathbf{E}_l} = 2\mathbf{c}\mathbf{E}_l, \quad \frac{\partial I_6}{\partial \mathbf{E}_l} = 2\mathbf{c}^2\mathbf{E}_l, \quad \frac{\partial I_9}{\partial \mathbf{E}_l} = \mathbf{a}_0, \quad \frac{\partial I_{10}}{\partial \mathbf{E}_l} = \mathbf{c}\mathbf{a}_0. \quad (39)$$

From Eq. (25) using the chain rule we have

$$\mathbf{D} = -J^{-1}(2\mathbf{bE}\Omega_4 + 2\mathbf{b}^2\mathbf{E}\Omega_5 + 2\mathbf{b}^3\mathbf{E}\Omega_6 + \mathbf{a}\Omega_9 + \mathbf{ba}\Omega_{10}). \quad (40)$$

For an incompressible material, Eq. (27.3) becomes

$$\mathbf{D} = -(2\mathbf{bE}\Omega_4 + 2\mathbf{b}^2\mathbf{E}\Omega_5 + 2\mathbf{b}^3\mathbf{E}\Omega_6 + \mathbf{a}\Omega_9 + \mathbf{ba}\Omega_{10}). \quad (41)$$

Some restrictions on the energy function Ω can be obtained by considering the undeformed state. If for the undeformed state with no external electric field, there is no residual stresses and no residual polarization, then we have

$$\boldsymbol{\tau} = \mathbf{0}, \quad \mathbf{D} = \mathbf{0}. \quad (42)$$

In this case the invariants (30.1,2)–(32) are given by

$$I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_5 = I_6 = I_9 = I_{10} = 0, \quad I_7 = I_8 = \mathbf{a}_0\cdot\mathbf{a}_0. \quad (43)$$

Let $\bar{\Omega}_i$ denote the function Ω_i evaluated with the above values for the invariants. Remembering that $\mathbf{F} = \mathbf{I}$, $\mathbf{E}_l = \mathbf{0}$ and $\mathbf{a} = \mathbf{F}\mathbf{a}_0 = \mathbf{a}_0$, then Eqs. (37) and (40) become

$$\boldsymbol{\tau} = 2(\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3)\mathbf{I} + 2(\bar{\Omega}_7 + 2\bar{\Omega}_8)\mathbf{a}_0\otimes\mathbf{a}_0, \quad (44)$$
$$\mathbf{D} = -(\bar{\Omega}_9 + \bar{\Omega}_{10})\mathbf{a}_0, \quad (45)$$

and in view of Eqs. (42) we need

$$\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3 = 0, \quad \bar{\Omega}_7 + 2\bar{\Omega}_8 = 0, \quad \bar{\Omega}_9 + \bar{\Omega}_{10} = 0. \quad (46.1\text{–}3)$$

In the incompressible case, Eq. (46.1), should be replaced by

$$2\bar{\Omega}_1 + 4\bar{\Omega}_2 - p = 0. \quad (47)$$

Piezoelectric materials produce polarization when deformed, even in the case there is no external field [17,25]. The reason why some materials like quartz produces a polarization field when deformed has to do with its particular atomic structure; a deformation produces an asymmetric arrangement of charges creating this polarization field. In general, we cannot expect the same phenomenon for transversely isotropic ES materials.

Consider the case when there is deformation but no applied external field, in such a case if $\mathbf{E} = \mathbf{0}$ we have the extra restriction $\mathbf{D} = \mathbf{0}$. Let $\breve{\Omega}_i$ denote the function Ω_i evaluated for $I_4 = I_5 = I_6 = I_9 = I_{10} = 0$ (these values for the invariants are consequence of $\mathbf{E} = 0$). With $\mathbf{D} = \mathbf{0}$ from Eq. (40) we have the restriction

$$\mathbf{I}\breve{\Omega}_9 + \mathbf{b}\breve{\Omega}_{10} = \mathbf{0}, \quad (48)$$

which, if $\mathbf{b} \neq \mathbf{I}$, implies

$$\breve{\Omega}_9 = \breve{\Omega}_{10} = 0. \quad (49)$$

We easily see that if Eq. (49) holds so does (46.3).

3.2 The electric displacement as the independent electric variable

If we choose to work with $\mathbf{D}_l$ as the independent variable, then by defining the energy potential Ω^* by using the partial Legrende transform

$$\Omega^*(\mathbf{F}, \mathbf{D}_l, \mathbf{a}_0) = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0) + \mathbf{D}_l \cdot \mathbf{E}_l, \tag{50}$$

it follows that

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}}, \quad \mathbf{E}_l = \frac{\partial \Omega^*}{\partial \mathbf{D}_l}. \tag{51.1,2}$$

For an incompressible material, we have

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}. \tag{52}$$

Consider the following set of invariants [23,24]:

$$I_1 = \operatorname{tr}\mathbf{c}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{c})^2 - \operatorname{tr}\mathbf{c}^2], \quad I_3 = \det\mathbf{c}, \tag{53}$$

$$K_4 = \mathbf{D}_l \cdot \mathbf{D}_l, \quad K_5 = \mathbf{D}_l \cdot \mathbf{c}\mathbf{D}_l, \quad K_6 = \mathbf{D}_l \cdot \mathbf{c}^2\mathbf{D}_l, \tag{54}$$

$$I_7 = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{a}_0, \quad I_8 = \mathbf{a}_0 \cdot \mathbf{c}^2\mathbf{a}_0, \quad K_9 = \mathbf{a}_0 \cdot \mathbf{D}_l, \quad K_{10} = \mathbf{a}_0 \cdot \mathbf{c}\mathbf{D}_l. \tag{55}$$

The derivative $\frac{\partial \Omega^*}{\partial \mathbf{F}}$ can be calculated using the invariants I_i and K_i defined as above, in which case the expression for the total stress (51.1) becomes[4]

$$\begin{aligned}\boldsymbol{\tau} = J^{-1}[&2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* + 2I_3\mathbf{I}\Omega_3^* + 2J^2\mathbf{D}\otimes\mathbf{D}\Omega_5^* \\ &+2J^2(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D})\Omega_6^* + 2\mathbf{a}\otimes\mathbf{a}\Omega_7^* + 2(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a}\otimes\mathbf{a})\Omega_8^* \\ &+J(\mathbf{a}\otimes\mathbf{D} + \mathbf{D}\otimes\mathbf{a})\Omega_{10}^*],\end{aligned} \tag{56}$$

where the connections $\mathbf{D}_l = J\mathbf{F}^{-1}\mathbf{D}$ and $\mathbf{a}_0 = \mathbf{F}^{-1}\mathbf{a}$ have been used. The corresponding expression for the incompressible case (52) is

$$\begin{aligned}\boldsymbol{\tau} = {}&2\mathbf{b}\Omega_1^* + 2(I_1\mathbf{b} - \mathbf{b}^2)\Omega_2^* - p^*\mathbf{I} + 2\mathbf{D}\otimes\mathbf{D}\Omega_5^* + 2(\mathbf{D}\otimes\mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D}\otimes\mathbf{D})\Omega_6^* \\ &+2\mathbf{a}\otimes\mathbf{a}\Omega_7^* + 2(\mathbf{a}\otimes\mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a}\otimes\mathbf{a})\Omega_8^* + (\mathbf{a}\otimes\mathbf{D} + \mathbf{D}\otimes\mathbf{a})\Omega_{10}^*.\end{aligned} \tag{57}$$

Finally, the expression for the electric field (51.2) becomes

$$\mathbf{E} = 2J\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + 2J\mathbf{D}\Omega_5^* + 2J\mathbf{b}\mathbf{D}\Omega_6^* + \mathbf{b}^{-1}\mathbf{a}\Omega_9^* + \mathbf{a}\Omega_{10}^*, \tag{58}$$

and the corresponding incompressible case is

$$\mathbf{E} = 2\mathbf{b}^{-1}\mathbf{D}\Omega_4^* + 2\mathbf{D}\Omega_5^* + 2\mathbf{b}\mathbf{D}\Omega_6^* + \mathbf{b}^{-1}\mathbf{a}\Omega_9^* + \mathbf{a}\Omega_{10}^*. \tag{59}$$

As in Sect. 3.1, we can find some restrictions on the form of the energy function if we assume that for the case when there is no deformation or external electric displacement, there is no residual stresses and residual electric field. Let $\bar{\Omega}_i^*$ denote the function Ω_i^* evaluated with the invariants (53)–(55) calculated using $\mathbf{F} = \mathbf{I}$ and $\mathbf{D}_l = \mathbf{0}$. From Eqs. (56) and (58), the conditions $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$ imply

$$\bar{\Omega}_1^* + 2\bar{\Omega}_2^* + \bar{\Omega}_3^* = 0, \quad \bar{\Omega}_7^* + 2\bar{\Omega}_8^* = 0, \quad \bar{\Omega}_9^* + \bar{\Omega}_{10}^* = 0. \tag{60.1–3}$$

In the incompressible case, (60.1) should be replaced by

$$2\bar{\Omega}_1^* + 4\bar{\Omega}_2^* - p = 0. \tag{61}$$

[4] The notation Ω_i^* means the derivative of Ω^* in I_i if $i = 1, 2, 3, 7, 8$, or K_i if $i = 4, 5, 6, 9, 10$.

A different restriction can be found if we assume now that whenever we have deformation but no external electric displacement, then from Eq. (58) the electric field is zero. Let $\breve{\Omega}_i^*$ denote the function Ω_i^* evaluated for $\mathbf{D}_l = \mathbf{0}$ but with $\mathbf{F}$ in general different to the identity tensor. From Eq. (58) we have

$$\mathbf{b}^{-1}\breve{\Omega}_9^* + \mathbf{I}\breve{\Omega}_{10}^* = \mathbf{0}, \tag{62}$$

which, if $\mathbf{b}^{-1} \neq \mathbf{I}$, implies

$$\breve{\Omega}_9^* = \breve{\Omega}_{10}^* = 0. \tag{63}$$

4 Derivation of the equations for the linear elastic case

To develop a linear theory through a linear expansion from the non-linear general formulation is a standard procedure. In electro-elasticity that has been done mainly by expanding directly, for example, the expressions (23), (25) as Taylor series in $\mathbf{c}$ (instead of $\mathbf{F}$) and $\mathbf{E}_l$ (see, for example, [9,17,26,27]). In this section, we want to obtain linear approximate expressions from, for example, (37) and (40), which would relate directly the different parameters and quantities that appear in the general non-linear formulation, and the parameters that appear in the classical linear theory.

Let us assume that

$$|\mathrm{Grad}\,\mathbf{u}| \ll 1, \quad |\mathbf{E}_l| \ll 1. \tag{64.1,2}$$

It is not problematic to define what we mean for 'small' in the case of the gradient of the displacement. The situation is more complicated for the electric field. The concept of 'small' is relative; in the case of the gradient of the displacement this is not a problem because this gradient of the displacement is dimensionless. For the electric field, we need to define the 'smallness' of the electric field $\mathbf{E}_l$ with respect to a 'reference value' for the field. Let us denote this value E_R, then the inequality (64.2) should be understood as $|\mathbf{E}_l|/E_R \ll 1$ or $|\mathbf{E}_l| \ll E_R$. In this section, we assume that $\mathbf{E}_l$ has been divided by E_R, and so $|\mathbf{E}_l| \ll 1$ actually means that $|\mathbf{E}_l| \ll E_R$. We do not use a different notation for this dimensionless electric field. As for E_R, this might seem to be an arbitrary value, but in fact it should have a physical meaning related to, for example, the behaviour of the polarization near the saturation point. We do not have enough experimental data for ES elastomers, and so we do not discuss any further about E_R.

The linear deformation tensor $\mathbf{e}$ is defined as $\mathbf{e} \equiv \frac{1}{2}(\mathrm{Grad}\,\mathbf{u} + \mathrm{Grad}\,\mathbf{u}^{\mathrm{T}})$. Let us determine the approximation of $\mathbf{b}$, $\mathbf{b}^2$, $\mathbf{bE}$ and $\mathbf{b}^2\mathbf{E}$ if $|\mathrm{Grad}\,\mathbf{u}|$ and $|\mathbf{E}_l|$ are of order δ with $\delta \ll 1$. From Eqs. (64) and the definitions (2), (7), and (3) we have

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} \approx \mathbf{I} + 2\mathbf{e}, \quad \mathbf{E}_l \approx \mathbf{E}, \quad \mathbf{bE} \approx \mathbf{E}. \tag{65}$$

As well as this, it is not difficult to show that $\mathbf{b}^2 \approx \mathbf{I} + 4\mathbf{e}$, $\mathbf{b}^2\mathbf{E} \approx \mathbf{E}$ and $I_3 = J^2 \approx 1$.

Using the above expressions for $\mathbf{b}$, $\mathbf{E}_l$, $\mathbf{bE}$, $\mathbf{b}^2$, $\mathbf{b}^2\mathbf{E}$ and J in (37), and neglecting terms of order δ^2 we get

$$\begin{aligned}\boldsymbol{\tau} \approx\ & 2[\Omega_1 + (I_1 - 1)\Omega_2 + \Omega_3]\mathbf{I} + 4[\Omega_1 + (I_1 - 2)\Omega_2]\mathbf{e} + 2[\Omega_7 + \Omega_8]\mathbf{a} \otimes \mathbf{a} \\ & + 4\Omega_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \Omega_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}).\end{aligned} \tag{66}$$

For the electric displacement (40), we have

$$\mathbf{D} \approx -(2[\Omega_4 + \Omega_5 + \Omega_6]\mathbf{E} + [\Omega_9 + \Omega_{10}]\mathbf{a} + 2\Omega_{10}\mathbf{ea}). \tag{67}$$

To be frame-indifferent, Ω is a function of $\mathbf{c}$ and $\mathbf{E}_l$, but since $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ we write $\Omega = \Omega(\mathbf{F}, \mathbf{E}_l, \mathbf{a}_0)$. Now let us approximate Ω_i in $\mathbf{c}$ and $\mathbf{E}$. We have

$$\Omega_i = \bar{\Omega}_i + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{c}} : (\mathbf{c} - \mathbf{I}) + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{E}} \cdot \mathbf{E} + \cdots,$$

where $\bar{f}$ denotes the function $f = f(\mathbf{c}, \mathbf{E}_l)$ evaluated with $\mathbf{F} = \mathbf{I}$ and $\mathbf{E}_l = \mathbf{0}$. Using the definition of the linear deformation tensor $\mathbf{e}$ we have the following approximation:

$$\Omega_i \approx \bar{\Omega}_i + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{c}} : 2\mathbf{e} + \frac{\partial \bar{\Omega}_i}{\partial \mathbf{E}} \cdot \mathbf{E}. \tag{68}$$

As a result, after neglecting the terms of order δ^2 we obtain for (66)

$$\begin{aligned}\boldsymbol{\tau} \approx\, & 2(\bar{\Omega}_1 + 2\bar{\Omega}_2 + \bar{\Omega}_3)\mathbf{I} + 4[(\bar{\boldsymbol{\Omega}}_{1c} + 2\bar{\boldsymbol{\Omega}}_{2c} + \bar{\boldsymbol{\Omega}}_{3c}) : \mathbf{e}]\mathbf{I} + 2[(\bar{\boldsymbol{\Omega}}_{1E} + 2\bar{\boldsymbol{\Omega}}_{2E} + \bar{\boldsymbol{\Omega}}_{3E}) \cdot \mathbf{E}]\mathbf{I} \\ & + 4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} + 2(\bar{\Omega}_7 + 2\bar{\Omega}_8)\mathbf{a} \otimes \mathbf{a} + 4[(\bar{\boldsymbol{\Omega}}_{7c} + 2\bar{\boldsymbol{\Omega}}_{8c}) : \mathbf{e}]\mathbf{a} \otimes \mathbf{a} \\ & + 2[(\bar{\boldsymbol{\Omega}}_{7E} + 2\bar{\boldsymbol{\Omega}}_{8E}) \cdot \mathbf{E}]\mathbf{a} \otimes \mathbf{a} + 4\bar{\Omega}_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \bar{\Omega}_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}),\end{aligned} \tag{69}$$

where we have used the notation $\bar{\boldsymbol{\Omega}}_{ic} \equiv \frac{\partial \bar{\Omega}_i}{\partial \mathbf{c}}$, $\bar{\boldsymbol{\Omega}}_{iE} \equiv \frac{\partial \bar{\Omega}_i}{\partial \mathbf{E}}$. Using Eqs. (46.1), and (46.2), the above equation simplifies to

$$\begin{aligned}\boldsymbol{\tau} \approx\, & 4[(\bar{\boldsymbol{\Omega}}_{1c} + 2\bar{\boldsymbol{\Omega}}_{2c} + \bar{\boldsymbol{\Omega}}_{3c}) : \mathbf{e}]\mathbf{I} + 2[(\bar{\boldsymbol{\Omega}}_{1E} + 2\bar{\boldsymbol{\Omega}}_{2E} + \bar{\boldsymbol{\Omega}}_{3E}) \cdot \mathbf{E}]\mathbf{I} + 4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} \\ & + 4[(\bar{\boldsymbol{\Omega}}_{7c} + 2\bar{\boldsymbol{\Omega}}_{8c}) : \mathbf{e}]\mathbf{a} \otimes \mathbf{a} + 2[(\bar{\boldsymbol{\Omega}}_{7E} + 2\bar{\boldsymbol{\Omega}}_{8E}) \cdot \mathbf{E}]\mathbf{a} \otimes \mathbf{a} \\ & + 4\bar{\Omega}_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \bar{\Omega}_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}).\end{aligned} \tag{70}$$

For the constitutive equation for the electric displacement (70) we get

$$\begin{aligned}\mathbf{D} \approx -\{ & 2(\bar{\Omega}_4 + \bar{\Omega}_5 + \bar{\Omega}_6)\mathbf{E} + (\bar{\Omega}_9 + \bar{\Omega}_{10})\mathbf{a} + 2[(\bar{\boldsymbol{\Omega}}_{9c} + \bar{\boldsymbol{\Omega}}_{10c}) : \mathbf{e}]\mathbf{a} \\ & + [(\bar{\boldsymbol{\Omega}}_{9E} + \bar{\boldsymbol{\Omega}}_{10E}) \cdot \mathbf{E}]\mathbf{a} + 2\bar{\Omega}_{10}\mathbf{ea}\},\end{aligned} \tag{71}$$

which in view of Eq. (46.3) reduces to

$$\mathbf{D} \approx -\{2(\bar{\Omega}_4 + \bar{\Omega}_5 + \bar{\Omega}_6)\mathbf{E} + 2[(\bar{\boldsymbol{\Omega}}_{9c} + \bar{\boldsymbol{\Omega}}_{10c}) : \mathbf{e}]\mathbf{a} + [(\bar{\boldsymbol{\Omega}}_{9E} + \bar{\boldsymbol{\Omega}}_{10E}) \cdot \mathbf{E}]\mathbf{a} + 2\bar{\Omega}_{10}\mathbf{ea}\}. \tag{72}$$

We calculate $\bar{\boldsymbol{\Omega}}_{ic}$ and $\bar{\boldsymbol{\Omega}}_{iE}$ in terms of the derivatives in the invariants, which are evaluated at the reference configuration for $\mathbf{E} = \mathbf{0}$. From the chain rule we have

$$\boldsymbol{\Omega}_{ic} = \frac{\partial^2 \Omega}{\partial I_i \partial I_j}\frac{\partial I_j}{\partial \mathbf{c}}, \quad \boldsymbol{\Omega}_{iE} = \frac{\partial^2 \Omega}{\partial I_i \partial I_j}\frac{\partial I_j}{\partial \mathbf{E}}.$$

Consider the following derivatives of the invariants:

$$\frac{\partial I_1}{\partial \mathbf{c}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{c}} = I_1\mathbf{I} - \mathbf{c}, \quad \frac{\partial I_3}{\partial \mathbf{c}} = I_3\mathbf{c}^{-1}, \quad \frac{\partial I_4}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial I_5}{\partial \mathbf{c}} = \mathbf{E} \otimes \mathbf{E}, \tag{73}$$

$$\frac{\partial I_6}{\partial \mathbf{c}} = \mathbf{E} \otimes \mathbf{cE} + \mathbf{cE} \otimes \mathbf{E}, \quad \frac{\partial I_7}{\partial \mathbf{c}} = \mathbf{a}_0 \otimes \mathbf{a}_0, \quad \frac{\partial I_8}{\partial \mathbf{c}} = \mathbf{a}_0 \otimes \mathbf{ca}_0 + \mathbf{ca}_0 \otimes \mathbf{a}_0, \tag{74}$$

$$\frac{\partial I_9}{\partial \mathbf{c}} = \mathbf{0}, \quad \frac{\partial I_{10}}{\partial \mathbf{c}} = \frac{1}{2}(\mathbf{a}_0 \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}_0), \tag{75}$$

and

$$\frac{\partial I_1}{\partial \mathbf{E}} = \frac{\partial I_2}{\partial \mathbf{E}} = \frac{\partial I_3}{\partial \mathbf{E}} = \mathbf{0}, \quad \frac{\partial I_4}{\partial \mathbf{E}} = 2\mathbf{E}, \quad \frac{\partial I_5}{\partial \mathbf{E}} = 2\mathbf{cE}, \quad \frac{\partial I_6}{\partial \mathbf{E}} = 2\mathbf{c}^2\mathbf{E}, \tag{76}$$

$$\frac{\partial I_7}{\partial \mathbf{E}} = \frac{\partial I_8}{\partial \mathbf{E}} = \mathbf{0}, \quad \frac{\partial I_9}{\partial \mathbf{E}} = \mathbf{a}_0, \quad \frac{\partial I_{10}}{\partial \mathbf{E}} = \mathbf{ca}_0. \tag{77}$$

Therefore, we have

$$\bar{\boldsymbol{\Omega}}_{ic} = (\bar{\Omega}_{i,1} + 2\bar{\Omega}_{i,2} + \bar{\Omega}_{i,3})\mathbf{I} + (\bar{\Omega}_{i,7} + 2\bar{\Omega}_{i,8})\mathbf{a} \otimes \mathbf{a}, \tag{78}$$

$$\bar{\boldsymbol{\Omega}}_{iE} = (\bar{\Omega}_{i,9} + \bar{\Omega}_{i,10})\mathbf{a}, \tag{79}$$

where we have used the notation $\Omega_{i,j} \equiv \frac{\partial^2 \Omega}{\partial I_i \partial I_j}$.

4.1 Approximation for the stress

Let us calculate separately the different terms that appear in (70). Since $\Omega_{i,j} = \Omega_{j,i}$, we have

$$4(\bar{\boldsymbol{\Omega}}_{1c} + 2\bar{\boldsymbol{\Omega}}_{2c} + \bar{\boldsymbol{\Omega}}_{3c}) = 4[(\bar{\Omega}_{1,1} + 4\bar{\Omega}_{1,2} + 2\bar{\Omega}_{1,3} + 4\bar{\Omega}_{2,2} + 4\bar{\Omega}_{2,3} + \bar{\Omega}_{3,3})\mathbf{I} + (\bar{\Omega}_{1,7} + 2\bar{\Omega}_{1,8} + 2\bar{\Omega}_{2,7} + 4\bar{\Omega}_{2,8} + \bar{\Omega}_{3,7} + 2\bar{\Omega}_{3,8})\mathbf{a} \otimes \mathbf{a}], \quad (80)$$

$$2(\bar{\boldsymbol{\Omega}}_{1E} + 2\bar{\boldsymbol{\Omega}}_{2E} + \bar{\boldsymbol{\Omega}}_{3E}) = 2(\bar{\Omega}_{1,9} + \bar{\Omega}_{1,10} + 2\bar{\Omega}_{2,9} + 2\bar{\Omega}_{2,10} + \bar{\Omega}_{3,9} + \bar{\Omega}_{3,10})\mathbf{a}, \quad (81)$$

$$4(\bar{\boldsymbol{\Omega}}_{7c} + 2\bar{\boldsymbol{\Omega}}_{8c}) = 4[(\bar{\Omega}_{7,1} + 2\bar{\Omega}_{7,2} + \bar{\Omega}_{7,3} + 2(\bar{\Omega}_{8,1} + 2\bar{\Omega}_{8,2} + \bar{\Omega}_{8,3}))\mathbf{I} + (\bar{\Omega}_{7,7} + 4\bar{\Omega}_{7,8} + 4\bar{\Omega}_{8,8})\mathbf{a} \otimes \mathbf{a}], \quad (82)$$

$$2(\bar{\boldsymbol{\Omega}}_{7E} + 2\bar{\boldsymbol{\Omega}}_{8E}) = 2[\bar{\Omega}_{7,9} + \bar{\Omega}_{7,10} + 2(\bar{\Omega}_{8,9} + \bar{\Omega}_{8,10})]\mathbf{a}. \quad (83)$$

Let us define

$$\alpha_1 = \bar{\Omega}_{1,1} + 4\bar{\Omega}_{1,2} + 2\bar{\Omega}_{1,3} + 4\bar{\Omega}_{2,2} + 4\bar{\Omega}_{2,3} + \bar{\Omega}_{3,3}, \quad (84)$$

$$\alpha_2 = \bar{\Omega}_{1,7} + 2\bar{\Omega}_{1,8} + 2\bar{\Omega}_{2,7} + 4\bar{\Omega}_{2,8} + \bar{\Omega}_{3,7} + 2\bar{\Omega}_{3,8}, \quad (85)$$

$$\alpha_3 = 2(\bar{\Omega}_{1,9} + \bar{\Omega}_{1,10} + 2\bar{\Omega}_{2,9} + 2\bar{\Omega}_{2,10} + \bar{\Omega}_{3,9} + \bar{\Omega}_{3,10}), \quad (86)$$

$$\alpha_4 = \bar{\Omega}_{7,7} + 4\bar{\Omega}_{7,8} + 4\bar{\Omega}_{8,8}, \quad (87)$$

$$\alpha_5 = 2[\bar{\Omega}_{7,9} + \bar{\Omega}_{7,10} + 2(\bar{\Omega}_{8,9} + \bar{\Omega}_{8,10})], \quad (88)$$

then (70) becomes

$$\boldsymbol{\tau} \approx 4[(\alpha_1\mathbf{I} + \alpha_2\mathbf{a} \otimes \mathbf{a}) : \mathbf{e}]\mathbf{I} + \alpha_3(\mathbf{a} \cdot \mathbf{E})\mathbf{I} + 4(\bar{\Omega}_1 + \bar{\Omega}_2)\mathbf{e} + 4[(\alpha_2\mathbf{I} + \alpha_4\mathbf{a} \otimes \mathbf{a}) : \mathbf{e}]\mathbf{a} \otimes \mathbf{a} + \alpha_5(\mathbf{a} \cdot \mathbf{E})\mathbf{a} \otimes \mathbf{a} + 4\bar{\Omega}_8(\mathbf{a} \otimes \mathbf{ea} + \mathbf{ea} \otimes \mathbf{a}) + \bar{\Omega}_{10}(\mathbf{a} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{a}). \quad (89)$$

Let us define $\beta_1 \equiv 4(\bar{\Omega}_1 + \bar{\Omega}_2)$ and let us consider the particular case $\mathbf{a} = \hat{\mathbf{i}}_3$, where $\hat{\mathbf{i}}_3$ is the unitarian vector in the direction 3 (Cartesian coordinates). We obtain

$$\tau_{11} = (4\alpha_1 + \beta_1)e_{11} + 4\alpha_1 e_{22} + 4(\alpha_1 + \alpha_2)e_{33} + \alpha_3 E_3, \quad (90)$$

$$\tau_{22} = 4\alpha_1 e_{11} + (4\alpha_1 + \beta_1)e_{22} + 4(\alpha_1 + \alpha_2)e_{33} + \alpha_3 E_3, \quad (91)$$

$$\tau_{33} = 4(\alpha_1 + \alpha_2)(e_{11} + e_{22}) + [4(\alpha_1 + 2\alpha_2 + \alpha_4) + \beta_1 + 8\bar{\Omega}_8]e_{33} + (\alpha_3 + \alpha_5 + 2\bar{\Omega}_{10})E_3, \quad (92)$$

$$\tau_{23} = (\beta_1 + 4\bar{\Omega}_8)e_{23} + \bar{\Omega}_{10}E_2, \quad (93)$$

$$\tau_{13} = (\beta_1 + 4\bar{\Omega}_8)e_{13} + \bar{\Omega}_{10}E_1, \quad (94)$$

$$\tau_{12} = \beta_1 e_{12}. \quad (95)$$

Defining $\gamma_1 = 4(\alpha_1 + 2\alpha_2 + \alpha_4) + \beta_1 + 8\bar{\Omega}_8$, and using the following vector notation for the stress and the deformation

$$\mathsf{T} = (T_1, T_2, T_3, T_4, T_5, T_6)^{\mathrm{T}} = (\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{13}, \tau_{12})^{\mathrm{T}}, \quad (96)$$

$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6)^{\mathrm{T}} = (e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{13}, 2e_{12})^{\mathrm{T}}, \quad (97)$$

we can rewrite Eqs. (90)–(95) as (see, for example, the form of the linear constitutive equations for a polarized ceramic [17])

$$\begin{pmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6 \end{pmatrix} = \begin{pmatrix} (4\alpha_1+\beta_1) & 4\alpha_1 & 4(\alpha_1+\alpha_2) & 0 & 0 & 0\\ 4\alpha_1 & (4\alpha_1+\beta_1) & 4(\alpha_1+\alpha_2) & 0 & 0 & 0\\ 4(\alpha_1+\alpha_2) & 4(\alpha_1+\alpha_2) & \gamma_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(\beta_1+4\bar{\Omega}_8) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}(\beta_1+4\bar{\Omega}_8) & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\beta_1 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\\ \mathcal{E}_4\\ \mathcal{E}_5\\ \mathcal{E}_6 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & \alpha_3\\ 0 & 0 & \alpha_3\\ 0 & 0 & (\alpha_3+\alpha_5+2\bar{\Omega}_{10})\\ 0 & \bar{\Omega}_{10} & 0\\ \bar{\Omega}_{10} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1\\ E_2\\ E_3 \end{pmatrix}. \tag{98}$$

4.2 Approximation for the electric displacement

We can repeat the same procedure for the electric displacement. From Eq. (72) we have

$$\begin{aligned} 2(\bar{\boldsymbol{\Omega}}_{9c}+\bar{\boldsymbol{\Omega}}_{10c}) &= 2[(\bar{\Omega}_{9,1}+2\bar{\Omega}_{9,2}+\bar{\Omega}_{9,3}+\bar{\Omega}_{10,1}+2\bar{\Omega}_{10,2}+\bar{\Omega}_{10,3})\mathbf{I}\\ &\quad+(\bar{\Omega}_{9,7}+2\bar{\Omega}_{9,8}+\bar{\Omega}_{10,7}+2\bar{\Omega}_{10,8})\mathbf{a}\otimes\mathbf{a}]\\ &= \alpha_3\mathbf{I}+\alpha_5\mathbf{a}\otimes\mathbf{a}, &(99)\\ \bar{\boldsymbol{\Omega}}_{9E}+\bar{\boldsymbol{\Omega}}_{10E} &= (\bar{\Omega}_{9,10}+\bar{\Omega}_{9,9}+\bar{\Omega}_{10,10}+\bar{\Omega}_{10,9})\mathbf{a}. &(100) \end{aligned}$$

Finally, let us define

$$\beta_2 = \bar{\Omega}_{9,9}+\bar{\Omega}_{10,10}+2\bar{\Omega}_{9,10},\quad \varepsilon_1 = 2(\bar{\Omega}_4+\bar{\Omega}_5+\bar{\Omega}_6). \tag{101}$$

Using Eq. (46.3) we get from (72)

$$D_1 = -[\varepsilon_1 E_1 + 2\bar{\Omega}_{10}\mathcal{E}_5], \tag{102}$$
$$D_2 = -[\varepsilon_1 E_2 + 2\bar{\Omega}_{10}\mathcal{E}_4], \tag{103}$$
$$D_3 = -[\varepsilon_1 E_3 + \beta_2 E_3 + \alpha_3\mathcal{E}_1 + \alpha_3\mathcal{E}_2 + (\alpha_3+\alpha_5)\mathcal{E}_3 + 2\bar{\Omega}_{10}\mathcal{E}_3], \tag{104}$$

which can be rewritten as

$$\begin{pmatrix} D_1\\ D_2\\ D_3 \end{pmatrix} = -\begin{pmatrix} \varepsilon_1 & 0 & 0\\ 0 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_1+\beta_2 \end{pmatrix} \begin{pmatrix} E_1\\ E_2\\ E_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & \bar{\Omega}_{10} & 0\\ 0 & 0 & 0 & \bar{\Omega}_{10} & 0 & 0\\ \alpha_3 & \alpha_3 & (\alpha_3+\alpha_5+2\bar{\Omega}_{10}) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\\ \mathcal{E}_4\\ \mathcal{E}_5\\ \mathcal{E}_6 \end{pmatrix}. \tag{105}$$

5 Boundary value problems

For transversely isotropic electro-active elastomers, we do not have the complete set of universal solutions as in the isotropic case [15,28]. In the next examples, in particular for the problems with cylindrical symmetry, we show that the controllability depends strongly on the particular alignment of the electro-active particles with respect to the given external electric field or electric displacement.

Remark To look for exact solutions of Eqs. (6), and (12), we need to consider the boundary conditions (15) and (17). These conditions are rather complex; the method used by Sing and Pipkin [15] and other authors [11–14,29] was to assume 'semi-infinite' geometries, see, for example, the paper by Bustamante et al. [30].

All the problems presented in this section correspond to incompressible materials.

5.1 Simple shear

Consider the simple shear deformation defined as

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{106}$$

which is applied to a slab of initial dimensions $0 \le X_1 \le A$, $0 \le X_2 \le B$ and $0 \le X_3 \le C$. We apply an electric field with components $\mathsf{E}_0 = \mathsf{E}_l = (0, E_o, 0)^{\mathrm{T}}$ and an alignment for the particles $\mathsf{a}_0 = (0, 1, 0)^{\mathrm{T}}$. In this problem, we reproduce theoretically what happens with the shear of a transversely isotropic slab, which has been studied experimentally for the magneto-elastic problem by Jolly et al. [5].

The matrix forms of the deformation gradient and the left and right Cauchy–Green deformation tensors are given as

$$\mathsf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{107}$$

We have $\det \mathbf{F} = 1$. The invariants I_i are given by Eqs. (30)–(32) as

$$I_1 = 3 + \gamma^2 = I_2, \quad I_4 = E_o^2, \quad I_5 = (1+\gamma^2)E_o^2, \quad I_6 = [\gamma^2 + (1+\gamma^2)]E_o^2, \tag{108}$$

$$I_7 = 1 + \gamma^2, \quad I_8 = \gamma^2 + (1+\gamma^2)^2, \quad I_9 = E_o, \quad I_{10} = (1+\gamma^2)E_o. \tag{109}$$

From Eqs. (38) and (41) we obtain

$$\tau_{11} = 2(1+\gamma^2)\Omega_1 + 2(2+\gamma^2)\Omega_2 - p + 2(\gamma E_o)^2\Omega_5 + 4\gamma^2(2+\gamma^2)E_o^2\Omega_6 + 2\gamma^2\Omega_7 + 4\gamma^2(2+\gamma^2)\Omega_8 + 2\gamma^2 E_o\Omega_{10}, \tag{110}$$

$$\tau_{22} = 2\Omega_1 + 4\Omega_2 - p + 2E_o^2\Omega_5 + 4(1+\gamma^2)E_o^2\Omega_6 + 2\Omega_7 + 4(1+\gamma^2)\Omega_8 + 2E_o\Omega_{10}, \tag{111}$$

$$\tau_{33} = 2\Omega_1 + 2(2+\gamma^2)\Omega_2 - p, \tag{112}$$

$$\tau_{12} = 2\gamma\Omega_1 + 2\gamma\Omega_2 + 2\gamma E_o^2\Omega_5 + 2\gamma(3+2\gamma^2)E_o^2\Omega_6 + 2\gamma\Omega_7 + 2\gamma(3+2\gamma^2)\Omega_8 + 2\gamma E_o\Omega_{10}, \tag{113}$$

$$\tau_{13} = \tau_{23} = 0, \tag{114}$$

and

$$D_1 = -[2 + \gamma E_o\Omega_4 + 2\gamma(2+\gamma^2)E_o\Omega_5 + 2\gamma(1+\gamma^2)(3+\gamma^2)E_o\Omega_6 + \gamma\Omega_9 + \gamma(2+\gamma^2)\Omega_{10}], \tag{115}$$

$$D_2 = -[2E_o\Omega_4 + 2(1+\gamma^2)E_o\Omega_5 + 2(1+3\gamma^2+\gamma^4)E_o\Omega_6 + \Omega_9 + (1+\gamma^2)\Omega_{10}], \tag{116}$$

$$D_3 = 0. \tag{117}$$

Let us define

$$\omega(\gamma, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10. \tag{118}$$

Using Eqs. (108)–(109) and the chain rule we can show that

$$\frac{\partial\omega}{\partial\gamma} = 2\gamma\Omega_1 + 2\gamma\Omega_2 + 2\gamma\Omega_5 E_o^2 + \Omega_6 2\gamma(3+\gamma^2)E_o^2 + 2\gamma\Omega_7 + \Omega_8 2\gamma(3+\gamma^2) + 2\gamma\Omega_{10}E_o, \tag{119}$$

$$\frac{\partial\omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5(1+\gamma^2)E_o + \Omega_6 2[\gamma^2 + (1+\gamma^2)^2]E_o + \Omega_9 + \Omega_{10}(1+\gamma^2), \tag{120}$$

and we get the connections

$$\tau_{12} = \frac{\partial\omega}{\partial\gamma}, \quad D_2 = -\frac{\partial\omega}{\partial E_o}. \tag{121}$$

The stress, the electric field and the electric displacement are constant, and as a result, they satisfy automatically, Eqs. (6) and (12). However, as we mentioned in the remark, the situation with the boundary conditions (15) is not simple. If we consider a 'finite' slab, then it is not difficult to see that to satisfy simultaneously the two boundary conditions (15) we would need a non-uniform field, which in general will depend strongly on the particular form of Ω. For the solution (110)–(114) and (115)–(117) to be valid we would need at least that two of the three dimensions of the slab to be infinite. Consider, for example, the initial dimensions for the slab, $-\infty \le X_1 \le \infty$, $0 \le X_2 \le B$ and $-\infty \le X_3 \le \infty$; this is the geometry of a infinite wall of width B. In this case, the only surfaces where we need to check the boundary conditions are the surfaces $X_2 = 0$ and $X_2 = B$. For a uniform external electric field of the form $(0, E_o, 0)^T$ the boundary conditions (15) are satisfied automatically.

5.2 Uniform extension of a bar

We consider the uniform extension of a cylindrical bar. The uniform extension of a cylinder was used to obtain some important results for MS and ES elastomers [3,4], where we can see the difference in the response for isotropic and transversely isotropic cases.

Consider a cylinder with initial dimensions $0 \le R \le R_o$ and $-\infty \le Z \le \infty$. In cylindrical coordinates the deformation is given as

$$r = \lambda^{-1/2} R, \quad \theta = \Theta, \quad z = \lambda Z. \tag{122}$$

We work with the external axial applied field $\mathsf{E}_0 = \mathsf{E}_l = (0, 0, E_o)^{\mathrm{T}}$ and the alignment for the particles in the reference configuration $\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}$. The matrix forms of the deformation gradient and left and right Cauchy–Green tensors are given by

$$\mathsf{F} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \tag{123}$$

then the invariants are, from Eqs. (30)–(32), given as

$$I_1 = 2\lambda^{-1} + \lambda^2, \quad I_2 = \lambda^{-2} + 2\lambda, \quad I_4 = E_o^2, \quad I_5 = \lambda^2 E_o^2, \quad I_6 = \lambda^4 E_o^2, \tag{124}$$
$$I_7 = \lambda^2, \quad I_8 = \lambda^4, \quad I_9 = E_o, \quad I_{10} = \lambda^2 E_o. \tag{125}$$

The components of the stress and the electric displacement (38) and (41) are

$$\tau_{rr} = \tau_{\theta\theta} = 2\lambda^{-1}\Omega_1 + 2(\lambda^{-2} + \lambda)\Omega_2 - p, \tag{126}$$
$$\tau_{zz} = 2\lambda^2\Omega_1 + 4\lambda\Omega_2 - p + 2\lambda^2 E_o^2\Omega_5 + 4\lambda^4 E_o^2\Omega_6 + 2\lambda^2\Omega_7 + 4\lambda^4\Omega_8 + 2\lambda^2 E_o\Omega_{10}, \tag{127}$$
$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{128}$$

and

$$D_r = D_\theta = 0, \tag{129}$$
$$D_z = -(2\lambda E_o\Omega_4 + 2\lambda^3 E_o\Omega_5 + 2\lambda^5 E_o\Omega_6 + \lambda\Omega_9 + \lambda^3\Omega_{10}). \tag{130}$$

The components of the Maxwell stress in the radial and the azimuthal directions are given by Eq. (18):

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{131}$$

From Eq. (17), we have

$$\tau_{rr} - \tau_{m_{rr}} = 0, \quad \tau_{\theta\theta} - \tau_{m_{\theta\theta}} = 0,$$

and as a result we obtain

$$p = 2\lambda^{-1}\Omega_1 + 2(2 + \lambda^{-2})\Omega_2 + \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{132}$$

Therefore, for (127) we get

$$\tau_{zz} = 2(\lambda^2 - \lambda^{-1})\Omega_1 + 2(\lambda - \lambda^{-2})\Omega_2 + 2\lambda^2 E_o^2 \Omega_5 + 4\lambda^4 E_o^2 \Omega_6 + 2\lambda^2 \Omega_7 + 4\lambda^4 \Omega_8 + 2\lambda^2 E_o \Omega_{10} - \frac{\varepsilon_o}{2}\lambda^{-2} E_o^2. \quad (133)$$

We can define

$$\omega(\lambda, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10. \quad (134)$$

Using the chain rule with (124) and (125) we can show that

$$\frac{\partial \omega}{\partial \lambda} = 2\Omega_1(\lambda - \lambda^{-2}) + 2\Omega_2(1 - \lambda^{-3}) + 2\Omega_5 \lambda E_o^2 + 4\Omega_6 \lambda^3 E_o^2 + 2\Omega_7 \lambda + 4\Omega_8 \lambda^3 + 2\Omega_{10} \lambda E_o, \quad (135)$$

$$\frac{\partial \omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5 \lambda^2 E_o + 2\Omega_6 \lambda^4 E_o + \Omega_9 + \Omega_{10}\lambda^2. \quad (136)$$

Thus

$$\tau_{zz} = \lambda \frac{\partial \omega}{\partial \lambda} - \frac{\varepsilon_o}{2}\lambda^{-2} E_o^2, \quad (137)$$

and

$$D_z = -\lambda \frac{\partial \omega}{\partial E_o}. \quad (138)$$

As in the simple shear problem, since the components of the stress and the electric field and electric displacement are constant, they satisfy automatically Eqs. (6) and (12). Regarding the boundary conditions (15) the cylinder has an infinite length ($L = \infty$), then the only surface where we would need to check the boundary conditions (15) is the surface $r = r_o$ and to satisfy (15) we only need to require the same uniform electric field outside and inside the cylinder.

5.3 Problems with cylindrical symmetry

Let us assume that $\boldsymbol{\tau} = \boldsymbol{\tau}(r, z)$, then in cylindrical coordinates the Eq. (12) becomes (see, for example, [31])

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \quad (139)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = 0, \quad (140)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} = 0. \quad (141)$$

If we assume $\mathbf{E} = \mathbf{E}(r, z)$ then Eq. (6.1) in cylindrical coordinates becomes

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(r E_\theta) = 0. \quad (142)$$

Finally, if we assume that $\mathbf{D} = \mathbf{D}(r, z)$ the simplified form of Eq. (6.2) is

$$\frac{1}{r}\frac{\partial}{\partial r}(r D_r) + \frac{\partial D_z}{\partial z} = 0. \quad (143)$$

We study three problems, the extension and inflation of a tube, the extension and torsion of a tube, and helical shear [11,14]. For the first two of these problems we want to find universal solutions, which will depend among other factors on the particular form of the field $\mathbf{a}_0$.

5.3.1 Extension and inflation of a tube

Consider the deformation given in cylindrical coordinates [14]

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{144}$$

where $a \le r \le b$, $0 \le \theta \le 2\pi$ and $-\infty \le z \le \infty$.

The matrix forms of the deformation gradient and the left and right Cauchy–Green tensors are given by

$$\mathsf{F} = \begin{pmatrix} (\lambda_z\lambda)^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad \mathsf{b} = \mathsf{c} = \begin{pmatrix} (\lambda_z\lambda)^{-2} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda_z^2 \end{pmatrix}, \tag{145}$$

where we have used the definition $\lambda = r/R$.

The first two invariants (30.1) and (30.2) are

$$I_1 = (\lambda_z\lambda)^{-2} + \lambda^2 + \lambda_z^2, \quad I_2 = \lambda_z^{-2} + \lambda^{-2} + (\lambda\lambda_z)^2. \tag{146}$$

Now, it is necessary to consider a particular form for the external applied electric field, and for the initial alignment of the electro-active particles. Consider two cases.

1. **Axial electric field and particle alignment**
Let us assume that the external electric field is $\mathsf{E}_l = (0, 0, E_o)^{\mathrm{T}}$, where E_o is constant. From Eq. (7.1) we have that $\mathsf{E} = (0, 0, \lambda_z^{-1}E_o)^{\mathrm{T}}$, and Eqs. (142) are satisfied automatically. As well as this, from (31) we have

$$I_4 = E_o^2, \quad I_5 = \lambda_z^2 E_o^2, \quad I_6 = \lambda_z^4 E_o^2. \tag{147}$$

With the above electric field, the non-zero components of the Maxwell stress tensor (18) are

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda^{-2}E_o^2, \quad \tau_{m_{zz}} = \frac{\varepsilon_o}{2}\lambda^{-2}E_o^2. \tag{148}$$

We assume that in the reference configuration the particles are aligned uniformly in the axial direction. We have $\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}$, the remaining invariants are given by (32) as

$$I_7 = \lambda_z^2, \quad I_8 = \lambda_z^4, \quad I_9 = E_o, \quad I_{10} = \lambda_z^2 E_o. \tag{149}$$

From (38) we get

$$\tau_{rr} = 2(\lambda_z\lambda)^{-2}\Omega_1 + 2(\lambda_z^{-2} + \lambda^{-2})\Omega_2 - p, \tag{150}$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1 + 2[\lambda_z^{-2} + (\lambda_z\lambda)^2]\Omega_2 - p, \tag{151}$$

$$\begin{aligned}\tau_{zz} = {} & 2\lambda_z^2\Omega_1 + 2[\lambda^{-2} + (\lambda_z\lambda)^2]\Omega_2 - p + 2(\lambda_z E_o)^2\Omega_5 + 4(\lambda_z^2 E_o)^2\Omega_6 \\ & + 2\lambda_z^2\Omega_7 + 4\lambda_z^4\Omega_8 + 2\lambda_z^2 E_o\Omega_{10},\end{aligned} \tag{152}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{153}$$

and from (41) we have

$$D_r = D_\theta = 0, \tag{154}$$

$$D_z = -(2\lambda_z E_o\Omega_4 + 2\lambda_z^3 E_o\Omega_5 + 2\lambda_z^5 E_o\Omega_6 + \lambda_z\Omega_9 + \lambda_z^3\Omega_{10}). \tag{155}$$

We have that $\lambda = \lambda(r)$; thus, the different invariants are function of r. Consider the decomposition of the components of the stress $\tau_{rr} = \tilde{\tau}_{rr} - p$, $\tau_{\theta\theta} = \tilde{\tau}_{\theta\theta} - p$, $\tau_{zz} = \tilde{\tau}_{zz} - p$. We have that $\tilde{\tau}_{rr} = \tilde{\tau}_{rr}(r)$, $\tilde{\tau}_{\theta\theta} = \tilde{\tau}_{\theta\theta}(r)$ and $\tilde{\tau}_{zz} = \tilde{\tau}_{zz}(r)$. As a result, Eq. (140) is satisfied automatically. From (141) we have

$$\frac{\partial}{\partial z}(\tilde{\tau}_{zz}(r) - p) = 0, \quad \Rightarrow \quad \frac{\partial p}{\partial z} = 0 \quad \Leftrightarrow \quad p = p(r), \tag{156}$$

and the deformation is controllable.
Regarding the electric displacement, from (155) we have that $D_z = D_z(r)$ and with (154) we conclude that (143) is satisfied trivially. As a result for this electric field and initial orientation of the electro-active

particles we conclude that (144) is universal (for an analysis of universal solutions in the context of isotropic electro- and magneto-elastic problems see [15,28]).
Let us consider the simplified form for the energy function

$$\omega(\lambda_z, \lambda, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots 10,$$

where from (146), (147) and (149) we have $I_i = I_i(\lambda_z, \lambda, E_o)$. Consider the partial derivatives

$$\frac{\partial \omega}{\partial \lambda_z} = 2\Omega_1(\lambda_z - \lambda_z^{-3}\lambda^{-2}) + 2\Omega_2(\lambda^2\lambda_z - \lambda_z^{-3}) + 2\Omega_5\lambda_z E_o^2 + 4\Omega_6\lambda_z^3 E_o^2 + 2\Omega_7\lambda_z + 4\Omega_8\lambda_z^3 + 2\Omega_{10}\lambda_z E_o, \tag{157}$$

$$\frac{\partial \omega}{\partial \lambda} = 2\Omega_1(\lambda - \lambda_z^{-2}\lambda^{-3}) + 2\Omega_2(\lambda\lambda_z^2 - \lambda^{-3}), \tag{158}$$

$$\frac{\partial \omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5\lambda_z^2 E_o + 2\Omega_6\lambda_z^4 E_o + \Omega_9 + \Omega_{10}\lambda_z^2. \tag{159}$$

From the above relations we can prove that

$$\tau_{zz} - \tau_{\theta\theta} = \lambda_z \frac{\partial \omega}{\partial \lambda_z} - \lambda \frac{\partial \omega}{\partial \lambda}, \tag{160}$$

and

$$D_z = -\lambda_z \frac{\partial \omega}{\partial E_o}. \tag{161}$$

2. **Radial electric displacement and alignment for the particles**
Consider an external electric displacement in vector form given by $\mathsf{D}_l = (D_o/R, 0, 0)^{\mathrm{T}}$, where D_o is constant. From (7.2) for an incompressible material we have $\mathsf{D} = (\lambda_z^{-1} D_o/r, 0, 0)^{\mathrm{T}}$. For this particular form of the electric displacement, Eq. (143) is satisfied automatically.
Consider a radial uniform orientation for the electro-active particles. So $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$, and as a result $\mathsf{a} = ((\lambda_z\lambda)^{-1}, 0, 0)^{\mathrm{T}}$. The invariants are given by (53)–(55), and we have

$$K_4 = (D_o/R)^2, \quad K_5 = (D_o/R)^2(\lambda_z\lambda)^{-2}, \quad K_6 = (D_o/R)^2(\lambda_z\lambda)^{-4}, \tag{162}$$

$$I_7 = (\lambda_z\lambda)^{-2}, \quad I_8 = (\lambda_z\lambda)^{-4}, \quad K_9 = D_o/R, \quad K_{10} = (D_o/R)(\lambda_z\lambda)^{-2}. \tag{163}$$

From (57) we obtain

$$\tau_{rr} = 2(\lambda_z\lambda)^{-2}\Omega_1^* + 2(\lambda^{-2} + \lambda_z^{-2})\Omega_2^* - p^* + 2\lambda_z^{-2}\left(\frac{D_o}{r}\right)^2\Omega_5^* + 4(\lambda_z\lambda)^{-4}\left(\frac{D_o}{R}\right)^2\Omega_6^* + 2(\lambda_z\lambda)^{-2}\Omega_7^* + 4(\lambda_z\lambda)^{-4}\Omega_8^* + 2(\lambda_z\lambda)^{-2}\frac{D_o}{R}\Omega_{10}^*, \tag{164}$$

$$\tau_{\theta\theta} = 2\lambda^2\Omega_1^* + 2[\lambda_z^{-2} + (\lambda_z\lambda)^2]\Omega_2^* - p^*, \tag{165}$$

$$\tau_{zz} = 2\lambda_z^2\Omega_1^* + 2[\lambda^{-2} + (\lambda_z\lambda)^2]\Omega_2^* - p^*, \tag{166}$$

$$\tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0, \tag{167}$$

and from (59) we get for the electric field

$$E_r = 2\lambda_z\lambda\frac{D_o}{R}\Omega_4^* + 2(\lambda_z\lambda)^{-1}\frac{D_o}{R}\Omega_5^* + 2(\lambda_z\lambda)^{-3}\frac{D_o}{R}\Omega_6^* + \lambda_z\lambda\Omega_9^*(\lambda_z\lambda)^{-1}\Omega_{10}^*, \tag{168}$$

$$E_\theta = E_z = 0. \tag{169}$$

The invariants are functions of λ_z, λ and D_o/R. The above electric field then satisfies (142). As well as this, by an argument similar to the one used in case 1, we can prove that this deformation is controllable, and that p^* can be calculated from (139) using the above components of the stress and the same decomposition used for the stress in 1.

Let us define $\xi = D_o/R$, and the simplified energy function

$$\omega(\lambda_z, \lambda, \xi) = \Omega^*(I_i, K_j),$$

and consider the derivatives

$$\frac{\partial\omega}{\partial\lambda_z} = 2\Omega_1^*(\lambda_z - \lambda_z^{-3}\lambda^{-2}) + 2\Omega_2^*(\lambda_z\lambda^2 - \lambda_z^{-3}) - 2\Omega_5^*\lambda_z^{-3}\lambda^{-2}\xi^2$$
$$-4\Omega_6^*\lambda_z^{-5}\lambda^{-4}\xi^2 - 2\Omega_7^*\lambda_z^{-3}\lambda^{-2} - 4\Omega_8^*\lambda^{-5}\lambda^{-4} - 2\Omega_{10}^*\lambda_z^{-3}\lambda^{-2}\xi, \tag{170}$$

$$\frac{\partial\omega}{\partial\lambda} = 2\Omega_1^*(\lambda - \lambda_z^{-2}\lambda^{-3}) + 2\Omega_2^*(\lambda\lambda_z^2 - \lambda^{-3}) - 2\Omega_5^*\lambda^{-3}\lambda_z^{-2}\xi^2$$
$$-4\Omega_6^*\lambda^{-5}\lambda_z^{-4}\xi^2 - 2\Omega_7^*\lambda_z^{-2}\lambda^{-3} - 4\Omega_8^*\lambda_z^{-4}\lambda^{-5} - 2\Omega_{10}^*\lambda_z^{-2}\lambda^{-3}\xi, \tag{171}$$

$$\frac{\partial\omega}{\partial\xi} = 2\Omega_4^*\xi + 2\Omega_5^*\xi(\lambda_z\lambda)^{-2} + 2\Omega_6^*\xi(\lambda_z\lambda)^{-4} + \Omega_9^* + \Omega_{10}^*(\lambda_z\lambda)^{-2}. \tag{172}$$

It is easy to show that

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\frac{\partial\omega}{\partial\lambda}, \quad \tau_{zz} - \tau_{rr} = \lambda_z\frac{\partial\omega}{\partial\lambda_z}, \tag{173}$$

and

$$\tau_{zz} - \tau_{\theta\theta} = \lambda_z\frac{\partial\omega}{\partial\lambda_z} - \lambda\frac{\partial\omega}{\partial\lambda}, \tag{174}$$

which is the same relation found in case 1. Finally, for the electric field we have

$$E_r = \lambda_z\lambda\frac{\partial\omega}{\partial\xi}. \tag{175}$$

Remark We could explore two more cases, an axial electric field with a radial alignment for the particles, and a radial electric displacement with an axial alignment for the particles. It is possible to prove that for these two cases $\tau_{rz} \neq 0$, as a result for these two cases the deformation (144) is not controllable.

5.3.2 Extension and torsion of a tube

Consider the deformation [29]

$$r = \lambda_z^{-1/2}R, \quad \theta = \Theta + \lambda_z\tau Z, \quad z = \lambda_z Z, \tag{176}$$

where λ_z and τ are constants, and $a \le r \le b$, $0 \le \theta < 2\pi$, and $-\infty \le z \le \infty$. Let us define $\gamma = \tau r$, then the matrix representations of the deformation gradient and the left and right Cauchy–Green tensors are given as

$$\mathsf{F} = \begin{pmatrix} \lambda_z^{-1/2} & 0 & 0 \\ 0 & \lambda_z^{-1/2} & \gamma\lambda_z \\ 0 & 0 & \lambda_z \end{pmatrix}, \tag{177}$$

$$\mathsf{b} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} + \gamma^2\lambda_z^2 & \gamma\lambda_z^2 \\ 0 & \gamma\lambda_z^2 & \lambda_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} \lambda_z^{-1} & 0 & 0 \\ 0 & \lambda_z^{-1} & \gamma\lambda_z^{1/2} \\ 0 & \gamma\lambda_z^{1/2} & (1+\gamma^2)\lambda_z^2 \end{pmatrix}. \tag{178}$$

We have that $\det \mathbf{F} = 1$.

The first and second invariants are given by Eqs. (30.1) and (30.2)

$$I_1 = 2\lambda_z^{-1} + (1+\gamma^2)\lambda_z^2, \quad I_2 = \lambda_z^{-2} + (2+\gamma^2)\lambda_z. \tag{179}$$

As in the previous boundary value problem, we must choose a field and an alignment for the electro-active particles. Two cases are considered.

1. **Axial uniform electric field and axial alignment for the electro-active particles.**
For this case, we consider the external electric field $\mathsf{E}_l = (0, 0, E_o)^{\mathrm{T}}$, and the alignment of the particles in the reference configuration $\mathsf{a}_0 = (0, 0, 1)^{\mathrm{T}}$. The rest of the invariants (31)–(32) are

$$I_4 = E_o^2, \quad I_5 = E_o^2(1+\gamma^2)\lambda_z^2, \quad I_6 = E_o^2[\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4], \tag{180}$$

$$I_7 = (1+\gamma^2)\lambda_z^2, \quad I_8 = \gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4, \quad I_9 = E_o, \quad I_{10} = E_o(1+\gamma^2)\lambda_z^2. \tag{181}$$

The non-zero components of the Maxwell stress (18) are

$$\tau_{m_{rr}} = \tau_{m_{\theta\theta}} = -\frac{\varepsilon_o}{2}\lambda_z^{-2}E_o^2, \quad \tau_{m_{zz}} = \frac{\varepsilon_o}{2}\lambda_z^{-2}E_o^2. \tag{182}$$

The components of the stress (38) and the components of the electric displacement (41) are

$$\tau_{rr} = -p + 2\lambda_z^{-1}\Omega_1 + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2, \tag{183}$$

$$\begin{aligned}\tau_{\theta\theta} = &-p + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1 + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2 + 2E_o^2\gamma^2\lambda_z^2\Omega_5 \\ &+4E_o^2\gamma^2\lambda_z[1 + (1+\gamma^2)\lambda_z^3]\Omega_6 + 2\gamma^2\lambda_z^2\Omega_7 + 4\gamma^2\lambda_z[1+(1+\gamma^2)\lambda_z^3]\Omega_8 \\ &+2E_o\gamma^2\lambda_z^2\Omega_{10},\end{aligned} \tag{184}$$

$$\begin{aligned}\tau_{zz} = &-p + 2\lambda_z^2\Omega_1 + 4\lambda_z\Omega_2 + 2E_o^2\lambda_z^2\Omega_5 + 4E_o^2\lambda_z^4(1+\gamma^2)\Omega_6 + 2\lambda_z^2\Omega_7 \\ &+4\lambda_z^4(1+\gamma^2)\Omega_8 + 2E_o\lambda_z^2\Omega_{10},\end{aligned} \tag{185}$$

$$\begin{aligned}\tau_{\theta z} = &2\gamma\lambda_z\{\lambda_z\Omega_1 + \Omega_2 + E_o^2\lambda_z\Omega_5 + E_o^2[1 + 2(1+\gamma^2)\lambda_z^3]\Omega_6 + \lambda_z\Omega_7 \\ &+[1+2(1+\gamma^2)\lambda_z^3]\Omega_8 + E_o\lambda_z\Omega_{10}\},\end{aligned} \tag{186}$$

$$\tau_{r\theta} = \tau_{rz} = 0. \tag{187}$$

$$D_r = 0, \tag{188}$$

$$\begin{aligned}D_\theta = &-\gamma\{2E_o\lambda_z\Omega_4 + 2E_o[1+(1+\gamma^2)\lambda_z^3]\Omega_5 + 2E_o[\lambda_z^{-1} + (1+2\gamma^2)\lambda_z^2 \\ &+(1+2\gamma^2+\gamma^4)\lambda_z^5]\Omega_6 + \lambda_z\Omega_9 + [1+(1+\gamma^2)\lambda_z^3]\Omega_{10}\},\end{aligned} \tag{189}$$

$$\begin{aligned}D_z = &-\lambda_z\{2E_o\Omega_4 + 2E_o\lambda_z^2(1+\gamma^2)\Omega_5 + 2E_o\lambda_z[\gamma^2 + (1+2\gamma^2+\gamma^4)\lambda_z^5]\Omega_6 \\ &+\Omega_9 + \lambda_z^2(1+\gamma^2)\Omega_{10}\}.\end{aligned} \tag{190}$$

We prove that the above deformation is controllable. As in the problem of Sect. 5.3.1, if we decompose τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz} as $\tau_{rr} = -p + \tilde{\tau}_{rr}$, $\tau_{\theta\theta} = -p + \tilde{\tau}_{\theta\theta}$ and $\tau_{zz} = -p + \tilde{\tau}_{zz}$; remembering that $\gamma = \tau r$, and considering (179)–(181), we can show that $\tilde{\tau}_{rr} = \tilde{\tau}_{rr}(r)$, $\tilde{\tau}_{\theta\theta} = \tilde{\tau}_{\theta\theta}(r)$, $\tilde{\tau}_{zz} = \tilde{\tau}_{zz}(r)$ and $\tau_{\theta z} = \tau_{\theta z}(r)$. Then, Eq. (140) is satisfied automatically, and from Eqs. (139) and (141) we have that

$$-\frac{\partial p}{\partial r} + \frac{\mathrm{d}\tilde{\tau}_{rr}}{\mathrm{d}r} + \frac{1}{r}(\tilde{\tau}_{rr} - \tilde{\tau}_{\theta\theta}) = 0, \quad \frac{\partial p}{\partial z} = 0,$$

from where it is easy to see that p is a function of r, and that it can be calculated directly from (139).
As well as this, from (188) to (190) we have that $D_\theta = D_\theta(r)$ and $D_z = D_z(r)$, and as a result (143) is also satisfied; thus this deformation is universal. Since $-\infty \leq z \leq \infty$, it can be easily proved, as in the previous problems, that the boundary conditions (15) are satisfied.
Consider now the simplified form for the energy function

$$\omega = \omega(\lambda_z, \gamma, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10.$$

From Eqs. (179) to (181) we have

$$\begin{aligned}\frac{\partial \omega}{\partial \lambda_z} = &2\Omega_1[(1+\gamma^2)\lambda_z - \lambda_z^{-2}] + \Omega_2(2+\gamma^2 - 2\lambda_z^{-3}) + 2\Omega_5E_o^2(1+\gamma^2)\lambda_z \\ &+\Omega_6E_o^2[\gamma^2 + 4(1+\gamma^2)^2\lambda_z^3] + 2\Omega_7(1+\gamma^2)\lambda_z + \Omega_8[\gamma^2 + 4(1+\gamma^2)^2\lambda_z^3] \\ &+2\Omega_{10}E_o(1+\gamma^2)\lambda_z,\end{aligned} \tag{191}$$

$$\frac{\partial \omega}{\partial \gamma} = 2\gamma\lambda_z\{\Omega_1\lambda_z + \Omega_2 + \Omega_5 E_o^2\lambda_z + \Omega_6 E_o^2[1 + 2(1+\gamma^2)\lambda_z^3] + \Omega_7\lambda_z + \Omega_8[1 + 2(1+\gamma^2)\lambda_z^3] + \Omega_{10}E_o\lambda_z\}, \tag{192}$$

$$\frac{\partial \omega}{\partial E_o} = 2\Omega_4 E_o + 2\Omega_5 E_o(1+\gamma^2)\lambda_z^2 + 2\Omega_6 E_o[\gamma^2\lambda_z + (1+\gamma^2)^2\lambda_z^4] + \Omega_9 + \Omega_{10}(1+\gamma^2)\lambda_z^2, \tag{193}$$

and then it is possible to derive the simple connections

$$\tau_{\theta z} = \frac{\partial \omega}{\partial \gamma}, \quad D_z = -\lambda_z \frac{\partial \omega}{\partial E_o}. \tag{194}$$

2. **Radial electric displacement and radial orientation for the electro-active particles**
Consider the case where the external electric displacement has the vector form $\mathsf{D}_l = (D_o/R, 0, 0)^{\mathrm{T}}$ in the reference configuration. As a result $\mathsf{D} = (\lambda_z^{-1/2} D_o/R, 0, 0)^{\mathrm{T}}$, so $\mathsf{D} = (\lambda_z^{-1} D_o/r, 0, 0)^{\mathrm{T}}$ and this field satisfies Eq. (143). From Eq. (54) we have

$$K_4 = \frac{D_o^2\lambda_z^{-1}}{r^2}, \quad K_5 = \frac{D_o^2\lambda_z^{-2}}{r^2}, \quad K_6 = \frac{D_o^2\lambda_z^{-3}}{r^2}. \tag{195}$$

We consider a radially uniform alignment for the particles in the reference configuration given by $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$, then $\mathsf{a} = (\lambda_z^{-1/2}, 0, 0)^{\mathrm{T}}$, and the rest of the invariants (55) are (the first and second invariants are given in (179))

$$I_7 = \lambda_z^{-1}, \quad I_8 = \lambda_z^{-2}, \quad K_9 = \frac{D_o}{R}, \quad K_{10} = \frac{\lambda_z^{-1} D_o}{R}. \tag{196}$$

Using $R = \lambda_z^{1/2} r$, the components of the total stress (57) and the electric field (59) are

$$\tau_{rr} = -p^* + 2\lambda_z^{-1}\Omega_1^* + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2^* + 2\lambda_z^{-2}\left(\frac{D_o}{r}\right)^2\Omega_5^* + 4\lambda_z^{-3}\left(\frac{D_o}{r}\right)^2\Omega_6^* + 2\lambda_z^{-1}\Omega_7^* + 4\lambda_z^{-2}\Omega_8^* + 2\lambda^{-3/2}\frac{D_o}{r}\Omega_{10}^*, \tag{197}$$

$$\tau_{\theta\theta} = -p^* + 2(\lambda_z^{-1} + \gamma^2\lambda_z^2)\Omega_1^* + 2[\lambda_z^{-2} + (1+\gamma^2)\lambda_z]\Omega_2^*, \tag{198}$$

$$\tau_{zz} = -p^* + 2\lambda_z^2\Omega_1^* + 4\lambda_z\Omega_2^*, \tag{199}$$

$$\tau_{r\theta} = \tau_{rz} = 0, \tag{200}$$

$$\tau_{\theta z} = 2\gamma\lambda_z^2\Omega_1^* + 2\gamma\lambda_z\Omega_2^*, \tag{201}$$

and

$$E_r = 2\frac{D_o}{r}\Omega_4^* + 2\frac{D_o}{r}\lambda_z^{-1}\Omega_5^* + 2\frac{D_o}{r}\lambda_z^{-2}\Omega_6^* + \lambda_z^{1/2}\Omega_9^* + \lambda_z^{-1/2}\Omega_{10}^*. \tag{202}$$

$$E_\theta = E_z = 0, \tag{203}$$

which by the same reasons described in the previous problems is also universal and satisfies (142). Let us define $\xi = D_o/R$, and let us consider the simplified form for the energy function

$$\omega = \omega(\lambda_z, \gamma, \xi) = \Omega^*(I_i, K_j).$$

Then we have

$$\frac{\partial\omega}{\partial\lambda_z} = 2[(1+\gamma^2)\lambda_z - \lambda_z^{-2}]\Omega_1^* + \Omega_2^*(2+\gamma^2-2\lambda_z^{-3}) - \Omega_5^*\lambda_z^{-2}\xi^2 - 2\Omega_6^*\lambda_z^{-3}\xi^2 - \Omega_7^*\lambda_z^{-2} - 2\Omega_8^*\lambda_z^{-3} - \Omega_{10}^*\lambda_z^{-2}\xi, \tag{204}$$

$$\frac{\partial\omega}{\partial\gamma} = 2\Omega_1^*\gamma\lambda_z^2 + 2\Omega_2^*\gamma\lambda_z, \tag{205}$$

$$\frac{\partial\omega}{\partial\xi} = 2\Omega_4^*\xi + 2\Omega_5^*\lambda_z^{-1}\xi + 2\Omega_6^*\lambda_z^{-2}\xi + \Omega_9^* + \Omega_{10}^*\lambda_z^{-1}, \tag{206}$$

from which it follows that

$$E_r = \lambda^{1/2}\frac{\partial\omega}{\partial\xi}, \quad \tau_{\theta z} = \frac{\partial\omega}{\partial\gamma}. \tag{207}$$

There are two possibilities that we may study. One is to consider a uniform axial electric field with a uniform radial alignment for the electro-active particles, and the other is to consider a radial electric displacement as in the above problem, but with a uniform axial alignment field for the electro-active particles. In any of these two extra cases is not difficult to show that a shear in the radial direction appears, which implies the arbitrary pressure p cannot be assumed to be a function of r only. As a result, these cases are not controllable and we do not consider them here.

5.3.3 Helical shear

Helical shear [18,32] has been studied in the context of isotropic ES elastomers. In this section, we check in which situation the non-linear universal relation found in [18] holds.

From [18] helical shear was defined in cylindrical coordinates by

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \tag{208}$$

where g and w are unknown functions of R, and $A \le R \le B$, $0 \le \Theta < 2\pi$ and $-\infty \le Z \le \infty$. The matrix forms of the deformation gradient and the left and right Cauchy–Green tensors are, respectively,

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} 1 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1+\kappa_\theta^2 & \kappa_\theta\kappa_z \\ \kappa_z & \kappa_\theta\kappa_z & 1+\kappa_z^2 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} 1+\kappa^2 & \kappa_\theta & \kappa_z \\ \kappa_\theta & 1 & 0 \\ \kappa_z & 0 & 1 \end{pmatrix}, \tag{209}$$

where $\kappa_\theta = rg'(r)$, $\kappa_z = w'(r)$ and $\kappa^2 = \kappa_\theta^2 + \kappa_z^2$.

There are many possibilities for the external electric field or electric displacement and for the alignment of the electro-active particles. We only consider one case, a uniform radial electric field, and a radial alignment for the particles.

In this case, the external electric field is $\mathsf{E}_l = (E_o, 0, 0)^{\mathrm{T}}$, where E_o is a constant. The alignment of the electro-active particles in the reference configuration is $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$. Therefore, the electric field and the particle orientation in the current configuration are $\mathsf{E} = (E_o, 0, 0)^{\mathrm{T}}$, $\mathsf{a} = (1, \kappa_\theta, \kappa_z)^{\mathrm{T}}$.

The invariants are given by (30)–(32):

$$I_1 = I_2 = 3 + \kappa^2, \quad I_4 = E_o^2, \quad I_5 = E_o^2(1+\kappa^2), \tag{210}$$

$$I_6 = E_o^2[\kappa^2 + (1+\kappa^2)^2], \quad I_7 = 1 + \kappa^2, \quad I_8 = \kappa^2 + (1+\kappa^2)^2, \tag{211}$$

$$I_9 = E_o, \quad I_{10} = E_o(1+\kappa^2). \tag{212}$$

From Eqs. (38) and (41) the components of the stress and the electric displacement are

$$\tau_{rr} = -p + 2\Omega_1 + 4\Omega_2 + 2E_o^2\Omega_5 + 4E_o^2(1+\kappa^2)\Omega_6 + 2\Omega_7 + 4(1+\kappa^2)\Omega_8 + 2E_o\Omega_{10}, \tag{213}$$

$$\tau_{\theta\theta} = -p + 2(1+\kappa_\theta^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_o^2\kappa_\theta^2\Omega_5 + 4E_o^2\kappa_\theta^2(2+\kappa^2)\Omega_6 + 2\kappa_\theta^2\Omega_7 + 4\kappa_\theta^2(2+\kappa^2)\Omega_8 + 2E_o\kappa_\theta^2\Omega_{10}, \tag{214}$$

$$\tau_{zz} = -p + 2(1+\kappa_z^2)\Omega_1 + 2(2+\kappa^2)\Omega_2 + 2E_o^2\kappa_z^2\Omega_5 + 4E_o^2\kappa_z^2(2+\kappa^2)\Omega_6 \\ +2\kappa_z^2\Omega_7 + 4\kappa_z^2(2+\kappa^2)\Omega_8 + 2E_o\kappa_z^2\Omega_{10}, \tag{215}$$

$$\tau_{r\theta} = 2\kappa_\theta[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2(3+2\kappa^2)\Omega_6 + \Omega_7 + (3+2\kappa^2)\Omega_8 + E_o\Omega_{10}], \tag{216}$$

$$\tau_{rz} = 2\kappa_z[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2(3+2\kappa^2)\Omega_6 + \Omega_7 + (3+2\kappa^2)\Omega_8 + E_o\Omega_{10}], \tag{217}$$

$$\tau_{\theta z} = 2\kappa_z\kappa_\theta[\Omega_1 + E_o^2\Omega_5 + 2E_o^2(2+\kappa^2)\Omega_6 + \Omega_7 + 2(2+\kappa^2)\Omega_8 + E_o\Omega_{10}], \tag{218}$$

and

$$D_r = -[2E_o\Omega_4 + 2E_o(1+\kappa^2)\Omega_5 + 2E_o(1+3\kappa^2+\kappa^4)\Omega_6 + \Omega_9 + (1+\kappa^2)\Omega_{10}], \tag{219}$$

$$D_\theta = -\kappa_\theta[2E_o\Omega_4 + 2E_o(2+\kappa^2)\Omega_5 + 2E_o(3+4\kappa^2+\kappa^4)\Omega_6 + \Omega_9 + (2+\kappa^2)\Omega_{10}], \tag{220}$$

$$D_z = -\kappa_z[2E_o\Omega_4 + 2E_o(2+\kappa^2)\Omega_5 + 2E_o(3+4\kappa^2+\kappa^4)\Omega_6 + \Omega_9 + (2+\kappa^2)\Omega_{10}]. \tag{221}$$

Regarding the total stress tensor, we can prove that its components satisfy the following non-linear universal relation (see, for example, [18,19,32])

$$(\tau_{\theta\theta} - \tau_{zz})\tau_{rz}\tau_{r\theta} = \tau_{\theta z}(\tau_{r\theta}^2 - \tau_{rz}^2). \tag{222}$$

This relation is also satisfied by the components of the stress if they are calculated from the constitutive equation (57) for the electric displacement $\mathsf{D}_l = (D_o/R, 0, 0)^{\mathrm{T}}$, and for a uniform radial field alignment for the particles $\mathsf{a}_0 = (1, 0, 0)^{\mathrm{T}}$.

Let us define

$$\omega = \omega(\kappa_\theta, \kappa_z, E_o) = \Omega(I_i), \quad i = 1, 2, \ldots, 10.$$

Then

$$\frac{\partial\omega}{\partial\kappa_\theta} = 2\kappa_\theta[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2\Omega_6(1+2\kappa^2) + \Omega_7 + \Omega_8(1+2\kappa^2)E_o\Omega_{10}], \tag{223}$$

$$\frac{\partial\omega}{\partial\kappa_z} = 2\kappa_z[\Omega_1 + \Omega_2 + E_o^2\Omega_5 + E_o^2\Omega_6(1+2\kappa^2) + \Omega_7 + \Omega_8(1+2\kappa^2)E_o\Omega_{10}], \tag{224}$$

$$\frac{\partial\omega}{\partial E_o} = 2E_o\Omega_4 + 2E_o\Omega_5(1+\kappa^2) + 2E_o\Omega_6(1+3\kappa^2+\kappa^4) + \Omega_9 + \Omega_{10}(1+\kappa^2), \tag{225}$$

from which we get the connections

$$\tau_{r\theta} = \frac{\partial\omega}{\partial\kappa_\theta}, \quad \tau_{rz} = \frac{\partial\omega}{\partial\kappa_z}, \tag{226}$$

and the connection for the radial component of the electric displacement

$$D_r = -\frac{\partial\omega}{\partial E_o}. \tag{227}$$

The boundary conditions (15) are satisfied trivially if $-\infty \leq Z \leq \infty$.

6 Conclusions

In the present paper, we developed a theory for transversely isotropic electro-sensitive elastomers, based on a previous work for isotropic ES elastomers [11]. Different experimental data [3–5] show that a preferred orientation for the electro-active particles may enhance significantly the properties of these materials in comparison with the isotropic case.

The different examples of boundary value problems show that the 'controllability' of a solution of the boundary value problem depends strongly on the relative alignment of the particles with respect to the electric field or electric displacement. It is necessary to point out again that the controllable solutions presented here were obtained for 'semi-infinite' geometries to work with the boundary conditions (15).

To work with an energy function that depends on ten invariants (see, for example, Eq. (37)) will cause great problems to find this function from experiments. It will be necessary to assume a simplified form for the function Ω (or Ω^*). This has been done for transversely MS elastomers by Bustamante and Ogden [16], where they chose to work with seven of these ten invariants. It is highly desirable in such a case to have a criterion to known in advance whether such simplifications are realistic, such a criterion is provided by the 'universal relations' [33]. For transversely isotropic electro-active elastomer some universal relations have been found for some simplified forms of the constitutive equation; for brevity, these results are not shown here and can be found in Chapter 8 of the thesis by Bustamante [20].

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