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A simple model of nonlinear viscoelasticity taking into account stress relaxation

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Abstract We present one of the simplest theory of nonlinear viscoelasticity taking into account stress relaxation. The model is a 3D nonlinear generalization of the standard *three-parameter model* of 1D classical viscoelasticity. In the framework of this theory, we examine some simple deformations. First of all, we consider a homogeneous deformation as a possible idealization of the usual triaxial test. By this analysis, we show that the model under investigation may be interesting to describe the mechanical behavior of materials like bitumen and hot mix asphalt. Moreover, we investigate the evolution of shearing motions (mainly in the quasi-static approximation) to point out several aspects of the rich mechanical response of constitutive theories in implicit form.

1 Introduction

The increasing use of polymeric solid materials calls for complex mathematical models going beyond infinitesimal strains and that, besides instantaneous elastic response to stress, take into account phenomena like delayed elastic response and stress relaxation. The same happens when we study, via continuum mechanics, many geo-materials like bitumen, hot mix asphalts and rocks.

Usually general models for 3D nonlinear viscoelasticity proposed in the literature are quite complex (see for example [1]) and for this reason any mathematical analysis is discouraged. Recently, a very simple model of nonlinear viscoelasticity which is a feasible 3D analogue of the classical Kelvin–Voigt solid have been considered by several authors. For example, some general mathematical features of this nonlinear Kelvin–Voigt model have been investigated by Quintanilla and Saccomandi [2].

It is well known that the Kelvin–Voigt viscoelasticity cannot account for stress relaxation. The aim of the present manuscript is to consider a model of 3D nonlinear viscoelasticity that may take into account the phenomena associated with stress relaxation. Toward this end, we consider a possible generalization of the

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standard three-parameters (one viscous and two elastic elements) model of classical 1D linear viscoelasticity [3]. Using the notation σ for the 1D stress and ε for the 1D strain, this classical model of linear viscoelasticity is given in the form

$$\sigma + \frac{\nu}{E'}\sigma_t = \nu\left(1 + \frac{E}{E'}\right)\varepsilon_t + E\varepsilon. \tag{1}$$

This represents a spring with stiffness E in parallel with a Maxwell element (dashpot ν and spring E'). Similarly, by considering a Kelvin–Voigt element (dashpot ν^* and spring E^*) in series with a spring $E^{*\prime}$ the same model can be expressed in the form

$$\sigma + \frac{\nu^*}{E'^* + E^*}\sigma_t = \frac{\nu^* E'^*}{E'^* + E^*}\varepsilon_t + \frac{E^* E^{*\prime}}{E'^* + E^*}\varepsilon. \tag{2}$$

As it is well known, both Eqs. (1) and (2) account for stress relaxation, the main objectives of this paper are to extend these basic 1D models in the framework of a 3D and nonlinear theory.

The plan of the paper is as follows: in Sect. 2, we develop the model. In Sect. 3, we study an idealization of the triaxial test, a common experiment in the framework of geo-materials. In Sect. 4, we consider the propagation of a transverse wave and, in the quasi-static approximation, the usual creep and recovery test based on the simple shear deformation. Small shearing motions superposed on large steady shear is explained in Sect. 5.

Our findings show that the model developed here is not only mathematically feasible but also allows to give an acceptable description of some experimental data. Therefore, we have found an interesting tool to investigate the complex, but intriguing, world of nonlinear viscoelasticity.

2 Basic equations

As is conventional in continuum mechanics, the motion of a body is described by a relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where $\mathbf{x}$ denotes the current coordinates of a point occupied by the particle of coordinates $\mathbf{X}$ in the reference configuration at the time t. The deformation gradient $\mathbf{F}(\mathbf{X}, t)$ and the spatial velocity gradient $\mathbf{L}(\mathbf{X}, t)$ associated with the motion are defined as $\mathbf{F} := \partial\mathbf{x}/\partial\mathbf{X}$ and $\mathbf{L} :=\text{grad}\mathbf{v}$, respectively ($\mathbf{v} = \partial\mathbf{x}/\partial t$). The other geometrical and kinematical quantities of interest are the left Cauchy–Green strain tensors $\mathbf{B} := \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and the stretching tensor $\mathbf{D} = (\mathbf{L} + \mathbf{L}^{\mathrm{T}})/2$. For incompressible materials, we have $\det\mathbf{F} = 1$ and trace $(\mathbf{D}) = 0$, therefore, the Cauchy stress $\mathbf{T}$ can be determined only within an arbitrary spherical part $-p\mathbf{I}$.

The equations of motions are given by the local form of the balance of linear momentum as

$$\rho\mathbf{a} = \text{div}\,(\mathbf{T}), \tag{3}$$

where ρ stands for the mass density, $\mathbf{a}$ the acceleration field and the notation div stands for the divergence operator with respect to current coordinates.

For the elastic part of the model, we shall consider a strain energy as in the generalized neo-Hookean case, i.e., $W = W(I_1)$ with $I_1 = \text{trace}\,(\mathbf{B})$. An example of possible strain energy is given by

$$W(I_1) = \frac{\mu_0}{2b}\left\{\left[1 + \frac{b}{n}(I_1 - 3)\right]^n - 1\right\}, \tag{4}$$

where μ_0 is the shear modulus, b a material constant, n a hardening parameter when $n > 1$, or a softening parameter when $n < 1$. When $n = 1$, from Eq. (4) we recover the classical neo-Hookean strain energy density function.

The constitutive model we consider is formulated considering for the Cauchy stress tensor the usual relation

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{5}$$

and for the extra-stress $\mathbf{S}$ the implicit representation

$$\mathbf{S} + \lambda_1 \overset{\triangledown}{\mathbf{S}} = \mu\mathbf{B} + 2\eta_1\mathbf{D}, \tag{6}$$

where $\mu = 2\partial W/\partial I_1$, λ_1 is a relaxation time and η_1 is the viscoelastic parameter. In Eq. (6), we have the upper-convected frame invariant derivative defined as

$$\overset{\triangledown}{\mathbf{S}} = \dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}, \tag{7}$$

where a superposed dot indicates the material time derivative. When $\lambda_1 = 0$, we recover the Kelvin–Voigt nonlinear viscoelastic model considered for example in [2].

Obviously, the viscoelastic moduli in Eq. (6) may depend on the deformation and/or on the rate of the deformation and they have to be considered constant only in the simplest situations. Here, we restrict to this simple case, because our main goal is to investigate the basilar effects of stress relaxation in a 3D and nonlinear framework. Whereas in the framework of non-Newtonian fluid-mechanics several examples of implicit models have been considered in detail (see for example [4]), it seems that this is not the case when we restrict our attention to solid mechanics [5]. To the best of our knowledge, we have to record a certain activity in this direction only in the framework of hypo-elasticity by Bernstein [6] or in the theory of internal variables. In very recent times, Rajagopal and coworkers have proposed a general thermodynamical framework that allows in a clear and efficient way to deduce the models of viscoelasticity and plasticity. This theory may deliver constitutive equations in implicit form [7]. All the above mentioned approaches are more general, but also more complex with respect to the one we are considering here. For this reason, this note is a modest contribution toward the study of implicit constitutive models.

3 A simple experiment

The first question is to understand if the model we are proposing is useful to describe some experimental data obtained from basic tests on real polymeric or geological materials. Toward this end, we consider the basic triaxial experiment that we idealize using a simple homogeneous motion. We remember that first of all we are interested in considering a nonlinear elastic behavior. For this reason, the stress relaxation time and the viscosity are considered as constant parameters.

Let r, θ and z be the cylindrical coordinates in the current configuration and R, Θ and Z the cylindrical coordinates in the reference configuration. We consider the homogeneous isochoric deformation

$$r = \frac{R}{\sqrt{\Lambda(t)}}, \quad \theta = \Theta, \quad z = \Lambda(t) Z. \tag{8}$$

Easily we compute

$$(\mathbf{F})_{ij} = \operatorname{diag}\left(\frac{1}{\sqrt{\Lambda(t)}}, \frac{1}{\sqrt{\Lambda(t)}}, \Lambda(t)\right), \quad (\mathbf{L})_{ij} = \frac{\Lambda'}{\Lambda}\operatorname{diag}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right). \tag{9}$$

Here an apex stands for the time derivative.

The triaxial test is quite common in the mechanical characterization of geological materials (see [8]) and today it is used also in the testing of asphalt mix (see, e.g., [9,10]).

By assuming a priori (as usual in the semi-inverse method) the extra-stress tensor (6) in diagonal form, we obtain the following equations:

$$S_{rr} + \lambda_1 \left(S'_{rr} + \Lambda'\Lambda^{-1} S_{rr}\right) = \mu\Lambda^{-1} - \eta_1 \Lambda'\Lambda^{-1}, \tag{10.1}$$

$$S_{\theta\theta} + \lambda_1 \left(S'_{\theta\theta} + \Lambda'\Lambda^{-1} S_{\theta\theta}\right) = \mu\Lambda^{-1} - \eta_1 \Lambda'\Lambda^{-1}, \tag{10.2}$$

$$S_{zz} + \lambda_1 \left(S'_{zz} - 2\Lambda'\Lambda^{-1} S_{zz}\right) = \mu\Lambda^{2} + 2\eta_1 \Lambda'\Lambda^{-1}. \tag{10.3}$$

We note that Eq. (10.1) [or (10.2)] can be rewritten, for $\Lambda(t) \neq 0$, as

$$\Lambda S_{rr} + \lambda_1 (\Lambda S_{rr})' = \mu - \eta_1 \Lambda', \tag{11}$$

and Eq. (10.3) as

$$\frac{S_{zz}}{\Lambda^2} + \lambda_1 \left(\frac{S_{zz}}{\Lambda^2}\right)' = \mu - \eta_1 \left(\frac{1}{\Lambda^2}\right)'. \tag{12}$$

During the experiment,[1] a specimen is mounted between the cross-heads which are then pulled (or compressed) at a constant speed, i.e.,

$$\Lambda(t) = 1 + \gamma t, \tag{13}$$

where γ is a constant. (We have $\gamma > 0$ for tension and $\gamma < 0$ for compression.) If no load is considered on the mantle (i.e., the lateral side) of the cylinder, then the relations $T_{rr} = T_{\theta\theta} = 0$ hold and

$$p = S_{rr}, \tag{14}$$

so that

$$T_{zz} = S_{zz} - S_{rr}. \tag{15}$$

It is convenient to work using the Piola–Kirchoff tensor (or nominal stress tensor) $\mathbf{P} = \mathbf{TF}^{-\mathrm{T}}$, given by

$$\begin{aligned} P_{RR} &= -p\sqrt{\Lambda} + S_{rr}\sqrt{\Lambda}, \\ P_{\Theta\Theta} &= -p\sqrt{\Lambda} + S_{\theta\theta}\sqrt{\Lambda}, \\ P_{ZZ} &= -p\Lambda^{-1} + S_{zz}\Lambda^{-1}. \end{aligned}$$

If Eq. (14) holds, the tensile force in the axial direction is given by

$$F = (S_{zz} - S_{rr})\,\Lambda^{-1}. \tag{16}$$

On the other hand, if in the triaxial test we consider a lateral confinement given by the load $\mathcal{L}$ (in the reference configuration), it must be $p = S_{rr} - \mathcal{L}\Lambda^{-1/2}$ and the tensile force is

$$F = (S_{zz} - S_{rr})\,\Lambda^{-1} + \mathcal{L}\Lambda^{-3/2}. \tag{17}$$

Moreover, we may assume that the initial conditions are $S_{zz}(0) = S_{rr}(0) = 0$.

As first step in our analysis, we introduce a dimensionless version of the various equations of interest. To this end, we introduce the quantities

$$t = t_c\widehat{t}, \quad S_{rr} = \mu_0\widehat{S}_{rr}, \quad S_{zz} = \mu_0\widehat{S}_{zz}, \quad F = \mu_0\widehat{F}, \quad \mathcal{L} = \mu_0\widehat{\mathcal{L}}. \tag{18}$$

The choice of t_c must be done carefully. Indeed, if we choose $t_c = \lambda_1$ we eliminate from the determining equations a parameter, and this can be used to simplify analytical computations. On the other hand, if we fix $t_c = \gamma^{-1}$ we perform a choice more useful to fit experimental data, since γ is a parameter a priori known. Here, we are interested to both choices, but we will address in a second moment the problems arising in the fitting of experimental data. Therefore, first of all we consider $t_c = \lambda_1$, then $\Lambda(\widehat{t}) = 1 + \lambda_1\gamma\widehat{t} = 1 + k\widehat{t}$, where we set $k = \gamma\,\lambda_1$. Hence,

$$\Lambda' = \frac{\mathrm{d}\Lambda}{\mathrm{d}\widehat{t}} = k, \quad \mathrm{d}\Lambda = k\mathrm{d}\hat{t}.$$

This choice allows to rewrite Eqs. (11), (12) and (17) as

$$\Lambda\widehat{S}_{rr} + \frac{\mathrm{d}\left(\Lambda\widehat{S}_{rr}\right)}{\mathrm{d}\widehat{t}} = \widehat{\mu} - \eta k, \tag{19.1}$$

$$\frac{\widehat{S}_{zz}}{\Lambda^2} + \frac{\mathrm{d}\left(\widehat{S}_{zz}\Lambda^{-2}\right)}{\mathrm{d}\widehat{t}} = \widehat{\mu} + 2\eta\,k\Lambda^{-3}, \tag{19.2}$$

$$\widehat{F} = \left(\widehat{S}_{zz} - \widehat{S}_{rr}\right)\Lambda^{-1} + \widehat{\mathcal{L}}\Lambda^{-3/2}, \tag{19.3}$$

where we set $\eta = \eta_1/(\lambda_1\mu_0)$ and $\widehat{\mu} = \mu/\mu_0$.

[1] We point out that the same experimental framework may be used in stress-relaxation tests keeping the deformation constant (i.e., $\Lambda' = 0$). Because λ_1 is constant, this kind of experiments will only reveal that the stress components will relax exponentially as in the linear case.

In the following, let us drop the hat symbol from the various formulas for simplicity of notation. Therefore, if we denote $\varsigma = \Lambda S_{rr}$, from Eq. (19.1) we obtain

$$\varsigma + \frac{\mathrm{d}\varsigma}{\mathrm{d}t} = \mu - \eta\, k \quad \rightarrow \quad \varsigma + k\frac{\mathrm{d}\varsigma}{\mathrm{d}\Lambda} = \mu - \eta\, k, \tag{20}$$

and, if we denote $\xi = S_{zz}/\Lambda^2$, from Eq. (19.2) we obtain

$$\xi + \frac{\mathrm{d}\xi}{\mathrm{d}t} = \mu + 2\eta\, k\Lambda^{-3} \quad \rightarrow \quad \xi + k\frac{\mathrm{d}\xi}{\mathrm{d}\Lambda} = \mu + 2\eta\, k\Lambda^{-3}. \tag{21}$$

The general solutions of Eqs. (20) and (21) are

$$\Lambda S_{rr} = \exp(-\Lambda/k)\left\{\int \exp(\Lambda/k)\,(\mu - \eta\, k)\,\mathrm{d}\Lambda + \text{const.}\right\} \tag{22}$$

and

$$\frac{S_{zz}}{\Lambda^2} = \exp(-\Lambda/k)\left\{\int \exp(\Lambda/k)\left(\mu + 2\eta\, k\Lambda^{-3}\right)\mathrm{d}\Lambda + \text{const.}\right\}, \tag{23}$$

respectively.

In the case of a neo-Hookean constitutive equation, we have $\mu = 1$ and by considering the initial conditions $S_{rr}(0) = S_{zz}(0) = 0$, in the case $k = 1$ we recover the closed form exact solutions as

$$\Lambda S_{rr} = (1-\eta)\,(1-\exp{(1-\Lambda)}), \tag{24}$$

$$\frac{S_{zz}}{\Lambda^2} = (1-\exp(1-\Lambda)) + \eta\,[Ei\,(1,-1) - Ei\,(1,-\Lambda)]\exp(-\Lambda) + \eta\left[2\exp(1-\Lambda) - \left(\frac{1}{\Lambda^2}+\frac{1}{\Lambda}\right)\right]. \tag{25}$$

(With Ei, we denote the usual error function.)

The exact solutions (24) and (25) are here recorded to illustrate the mathematical feasibility of the model, but they are not very useful: the corresponding diagram $T = T(\Lambda)$ has a simple quasi-linear behavior which seems to be not realistic for usual polymeric or geologic materials.

For this reason, we consider the dimensionless version of the nonlinear elastic strain energy in Eq. (4):

$$\mu = \left[1 + \frac{b}{n}\left(\frac{2}{\Lambda} + \Lambda^2 - 3\right)\right]^{n-1}.$$

Clearly in this case Eqs. (20) and (21) must be solved numerically using a standard numerical method for ODEs.

In Figs. 1, 2 and 3, we show the different abilities of the model (6) with strain energy (4), by reporting, for the motion under investigation, the nominal axial stress. We point out that in nonlinear elasticity it is usual to fix $b = 1$ and to enforce the restriction $n > 1/2$ in order to ensure a monotonic response in simple shear.

In [11], it is possible to find some data for a constant rate extension test at 10°C for bitumen (a standard material with penetration grade 50). Cheung and Cebon [11] propose several complex but 1D constitutive equations. The same data have been studied by Murali Krishnan and Rajagopal [12,13] via a theory with more than one natural configuration that yields a very careful theoretical description of the various data set. In Fig. 4, we plot the data set in [11] obtained for the rate $\gamma = 0.05\ \mathrm{s}^{-1}$ and the prediction by the model here considered when the (full dimension) parameters are

$$\mu_0 = 1\ \text{MPa}, \quad \eta_1 = 2\ \text{MPa s}, \quad \lambda_1 = 0.4\ \text{s}, \quad n = 0.55, \quad b = 7.5.$$

The theoretical curve fits very well the behavior of the data for a fixed strain rate. The problem appears when we try to describe the data associated with various strain rates using the same material parameters. The model we are investigating is not able to describe carefully all sets of data associated with the different strain rates. We deduce that to improve the model, at least λ_1 must depend on the strain rate. In so doing the mathematical complexity of the model will increase dramatically.

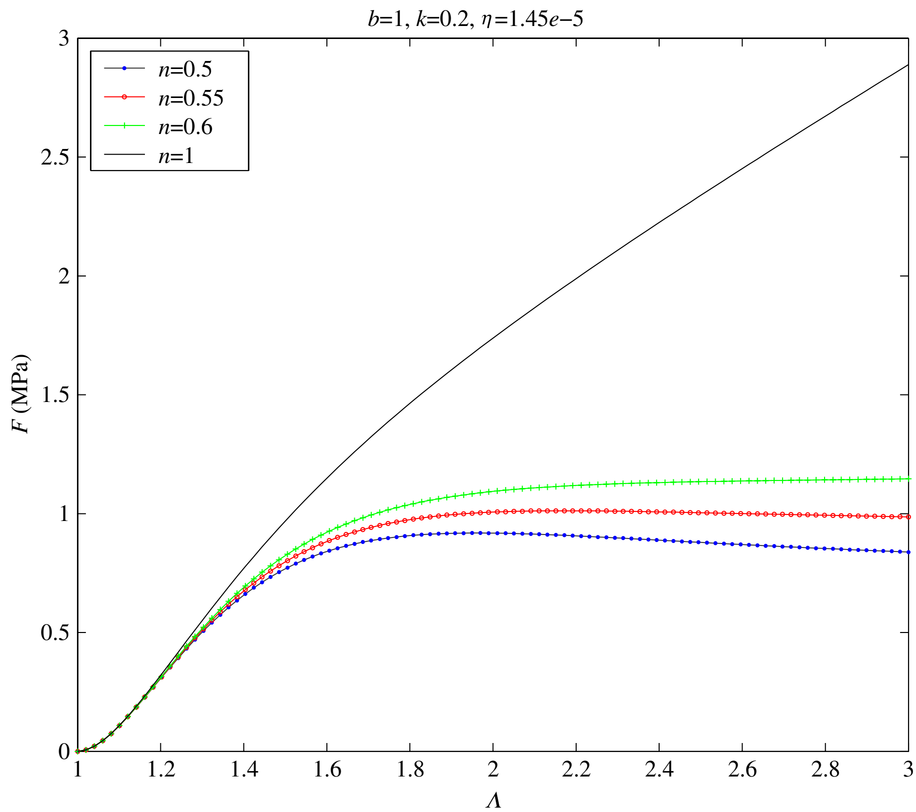

Fig. 1 We point out that a nonmonotonic behavior for the nominal stress F may be described by the model by considering $n \leq 0.55$

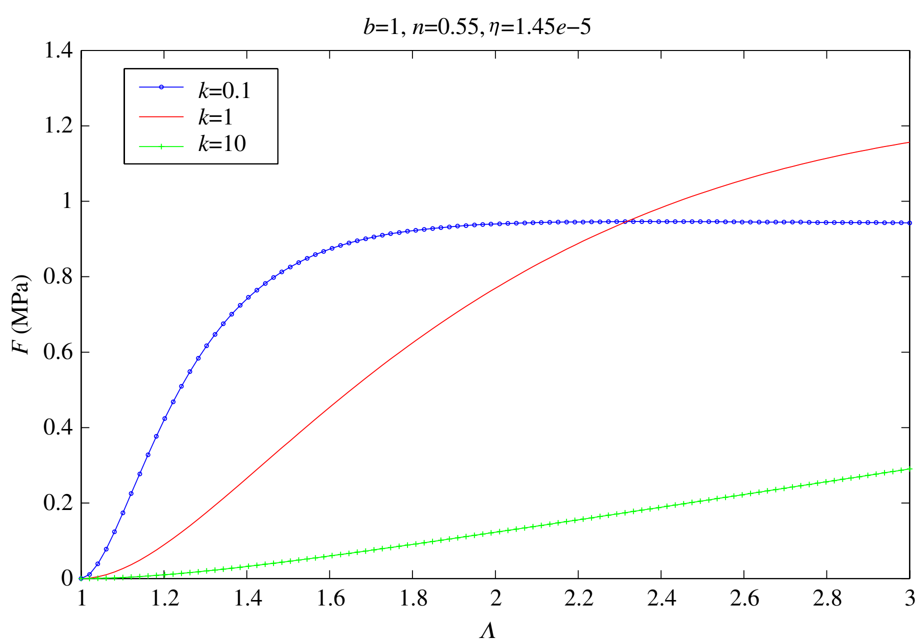

Fig. 2 The stress relaxation time influences in a sensitive way the rate of growth of the nominal stress

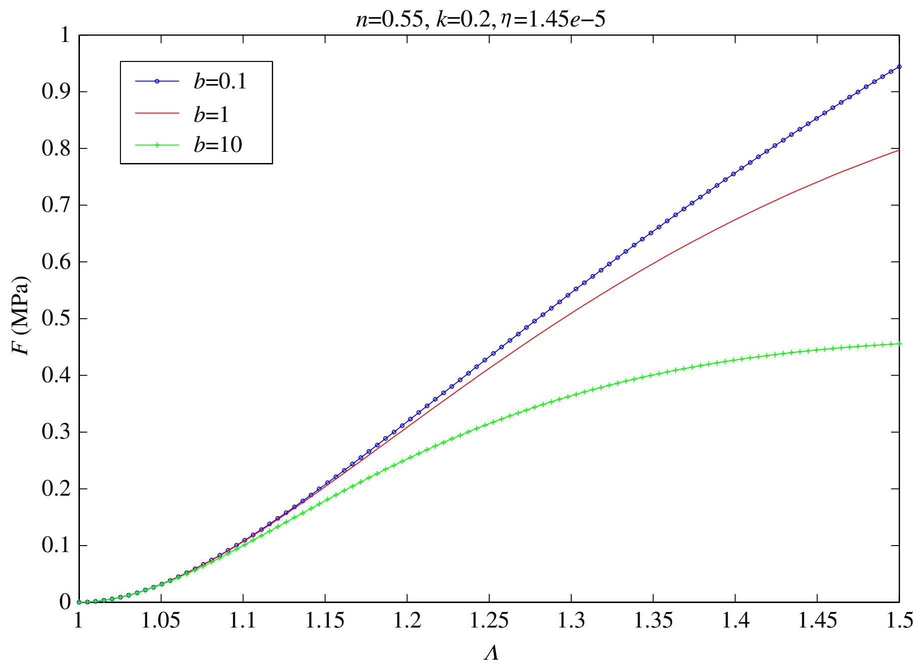

Fig. 3 The parameter b which in elasticity is usually fixed at $b = 1$ is important to control the maximum nominal stress

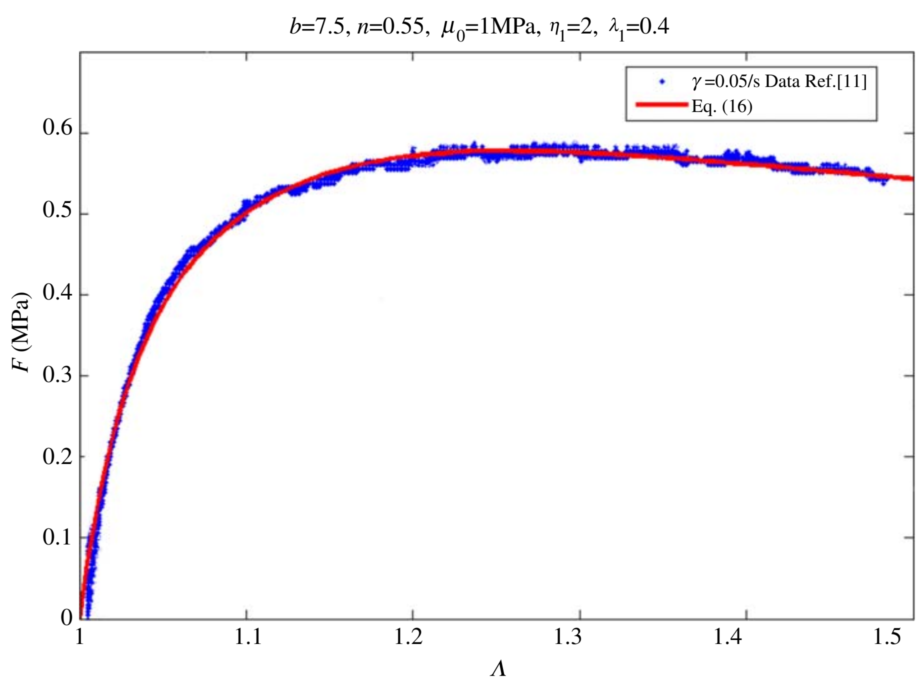

Fig. 4 The nominal stress displayed is typical of triaxial extension test for material like bitumen. Data from Cheung and Cebon [11]

In Fig. 5, we show that it is also possible to tune the various parameters to describe a typical curve of an extension experiment for hot mix asphalt (see for example [10]). It is easy to show that the model may be applied successfully also to other materials like plasticine [14].

Therefore, our conclusion is that the model here considered is quite interesting and it is able to describe with a certain degree of accuracy various sets of experimental data for real materials. This means that our

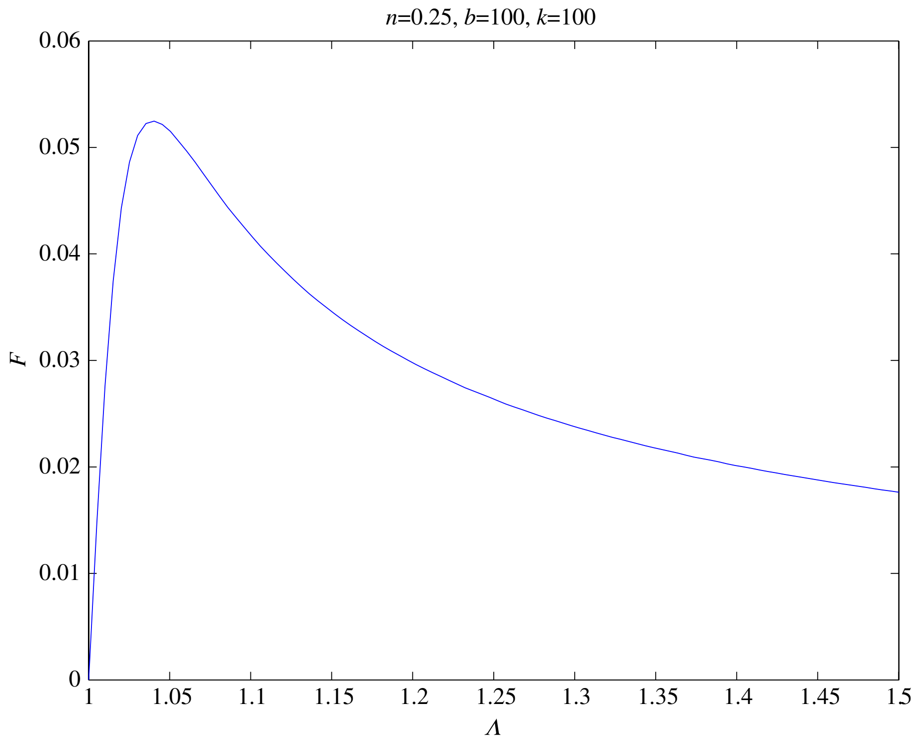

Fig. 5 A typical nonmonotonic curve for a tension test for hot mix asphalt, obtained by considering a softening elastic constitutive equation

model is more realistic than the simple generalization of the Kelvin–Voigt model proposed for example by Quintanilla and Saccomandi [2].

4 Shearing motions

Shearing motions are more complex than the homogeneous deformation just considered. To study these motions, we introduce rectangular Cartesian coordinates in the reference and current configurations and we write

$$x = X + f(Y,t), \quad y = Y, \quad z = Z, \tag{26}$$

such that we have

$$(\mathbf{F})_{ij} = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\mathbf{F}^{-1}\right)_{ij} = \begin{pmatrix} 1 & -K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

where the amount of shear is defined by $K = f_Y$ (the subscript indicates a partial derivative).

We consider this motion to investigate the mathematical feasibility of the model. Indeed, we show that there may be physical situations where the determining equations for the amount of shear, also in the quasi-static approximation, is more involved than those in the case of explicit constitutive laws. These difficulties are not present in the linear theory of the standard 3D solids, because in this case all the stress components are decoupled.

By a simple computation we have

$$(\mathbf{B})_{ij} = \begin{pmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{28}$$

and since

$$(\mathbf{L})_{ij} = \begin{pmatrix} 0 & K' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(\mathbf{L}^{\mathrm{T}}\right)_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ K' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

we have

$$(\mathbf{D})_{ij} = \frac{1}{2} \begin{pmatrix} 0 & K' & 0 \\ K' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Consequently, all the components of the extra-stress $\mathbf{S}$ may be functions only of y and t and from the balance equations (3) we obtain

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} + \frac{\partial S_{12}}{\partial y} = \rho \frac{\partial^2 f}{\partial t^2}, \\ -\frac{\partial p}{\partial y} + \frac{\partial S_{22}}{\partial y} = 0, \\ -\frac{\partial p}{\partial z} = 0. \end{aligned}\right\} \tag{31}$$

On the other hand from (6) and, in accordance with the semi-inverse method, setting $S_{13} = S_{23} = 0$ we obtain

$$\left.\begin{aligned} S_{11} + \lambda_1 \left[S'_{11} - 2K' S_{12}\right] = \mu \left(1 + K^2\right), \\ S_{22} + \lambda_1 S'_{22} = \mu, \\ S_{33} + \lambda_1 S'_{33} = \mu, \end{aligned}\right\} \tag{32}$$

and

$$S_{12} + \lambda_1 \left[S'_{12} - K' S_{22}\right] = \mu K + \eta_1 K'. \tag{33}$$

From (31) we may obtain the pressure field and a compatibility equation given by

$$\frac{\partial^2 S_{12}}{\partial Y^2} = \rho K''. \tag{34}$$

It is possible to rewrite the Eqs. (32)–(34) in dimensionless form by considering

$$Y = L\widehat{Y}, \quad S_{ij} = S_{12}(0)\widehat{S}_{ij}, \quad t = \lambda_1 \widehat{t}, \quad \mu = S_{12}(0)\widehat{\mu}. \tag{35}$$

In these new variables, we have

$$\begin{aligned} \frac{\partial^2 \widehat{S}_{12}}{\partial \widehat{Y}^2} &= \frac{L^2 \rho}{S_{12}(0)\lambda_1^2} K'', \\ \widehat{S}_{11} + \left[\widehat{S}'_{11} - 2K'\widehat{S}_{12}\right] &= \widehat{\mu}\left(1 + K^2\right), \\ \widehat{S}_{12} + \left[\widehat{S}'_{12} - K'\widehat{S}_{22}\right] &= \widehat{\mu} K + \tfrac{\eta_1}{S_{12}(0)\lambda_1} K', \\ \widehat{S}_{22} + \widehat{S}'_{22} &= \widehat{\mu}, \\ \widehat{S}_{33} + \widehat{S}'_{33} &= \widehat{\mu}. \end{aligned} \tag{36}$$

In the following, we introduce the notation

$$\iota = \frac{L^2 \rho}{S_{12}(0)\lambda_1^2}, \quad \eta = \frac{\eta_1}{S_{12}(0)\lambda_1}, \tag{37}$$

and we drop the hat in all the variables involved.

4.1 The neo-Hookean case

In the neo-Hookean case, we have that the dimensionless μ is simply given by $\mu_0/S_{12}(0)$. In this case, we may or may not have relaxation for the S_{22} component of the extra-stress tensor.

In the first case (no normal relaxation) we have that it must be $S_{22} = \mu$ and, therefore, the equations in the quasi-static equations are similar to the case of a Kelvin–Voigt solid. Stress relaxation will affect only initial conditions. Indeed, we have from Eq. (36)

$$S_{12} + S_{12}' = \mu K + (\eta + \mu)\, K'. \tag{38}$$

By a straightforward combination of $S_{12,YY} = \iota K_{tt}$ and Eq. (38), we deduce the third order equation

$$\iota\, (K_{tt} + K_{ttt}) = \mu K_{YY} + (\eta + \mu)\, K_{tYY}. \tag{39}$$

This equation has exactly the same structure as Eq. (21) in [15]. Therefore, this is a well known equation and, just for the sake of completeness, we consider limit $\iota = 0$ of Eq. (39) to study the classical creep test. In this case, a constant stress $S_{12}(0)$ is applied at time $t = 0$ and suddenly removed at time $t = t_0$. Using the classical Heaviside step function, the stress component before adimensionalization is given by $S_{12}(t) = S_{12}(0)\,[H(t) - H(t - t_0)]$ and, therefore, $\widehat{S}_{12}(t) = H(t) - H(t - t_0)$. Hence, we have that the ordinary differential equation for $K(t)$ is given by

$$\mu K + (\eta + \mu)\, K_t = 1, \tag{40}$$

and must be solved in $[0, t_0]$ with the initial condition $K(0) = (\eta + \mu)^{-1}$, while in $[t_{0,}\infty)$ the ordinary differential equation is given by

$$\mu K + (\eta + \mu)\, K_t = 0, \tag{41}$$

with the initial condition in t_0 given by

$$\lim_{t \to t_0^+} K(t) = \lim_{t \to t_0^-} K(t) - \frac{1}{\eta + \mu}. \tag{42}$$

Let

$$\Phi(t) = \frac{1}{\mu}\left[1 - \frac{\eta}{\eta + \mu}\exp\left(-\frac{\mu}{\eta + \mu}t\right)\right]. \tag{43}$$

Therefore, the solution of the creep problem can be written as

$$K(t) = \Phi(t)H(t) - \Phi(t - t_0)H(t - t_0). \tag{44}$$

The possibility of stress relaxation for the S_{22} component of the extra-stress is more realistic. This means that

$$S_{22}(t) = \mu + A\exp(-t), \tag{45}$$

where the integration constant A is given by the equation $S_{22}(0) = \mu + A$.

We remember that what we observe in the real world is not the extra-stress, but the full stress tensor (5). To give some insight on the meaning of $S_{22}(0)$, it is necessary to say some more details on the geometry and the loading conditions. If, for example, we consider no loading in the direction perpendicular to the 1, 2-plane, it must be that $T_{33} = 0$, i.e., $p = S_{33}$, and, therefore, we have

$$T_{22} = S_{22} - S_{33}, \tag{46}$$

and $S_{22}(0) = T_{22}(0) + S_{33}(0)$. In the simplest case, we can choose $S_{33} = \mu$.

In this case, Eq. (45) implies

$$S_{12} + S_{12}' = \mu K + (\eta + \mu)\, K' + A\exp(-t)K' \tag{47}$$

and the partial differential equation for the amount of shear K is given by

$$\iota\, (K_{tt} + K_{ttt}) = \mu K_{YY} + (\eta + \mu)\, K_{tYY} + A\exp(-t)K_{tYY}. \tag{48}$$

This simple example shows the difference between a 1D equation and a 3D model derived by the use of the standard methods of continuum mechanics. The interplay between the shear-stress and the normal stresses is clearly evident.

Let us take $\iota = 0$ (quasi-static case) and let us consider once again the creep test. In the loading phase (i.e., $[0, t_0]$), we have to solve the nonautonomous ordinary differential equation

$$\mu K + \left[\eta + \mu + A \exp(-t)\right] K_t = 1, \tag{49}$$

subject to the initial condition

$$K(0) = \frac{1}{\eta + \mu + A}. \tag{50}$$

In this case, the solution of the Cauchy problem is given by

$$K(t) = \frac{1}{\mu}\left\{1 - (\eta + \mu + A)^{-\frac{\eta}{\eta+\mu}} (\eta + A) \left[(\eta + \mu)\exp(t) + A\right]^{-\frac{\mu}{\eta+\mu}}\right\}. \tag{51}$$

When $A \to 0$ from Eq. (49) we recover Eq. (40) and from Eq. (51) the corresponding solution in (44). For $A \neq 0$, are introduced some differences that may be relevant in situation of transient loads, as those present in a road where shear stresses are always coupled with a vertical loading condition.

We point out that at the instant t_0 when the creep phase is finished, for the recovery phase we must solve the problem

$$\mu K + \left[\eta + \mu + A \exp(-t)\right] K_t = 0, \tag{52}$$

with the initial condition

$$K^+(t_0) = K^-(t_0) - \frac{1}{\eta + \mu + A \exp(-t_0)}.$$

Here $K^-(t_0)$ is computed from Eq. (51).

We do not discuss the resolution of (48) when $\iota \neq 0$, since this is obtained by Morrison [15], by a technique based on Laplace transforms and sufficiently cumbersome from the algebraic point of view.

In Fig. 6, we show an example of the creep-recovery phenomena for different values of the initial parameter A. From these simple solutions it is clear that a transient S_{22} component of the extra-stress may change the characteristic creep time. Therefore, the evolution of the shear is affected by the time-dependence of the normal stresses and this especially in the creep phase. In the recovery phase this difference is smeared by the factor $\exp(-t_0)$ in front of the integration constant A in the initial condition of Eq. (52).

4.2 The generalized neo-Hookean case

In the generalized neo-Hookean case (4), we have

$$S_{22} + S'_{22} = \mu(K^2). \tag{53}$$

To find nontrivial solutions, we have in any case to consider a nonconstant S_{22}, that may be computed by

$$S_{22}(t, Y) = S_{22}(0, Y)\exp(-t) + \exp(-t)\int_0^t \exp(s)\mu(K^2)\mathrm{d}s. \tag{54}$$

Therefore, we have the new equations

$$S_{12} + S'_{12} = \mu K + (\eta + S_{22}(t, Y))\, K', \tag{55}$$

$$\iota\,(K_{tt} + K_{ttt}) = \frac{\partial^2}{\partial Y^2}\left\{\mu(K^2)K + (\eta + S_{22}(t, Y))\, K'\right\}, \tag{56}$$

which, because of Eq. (54), is an integro-differential equation.

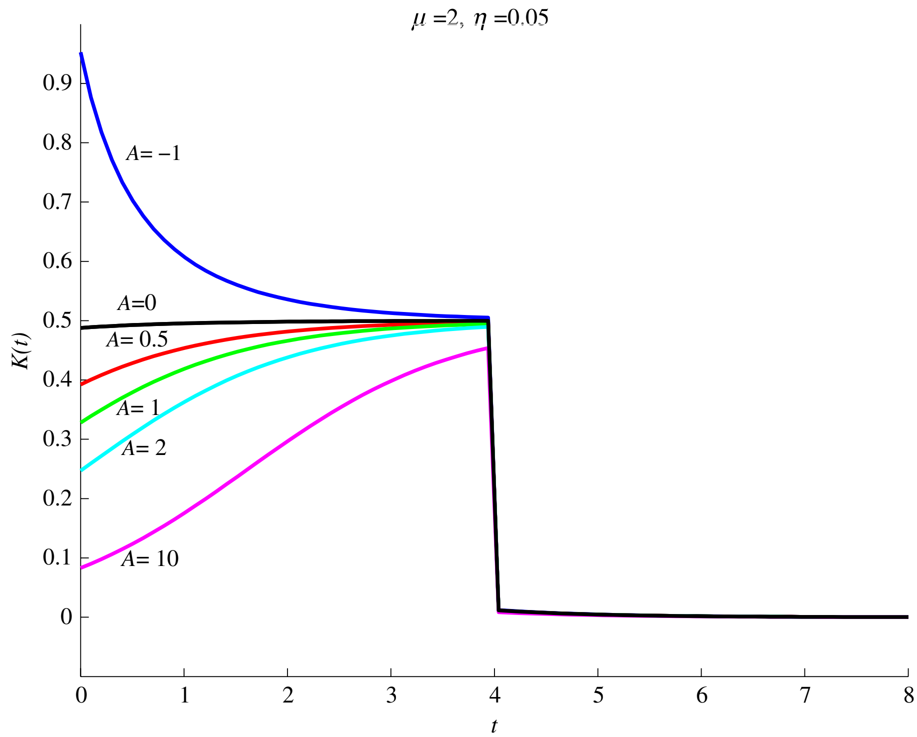

Fig. 6 Note that when $A \neq 0$, the various curves need more time to approach the asymptotic steady state in the creep phase. In the recovery phase, this difference is not so strong

If we consider the quasi-static case ($\iota = 0$), Eqs. (57) and (56) become the system of ordinary differential equations

$$\begin{aligned} \mu K + (\eta + S_{22})\, K' &= 1, \\ S_{22} + S_{22}' &= \mu, \end{aligned} \tag{57}$$

or the corresponding higher order equation.

In Fig. 7, we show the solutions of Eqs. (57) in the case that $\mu = \mu_0 + \mu_1 K^2$ and $S_{22}(0) = \mu_0$. We remark that only for the linear case we have that the normal extra-stress component remains constant.

The example of the shearing motion shows that for implicit constitutive equations there is a more deep interplay between normal and shearing stresses, that is usual in nonlinear theories of elasticity.

5 Small shearing motion superposed on large steady shear

Following Bernstein [16], we consider a nonlinear static simple shear deformation characterized by an amount of shear K_0 to obtain further analytical results which can be useful in the context of complex theories.

It is well known that in the static case for the various extra stress components it must be

$$\begin{aligned} &S^0_{11} = \mu\,(K_0)\left(1 + K_0^2\right), \quad S^0_{12} = \mu\,(K_0)\,K_0, \\ &S^0_{22} = S^0_{33} = \mu\,(K_0)\,. \end{aligned} \tag{58}$$

Moreover, in this case we have the celebrated universal relation

$$S^0_{11} - S^0_{22} = S^0_{12} K_0, \tag{59}$$

which tells us in a clear and direct way that a simple shear is always coupled with a normal stress difference. We also note that $S^0_{12} = S^0_{22} K_0$. By imposing that on the large steady shear, a small shearing motion is superposed, and we write

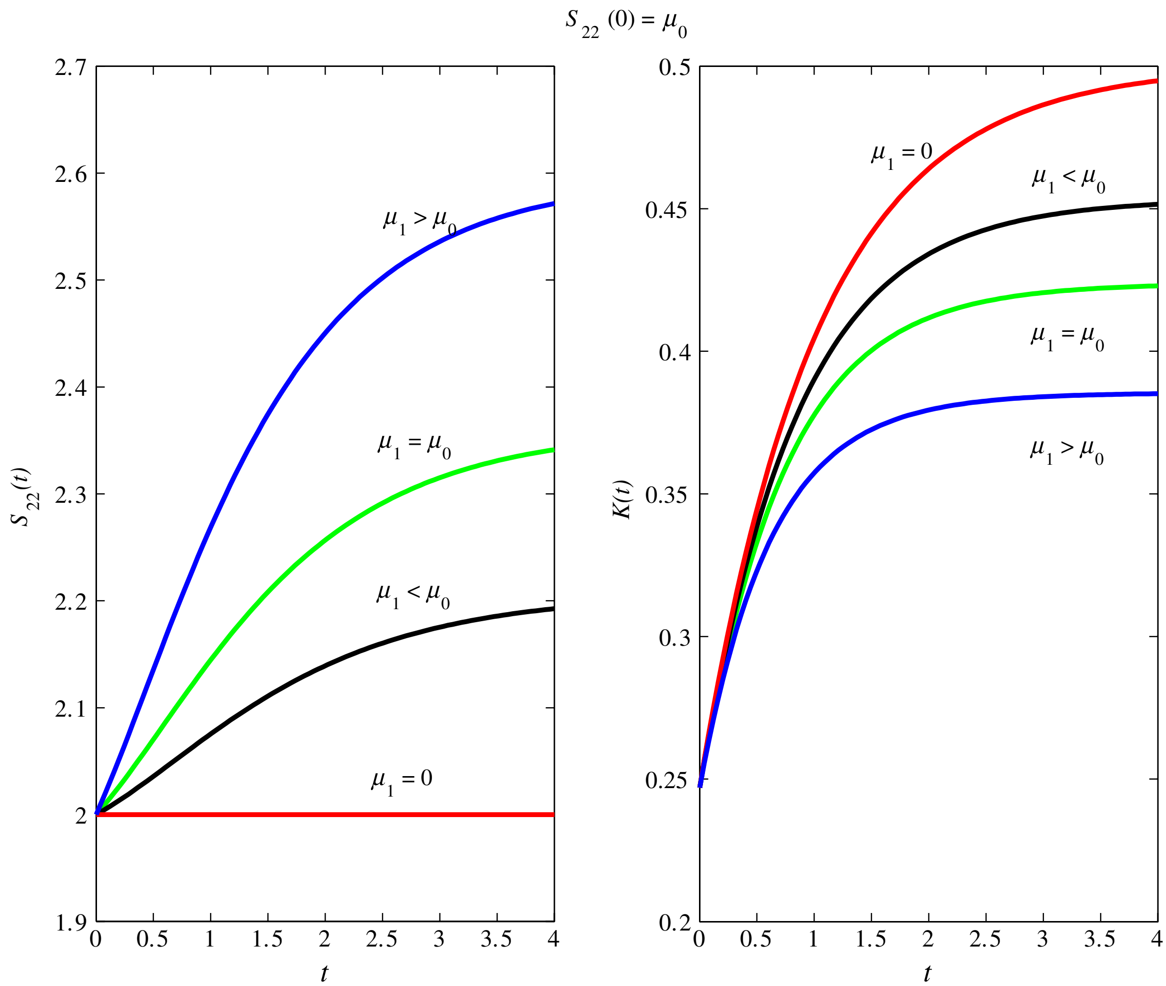

Fig. 7 In the generalized neo-Hookean case, the transient is always more important than in the neo-Hookean case

$$K(Y,t) = K_0 + \varepsilon K_1(Y,t), \quad S_{ij} = S_{ij}^0 + \varepsilon S_{ij}^1(Y,t),$$

where $\varepsilon \ll 1$. In this framework, it is possible to deduce some $O(\varepsilon)$ approximate equations that are quite interesting to understand various phenomena associated with our constitutive model. For the balance equation, we obtain the approximate equation $\frac{\partial^2 S_{12}^1}{\partial Y^2} = \iota K_1''$. By taking into account Eq. (58) and by introducing the approximation

$$\mu \approx \mu(K_0^2) + 2\varepsilon K_0 \left.\frac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} K_1,$$

at the order ε we find the following determining equations for the extra stress tensor:

$$\begin{aligned}
S_{11}^1 + \left(S_{11}^1\right)' &= 2K_0\left[\left(1+K_0^2\right)\left.\tfrac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} + \mu(K_0^2)\right]K_1 + 2S_{12}^0 K_1',\\
S_{12}^1 + \left(S_{12}^1\right)' &= \left[2K_0^2\left.\tfrac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} + \mu(K_0^2)\right]K_1 + \left(\eta + S_{22}^0\right)K_1',\\
S_{22}^1 + \left(S_{22}^1\right)' &= 2K_0\left.\tfrac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} K_1,\\
S_{33}^1 + \left(S_{33}^1\right)' &= 2K_0\left.\tfrac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} K_1.
\end{aligned} \tag{60}$$

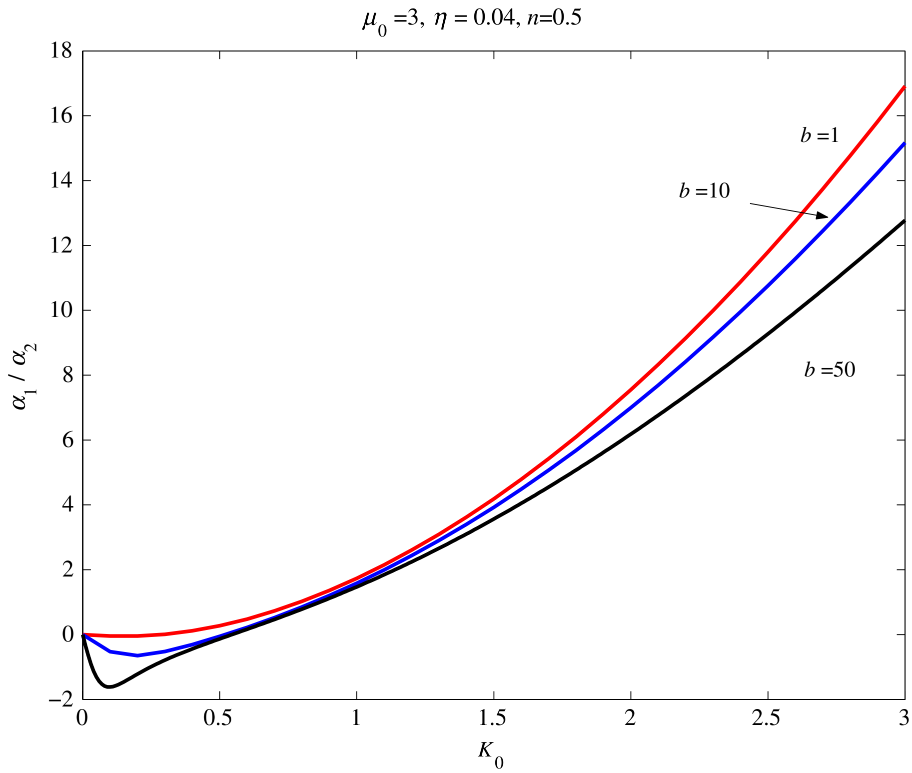

Fig. 8 The characteristic time α_1/α_2 for shearing motion at the order ε for variations of the parameter b in the strain energy model (4)

From these equations, using (59) in the case $\eta = 0$, we have the relation

$$S^1_{11} - S^1_{22} + \left(S^1_{11} - S^1_{22}\right)' = K_0 \left[S^1_{12} + \left(S^1_{12}\right)'\right] + S^0_{12} K'_1,$$

or equivalently

$$S^1_{11} - S^1_{22} = K_0 S^1_{12} + K_1 S^0_{12} - S^0_{12} \left(\exp(-t) \int \exp(s) K_1 \mathrm{d}s\right). \tag{61}$$

Equation (60) is in some way an analogue of the classical universal relation, and it is a good relationship to understand the evolution of the normal stress difference at the order ε. We point out that Eq. (61) is not a universal relation, because the dimensionless time we have introduced depends on the parameter λ_1.

From Eqs. (60) it is possible to obtain the partial differential equation

$$\iota \left(\frac{\partial^2 K_1}{\partial t^2} + \frac{\partial^3 K_1}{\partial t^3}\right) = \left[2K_0^2 \left.\frac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} + \mu(K_0^2)\right] \frac{\partial^2 K_1}{\partial Y^2} + \left[\eta + \mu\left(K_0^2\right)\right] \frac{\partial^3 K_1}{\partial Y^2 \partial t}. \tag{62}$$

On the other hand, from the equation

$$\left[2K_0^2 \left.\frac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} + \mu(K_0^2)\right] K_1 + \left[\eta + \mu\left(K_0^2\right)\right] K'_1 = h, \tag{63}$$

where h is an integration constant, it is possible to deduce the creep and the recovery rates in the classical corresponding experiments. It is interesting to note the similarity with the neo-Hookean case studied in the previous section.

By considering $S^1_{12}(t) = S^1_{12}(0)\left[H(t) - H(t - t_0)\right]$, and by introducing the notation

$$\alpha_1 = 2K_0^2 \left.\frac{\mathrm{d}\mu}{\mathrm{d}K^2}\right|_{\varepsilon=0} + \mu(K_0^2), \quad \alpha_2 = \eta + \mu\left(K_0^2\right),$$

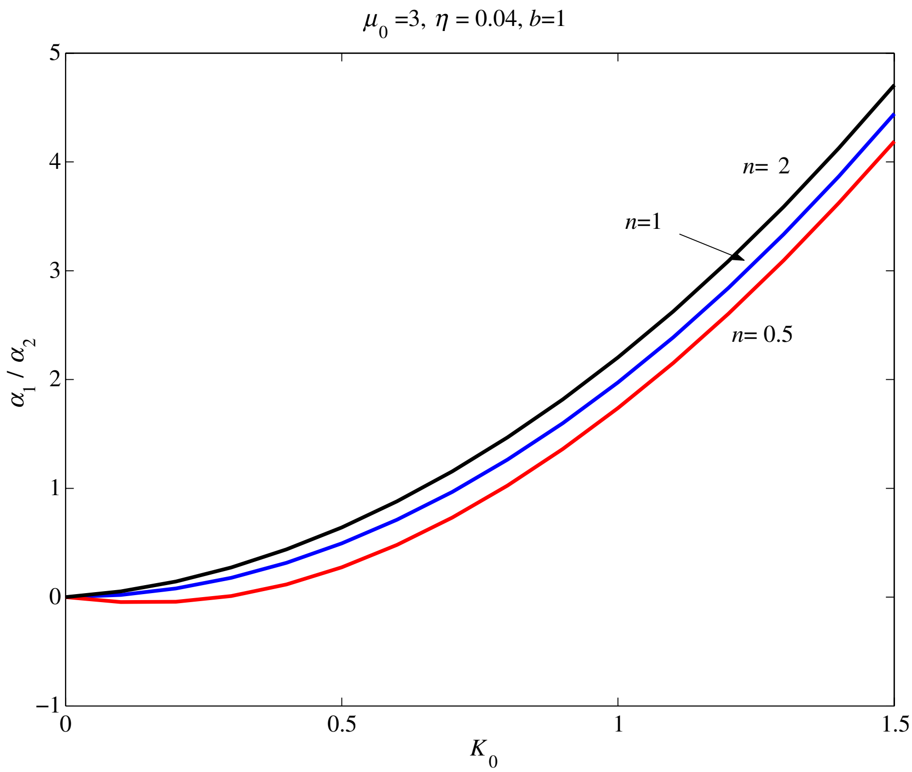

Fig. 9 The characteristic time for order ε shearing motion for variations of the parameter n

we have the solution

$$K_1(t) = \frac{1}{\alpha_1}\left\{\left[1 - \frac{1}{\alpha_2}\exp\left(-\frac{\alpha_1}{\alpha_2}t\right)\right]H(t) - \left[1 - \frac{1}{\alpha_2}\exp\left(-\frac{\alpha_1}{\alpha_2}(t-t_0)\right)\right]H(t-t_0)\right\}. \quad (64)$$

To investigate a specific case we consider again the strain energy in Eq. (4) and we obtain

$$\mu = \mu_0\left(1 + \frac{b}{n}K_0^2\right)^{n-1} + 2\varepsilon K_0\mu_0\frac{(n-1)\,b}{n}\left(1 + \frac{b}{n}K_0^2\right)^{n-2}K_1.$$

From Eq. (59), we have

$$S_{12}^0 = \mu_0\left(1 + \frac{b}{n}K_0^2\right)^{n-1}K_0,$$

and then we compute the value of K_0 when a certain amount of shear stress is given. In Figs. 8 and 9 we plot the quantity α_1/α_2 for various values of the material moduli in dependence of K_0. It is clear from these simple computations that this quantity is not very sensitive with respect to the various material parameters. On the other hand, the underlying static amount of shear plays an important role on the ratio α_1/α_2. This is an interesting feature in materials that may suffer important pre-deformations (as, e.g., hot mix asphalt in roads where the rutting phenomenon occurs).

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