Acta Mech 203, 1–11 (2009)
DOI 10.1007/s00707-008-0039-1

Zhen-Bang Kuang

Variational principles for generalized dynamical theory of thermopiezoelectricity

Received: 6 March 2008 / Revised: 7 April 2008 / Published online: 13 June 2008

Abstract The universal thermodynamic variational principle proposed in the previous papers for nonlinear dielectrics is extended to the thermopiezoelectricity and it is used as a fundamental physical principle to derive the simple complete governing equations of the generalized dynamical theory of thermopiezoelectricity in this paper. In the generalized dynamical theory it is assumed that the acceleration of the temperature needs the extra increment of the heat and the inertial entropy is proposed.

1 Introduction

The variational principles in classical thermoelasticity and thermopiezoelectricity have been established for a long time [1–3]. In previous theories authors looked for a functional with a mathematical action function and used its extremum condition to get the momentum equation and the heat conduction equation. The irreversible thermodynamics was not fully applied and the physical meaning of the mathematical action function is not clear. Usually the variational principles were constrained in a small temperature variation range.

In the classical thermoelasticity and thermopiezoelectricity the temperature field is coupled with the elastic or electroelastic field, but the temperature change does not propagate in a wave form. However, the temperature wave at low temperature in the form of heat pulses propagates with a finite velocity. So many generalized thermoelastic and thermopiezoelectric theories were proposed to allow a finite velocity for the propagation of a thermal wave. There were mainly three generalized theories: Lord–Shulman theory [4], Green–Lindsay theory [5] and Green–Naghdi theory [6]. The Lord–Shulman theory with one relaxation time was based on a new viscous heat conduction law to replace Fourier's law. The Green–Lindsay theory with two relaxation times was based on modifying the Clausius-Duhemin inequality and the energy equation; In their theory they used a new temperature function $\phi(T, \dot{T})$ to replace the usual temperature T. The Green–Naghdi theory was based on that there is no energy dissipation in the heat conductive process. Chandrasekharaiah [7] discussed also the governing equations of a temperature-rate-dependent theory of thermopiezoelasticity

In the papers [8,9] we proposed a universal thermodynamic variational principle for nonlinear dielectrics and pointed out that which is included in the first law of the thermodynamics. This universal thermodynamic variational principle is a fundamental physical principle and we can use it to establish other variational principles and the governing equations in the mechanics. In this paper the entropy flow is used as an independent vector variable to replace the heat flow. It is proposed that the acceleration of the temperature (the second time derivative of the temperature) needs the extra increment of the heat and the inertial entropy is introduced. Using these concepts the generalized theory with a finite velocity for the propagation of a thermal wave can simply be established and the physical meaning is clear. This theory is fully compatible with the classical irreversible thermodynamics. Based on this theory the variational principles introduced in this paper are exact and the governing equations derived by these variational principles are reasonable.

Z.-B. Kuang (✉)
Shanghai Jiaotong University, Shanghai 200240, People's Republic of China
E-mail: ZbKuang@mail.sjtu.edu.cn

Together with the first law of thermodynamics, the known facts show that the following universal thermodynamic variational principle also holds [8,9]:

$$\delta\Pi = \delta U - \delta W - \delta Q = 0, \tag{1}$$

where δ is the variation sign, U is the total internal energy of the body, W is the work done by the external force and electric field, Q is the heat absorbed from the external heat source. The meaning of Eq. (1) is that among all virtual general displacements (such as mechanical displacement, electric displacement), the true general displacements make Eq. (1) zero. From the first law of thermodynamics it is obviously correct for the true displacement. Therefore all other virtual general displacements can not fit Eq. (1) except at some branch points on the moving path, but the branch points should be discussed by stability or other theories.

According to the universal thermodynamic variational principle (1), we need to give the expressions of δU, δW, δQ. If we write down the expressions of δU, δW, δQ, then we obtain a physical variational principle, and can get the governing equations from it. We do not need to look for another variational functional. In the irreversible process in order to get the relation between the irreversible force and the irreversible flux (sometimes we call this relation the second kind of the constitutive equation) and the corresponding boundary condition, the dissipative energy and the inner heat produced from it should simultaneously enter the variational formula in an appropriate version. The thermodynamic variational principle is a natural result of the first law of thermodynamics and can be considered as the extension of the virtual work principle. Substantially it is an instantaneous principle in current configuration, so in the dynamics the inertial force should be included in the body force, i.e. D'Alembert's principle should be used.

Alternatively the electric Gibbs free energy density g can also be used in the thermodynamic variational principle, i.e.,

$$g = u - Ts - E_i D_i, \qquad \delta\Pi' = \delta G - \delta W^* - \delta Q^* = 0, \tag{2}$$

where G is the total electric Gibbs free energy of the body, W^* is the sum of the work of the external force on the medium and the complement work of the medium on the electric field, $-Q^*$ is the complement heat absorbed from the external heat source, u, T and s are the internal energy density (per volume), temperature and the entropy density (per volume), respectively. Other thermodynamic characteristic functions can also be used.

2 Inertial entropy

In the classical thermoelasticity and thermopiezoelectricity the temperature field is coupled with the elastic or electroelastic field, but the temperature change does not propagate in a wave form. There are many methods to establish a temperature wave theory with finite propagation velocity [4–7]. In this paper we use a simple method with inertial entropy to obtain it. We assume that the variation of the temperature with time will affect the heat flow and needs the extra heat. This extra heat is equal to the product of the temperature and the inertial entropy with negative sign. We further assume that the rate of the inertial entropy $\dot{s}^{(a)}$ is proportional to the acceleration of the temperature (the second time derivative of the temperature), i.e., $\dot{s}^{(a)} = C\varpi\ddot{T}/T_0$ and $s^{(a)} = C\int_0^{\mathrm{t}}\varpi\ddot{T}\mathrm{d}\tau/T_0 = C\varpi\dot{T}/T_0$, where C is the specific heat, T_0 is the reference temperature and ϖ is a constant coefficient with dimension time. The inertial entropy is not a true entropy, just like the inertial force in mechanics. So the increment of the heat produced by the acceleration of the temperature is equal to $CT\delta(\varpi\dot{T})/T_0 = C\varpi T\delta\dot{T}/T_0$. When the electric Gibbs function is adopted we should use the extra increment of the complement heat $C\varpi\dot{T}\delta T/T_0$. The advantage of this method is that Fourier's law is not changed, and only the heat conductive process is the irreversible process.

When the D'Alembert's principle is used, the first law of thermodynamics for continuum mechanics in the classical theory is

$$\dot{U} = \dot{W} + \dot{Q}, \tag{3}$$

$$\begin{aligned}
&U = \int\limits_V u\mathrm{d}V, \quad \dot{Q} = \int\limits_V \dot{r}\mathrm{d}V - \oint\limits_a \mathbf{q}\cdot\mathbf{n}\mathrm{d}a = \int\limits_V (\dot{r} - q_{i,i})\mathrm{d}V,\\
&\dot{W} = \int\limits_{\mathbf{V}} (\mathbf{f} - \rho\ddot{\mathbf{u}})\cdot\mathbf{v}\mathrm{d}V + \int\limits_a \mathbf{T}\cdot\mathbf{v}\mathrm{d}a + \int\limits_V \varphi\dot{\rho}_e\mathrm{d}V + \int\limits_a \varphi\dot{\sigma}_e\mathrm{d}a,
\end{aligned} \tag{4}$$

where $\rho, u_k, v_k, f_k, T_k, \varphi, \rho_{\mathrm{e}}, \sigma_{\mathrm{e}}, r$ and q_k are the density, displacement, velocity, mechanical body force, surface traction, electric potential, body electric charge density, surface electric charge density, body heat source strength and the heat flow, respectively.

In our inertial entropy theory of the heat propagation dynamics the expression of the heat in Eq. (4) should be modified to

$$\int_V C\frac{T}{T_0}\varpi\delta\dot{T}\mathrm{d}V = \int_V \delta r\,\mathrm{d}V - \int_a T\delta\boldsymbol{\eta}\cdot\mathbf{n}\mathrm{d}a - \delta Q, \quad \boldsymbol{\eta} = \int_0^t \frac{\mathbf{q}}{T}\mathrm{d}\tau, \quad \text{or}$$
$$\dot{Q} = \int_V (\dot{r} - T\dot{s}^{(a)})\mathrm{d}V - \int_a T\dot{\boldsymbol{\eta}}\cdot\mathbf{n}\mathrm{d}a, \quad \dot{s}^{(a)} = C\frac{1}{T_0}\varpi\ddot{T}, \quad \dot{\boldsymbol{\eta}} = \frac{\mathbf{q}}{T}, \tag{5}$$

where $\boldsymbol{\eta}$ is the entropy displacement vector, $\dot{\boldsymbol{\eta}}$ is the entropy flow vector. Equation (5) shows that the heat supplied by the environment is used to balance the heat increment δQ absorbed by the body and the variation of temperature with time in the medium. It is somewhat similar to the relation of the displacement acceleration to the external force. We can call $(-Ts^{(a)})$ the inertial heat, which just resembles the inertial force $(-\rho\ddot{\mathbf{u}})$ in the mechanical process.

In the irreversible thermodynamic process the internal dissipation energy in the medium will be transformed to the dissipation heat and produces the irreversible entropy [10–12]. So the total entropy s consists of the reversible part $s^{(r)}$ and the irreversible part $s^{(i)}$, i.e.,

$$\dot{s} = \dot{s}^{(r)} + \dot{s}^{(i)}. \tag{6}$$

From Eq. (5) we assume [10–12]

$$\dot{S}^{(r)} + \dot{S}^{(a)} = \int_{\mathrm{V}} \frac{\dot{r}}{T}\mathrm{d}V - \int_a \dot{\boldsymbol{\eta}}\cdot\mathbf{n}\mathrm{d}a, \quad \dot{S}^{(a)} = \int_V \frac{C}{T_0}\varpi\ddot{T}\mathrm{d}V,$$
$$\dot{s}^{(r)} + \dot{s}^{(a)} = \frac{\dot{r}}{T} - \dot{\eta}_{i,i} = \frac{1}{T}(\dot{r} - q_{i,i} + \dot{\eta}_i T_{,i}). \tag{7}$$

From Eqs. (6) and (7) we get

$$\dot{S}^{(i)} = \int_{\mathrm{V}} \dot{s}^{(i)}\mathrm{d}V = \int_{\mathrm{V}} \dot{s}\mathrm{d}V - \int_{\mathrm{V}} \dot{s}^{(r)}\mathrm{d}V \geq 0,$$
$$\dot{s}^{(i)} = \dot{s} - \dot{s}^{(r)} = \dot{s} + \dot{s}^{(a)} - \frac{\dot{r}}{T} + \frac{q_{i,i}}{T} - \frac{q_i T_{,i}}{T^2} \geq 0, \quad \text{or} \tag{8}$$
$$T\dot{s}^{(i)} = T\dot{s} + T\dot{s}^{(a)} - \dot{r} + q_{i,i} - \dot{\eta}_i T_{,i} \geq 0.$$

From Eq. (7) it is seen that in our inertial entropy theory of the heat propagation dynamics, the entropy rate produced by the environment is used to balance the reversible entropy rate $T\dot{s}^{(r)}$ of the body and the inertial entropy rate $T\dot{s}^{(a)}$ produced by the variation of the temperature in the medium.

In this paper the mechanical and electrical processes are all assumed to be reversible. Because the internal energy and the entropy are all assumed to be the state function, we can assume that the internal energy is a function of strain, electric displacement and entropy, i.e.,

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \mathbf{E}\cdot\dot{\mathbf{D}} + T\dot{s}. \tag{9}$$

From the entropy inequality (8) it is assumed that

$$T\dot{s} + T\dot{s}^{(a)} = \dot{r} - q_{i,i} = \dot{r} - (T\dot{\eta}_i)_{,i}, \quad T\dot{s} + C\frac{T}{T_0}\varpi\ddot{T} = \dot{r} - q_{i,i}, \quad \dot{h} = T\dot{s}^{(i)} = -T_{,i}\dot{\eta}_i, \tag{10}$$

where $\dot{h}$ is the dissipative energy rate and $\dot{s}^{(i)}$ is the entropy production. Using a Legendre transformation we get the electric Gibbs function g and the complement dissipative energy rate $\dot{h}'$ as

$$g = u - Ts - \mathbf{E}\cdot\mathbf{D}, \quad \dot{g} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \mathbf{D}\cdot\dot{\mathbf{E}} - s\dot{T}, \quad \dot{h}' = \eta_i\dot{T}_{,i}. \tag{11}$$

Using the theory of the usual irreversible thermodynamics [10–12], from the last equations in Eqs. (10) and (11) we assume

$$\begin{aligned}&\dot{\eta}_i = \dot{\eta}_i(T_{,j}), \quad \text{or} \quad \dot{\eta}_i = -\lambda_{ij}T^{-1}T_{,j}, \quad T\dot{\eta}_i = q_i = -\lambda_{ij}T_{,j},\\ &T_{,j} = -\widehat{\lambda}_{ij}T\dot{\eta}_i = -\widehat{\lambda}_{ij}q_i, \quad \widehat{\lambda}_{ij} = \lambda_{ij}^{-1},\end{aligned} \tag{12}$$

where λ is the usual heat conductive coefficient. Eq. (12) is just the Fourier's law,

3 Internal energy and electric Gibbs function

Since $u, \dot{h}$ and $g, \dot{h}'$ must be invariants in a rigid body rotation, in the case of small deformation theory they should be in the following form due to the theorem on invariant functions:

$$\begin{aligned}&u(\varepsilon_{kl}, D_k, s) = \frac{1}{2}C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} - h_{kij}D_k\varepsilon_{ij} + \frac{1}{2}\beta_{ij}D_iD_j - \widehat{\alpha}_{ij}\varepsilon_{ji}s - \widehat{\tau}_iD_is + \frac{T_0}{2}\widehat{C}s^2,\\ &\delta h = \widehat{\lambda}_{ij}T\dot{\eta}_j\delta\eta_i,\\ &C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad h_{kij} = h_{kji}, \quad \beta_{kl} = \beta_{lk}, \quad \widehat{\alpha}_{ij} = \widehat{\alpha}_{ji}, \quad \widehat{\lambda}_{ij} = \widehat{\lambda}_{ji},\end{aligned} \tag{13}$$

$$\begin{aligned}&g(\varepsilon_{kl}, E_k, \vartheta) = \frac{1}{2}C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} - \mathrm{e}_{kij}E_k\varepsilon_{ij} - \frac{1}{2}\in_{ij}E_iE_j - \alpha_{ij}\varepsilon_{ij}\vartheta - \tau_iE_i\vartheta - \frac{1}{2T_0}C\vartheta^2,\\ &\delta h' = -\left(\int_0^t \lambda_{ij}\frac{1}{T}\vartheta_{,i}\mathrm{d}\tau\right)\delta\vartheta_{,j} = \eta_j\delta\vartheta_{,j},\\ &C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad \mathrm{e}_{kij} = \mathrm{e}_{kji}, \quad \in_{kl}=\in_{lk}, \quad \alpha_{ij} = \alpha_{ji}, \quad \lambda_{ij} = \lambda_{ji},\end{aligned} \tag{14}$$

where C is the specific heat at constant deformation and constant electric field, $\widehat{C}$ is the specific heat at constant deformation and constant electric displacement, $C_{ijkl}, h_{kij}, \mathrm{e}_{kij}, \beta_{ij}, \in_{ij}, \alpha_{ij}, \widehat{\alpha}_{ij}, \tau_i, \widehat{\tau}_i$ are material constants. The following facts are emphasized: It is assumed that in Eq. (13) $s = 0$ when $T = 0$ and $s = s_0$ when $T = T_0$. But in Eq. (14) it is assumed that $s = 0$ when $T = T_0$ or $\vartheta = 0$, where $\vartheta = T - T_0$; T_0 is the reference temperature of the environment. It is obvious that $T_{,j} = \vartheta_{,j}, \dot{T} = \dot{\vartheta}$. The constitutive equations derived from Eq. (13) are

$$\begin{aligned}\sigma_{ij} &= \frac{\partial u}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - h_{kij}D_k - \widehat{\alpha}_{ij}s,\\ E_i &= \frac{\partial u}{\partial D_i} = \beta_{ij}D_j - h_{kij}\varepsilon_{kl} - \widehat{\tau}_is,\\ T &= \frac{\partial u}{\partial s} = -\widehat{\alpha}_{ij}\varepsilon_{ji} - \widehat{\tau}_iD_i + T_0\widehat{C}s,\end{aligned} \tag{15a}$$

and the heat conduction equation (Fourier's law or the second kind of the constitutive equation)

$$T_{,i} = -\frac{\partial h}{\partial \eta_i} = -\widehat{\lambda}_{ij}T\dot{\eta}_j = -\widehat{\lambda}_{ij}q_j, \qquad \int_0^t T_{,i}\mathrm{d}\tau = -\int_0^t \widehat{\lambda}_{ij}T\dot{\eta}_j\mathrm{d}\tau. \tag{15b}$$

The constitutive equations derived from Eq. (14) are

$$\begin{aligned}\sigma_{ij} &= \frac{\partial g}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k - \alpha_{ij}\vartheta,\\ D_i &= -\frac{\partial g}{\partial E_i} = \in_{ij}E_j + e_{kij}\varepsilon_{kl} + \tau_i\vartheta,\\ s &= -\frac{\partial g}{\partial \vartheta} = \alpha_{ij}\varepsilon_{ij} + \tau_iE_i + C\vartheta/T_0,\end{aligned} \tag{16a}$$

and the heat conduction equation (Fourier's law or the second kind of the constitutive equation)

$$\eta_i = \frac{\partial h'}{\partial \vartheta_{,i}} = -\int_0^t \lambda_{ij} \frac{1}{T} \vartheta_{,j} \mathrm{d}\tau, \qquad T\dot{\eta}_i = q_i = -\lambda_{ij}\vartheta_{,j}. \tag{16b}$$

4 Electric Gibbs function variational principle

It is assumed that the mechanical displacement u_i, the electric potential φ and the temperature T satisfy their own boundary conditions $u_i = u_i^*$, $\varphi = \varphi^*$ and $T = T^*$ (or $\vartheta = \vartheta^*$) on a_u, a_φ and a_T, respectively. In the medium we also have $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, $E_i = -\varphi_{,i}$ and the constitutive equations (16a) and (16b). From the universal thermodynamic variational principle the electric Gibbs function variational principle with the inertial entropy for thermopiezoelectricity is

$$\begin{aligned}
\delta\Pi' &= \int_V \delta(g + h')\mathrm{d}V - \delta Q^* - \delta w^* = 0,\\
\delta Q^* &= -\int_0^t \int_V \left(\frac{\dot{r}}{T}\right) \delta\vartheta \mathrm{d}V \mathrm{d}\tau + \int_0^t \int_{a_q} \dot{\eta}^* \delta\vartheta \mathrm{d}a \mathrm{d}\tau - \int_0^t \int_V \dot{s}^{(i)} \delta\vartheta \mathrm{d}V \mathrm{d}\tau\\
&\quad + \int_0^t \int_V C \frac{1}{T_0} \varpi \ddot{\vartheta} \delta\vartheta \mathrm{d}V \mathrm{d}\tau,\\
\delta W^* &= \int_V (f_k - \rho \ddot{u}_k)\delta u_k \mathrm{d}V - \int_V \rho_e \delta\varphi \mathrm{d}V + \int_{a_\sigma} T_k^* \delta u_k \mathrm{d}a - \int_{a_D} \sigma^* \delta\varphi \mathrm{d}a,
\end{aligned} \tag{17}$$

where f_k, T_k^*, ρ_e, σ^* and $\dot{\eta}_i^*(\dot{\eta}^* = \dot{\eta}_i^* n_i)$ are the given mechanical body force, surface traction, body electric charge density, surface electric charge density and surface entropy flow, respectively. In Eq. (17) the term $\int_0^t \dot{s}^{(i)} \delta\vartheta \mathrm{d}\tau$ is the complement heat per unit volume corresponding to the inner complement dissipation energy $\delta h'$. This is just the request by the thermodynamic variational principle, and this is consistent with the first law of thermodynamics. In the previous variational principles this fact was not considered. In order to obtain the heat conduction equation and the boundary condition of the heat flow from the variational principle, the complement dissipation energy $\int_V \delta h' \mathrm{d}V$ in $\delta\Pi$ and the inner irreversible complement heat $\int_0^t \int_V \dot{s}^{(i)} \delta\vartheta \mathrm{d}V \mathrm{d}\tau$ in δQ^* should be simultaneously included in the variational functional. In Eq. (17) there are two kinds of variational formulas. The first kind is $\int_V \delta g \mathrm{d}V - \delta w^*$, in which the integrands contain variables themselves, so it need not integrate with time t. The second kind is $\int_V \delta h' \mathrm{d}V - \delta Q^*$, in which the integrands contain the time derivatives of variables, so it needs to integrate with time t, which is the feature of the irreversible process. In the irreversible process the integral is dependent to the integral path. In Eq. (17) the independent variables are $\mathbf{u}$, φ and T (or ϑ).

It is noted that

$$\begin{aligned}
\delta \int_V g \mathrm{d}V &= \int_V (C_{ijkl}\varepsilon_{kl} + \mathrm{e}_{kij}\varphi_{,k} - \alpha_{ij}\vartheta)\delta u_{i,j} \mathrm{d}V + \int_V (\mathrm{e}_{kij}\varepsilon_{ij} + \in_{ik} E_i + \tau_k \vartheta)\delta\varphi_{,k} \mathrm{d}V\\
&\quad - \int_V (\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + C\vartheta/T_0)\delta\vartheta \mathrm{d}V\\
&= \int_a \sigma_{ij} n_j \delta u_i \mathrm{d}a - \int_V \sigma_{ij,j} \delta u_i \mathrm{d}V + \int_a D_k n_k \delta\varphi \mathrm{d}a - \int_V D_{k,k} \delta\varphi \mathrm{d}V - \int_V s \delta\vartheta \mathrm{d}V,\\
\int_V \delta h' \mathrm{d}V &= -\int_a \left(\int_0^t \lambda_{ij} \frac{1}{T} \vartheta_{,i} n_j \mathrm{d}\tau\right) \delta\vartheta \mathrm{d}a + \int_V \left[\int_0^t \left(\lambda_{ij} \frac{1}{T} \vartheta_{,i}\right)_{,j} \mathrm{d}\tau\right] \delta\vartheta \mathrm{d}V.
\end{aligned} \tag{18}$$

Finishing the variational calculation, we have

$$\begin{aligned}\delta\Pi' = & \int\limits_{a_\sigma}(\sigma_{ij}n_j - T_i^*)\delta u_i\mathrm{d}a - \int\limits_V(\sigma_{ij,j} + f_i - \rho\ddot{u}_i)\delta u_i\mathrm{d}V \\ & + \int\limits_{a_D}(\sigma^* + D_k n_k)\delta\varphi\mathrm{d}a - \int\limits_V(D_{k,k} - \rho_e)\delta\varphi\mathrm{d}V - \int\limits_{a_q}\left[\int\limits_0^t \frac{1}{T}\lambda_{ij}\vartheta_{,i}n_j\mathrm{d}\tau + \eta^*\right]\delta\vartheta\mathrm{d}a \\ & - \int\limits_V\left\{s + \int\limits_0^t\left[-\frac{\dot{r}}{T} - \left(\lambda_{ij}\frac{1}{T}\vartheta_{,i}\right)_{,j} - \dot{s}^{(i)}\right]\mathrm{d}\tau\right\}\delta\vartheta\mathrm{d}V - \int\limits_0^t\int\limits_V\frac{C}{T_0}\varpi\ddot{\vartheta}\mathrm{d}V\mathrm{d}\tau\right\}\delta\vartheta\mathrm{d}V = 0.\end{aligned} \tag{19}$$

Due to the arbitrariness of $\delta u_i,\ \delta\varphi$ and δT, from Eq. (19) we get

$$\begin{aligned}&\sigma_{kl,l} + f_k = \rho\ddot{u}_k, \quad D_{k,k} = \rho_e, \\ &s + \frac{C}{T_0}\varpi\dot{\vartheta} = \int\limits_0^t\left[\frac{\dot{r}}{T} + \left(\lambda_{ij}\frac{1}{T}\vartheta_{,i}\right)_{,j} + \dot{s}^{(i)}\right]\mathrm{d}\tau, \quad \text{or} \quad \dot{s} + \frac{C}{T_0}\varpi\ddot{\vartheta} = \frac{\dot{r}}{T} - \frac{q_{i,i}}{T} \quad \text{in the medium;} \\ &\sigma_{kl}n_l = T_k^* \quad \text{on} \quad a_\sigma; \quad D_k n_k = -\sigma^* \quad \text{on} \quad a_D; \\ &\int\limits_0^t \lambda_{ij}\frac{1}{T}\vartheta_{,i}n_j\mathrm{d}\tau = -\eta^*, \quad \text{or} \quad \dot{\eta}_i = \dot{\eta}_i^*, \quad \text{or} \quad q_i = q_i^* \quad \text{on} \quad a_q.\end{aligned} \tag{20}$$

The above variational principle requests prior that the displacement, the electric potential and the temperature satisfy the boundary conditions, so in the governing equations the following equations should also be added:

$$\mathbf{u} = \mathbf{u}^* \quad \text{on} \quad a_u; \quad \varphi = \varphi^* \quad \text{on} \quad a_\varphi; \quad T = T^*(\text{or } \vartheta = \vartheta^*) \quad \text{on} \quad a_T. \tag{21}$$

From Eq. (20) we get the thermal conduction (energy) equation

$$\dot{s} + \frac{C}{T_0}\varpi\ddot{\vartheta} = \frac{\dot{r}}{T} - \frac{q_{i,i}}{T}, \quad (\lambda_{ij}\vartheta_{,j})_{,i} = T\left(\alpha_{ij}\dot{\varepsilon}_{ij} + \tau_i\dot{E}_i + \frac{C}{T_0}\dot{\vartheta} + \frac{C}{T_0}\varpi\ddot{\vartheta}\right) - \dot{r}. \tag{22a}$$

The above equation is a temperature wave equation with one parameter ϖ and a finite propagation velocity. This equation can also be obtained from Eqs. (10) and (16) .

If ϑ is much less than T_0, $\vartheta \ll T_0$, then the instant temperature T in Eq. (17) can be replaced by T_0. Eq. (22a) is reduced to

$$\lambda_{ij}\vartheta_{,ji} = T_0\alpha_{ij}\dot{\varepsilon}_{ij} + T_0\tau_i\dot{E}_i + C(\dot{\vartheta} + \varpi\ddot{\vartheta}) - \dot{r}. \tag{22b}$$

If we also use $\delta h' = \eta_j\delta\vartheta_{,j}$ in Eq. (17), then Eq. (17) is reduced to

$$\begin{aligned}\delta\Pi' &= \int\limits_V(\delta g + \eta_j\delta\vartheta_{,j})\mathrm{d}V - \delta Q^* - \delta w^* = 0, \\ \delta Q^* &= -\frac{1}{T_0}\int\limits_V r\delta\vartheta\mathrm{d}V + \int\limits_{a_q}\eta_0^*\delta\vartheta\mathrm{d}a - \int\limits_V s^{(i)}\delta\vartheta\mathrm{d}V + \int\limits_V C\frac{1}{T_0}\varpi\dot{\vartheta}\delta\vartheta\mathrm{d}V, \\ \delta W^* &= \int\limits_V(f_k - \rho\ddot{u}_k)\delta u_k\mathrm{d}V - \int\limits_V\rho_e\delta\varphi\mathrm{d}V + \int\limits_{a_\sigma}T_k^*\delta u_k\mathrm{d}a - \int\limits_{a_D}\sigma^*\delta\varphi\mathrm{d}a,\end{aligned} \tag{23}$$

where $\eta_0^* = (1/T_0)\int_0^t q^*dt$. Equation (23) there does not contain any term containing the time integral.

5 Internal energy variational principle

It is assumed that the mechanical displacement u_i, the electric displacement D_i and the entropy flow $\dot{\boldsymbol{\eta}}$ or the heat flow $\mathbf{q}$ satisfy their own boundary conditions $\mathbf{u} = \mathbf{u}^*$, $\mathbf{D} \cdot \mathbf{n} = -\sigma^*$ and $\dot{\eta}_i n_i = \dot{\eta}^*$, *or* $q_i n_i = q^*$ on the boundaries a_u, a_D and a_q, respectively. In the medium we have $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, $\rho_e = D_{i,i}$, the constitutive equation (15a) and the thermal conduction equation (22a). From the universal thermodynamic variational principle the internal energy variational principle for the thermopiezoelectricity is

$$
\begin{aligned}
\delta\Pi &= \int\limits_V \delta(u-h)\mathrm{d}V - \delta Q - \delta w = 0,\\
\delta Q &= \int\limits_V \delta r \mathrm{d}V - \int\limits_{a_T} T^*\delta\eta \mathrm{d}a + \int\limits_V T_{,i}\delta\eta_i \mathrm{d}V - \int\limits_V \frac{C}{T_0}\varpi T\delta\dot{T}\mathrm{d}V,\\
\delta W &= \int\limits_V (f_k - \rho\ddot{u}_k)\delta u_k \mathrm{d}V + \int\limits_V \varphi\delta\rho_e \mathrm{d}V + \int\limits_{a_\sigma} T_k^*\delta u_k \mathrm{d}a + \int\limits_{a_\varphi} \varphi^*\delta\sigma \mathrm{d}a,
\end{aligned}
\tag{24}
$$

where $\delta\eta = n_i\delta\eta_i$. In Eq. (24) the variable T can be expressed by s, so the independent variables are $\mathbf{u}$, $\mathbf{D}$ and $\boldsymbol{\eta}$. In order to obtain Fourier's law (15b) and the boundary condition of the temperature from the variational principle, the dissipative energy $\int_V \delta h \mathrm{d}V$ in $\delta\Pi$ and the inner irreversible heat $-\int_V T_{,i}\delta\eta_i \mathrm{d}V$ in δQ^* should be included in the variational functional. It is noted that

$$
\begin{aligned}
\delta\int\limits_V u\mathrm{d}V &= \int\limits_V (C_{ijkl}\varepsilon_{kl} - h_{kij}D_k - \widehat{\alpha}_{ij}s)\delta u_{i,j}\mathrm{d}V + \int\limits_V (-h_{kij}\varepsilon_{ij} + \beta_{kl}D_l - \widehat{\tau}_k s)\delta D_k \mathrm{d}V\\
&\quad - \int\limits_V (\widehat{\alpha}_{ij}\varepsilon_{ji} + \widehat{\tau}_i D_i - T_0\widehat{C}s)\delta s \mathrm{d}V\\
&= \int\limits_a \sigma_{ij}n_j\delta u_i \mathrm{d}a - \int\limits_V \sigma_{ij,j}\delta u_i \mathrm{d}V + \int\limits_V E_k\delta D_k \mathrm{d}V + \int\limits_V T\delta s \mathrm{d}V,\\
\delta\int\limits_V h\mathrm{d}V &= \int\limits_V \widehat{\lambda}_{ij}T\dot{\eta}_j\delta\eta_i \mathrm{d}V.
\end{aligned}
\tag{25}
$$

Substituting Eq. (25) into Eq. (24) and using $\rho_e = D_{i,i}$ we obtain

$$
\begin{aligned}
\delta\Pi &= \int\limits_a \sigma_{ij}n_j\delta u_i \mathrm{d}a - \int\limits_V \sigma_{ij,j}\delta u_i \mathrm{d}V + \int\limits_V E_k\delta D_k \mathrm{d}V + \int\limits_V T\delta s \mathrm{d}V\\
&\quad - \int\limits_V \widehat{\lambda}_{ij}T\dot{\eta}_j\delta\eta_i \mathrm{d}V - \int\limits_V \delta r \mathrm{d}V + \int\limits_{a_T} T^*\delta\eta \mathrm{d}a - \int\limits_a Tn_i\delta\eta_i \mathrm{d}a + \int\limits_V T\delta\eta_{i,i}\mathrm{d}V\\
&\quad + \int\limits_V C\frac{T}{T_0}\varpi\delta\dot{T}\mathrm{d}V - \int\limits_V (f_k - \rho\ddot{u}_k)\delta u_k \mathrm{d}V - \int\limits_a \varphi n_i\delta D_i \mathrm{d}V + \int\limits_V \varphi_{,i}\delta D_i \mathrm{d}V\\
&\quad - \int\limits_{a_\sigma} T_k^*\delta u_k \mathrm{d}a - \int\limits_{a_\varphi} \varphi^*\delta\sigma \mathrm{d}a = \int\limits_{a_\sigma} (\sigma_{ij}n_j - T_i^*)\delta u_i \mathrm{d}a\\
&\quad - \int\limits_V (\sigma_{ij,j} + f_k - \rho\ddot{u}_k)\delta u_i \mathrm{d}V + \int\limits_V (E_k + \varphi_{,k})\delta D_k \mathrm{d}V + \int\limits_{a_\varphi} (\varphi - \varphi^*)\delta\sigma \mathrm{d}a\\
&\quad - \int\limits_{a_T} (T - T^*)\delta\eta \mathrm{d}a + \int\limits_V (T\delta s + T\delta s^{(a)} - T\widehat{\lambda}_{ij}\dot{\eta}_j\delta\eta_i + T\delta\eta_{i,i} - \delta r)\mathrm{d}V = 0.
\end{aligned}
\tag{26}
$$

Using the relation

$$\begin{gathered}
T\delta s + T\delta s^{(a)} - T\widehat{\lambda}_{ij}\dot{\eta}_j\delta\eta_i + T\delta\eta_{i,i} - \delta r = -(T\delta\eta_i)_{,i} - T\widehat{\lambda}_{ij}\dot{\eta}_j\delta\eta_i + T\delta\eta_{i,i} \\
\int\limits_V [-(T\delta\eta_i)_{,i} + T\delta\eta_{i,i} - T\widehat{\lambda}_{ij}\dot{\eta}_j\delta\eta_i]\mathrm{d}V = \int\limits_V (-T_{,i} - T\widehat{\lambda}_{ij}\dot{\eta}_j)\delta\eta_i\mathrm{d}V
\end{gathered}$$

and the arbitrariness of δu_i, δD_i, and $\delta\eta_i$ we get

$$\begin{aligned}
&\sigma_{ij,j} + f_i - \rho\ddot{u}_i = 0, \quad E_i = -\varphi_{,i} \quad \text{in the medium;} \\
&T_{,i} = -T\widehat{\lambda}_{ij}\dot{\eta}_j \quad \text{in the medium;} \\
&\sigma_{ij}n_j = T_i^* \quad on \quad a_\sigma; \quad \varphi = \varphi^* \quad \text{on} \quad a_\varphi; \quad T = T^* \quad \text{on} \quad a_T.
\end{aligned} \tag{27}$$

If we let $T_{,i} = -T\widehat{\lambda}_{ij}\dot{\eta}_j$ hold prior, then we have

$$-T\widehat{\lambda}_{ij}\dot{\eta}_j\delta\eta_i + T\delta\eta_{i,i} = T_{,i}\delta\eta_i + T\delta\eta_{i,i} = (T\delta\eta_i)_{,i} = (T\dot{\eta}_t\delta t) = \delta q_{i,i}.$$

So the last term in Eq. (26) becomes

$$T\delta s + T\delta s^{(a)} + \delta q_{i,i} - \delta r = 0 \quad \text{or} \quad \dot{s} + \dot{s}^{(a)} = \frac{\dot{r}}{T} - \frac{q_{i,i}}{T},$$

which is just the thermal conduction equation (22a). However, the entropy wave equation is very complex.

The above variational principle requests prior that the displacement, the electric potential and the temperature satisfy he boundary conditions, so in the governing equations the following equations should also be added:

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad a_u; \quad D_i n_i = -\sigma^* \quad \text{on} \quad a_D; \quad \eta_i n_i = \eta^* \text{ (or } q_i n_i = q^*) \quad \text{on} \quad a_\eta. \tag{28}$$

Comparing Eq. (24) with Eq. (17), it is found that the number of the independent variables in Eq. (24) is more than that in the electric Gibbs function variational principle. So the application of the internal energy variational principle in engineering is more difficult than that of the electric Gibbs function variational principle. It is also noted that the internal energy variational principle is not obtained in previous literature though the variational principles in thermoelasticity and the thermopiezoelectricity have been researched for a long time.

6 Gibbs function variational principle

The Gibbs function is defined as

$$G = u - \sigma_{ij}\varepsilon_{ij} - Ts - E_i D_i, \quad \delta G = -(\varepsilon_{ij}\delta\sigma_{ij} + D_i\delta E_i + s\delta\vartheta) \tag{29}$$

if the mechanical stress σ_{ij}, the electric potential φ and the temperature ϑ satisfy their own boundary conditions $\sigma_{ij}n_j = T_i^*, \varphi = \varphi^*, \vartheta = \vartheta^*(= 0)$ on a_σ, a_φ and a_T, respectively. In the medium we require $\sigma_{ij,j} = -(f_i - \rho\ddot{u}_{i_i})$, $E_i = -\varphi_{,i}$ and Fourier's law $T\dot{\eta}_i = q_i = -\lambda_{ij}\vartheta_{,j}$. It is also assumed that the constitutive equation can be derived from the Gibbs function G:

$$\varepsilon_{ij} = -\frac{\partial G}{\partial\sigma_{ij}}, \quad D_i = -\frac{\partial G}{\partial E_i}, \quad s = -\frac{\partial G}{\partial\vartheta}. \tag{30}$$

Let G be the symmetric function of σ_{ij}, so ε_{ij} is a symmetric tensor.

From the universal thermodynamic variational principle the Gibbs function variational principle can be expressed as

$$\begin{aligned}
\delta\Pi'' &= \int\limits_V \delta(G+h')\mathrm{d}V - \delta Q^* - \delta w^{**} = 0,\\
\delta Q^* &= -\int\limits_0^t\int\limits_V \left(\frac{\dot{r}}{T}\right)\delta\vartheta\,\mathrm{d}V\mathrm{d}\tau + \int\limits_0^t\int\limits_{a_q} \dot{\eta}^*\delta\vartheta\,\mathrm{d}a\mathrm{d}\tau - \int\limits_0^t\int\limits_V \dot{s}^{(i)}\delta\vartheta\,\mathrm{d}V\mathrm{d}\tau + \int\limits_0^t\int\limits_V C\frac{1}{T_0}\varpi\ddot{\vartheta}\delta\vartheta\,\mathrm{d}V\mathrm{d}\tau, \qquad (31)\\
\delta W^{**} &= -\int\limits_V u_k\delta(f_k-\rho\ddot{u}_k)\mathrm{d}V - \int\limits_V \rho_e\delta\varphi\,\mathrm{d}V - \int\limits_{a_u} u_k^*\delta T_k\mathrm{d}a - \int\limits_{a_D}\sigma^*\delta\varphi\,\mathrm{d}a.
\end{aligned}$$

Using

$$\begin{aligned}
&\delta\int\limits_V G\mathrm{d}V = -\int\limits_V \varepsilon_{ij}\delta\sigma_{ij}\mathrm{d}V + \int\limits_a D_k n_k\delta\varphi\,\mathrm{d}a - \int\limits_V D_{k,k}\delta\varphi\,\mathrm{d}V - \int\limits_V s\delta\vartheta\,\mathrm{d}V,\\
&\int\limits_V \delta h'\mathrm{d}V = -\int\limits_a \left(\int\limits_0^t \lambda_{ij}\frac{1}{T}\vartheta_{,i}n_j\mathrm{d}\tau\right)\delta\vartheta\,\mathrm{d}a + \int\limits_V\left[\int\limits_0^t\left(\lambda_{ij}\frac{1}{T}\vartheta_{,i}\right)_{,j}\right]\delta\vartheta\,\mathrm{d}\tau\mathrm{d}V, \qquad (32)\\
&\int\limits_V u_k\delta(f_k-\rho\ddot{u}_k)\mathrm{d}V = -\int\limits_V u_k\delta\sigma_{kj,j}\mathrm{d}V = -\int\limits_a u_k\delta T_k\mathrm{d}a + \int\limits_V u_{k,j}\delta\sigma_{kj}\mathrm{d}V.
\end{aligned}$$

Equation (31) is reduced to

$$\begin{aligned}
\delta\Pi'' = &\int\limits_{a_u}(u_k^*-u_k)\delta T_k\mathrm{d}a + \int\limits_V (u_{i,j}-\varepsilon_{ij})\delta\sigma_{ij}\mathrm{d}V + \int\limits_{a_D}(D_k n_k+\sigma^*)\delta\varphi\,\mathrm{d}a\\
&+\int\limits_V(\rho_e-D_{k,k})\delta\varphi\,\mathrm{d}V - \int\limits_0^t\int\limits_{a_q}\left(\dot{\eta}^*+\lambda_{ij}\frac{1}{T}\vartheta_{,i}n_j\mathrm{d}\tau\right)\delta\vartheta\,\mathrm{d}a\mathrm{d}\tau \qquad (33)\\
&-\int\limits_V s\delta\vartheta\,\mathrm{d}V + \int\limits_V\left\{\int\limits_0^t\left[\left(\lambda_{ij}\frac{1}{T}\vartheta_{,i}\right)_{,j} + \frac{\dot{r}}{T} + \dot{s}^{(i)} - C\frac{1}{T_0}\varpi\ddot{\vartheta}\right]\mathrm{d}\tau\right\}\delta\vartheta\,\mathrm{d}V.
\end{aligned}$$

Due to the arbitrariness of $\delta\sigma_{ij},\ \delta\varphi$ and δT, from Eq. (33) we get

$$\begin{aligned}
&\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}),\quad D_{k,k}=\rho_e,\\
&s+\frac{C}{T_0}\varpi\dot{\vartheta} = \int\limits_0^t\left[\frac{\dot{r}}{T}+\left(\lambda_{ij}\frac{1}{T}\vartheta_{,i}\right)_{,j}+\dot{s}^{(i)}\right]\mathrm{d}\tau,\quad \text{or}\quad \dot{s}+\frac{C}{T_0}\varpi\ddot{\vartheta} = \frac{\dot{r}}{T}-\frac{q_{i,i}}{T}\quad \text{in the medium;}\\
&u_k = u_k^*\quad\text{on}\quad a_u,\quad D_k n_k = -\sigma^*\quad\text{on}\quad a_D; \qquad (34)\\
&\int\limits_0^t \lambda_{ij}\frac{1}{T}\vartheta_{,i}n_j\mathrm{d}\tau = -\eta^*\quad\text{or}\quad \dot{\eta}_i=\dot{\eta}_i^*\quad\text{or}\quad q_i=q_i^*\quad\text{on}\quad a_q,
\end{aligned}$$

where the symmetric behavior of the strain is used. Obviously we can use $-G$ to replace G and the corresponding modifications for other formulas in Eq. (32) to get a more succinct form.

7 Some remarks

(1) **The variational principles for classical theory of thermopiezoelectricity** If we omit the terms containing the inertial entropy in the above theories, the variational principles for classical theory of thermopiezoelectricity will be obtained from Eqs. (17), (24) and (31). Comparing these variational principles with theories in previous literatures we can find that the variational principles obtained in this paper are exact for all temperature range. This is one of the advantages by using the universal thermodynamic variational principle.

(2) **Other forms of the variational principles** In thermopiezoelectricity the variables can be divided into three groups, i.e. $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})$, $(\mathbf{E}, \mathbf{D})$ and (T, s), so there are eight characteristic functions of the thermodynamics. Therefore there are eight fundamental variational principles. Using the Lagrange multiplier method [9,12,13], we can get a lot of general or mixed variational principles. All the six variables $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{E}$, $\mathbf{D}$, T, s have been separately appeared in the above discussed different variational principles as independent variables, so there is no difficulty to establish any variational principle in thermopiezoelectricity. It can also be seen that the electric Gibbs function variational principle, Eq. (17), has the least number of the independent variables $\mathbf{u}$, φ and T. Another variational principle may have the least number of the independent variables $\mathbf{u}$, φ and s, but the governing equations with these independent variables are more complex.

(3) **The forms of the thermodynamic characteristic functions** In the above discussions the expressions of u and g are given and the constitutive equations are all in linear forms. The particular forms of the thermodynamic characteristic functions are not substantially for the derivation of the variational principle, provided the constitutive equations can be derived from these characteristic functions as shown in Eq. (31) for G. However, the thermal wave equation is dependent to the concrete expressions of the thermodynamic characteristic functions and the constitutive equations.

(4) **Some discussions on the theory** In Lord–Shulman theory for an isotropic material they used the viscous heat conduction law $q_i + \varpi\dot{q}_i = -\kappa\vartheta_{,j}$ to replace Fourier's law, and the relation $T\dot{s} = \dot{r} - q_{i,i}$ is still used. So from their theory we get $\kappa T_{,ii} = T\dot{s} + \varpi(T\dot{s})^{\cdot}$ (in our notation and $\dot{r} = 0$). This formula shows that the entropy possesses viscous property. If the entropy possesses viscous property, then is the entropy a state function? When the variation of temperature is small and many small terms can be neglected they got the linearized equation for the isotropic thermoelasticity

$$\kappa\vartheta_{,kk} = C(\dot{\vartheta} + \varpi\ddot{\vartheta}) + (3\lambda + 2\mu)\alpha T_0(\dot{\varepsilon}_{kk} + \varpi\ddot{\varepsilon}_{kk}), \tag{35}$$

which is widely used in literature. In Eq. (35) λ and μ are the usual elastic constants, κ is the isotropic thermal conduction coefficient. Comparing the above equation with Eq. (22) for an isotropic material it is found that the difference between two equations is only in a term $(3\lambda + 2\mu)\alpha T_0\varpi\ddot{\varepsilon}_{kk}$, which is very small and can be neglected [4]. Though the starting points of two theories are different, but the final equations of the temperature propagation are similar. However, Eq. (22) is not limited to a small variation of the temperature.

In all the present theories of anisotropic thermoelasticity and thermopiezoelectricity, there is only one relaxation time constant in the temperature wave equation. Though there are two relaxation time constants in Green–Lindsay theory [5], but the second relaxation time constant does not appear in the temperature wave equation [7,14]. All these theories support the theory with one relaxation time constant in the thermal wave propagation and this is consistent with the inertial entropy theory.

(5) **A simple example for the pure heat conduction** For a pure heat conduction problem of an isotropic material when the variation of the temperature is small the heat conduction equation deduced from Eq. (22b) is

$$\lambda\vartheta_{,ii} = C(\dot{\vartheta} + \varpi\ddot{\vartheta}) - \dot{r}. \tag{36}$$

Now we discuss the temperature wave propagating along the x direction with $r = 0$. In this case we let

$$\vartheta = \Theta\exp[i(kx - \omega t)]. \tag{37}$$

Substituting Eq. (37) into (36) we get

$$\begin{aligned} &k^2 = C\lambda^{-1}\omega^2(i\omega^{-1} + \varpi), \\ &k = \pm(C\lambda^{-1}\varpi)^{\frac{1}{2}}\omega\left[\sqrt{\frac{1}{2}\left(\sqrt{1+\omega^{-2}\varpi^{-2}} + 1\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{1+\omega^{-2}\varpi^{-2}} - 1\right)}\right]. \end{aligned} \tag{38}$$

Substituting Eq. (38) into (37) we get

$$\begin{aligned}
&\vartheta = \Theta\exp[i(kx-\omega t)] = \Theta\exp(-k_2 x)\exp[i(k_1 x-\omega t)],\\
&k_1 = \pm(C\lambda^{-1}\varpi)^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}\left(\sqrt{1+\omega^{-2}\varpi^{-2}}+1\right)}, \qquad k_2 = (C\lambda^{-1}\varpi)^{\frac{1}{2}}\omega\sqrt{\frac{1}{2}\left(\sqrt{1+\omega^{-2}\varpi^{-2}}-1\right)},\\
&v = \frac{\omega}{k_1} = \sqrt{\frac{\lambda}{C\varpi}}\Bigg/\sqrt{\frac{1}{2}\left(1+\sqrt{1+\omega^{-2}\varpi^{-2}}\right)},
\end{aligned} \tag{39}$$

where v is the phase velocity. So the temperature wave is an attenuate dispersive wave. If $\varpi \to \infty$ and $C\varpi$ is finite, then $v \to \sqrt{\lambda/C\varpi}$ and the attenuate coefficient $k_2 = 0$. This is a wave solution without attenuation. If ϖ is very small and $\varpi\omega \to 0$, but C is finite, then the phase velocity of the temperature wave approaches $\sqrt{2\lambda\omega/C}$ and the attenuate coefficient $k_2 = \sqrt{C/2\lambda\omega}$. In this case the wave solution without attenuation is not existed.

8 Conclusions

In this paper the universal thermodynamic variational principle recommended in [8,9] is extended to thermopiezoelectricity. Combining the irreversible thermodynamics with the inertial entropy the internal energy, electric Gibbs function and Gibbs function variational principles are derived from the universal thermodynamic variational principle. The variational principles derived in this paper are exact. The assumed expression of the inertial entropy is just assumed to make the heat conduction equation a wave equation, so the inertial entropy should be further examined, but it is a reasonable idea, which lets us solve the temperature wave problem with a finite propagation velocity in a very simple manner. The governing equations for thermopiezoelectricity recommended in this paper are the simplest among all relative theories.

References

1. Nowacki, W.: Some general theorems of thermopiezoelectricity. J. Therm. Stress **1**, 171–182 (1978)
2. Shen, S.P., Kuang, Z.-B.: An active control model of laminated piezothermoelastic plate. Int. J. Solids Struct. **36**, 1925–1947 (1999)
3. Chen, C., Shen, Y., Tian, X.: Variational principles of nonlinear piezothermoelastic media. Acta Mech. Solida Sin. **11**, 13–27 (1998)
4. Lord, H.W., Shulman, Y.: A generalized dynamical theory of thermoelasticity. J. Mech. Phys. Solids **15**, 299–309 (1967)
5. Green, A.E., Lindsay: Thermoelasticity. J. Elast. **2**, 1–7 (1972)
6. Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation. J. Elast. **31**, 189–208 (1993)
7. Chandrasekharaiah, D.S.: A temperature-rate-dependent theory of thermopiezoelasticity. J. Therm. Stress **7**, 293–306 (1984)
8. Kuang, Z.-B.: Some problems in electrostrictive and magnetostrictive materials. Acta Mech. Solida Sin. **20**, 219–227 (2007)
9. Kuang, Z.-B.: Some variational principles in elastic dielectric and elastic magnetic materials. Eur. J. Mech. A/Solids **27**, 504–514 (2007)
10. De Groet, S.R.: Thermodynamics of Irreversible Processes. North-Holland Publishing Company, Amsterdam (1952)
11. Germain, P., Nguyen, Q.S., Suquet, P.: Continuum thermodynamics. J. Appl. Mech. **50**, 1010–1020 (1983)
12. Kuang, Z.-B.: Nonlinear Continuum Mechanics. Shanghai Jiaotong University Press, Shanghai (2002, in Chinese)
13. Kuang, Z.B.: General variational principle and the theory of the plate. In: Scientific report of Xi'an Jiaotong University, Xi'an (1964, in Chinese)
14. Singh, B.: On the theory of generalized thermoelasticity for piezoelectric materials. Appl. Math. Comput. **171**, 398–405 (2005)