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On the Lagrangian electrostatics of elastic solids

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Abstract The equilibrium equations and the Maxwell equations of electrostatics are derived here through a variational procedure. All the treatment is expounded in the reference configuration of the deformed body and the electromagnetic quantities and equations of interest are expressed in Lagrangian form. The total stress tensor, which includes the electrical effects, emerges in a natural way in this framework. Along with this stress, an additional electromechanical tensor is introduced, the Eshelby stress tensor. This tensor, which plays a relevant role in the presence of inhomogeneities or material defects, addresses remarkable integral identities that are valuable in boundary value problems. In this framework, a Lagrangian Electrostatics emerges as a self-consistent topic that parallels the classical electrostatics. Eventually, simple constitutive relationships are discussed.

1 Introduction

Electromagnetic fields in Lagrangian form are preliminarily introduced. These fields are related to the classical ones through the deformation, in such a way that the Maxwell equations preserve their form, passing from the current to the reference configuration of the solid [1–7]. The form invariance of the Maxwell equations in the various configurations essentially accounts for the conservation of the total electric charge. In this context, a Lagrangian electrostatics, as opposed to a classical electrostatics, can be envisaged as a self-consistent topic either for dielectrics or for electric conductors, if the deformation in known.

The coupling of the mechanical and electrostatic equations can be established in a natural way through a variational procedure, by introducing the notion of electric enthalpy [8,9]. As this quantity includes the electric energy density, which is present inside and outside the material of interest, the enthalpy density is also defined in a vacuum. As a result of the variational procedure, the Maxwell equations and the equilibrium conditions for the stress tensor are recovered also in a vacuum. Jump conditions across the material boundary also stem in a natural way from this approach. It is worth mentioning that the total stress tensor in the material includes the Maxwell stress, which pertains otherwise to a vacuum.

An electrostatic Eshelby-like stress tensor is also introduced in this framework. As is known, the Eshelby stress tensor accounts for the presence of material inhomogeneities in elastic materials [3,4,10,11]. In electromagnetic materials, this tensor also accounts for the presence of inhomogeneities in a broader context, which includes the electric free charge that plays the role of an inhomogeneity for the Maxwellian fields. The general expression of this tensor for a deformable electromagnetic body can be found in [3–6,12–15]. It is here re-proposed in the framework of electrostatics in two slightly different forms. One of these two forms includes the Maxwell stress tensor, which by contrast does not appear in the second reduced form. Although the difference among the two forms is only apparent, it becomes relevant in the presence of electric free-charges.

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This relevance specially emerges in boundary value problems [9]. In fact, by adopting the Eshelby tensor in the reduced form, the polarisation and the 'elastic part of the stress' (which is a well defined quantity) are the quantities that are naturally involved at the boundary of the elastic dielectric. Differently, the electric displacement, which is intimately related with the free charges, and the total traction naturally appear in the boundary conditions, if one refers to the alternate form. Thus, the two different forms of the Eshelby tensor address in a natural way different sets of boundary conditions.

Constitutive relationships in the Lagrangian electrostatics are also discussed. In analogy with the classic electrostatics, the Lagrangian electrostatics deals with the Lagrangian electrostatic potential and with the Lagrangian electric displacement. However, the Lagrangian electrostatics does not constitute a *replica* of the classical electrostatics [16–18]. The main reason is due to an irrecoverable lack of invariance for the constitutive relations in the various configurations. For instance, simple constitutive relationships that hold true in the reference configuration of an isotropic and homogeneous configuration of a dielectric fail in the current configuration. As a result, the Lagrangian electrostatic potential is a harmonic function in the reference configuration, whereas the corresponding classical electrostatic potential in the current configuration is generally not. In addition, the derivative of this potential with respect to the unit normal of the current boundary of a conductor does not necessarily express the surface free charge, as in the classical electrostatics.

As a final remark, it is worthwhile to note that the right Cauchy-Green deformation tensor plays the role of a metric tensor in the relationship between the Lagrangian electric displacement and the Lagrangian electric field. It is noticeable that this remark extends to a vacuum.

2 The Lagrangian fields of electrostatics

In the following, V and $\mathcal{V}$ denote the region occupied by a dielectric in its reference and in its current configuration, respectively. $V \subseteq E_3$ and $\mathcal{V} \subseteq E_3$, E_3 representing the Euclidean space. $\mathcal{V}$ is the image of V through the deformation $\boldsymbol{\kappa}: \mathbf{X} \rightarrow \mathbf{x}, \mathbf{X} \in V$ and $\mathbf{x} \in V$. $\boldsymbol{\kappa} \in C^3(V, \mathcal{V})$. If the dielectric is of finite extent, ∂V and $\partial \mathcal{V}$ represent the boundary of V and $\mathcal{V}$, respectively. $\mathbf{N}$ and $\mathbf{n}$ denote the outward unit normal to these boundaries, respectively. $\mathbf{F}$ denotes the deformation gradient and $\mathbf{F}^{\mathrm{T}}$ its transpose. $J \equiv (\det \mathbf{F})$ is assumed to be strictly positive, as usual [19–21]. $\mathbf{E}(\mathbf{x})$ represents the classical electric field and $\mathbf{D}(\mathbf{x})$ the electric displacement. These are the only fields that govern electrostatics, having assumed that all other Maxwellian fields identically vanish. According to the Maxwell equations, $\mathbf{E}$ is an irrotational field, whereas $\mathbf{D}$ is solenoidal in the absence of free charges [16–18]. Along with these classical fields, the following Lagrangian fields are introduced:

$$\boldsymbol{\mathcal{E}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \tag{1}$$

$$\boldsymbol{\mathcal{D}} = \mathbf{J}\mathbf{F}^{-1}\mathbf{D}, \tag{2}$$

$$\mathbb{P} = \mathbf{J}\mathbf{F}^{-1}\mathbf{P}. \tag{3}$$

These fields satisfy the Maxwell equations in Lagrangian form, which are

$$\mathrm{Div}\boldsymbol{\mathcal{D}} = J\rho_{\mathrm{e}}, \tag{4}$$

$$\mathrm{Curl}\boldsymbol{\mathcal{E}} = 0 \quad \text{in } E_3 - \partial V; \tag{5}$$

$$[\boldsymbol{\mathcal{D}}] \bullet \mathbf{N} = \Sigma, \tag{6}$$

$$[\boldsymbol{\mathcal{E}}] \times \mathbf{N} = 0 \quad \text{across } \partial V. \tag{7}$$

$(J\rho_{\mathrm{e}})$ is defined in a domain $V_{\mathrm{e}} \subset E_3$ and represents the electric free charge density per unit volume of the reference configuration. Σ is the electric free charge density per unit surface. This surface charge is possibly present on part of the dielectric boundary $\partial V_1 \subseteq \partial$V. Square brackets in the Eqs. (6) and (7) denote the finite discontinuity of the quantities included. Basing on the Eq. (5), the Lagrangian electric potential Φ is introduced so that

$$\boldsymbol{\mathcal{E}} = -\mathrm{Grad}\,\Phi. \tag{8}$$

It is worth remarking that the differential operators Div, Curl and Grad act in the reference configuration. In analogy, div, curl and grad denote hereafter the corresponding differential operators in the current configuration. The relationships among the above introduced material electrostatic fields deserve a special attention and will be discussed in a subsequent section. It is reported here only the classical link that relates **E**, **D** and the polarisation **P,** in S.I. units:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad \text{in } E_3 - \mathcal{V}, \tag{9}$$
$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{in } \mathcal{V}. \tag{10}$$

ε_0 represents the electric permeability or permittivity of a vacuum.

As **P** is a feature of the material, it has to be understood as defined only inside the material. Consistency with Eq. (9) suggests the relationship between $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{E}}$ in $(E_3 - V)$. This relationship, which is found by taking into account the expressions (1) and (2), reads

$$\boldsymbol{\mathcal{D}} = \varepsilon_0 J \mathbf{C}^{-1} \boldsymbol{\mathcal{E}} + \mathbb{P}, \tag{11}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the right Cauchy-Green deformation tensor. One can note that in the formula (11), $\mathbf{C}^{-1}$ acts like a metric tensor on the material electric field and this role of metric tensor survives also in the absence of polarisation. This remark warns against a straightforward identification of the electric displacement with the electric field in a vacuum.

3 Mechanical and Electrical equilibrium

In order to proceed with a variational approach, the notion of electric enthalpy is introduced. Its density per unit volume of the reference configuration V is

$$H = W(\mathbf{F}, \mathbf{P}, \mathbf{X}) - {}^1\!/\!{}_2 \varepsilon_0 J \mathbf{E}^2 - J\mathbf{E} \bullet \mathbf{P}, \tag{12}$$

where W is the store energy of the hyperelastic dielectric material [8,22].

In the expression (12) the fields **E** and **P** are assumed to be independent fields. Alternatively, they can be assumed to be dependent fields and understood specifically as $\mathbf{E} = \boldsymbol{E}(\mathbf{F}, \mathrm{Grad}\Phi)$ and $\mathbf{P} = \boldsymbol{P}(\mathbf{F}, \mathbb{P})$. In this view, H turns out to represent a reduced form of a general Lagrangian that is discussed and exploited in [5,6,23,24]. Therefore, the enthalpy H can be re-written as

$$L = {}^1\!/\!{}_2 \varepsilon_0 J \boldsymbol{\mathcal{E}} \bullet \mathbf{C}^{-1} \boldsymbol{\mathcal{E}} + \mathbb{P} \bullet \boldsymbol{\mathcal{E}} - \hat{\mathrm{W}}(\mathbf{F}, \mathbf{F}\mathbb{P}, \mathbf{X}). \tag{13}$$

Here, the stored energy $\hat{\mathrm{W}}$ depends on the polarisation density per unit volume of the reference configuration, through the quantity $\mathbf{F}\mathbb{P} \equiv \mathbf{JP}$.

In this context, one of the Euler-Lagrange equations that is derived from L is

$$\mathrm{Div}(\partial L / \partial \boldsymbol{\mathcal{E}}) = 0. \tag{14}$$

Note that this equation corresponds to the Eq. (4) in the absence of electric free charge density, as

$$(\partial L / \partial \boldsymbol{\mathcal{E}}) = \varepsilon_0 \mathrm{J} \mathbf{C}^{-1} \boldsymbol{\mathcal{E}} + \mathbb{P} \equiv \boldsymbol{\mathcal{D}}. \tag{15}$$

A second Euler-Lagrange equation is

$$\mathrm{Div}(\partial L / \partial \mathbf{F}) = 0. \tag{16}$$

This represents the equation of the mechanical equilibrium, if the tensor valued quantity $-(\partial \mathrm{L} / \partial \mathbf{F}) \equiv \mathbf{T_R}$ is identified with the first Piola-Kirchhoff stress tensor.

Equation (16) extends to a vacuum, where $\mathbb{P}$ and W identically vanish and the enthalpy reduces to the Lagrangian electrostatic energy $({}^1\!/\!{}_2 \varepsilon_0 J \boldsymbol{\mathcal{E}} \bullet \mathbf{C}^{-1} \boldsymbol{\mathcal{E}})$. In this case, the following stress tensor is defined in a vacuum and reads

$$-(\partial L / \partial \mathbf{F}) = J \mathbf{T}_{\mathrm{M}} \mathbf{F}^{-\mathrm{T}}, \tag{17}$$

where

$$\mathbf{T}_{\mathrm{M}} \equiv \quad \varepsilon_0 (\mathbf{E} \otimes \mathbf{E} - {}^1\!/\!{}_2 \mathrm{E}^2 \mathbf{I}) \tag{18}$$

is the Maxwell stress tensor, and **I** denotes the second order identity tensor.

The formulae (17) and (18) are interesting, as they provide the Maxwell stress tensor due to an electric field that pervades a vacuum. The procedure can be considered rather artificial, as an unlike deformation of a vacuum underlies the treatment. However, nothing prevents one to conceive a vacuum as a deformable material, whose mass density tends toward zero.

In this context, the following equation

$$[(\partial L/\partial \mathbf{F})]\mathbf{N} = 0 \quad \text{across } \partial \mathrm{V}, \tag{19}$$

along with the Eq. (6), is also derived from L through the variational procedure.

The third Euler Lagrange equation is

$$(\partial L/\partial \mathbb{P}) = 0 \tag{20}$$

or, equivalently,

$$(\partial \hat{\mathrm{W}}/\partial \mathbb{P}) = \boldsymbol{\mathcal{E}}. \tag{21}$$

This equation has to be understood as a constitutive relationship.

It is worth reporting the total Cauchy stress tensor $\mathbf{T} = -J^{-1}\mathbf{T}_{\mathrm{R}}\mathbf{F}^{\mathrm{T}}$. Its explicit expression, which is found by taking into account all the previous formulas, is the following [1–6, 14, 23]:

$$\mathbf{T} = \mathbf{T}_{\mathrm{M}} + \mathbf{E} \otimes \mathbf{P} + J^{-1}(\partial \hat{W}(\mathbf{F}, \mathbf{F}\mathbb{P})/\partial \mathbf{F})\mathbf{F}^{\mathrm{T}}. \tag{22}$$

4 The Eshelby tensor

In the presence of electric fields, the Eshelby stress, a material or configurational stress tensor is defined as

$$\boldsymbol{b} = -L\mathbf{I} + (\mathrm{Grad}\,\Phi) \otimes (\partial L/\partial \boldsymbol{\mathcal{E}}) + \mathbf{F}^{\mathrm{T}}(\partial L/\partial \mathbf{F}) \tag{23}$$

or, equivalently,

$$\boldsymbol{b} = -L\mathbf{I} + \boldsymbol{\mathcal{E}} \otimes \boldsymbol{\mathcal{D}} + \mathbf{F}^{\mathrm{T}}(\partial L/\partial \mathbf{F}), \tag{24}$$

if the formulae (8) and (15) are taken into account [3–6, 12–15, 24].

The tensor $\boldsymbol{b}$ is defined in a referential frame of the dielectric and is not related to the physical stress. In fact, the quantity $(\boldsymbol{b}\mathbf{N})\mathrm{dS}$, dS denoting the surface element of $\partial \mathrm{V}$, does not represents the physical traction in the reference configuration. By contrast the stress tensors $\mathbf{T}$ and $\mathbf{T}_{\mathrm{M}}$ that have been introduced in the previous section through the Eqs. (2) and (18), respectively, are naturally defined in the current frame of reference and are related to the Cauchy-traction.

It is not difficult to check that the following identity holds true along the solution of the electro-mechanical problem:

$$\mathrm{Div}\,\boldsymbol{b} = -(\partial L/\partial \mathbf{X}) \equiv (\partial \hat{W}/\partial \mathbf{X}) \quad \text{in V}. \tag{25}$$

In the absence of inhomogeneities the right hand side of the formula (25) identically vanishes. However, if electric free charges are present in a domain V_{e}, the relationship (25) turns into

$$\mathrm{Div}\,\boldsymbol{b} = J\rho_{\mathrm{e}}\boldsymbol{\mathcal{E}} \quad \text{in } \mathrm{V}_{\mathrm{e}}. \tag{26}$$

This result, which is found by virtue of the Eq. (4), shows in evidence that the electric free charge ρ_{e} represents a source of an inhomogeneity force $(J\rho_{\mathrm{e}}\boldsymbol{\mathcal{E}})$, in fact the electrostatic force in the reference configuration.

4.1 Reduced form of the Eshelby tensor

Along with the definition (24) of the Eshelby tensor, an alternate form for this tensor is the following [5,6,12–14]:

$$\mathbf{b} = (\mathrm{W} - \boldsymbol{\mathcal{E}} \bullet \mathbb{P})\mathbf{I} + \boldsymbol{\mathcal{E}} \otimes \mathbb{P} - \mathbf{F}^{\mathrm{T}}(\partial\hat{\mathrm{W}}(\mathbf{F}, \mathbf{F}\mathbb{P})/\partial\mathbf{F}). \tag{27}$$

It is not difficult to check that, in the absence of inhomogeneities, $\mathbf{b}$ satisfies the identity

$$\mathrm{Div}\mathbf{b} = 0 \quad \text{in V}. \tag{28}$$

The formula (27) stems directly from the definition, by developing explicitly the last term at right hand side of the formula (27) as follows:

$$\mathbf{F}^{\mathrm{T}}(\partial L/\partial\mathbf{F}) = \{{}^{1}\!/_{2}\varepsilon_{\mathrm{o}} J\boldsymbol{\mathcal{E}} \bullet \mathbf{C}^{-1}\boldsymbol{\mathcal{E}}\}\mathbf{I} + \boldsymbol{\mathcal{E}} \otimes (\varepsilon_{\mathrm{o}} J\mathbf{C}^{-1}\boldsymbol{\mathcal{E}}) - \mathbf{F}^{\mathrm{T}}(\partial\hat{\mathrm{W}}(\mathbf{F}, \mathbf{F}\mathbb{P})/\partial\mathbf{F}). \tag{29}$$

The expression (24) is readily recovered by substituting the formulas (11) and (29) into the formula (24). Thus, $\mathbf{b}$ and $\boldsymbol{b}$ represent the same quantity though expressed through different fields. However, according to the formula (27), $\mathbf{b}$ is only defined inside the dielectric, whereas the expression (24) suggests that $\boldsymbol{b}$ also extends to a vacuum.

On the base of these remarks, one foresees the possibility of prescribing different sets of conditions in boundary value problems. This possibility is discussed in the following section.

5 Eshelby tensor in boundary value problems: remarkable identities

The introduction of the Eshelby tensor addresses a few identities that are reported below.

A first identity is trivially achieved by integrating Eq. (28) and by appealing to the Gauss theorem:

$$\int_{\partial V} \mathbf{bN}\mathrm{d}S = \int_{\partial V} \left\{(\mathrm{W} - \boldsymbol{\mathcal{E}} \bullet \mathbb{P})\mathbf{N} + \boldsymbol{\mathcal{E}}(\mathbb{P} \bullet \mathbf{N}) - \mathbf{F}^{\mathrm{T}}(\partial\hat{\mathrm{W}}/\partial\mathbf{F})\mathbf{N}\right\} \mathrm{d}S. \tag{30}$$

Notice that in this equation only the 'elastic part of the stress' $(\partial\hat{\mathrm{W}}/\partial\mathbf{F})$, which is part of the total stress, is involved at the boundary. Also notice that the polarisation charge density at the surface-boundary of the reference configuration $(\mathbb{P} \bullet \mathbf{N})$ appears at right hand side of the equality (30). It can be readily shown that $(\mathbb{P} \bullet \mathbf{N})\mathrm{d}S = (\mathbf{P} \bullet \mathbf{n})\mathrm{d}s$, $\mathrm{d}s$ representing the elementary surface area of $\partial\mathcal{V}$ and $(\mathbf{P} \bullet \mathbf{n})$ the classical surface polarisation charge density. It is worth remarking that in classical electrostatics, this polarisation is related to the jump of the normal component of the electric field across $\partial\mathcal{V}$ [16–18]. This may not be the case in Lagrangian electrostatics, as we will see in the next section.

Following a similar procedure, one can perform the integration of equation (24) and write a second integral identity

$$\int_{\partial V} [\boldsymbol{b}\mathbf{N}]\mathrm{d}S = -\int_{\partial V} \left\{[L]\mathbf{N} - [\boldsymbol{\mathcal{E}}](\boldsymbol{\mathcal{D}} \bullet \mathbf{N}) + [\mathbf{F}^{\mathrm{T}}(\partial L/\partial\mathbf{F})]\mathbf{N}\right\} \mathrm{d}S, \tag{31}$$

under classical standard assumptions [19]. Note that the quantity $(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})$ is continuous across ∂V in the absence of surface free charges, in accordance with Eq. (6).

By taking into account the mechanical equilibrium condition (19) and the Eq. (7), the identity (31) also reads

$$\int_{\partial V} [\boldsymbol{b}\mathbf{N}]\mathrm{d}S = -\int_{\partial V} \left\{[L + \mathbf{F} \bullet \langle(\partial L/\partial\mathbf{F})\rangle] - [\boldsymbol{\mathcal{E}} \bullet \mathbf{N}](\boldsymbol{\mathcal{D}} \bullet \mathbf{N})\right\} \mathbf{N}\mathrm{d}S. \tag{32}$$

where $\langle(\partial L/\partial\mathbf{F})\rangle \equiv {}^{1}\!/_{2}\{(\partial L/\partial\mathbf{F})^{+} + (\partial L/\partial\mathbf{F})^{-}\}$. Superscripts + and − denote the limit of the quantities included in the round brackets $(\bullet)^{+}$ and $(\bullet)^{-}$, as the oriented boundary ∂V is approached from the exterior and from the interior domain, respectively.

If surface free charges Σ are present on part of the boundary ∂V_1, Eq. (32) modifies as follows:

$$\int_{\partial V}[\boldsymbol{b}\mathbf{N}]\mathrm{d}S=-\int_{\partial V1}\Big\{\{[L+\mathbf{F}\bullet\langle(\partial L/\partial\mathbf{F})\rangle]-[\boldsymbol{\mathcal{E}}\bullet\mathbf{N}]\langle\boldsymbol{\mathcal{D}}\bullet\mathbf{N}\rangle\}\mathbf{N}-\langle\boldsymbol{\mathcal{E}}\rangle\Sigma\Big\}\,\mathrm{d}S$$
$$-\int_{\partial V-\partial V1}\Big\{[L+\mathbf{F}\bullet\langle(\partial L/\partial\mathbf{F})\rangle-[\boldsymbol{\mathcal{E}}\bullet\mathbf{N}](\boldsymbol{\mathcal{D}}\bullet\mathbf{N})\Big\}\,\mathbf{N}\mathrm{d}S. \tag{33}$$

In the formula (33), the Eq. (6) across ∂V_1 has been taken into account, along with the identity

$$[\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})]=[\boldsymbol{\mathcal{E}}]\langle\boldsymbol{\mathcal{D}}\bullet\mathbf{N}\rangle+\langle\boldsymbol{\mathcal{E}}\rangle[\boldsymbol{\mathcal{D}}\bullet\mathbf{N}], \tag{34}$$

where $\langle\boldsymbol{\mathcal{D}}\bullet\mathbf{N}\rangle\equiv 1/2\{(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})^{+}+(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})^{-}\}$.

The identities (32) and (33) will be discussed below, after establishing another integral identity.

From the definition (24) of the Eshelby tensor the following formula can be readily derived:

$$\mathrm{Div}(\boldsymbol{b}\mathbf{X})=\mathrm{tr}\,\boldsymbol{b}, \tag{35}$$

where (tr $\boldsymbol{b}$) denotes the spur of $\boldsymbol{b}$.

Integration of the formula (35) in the domain V occupied by the dielectric leads to the following remarkable integral equality:

$$3\int_{V}L\mathrm{d}V=\int_{\partial V}\big\{(L)(\mathbf{N}\bullet\mathbf{X})-(\Phi-(\mathrm{Grad}\,\Phi)\bullet\mathbf{X})(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})-(\partial L/\partial\mathbf{F})\mathbf{N}\bullet(\mathbf{x}-\mathbf{F}\mathbf{X})\big\}\,\mathrm{d}S. \tag{36}$$

It is worth mentioning that this equality generalizes the Green's identity for elastic bodies [9,25].

From the physical standpoint, it seems feasible to consider **b** as the proper representative of the Eshelby stress tensor, as it is expressed in terms of fields that are solely defined in the dielectric. However, the definition of $\boldsymbol{b}$, such as expressed by the formula (24) is very useful. In fact, this naturally leads to the quantity $(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})$ that appears at right hand side of the integral formulas (32), (33) and (36). The prescription of the quantity $(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})$ represents the natural electrostatic boundary conditions either for a dielectric or of a conductor. More specifically, either $(\boldsymbol{\mathcal{D}}\bullet\mathbf{N})$ is continuous across the dielectric boundary, as in the case of the formula (32), or its jump $[\boldsymbol{\mathcal{D}}\bullet\mathbf{N}]$ is equal to the surface density of the free charge, in accordance with the Eq. (6).

It is worth noting that the identity (36) represents a suitable premise for establishing uniqueness results in elastic dielectrics [9]. It is worth reporting here the Ericksen's view on stresses and configurational stresses in electrostatics [26].

6 Simple constitutive relationships: the electrostatic problem

A slightly different though equivalent approach is proposed hereafter in introducing the constitutive relationships. Consider a homogeneous and isotropic rigid dielectric. The related classical constitutive relationships in S.I. units are

$$\mathbf{P}=\varepsilon_{o}\chi\mathbf{E}, \tag{37}$$
$$\mathbf{D}=\varepsilon_{o}\varepsilon\mathbf{E}, \tag{38}$$

where $\varepsilon=1+\chi.\varepsilon$ and χ represent the relative electric permeability and the electric susceptibility of the dielectric, respectively [16–18]. It is worth recalling that in a vacuum χ identically vanish and that ε reduces to unity. Assume that the dielectric that occupies a domain $\mathcal{V}$ of finite extent is immersed in a given external electric field, whose remote source is at infinity. The electric field is an irrotational field, $\mathbf{E}=\mathrm{grad}\varphi$, apart that at the boundary where it suffers a discontinuity. The mathematical problem reduces to the following one for the electrostatic potential φ:

$$\Delta\varphi=0\quad\text{in } E_3-\partial\mathcal{V}, \tag{39}$$
$$[\varphi]=0\quad\text{across }\partial\mathcal{V}, \tag{40}$$

$$\varepsilon_o(\partial\varphi/\partial\mathbf{n})^+ = \varepsilon_o\varepsilon(\partial\varphi/\partial\mathbf{n})^- \quad \text{across } \partial\mathcal{V}, \tag{41}$$

$$\varphi|_\infty \quad \text{given.} \tag{42}$$

Δ represents the Laplace operator .

The condition (42) expresses the physical fact that the sources, namely the electric charges, are located at remote distances from the dielectric. The problem has a unique solution, which is not identically vanishing provided that $\varphi|_\infty$ does not identically vanish.

In this context the stress tensor is

$$\mathbf{T} = \varepsilon\mathbf{T}_M \quad \text{in } \mathcal{V} \tag{43}$$

$$\mathbf{T} = \mathbf{T}_M \quad \text{in } E_3 - \mathcal{V}. \tag{44}$$

The overall traction across $\partial\mathcal{V}$ is given by the jump condition for $\mathbf{Tn}$, $\mathbf{n}$ being the outward unit normal to $\mathcal{V}$. Note that the resultant force is possibly not balanced in this problem.

If the dielectric is elastic but its behaviour is still described by the constitutive equations (37) or (38), the equations (39)–(42) should be satisfied along with the equilibrium equations, which involve the deformation and the total stress such as given by the formula (17). In addition, boundary conditions (either placements or traction or mixed conditions) need to be established for the mechanical equilibrium.

If the deformation is known, the electrostatic problem can be re-proposed in the reference configuration of the elastic dielectric. In this framework, the constitutive relationship (37) or (38) can be re-written for the Lagrangian fields in accordance with all relationships of Sect. 2:

$$\mathbb{P} = J\varepsilon_o\chi\mathbf{C}^{-1}\boldsymbol{\mathcal{E}}, \tag{45}$$

$$\boldsymbol{\mathcal{D}} = J\varepsilon_o\varepsilon\mathbf{C}^{-1}\boldsymbol{\mathcal{E}}. \tag{46}$$

These equations show in evidence the fact that the homogeneity and the isotropy of the dielectric in the current configuration are not preserved in the reference configuration. If Eqs. (4)–(8) and (46) are taken into account, the equation that governs the electrostatic potential Φ for a given deformation is

$$\text{Div}(J\mathbf{C}^{-1}\text{Grad}\Phi) = 0 \quad \text{in } E_3 - \partial V. \tag{47}$$

Clearly Φ is not a harmonic function as φ is.

Alternatively, one can consider the case of a dielectric, in which the following very simple constitutive equations hold true

$$\boldsymbol{\mathcal{D}} = \varepsilon_o\varepsilon\boldsymbol{\mathcal{E}} \tag{48}$$

$$\mathbb{P} = \varepsilon_o\chi\boldsymbol{\mathcal{E}}, \tag{49}$$

in a rather formal analogy with the Eqs. (37) and (38). Different from to the previous case, the dielectric that is described by Eq. (48) or Eq. (49) is a special case of homogeneous and isotropic material only in the undeformed configuration [7,22]. This dielectric is a different material with respect to the previous one.

Basing on Eq. (8), the electrostatic potential $\Phi(\mathbf{X})$ turns out to be a harmonic function for this dielectric, contrary to $\varphi(\mathbf{x})$, which has to satisfy a more involved field equation in this case. The relationship between Φ and φ is given by the formula (1) and explicitly reads

$$\text{Grad}\Phi = \mathbf{F}^T\text{grad}\varphi. \tag{50}$$

If the deformation is a smooth mapping, apart possibly that at the boundary, Eq. (50) entails the relation

$$\Phi(\mathbf{X}) = \varphi(\mathbf{x}(\mathbf{X})), \tag{51}$$

according to which the electrostatic potentials Φ and φ have the same value in each point of E_3, but different forms: one addresses a Lagrangian description, whereas the other accounts for an Eulerian description.

The formula (51) may be misleading in some circumstances or become a source of misunderstandings, if out of the appropriate context. For instance, the electric field at the current boundary of a conductor is given by the quantity $\{-(\partial\varphi/\partial\mathbf{n})\}$, as the electrostatic potential has a constant value on this boundary. Accordingly, the related free charge surface density σ on that part of the conductor, which is possibly in contact with the dielectric boundary, is [16–18]

$$\sigma = [\mathbf{D}] \bullet \mathbf{n} = -\varepsilon_o \varepsilon (\partial \varphi / \partial \mathbf{n}). \tag{52}$$

One can easily check that this result does not hold true in the Lagrangian electrostatics for a dielectric that is governed by the Eq. (48) or the Eq. (49). The Lagrangian potential has a constant value on the undeformed boundary of the conductor, but the free charge surface density Σ does not correspond to the normal derivative of Φ in this configuration. Therefore, on that part of the undeformed dielectric boundary, which is in contact with the conductor and where surface charges are present, this inequality holds true:

$$\Sigma = [\boldsymbol{\mathcal{D}}] \bullet \mathbf{N} \neq -\varepsilon_o \varepsilon (\partial \Phi / \partial \mathbf{N}), \tag{53}$$

contrary to the case of Eq. (52).

Final comments

Conservation of the electric charge addresses the form-invariance of the Maxwell equations in the various configurations of a deformable electromagnetic material. Consistent with this invariance, the classical fields of electrostatics transform accordingly in Lagrangian fields and a Lagrangian Electrostatics can be established on this basis. In this framework, valuable and interesting identities, which hold true at the electro-mechanical equilibrium, are introduced and discussed in Sects. 4 and 5.

The aforementioned form invariance fails for the constitutive relationships, even in a vacuum, where the constitutive-like relationship $\boldsymbol{\mathcal{D}} = J\varepsilon_o \mathbf{C}^{-1} \boldsymbol{\mathcal{E}}$ is induced by the deformed body. This remark suggests that the identification between the electric displacement and the electric field in a vacuum (which is frequently found in the physics literature) is possible only in the Euclidean space, as a metric tensor underlies the relationship between them.

References

1. Nelson, D.F.: Electric, Optic and Acoustic Interactions in Dielectrics. Wiley, New York (1979)
2. Eringen, C.A., Maugin, G.A.: Electrodynamics of Continua. Springer, New York (1990)
3. Maugin, G.A., Trimarco, C.: Pseudomomentum and material forces in electromagnetic solids. Int. J. Electromag. Mater. **2**, 207–216 (1991)
4. Maugin, G.A.: Material Inhomogeneities in Elasticity. In: Applied Mathematics and Mathematical Computation, vol. 3. Chapman & Hall, London (1993)
5. Trimarco, C., Maugin, G.A.: Material Mechanics of Electromagnetic Bodies. In: Configurational Mechanics of Materials, CISM Courses and Lectures, vol. 427. Springer, Wien (2001)
6. Trimarco, C.: A Lagrangian approach to electromagnetic bodies. Tech. Mech. **22**(3), 175–180 (2002)
7. Dorfmann, A., Ogden, R.W.: Nonlinear electroelasticity. Acta Mech. **174**(3), 167–183 (2005)
8. Mindlin, R.D.: Elasticity, piezoelasticity and crystal lattice dynamics. J. Elast. **2**, 17–282 (1972)
9. Knops, R.J., Trimarco, C.: On uniqueness in the affine boundary value problem of the nonlinear dielectric. J. Mech. Mater. Struct. **1**(5), 925–936 (2006)
10. Eshelby, J.D.: Elastic energy–momentum tensor. J. Elast. **5**, 321–335 (1976)
11. Maugin, G.A., Trimarco, C.: Elements of Field Theory in Inhomogeneous and Defective Materials. In: Configurational Mechanics of Materials, CISM Courses and Lectures, vol. 427. Springer, Wien (2001)
12. Maugin, G.A., Epstein, M.: The electroelastic energy–momentum tensor. Proc. R. Soc. Lond. A **433**, 299–312 (1991)
13. Trimarco, C.: Stresses and momenta in electromagnetic materials. Mech. Res. Commun. **29**(6), 485–492 (2002)
14. Trimarco, C.: On the material energy–momentum tensor in electrostatics and in magnetostatics. Advances in Mechanics and Mathematics. In: Steinmann, P., Maugin, G.A. (eds.) Mechanics of Material Forces, chap. 16, vol. 11, pp. 161–171. Springer, New York (2005)
15. Vu, D.K., Steinmann, P.: Nonlinear electro- and magneto-elastostatics: material and spatial settings. Int. J. Solids Struct. **44**(24), 7891–7905 (2007)
16. Becker, R., Sauter, F.: Electromagnetic Interactions, vol. 1. Blackie & Sons, London (1964)
17. Jackson, J.D.: Classical Electrodynamics. Wiley, New York (1975)
18. Stratton, J.A.: Electromagnetic Theory. McGraw-Hill, New York (1941)
19. Truesdell, C.A., Toupin, R.A.: The classical field theories. In: S., Flügge. Handbuch der Physik Bd.III/1., Springer, Berlin (1960)
20. Truesdell, C.A., Noll, W.: The non-linear field theories. In: S., Flügge. Handbuch der Physik Bd.III/3., Springer, Berlin (1965)
21. Ogden, R.W.: Non-linear Elastic Deformations. Ellis Horwood Ltd, Chichester (1984)
22. Toupin, R.A.: The Elastic Dielectrics. J. Ration. Mech. An. **5**, 849–915 (1956)

23. Trimarco, C.: The total kinetic energy of an electromagnetic body. Philos. Mag. **85**(33–35), 4277–4287 (2005)
24. Trimarco, C.: Configurational forces and gauge conditions in electromagnetic bodies. Int. J. Fract. **147**, 13–19 (2007)
25. Green, A.E.: On some general formulae in finite elasticity. Arch. Ration. Mech. Anal. **50**, 73–80 (1973)
26. Ericksen, J.L.: Theory of Elastic Dielectrics Revisited. Arch. Ration. Mech. Anal. **183**, 299–313 (2007)