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Application and extension of the stochastic linearization by Anh and Di Paola

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Abstract This study deals with the stochastic linearization technique in a new setting. First of all, the usual minimum mean-square difference requirement between the original nonlinear force and its linear counterpart is replaced by the orthogonal condition. Additionally, another recently developed idea of first replacing the nonlinear terms by higher order terms, prior to its ordinary reduction to linear ones, is super-imposed with the above condition. The results are checked on several nonlinear oscillators. In the Atalik and Utku oscillator, instead of 14% error obtained with classical linearization, the error is reduced to about 3%. In the Lutes and Sarkani oscillator the error is reduced from 22.85 to 1.23%, nearly 18-fold. In the latter case the optimal number of "regulation" steps is shown to be 2.

1 Introduction

In terminology of [5], "the method of statical linearization has remained a surprisingly popular tool over the many years since it was first formulated." The method is based on replacing the original nonlinear system by a linear one, that is equivalent to the original one in some probabilistic sense. Several criteria have been suggested to arrive at the expressions of the equivalent stiffness and equivalent damping.

Anh and Di Paola [3] suggested new realization of the stochastic linearization, that appears to be extremely unusual at the first glance. Instead of simplifying a nonlinear expression appearing in the differential equation, they, in essence, suggested to seemingly first complicate it by replacing it by higher order terms. These higher order terms then were replaced by the linear approximation, in several steps. This indirect linearization certainly prolongs, as it were, the linearization process. Yet it takes into account the higher order statistics and, as such, has more of a possibility to capture the behavior of the system. It turned out that this long way towards linearization leads to results that are closer to those obtained via the exact solution, when the latter is available, or Monte Carlo simulation, when the exact solution is not available. Commenting on this method as exemplified on a Duffing oscillator, [1] stresses that "in the [usual] linearization we go from X^3 [term] to X. That will yield some error, and we should do something to balance. For regulated Gaussian equivalent linearization (RGEL) we should go back [to balance the error]. Since [the difference of the powers of the original cubic and replacing linear terms is] $3 - 1 = 2$, so we go back also 2 degrees, i.e. from X^3 to X^5 and come back to the first place X^3 but with regulated coefficient (7/9 in this case)".

For the details of implementation of RGEL for the Duffing oscillator the readers may consult the study by Anh and Di Paola [3]. [2] provides an additional justification of this method: "The natural explanation [of

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RGEL] is that when we want to go through a thing ahead we should move the hand back as how far ahead so far back. That is why we go first from X^3 to X^5." Since this method produces more accurate results than the classic linearization, one way metaphorically refers as a "long shorter way," versus classical technique, that can be dubbed as a "short longer way." This metaphor is stemming from a folk story about a young boy who was asked by a stranger how to find the road to the big city. The boy asked: "Do you want a long shorter way, or a short longer way?" The stranger chose the latter, since the first adjective was a word "short." Yet, after several hours of wondering the man returned to the boy and told him: "The way is short, but there are unsurpassable rocks. Tell me the whereabouts of the long shorter way." This time the stranger succeeded to get to his destination.

Anh and Di Paola's [3] derivation can be viewed as a " long shorter way " for it yields much more satisfactory results than the direct linearization technique; the latter being a "short longer way." Recently, [4] demonstrated that the expressions for the equivalent stiffness and damping coefficients adopted in the literature can be obtained by alternative means, namely via modified orthogonality criterion.

These two ideas, those by Anh and Di Paola [3] and by [4] are combined in this study. We first apply the Anh and Di Paola procedure to the Atalik and Utku oscillator with attendant dramatic decrease in error in comparison with the classical stochastic linearization. Then we extend the Anh and Di Paola methodology to two-step regulation. The latter extension shows considerable improvement of the results in comparison with both the classical scheme as well as the single step regulation, in the Lutes and Sarkani oscillator.

2 Derivation of results by Anh and Di Paola by orthogonality condition

Anh and Di Paola [3] studied the following nonlinear random vibration problem

$$\ddot{X} + 2h\dot{X} + \omega_0^2 X + \varepsilon\, g\left(X, \dot{X}\right) = f(t) \tag{1}$$

where $X(t)$ is the displacement, $\dot{X}(t)$ is the velocity, $\ddot{X}(t)$ = acceleration of a single degree of freedom system, h = damping coefficient, ω_0 = natural frequency of the system obtained when $h \equiv 0, \varepsilon \equiv 0, f(t) \equiv 0$; $g\left(X, \dot{X}\right)$ is a nonlinear function, $\varepsilon = 3$ amplitude of nonlinearity, $f(t)$ = random excitation. Let $g\left(X, \dot{X}\right)$ be a polynomial expression of X and $\dot{X}$. The nonlinear function $g\left(X, \dot{X}\right)$ then becomes

$$g\left(X, \dot{X}\right) = \sum_{k=0}^{N}\sum_{j=0}^{N}\left(\alpha_{kj}\dot{X}^{2k}X^{2j+1} + \beta_{kj}X^{2k}\dot{X}^{2j+1}\right), \tag{2}$$

Classical linearization would perform the following replacements of the nonlinear terms by the linear ones:

$$\alpha_{kj}\dot{X}^{2k}X^{2j+1} \rightarrow \lambda_{kj}X \tag{3}$$

$$\beta_{kj}X^{2k}\dot{X}^{2j+1} \rightarrow \mu_{kj}\dot{X} \tag{4}$$

Instead, most unusually, at least at the first glance, Anh and Di Paola [3] suggested to replace non-linear terms by *higher-order nonlinear* ones,

$$\begin{aligned}\alpha_{kj}\dot{X}^{2k}X^{2j+1} &\rightarrow c_{kj}\left(\dot{X}^{2k}X^{2j+1}\right)\left(\dot{X}^{2k}X^{2j}\right)\\ &= c_{kj}\dot{X}^{4k}X^{4j+1},\end{aligned} \tag{5}$$

$$\begin{aligned}\beta_{kj}X^{2k}\dot{X}^{2j+1} &\rightarrow d_{kj}\left(X^{2k}\dot{X}^{2j+1}\right)\left(X^{2k}\dot{X}^{2j}\right)\\ &= d_{kj}X^{4k}\dot{X}^{4j+1}\end{aligned} \tag{6}$$

where the authors used the mean-square criterion for obtaining the coefficients d_{kj} and c_{kj}:

$$c_{kj} = \alpha_{kj}E\left[\dot{X}^{6k}X^{6j+2}\right]\Big/E\left[\dot{X}^{8k}X^{8j+2}\right], \tag{7}$$

$$d_{kj} = \beta_{kj}E\left[X^{6k}\dot{X}^{6j+2}\right]\Big/E\left[X^{8k}\dot{X}^{8j+2}\right]. \tag{8}$$

Anh and Di Paola [3] then replaced higher-order non-linear terms into the original non-linear terms

$$c_{kj}\dot{X}^{4k}X^{4j+1} \rightarrow q_{kj}\,\dot{X}^{2k}X^{2j+1}, \tag{9}$$

$$d_{kj}X^{4k}\dot{X}^{4j+1} \rightarrow b_{kj}\,X^{2k}\dot{X}^{2j+1} \tag{10}$$

where

$$b_{kj} = d_{kj}E\left[X^{6k}\dot{X}^{6j+2}\right]\Big/E\left[X^{4k}\dot{X}^{4j+2}\right], \tag{11}$$

$$q_{kj} = c_{kj}E\left[\dot{X}^{6k}\,X^{6j+2}\right]\Big/E\left[\dot{X}^{4k}\,X^{4j+2}\right]. \tag{12}$$

This step is followed by the conventional replacement

$$b_{kj}X^{2k}\dot{X}^{2j+1} \rightarrow h_{kj}\dot{X}, \tag{13}$$

$$q_{kj}\dot{X}^{2k}X^{2j+1} \rightarrow l_{kj}X \tag{14}$$

where

$$h_{kj} = b_{kj}E\left[X^{2k}\dot{X}^{2j+2}\right]\Big/E\left[\dot{X}^{2}\right], \tag{15}$$

$$l_{kj} = q_{kj}E\left[\dot{X}^{2k}X^{2j+2}\right]\Big/E\left[X^{2}\right]. \tag{16}$$

3 Modified stochastic linearization

Let us show that the procedure by Anh and Di Paola [3] can be directly obtained via modified stochastic linearization technique. Indeed, we demand statistical orthogonality of the difference of the left and right hand sides in Eq. (6):

$$e_1 = \beta_{kj}X^{2k}\dot{X}^{2j+1} - d_{kj}X^{4k}\dot{X}^{4j+1} \tag{17}$$

with $\dot{X}^{4k}X^{4j+1}$, i.e. we require

$$\left(e_1, \dot{X}^{4k}X^{4j+1}\right) = 0 \tag{18}$$

where (. , .) is the inner product defined as

$$(\varphi, \psi) = E\left[\varphi, \psi\right]. \tag{19}$$

Thus, Eq. (18) becomes:

$$E\left[\left(\beta_{kj}X^{2k}\dot{X}^{2j+1} - d_{kj}X^{4k}\dot{X}^{4j+1}\right)\dot{X}^{4k}X^{4j+1}\right] = 0, \tag{20}$$

yielding the expression (8) for d_{kj}. Analogously, the orthogonality requirement

$$\left(e_2, \dot{X}^{4k}X^{4j+1}\right) = 0 \tag{21}$$

where e_2 is the difference between the left and right hand sides in Eq. (5),

$$e_2 = \alpha_{kj}\dot{X}^{2k}X^{2j+1} - c_{kj}\dot{X}^{4k}X^{4j+1}, \tag{22}$$

yields Eq. (8).

The result of the second step is likewise deducible from the requirements

$$\left(e_3, X^{2k}\dot{X}^{2j+1}\right) = 0, \tag{23}$$

$$\left(e_4, \dot{X}^{2k}X^{2j+1}\right) = 0 \tag{24}$$

where e_3 is the difference between the left and the right sides of Eq. (10):

$$e_3 = d_{kj} X^{4k} \dot{X}^{4j+1} - b_{kj} X^{2k} \dot{X}^{2j+1} \tag{25}$$

and e_4 is the difference between the left and the right hand sides of Eq. (9):

$$e_4 = c_{kj} \dot{X}^{4k} X^{4j+1} - q_{kj} \dot{X}^{2k} X^{2j+1}. \tag{26}$$

Equations (23) and (24) lead to Eqs. (11) and (12), respectively. In perfect analogy, Eqs. (15) and (10) are obtained by postulating the following conditions:

$$\left(e_5, \dot{X}\right) = 0, \tag{27}$$

$$(e_6, X) = 0 \tag{28}$$

where

$$e_5 = b_{kj} x^{2k} \dot{X}^{2j+1} - h_{kj} \dot{X}, \tag{29}$$

$$e_6 = q_{kj} \dot{X}^{2k} X^{2j+1} - l_{kj} X. \tag{30}$$

As is seen, Eqs. (11), (12), (15) and (16) are obtained by stochastic Galerkin-type orthogonality conditions. As a result, the final, linear replacement takes place:

$$g\left(X, \dot{X}\right) = \sum_{k=0}^{N} \sum_{j=0}^{N} \left(h_{kj} \dot{X} + l_{kj} X\right) \tag{31}$$

where

$$h_{kj} = \frac{E\left[X^{2k} \dot{X}^{2j+2}\right]}{E[\dot{X}^2]} \frac{E\left[X^{6k} \dot{X}^{6j+2}\right]}{E\left[X^{4k} \dot{X}^{4j+2}\right]} \frac{E\left[X^{6k} \dot{X}^{6j+2}\right]}{E\left[X^{8k} \dot{X}^{8j+2}\right]} \beta_{kj}, \tag{32}$$

$$l_{kj} = \frac{E\left[\dot{X}^{2k} X^{2j+2}\right]}{E[X^2]} \frac{E\left[\dot{X}^{6k} X^{6j+2}\right]}{E\left[\dot{X}^{4k} X^{4j+2}\right]} \frac{E\left[\dot{X}^{6k} X^{6j+2}\right]}{E\left[\dot{X}^{8k} X^{8j+2}\right]} \alpha_{kj}. \tag{33}$$

Anh and Di Paola [3] evaluated by their method several oscillators. For the Duffing oscillator in the investigated numerical range, the numerical results led to roughly half the percentagewise error than that resulting by the conventional stochastic linearization technique, i.e. without recourse to amending the original system by the higher non-linearity degree. As noted before, Anh and Di Paola [3] call their method "a regulated Gaussian equivalent linearization (RGEL)." As is seen, RGEL can be interpreted as a multiple orthogonalization technique.

4 Atalik and Utku oscillator

Consider the following nonlinear system:

$$\ddot{X}(t) + \beta \dot{X}(t) + \alpha X^3(t) = F(t) \tag{34}$$

where β is the damping constant, α is the nonlinear stiffness constant and $F(t)$ is a Gaussian white noise process with

$$E[F(t)] = 0, E[F(t)F(t+\tau)] = 2d\beta\delta(\tau). \tag{35}$$

The exact stationary probability density function of the above system, obtained by the Fokker–Planck approach, is

$$p(x) = p_0 \exp\left(-\frac{\alpha}{4d} x^4\right) \tag{36}$$

where p_0 is the normalization constant. To obtain the exact mean square displacement,

$$\sigma_x^2 = E[X^2] = \int_{-\infty}^{+\infty} x^2 p(x)\ dx, \tag{37}$$

we use the integration formula

$$\int_0^{+\infty} x^{s-1} \exp\left(-ax^h\right)\ dx = \left(h^{-1}\right)\left(a^{-s/h}\right)\Gamma(s/h) \tag{38}$$

where $\Gamma(\bullet)$ is the Gamma function. The mean square displacement becomes

$$\sigma_x^2 = \frac{(1/4)\,(\alpha/4d)^{-3/4}\,\Gamma(3/4)}{(1/4)\,(\alpha/4d)^{-1/4}\,\Gamma(1/4)} \approx 0.6760\,(d/\alpha)^{1/2}. \tag{39}$$

The equivalent linear system to Eq. (34) can be written as

$$\ddot{X}(t) + \beta\dot{X}(t) + k_{eq}\,X(t) = F(t) \tag{40}$$

where the equivalent linear spring constant k_{eq} is found by processing the conventional linearization as equal to

$$k_{eq} = E\left[\frac{d}{dx}(\alpha X^3)\right] = 3\alpha E\left[X^2\right]. \tag{41}$$

The mean-square value of the displacement of the linearized system is

$$E[X^2] = d/k_{eq}. \tag{42}$$

Thus, we obtain the approximate solution as

$$\sigma_{x_e}^2 = (d/3\alpha)^{1/2} \approx 0.5776\ (d/\alpha)^{1/2}. \tag{43}$$

The percentagewise error committed by using the classical equivalent linearization technique in evaluating the mean-square displacement is

$$(0.6760 - 0.5776)\,/0.6760 = 14.6\%. \tag{44}$$

Let us apply the RGEL method, proposed by Anh and Di Paola. The scheme of the process can be read as follows:

$$\alpha X^3(t) \to k_1 X^5(t) \to k_2 X^3(t) \to k_{eqI}\,X(t). \tag{45}$$

We can readily utilize the results obtained by Anh and Di Paola for the Duffing oscillator of which the Atalik and Utku oscillator is a particular case:

$$\alpha X^3(t) \to \frac{\alpha}{9E[X^2(t)]}X^5(t) \to \frac{7\alpha}{9}X^3(t) \to \frac{7\alpha}{3}X(t). \tag{46}$$

One gets the equivalent linearized equation

$$\ddot{X}(t) + \beta\dot{X}(t) + \frac{7}{3}\alpha\,E[X^2(t)]\,X(t) = F(t). \tag{47}$$

The mean-square value of $X(t)$ is evaluated by the following expression:

$$E[X^2(t)] = \sqrt{\frac{3d}{7\alpha}} \approx 0.6546\left(\frac{d}{\alpha}\right)^{1/2}. \tag{48}$$

Now, the percentagewise error found by using the RGEL linearization technique to calculate the mean-square displacement is

$$(0.6760 - 0.6546)\,/0.6760 = 3.17\%. \tag{49}$$

We note a significant, over four-fold improvement, which demonstrates the extreme efficiency of the RGEL method. Naturally, the question of continuing the process to greater order arises. However, calculation of such process beyond the first step in Eq. (45), namely,

$$\begin{aligned}\alpha X^3(t) &\to \frac{\alpha}{9E[X^2(t)]}\,X^5(t) \to \frac{\alpha}{117\sigma_x^2 E[X^2(t)]}\,X^7(t)\\ &\to \frac{11\,\alpha}{117E[X^2(t)]}\,X^5(t) \to \frac{77\,\alpha}{117}\,X^3(t) \to \frac{77\,\alpha}{39}E[X^2(t)]\,X(t),\end{aligned} \tag{50}$$

leads to numerically worse results than the one previously found in Eq. (48). Hence, for the Atalik and Utku oscillator, the optimum number of regulation steps is unity. It is important to note that the evaluation of the two steps in the Duffing oscillator leads to the same conclusion. It should be emphasized that the study of a half-degree-of-freedom oscillator with a cubic spring, rather than a single-degree-of-freedom oscillator in Eq. (34), would have been more instructive to demonstrate the power of the Anh/Di Paola method. This idea was kindly suggested by the reviewer. Such a system will be addressed in the next section. An additional question arises if there is an oscillator where the optimum number of regulation steps is greater than one. The reply to this question is shown in the next section to be affirmative.

5 Lutes and Sarkani oscillator: exact solution

Consider the nonlinear oscillator by [6]

$$\dot{X}(t) + k\,|X(t)|^a\ \operatorname{sgn}[X(t)] = F(t) \tag{51}$$

where a is a real number, $F(t)$ is a zero-mean, stationary Gaussian white noise with spectral density S_0. [6] derives the exact probability density of the response

$$p_{X(t)} = A\exp\left[-\frac{k\,u^{a+1}}{(a+1)\ \pi\ S_0}\right] \tag{52}$$

where

$$A = \left(\frac{k}{(a+1)\ \pi\ S_0}\right)^{\frac{1}{a+1}}\left(\frac{a+1}{2}\right)\left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{53}$$

The variance of the response

$$\sigma_X^2 = 2A\int_0^\infty u^2\exp\left[-\frac{k\,u^{a+1}}{(a+1)\ \pi\ S_0}\right]du \tag{54}$$

is obtained exactly,

$$\sigma_{X,\text{exact}}^2 = \left(\frac{\pi\ S_0}{k}\right)^{\frac{2}{a+1}}(a+1)^{\frac{2}{a+1}}\ \Gamma\left(\frac{3}{a+1}\right)\left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{55}$$

[6] also derived the approximate response via the classical stochastic linearization technique as follows:

$$\sigma_{X,\text{approx}}^2 = \left(\frac{\pi\ S_0}{k}\right)^{\frac{2}{a+1}}\left[\frac{(2\pi)^{1/2}}{2^{a/2}a\Gamma\left(a/2\right)}\right]^{\frac{2}{a+1}}. \tag{56}$$

Table 1 Error incurred by using a single-step in the Anh and Di Paola regulation

a	$\sigma^2_{X,\text{exact}}$	$\sigma^2_{\text{classical } X,\text{approx}}$	Error, %	$E[X^2_{\text{regulated}}(t)]_I$	Error, %
1	1	1	0	1	0
2	0.7765	0.7323	5.6877	0.7824	0.7713
3	0.6760	0.5774	14.5904	0.6547	3.1546
4	0.6175	0.4764	22.8490	0.5620	8.9861
5	0.5786	0.4055	29.9225	0.4917	15.0206
6	0.5505	0.3529	35.8981	0.4367	20.6846
7	0.5291	0.3124	40.9630	0.3925	25.8224

The error η between exact and approximate solutions defined as

$$\eta = \frac{|\sigma^2_{X,\text{exactx}} - \sigma^2_{X,\text{approx}}|}{\sigma^2_{X,\text{exactx}}} \times 100\% \tag{57}$$

is shown in Table 1. [6] concluded that "the statistical linearization gives a good approximation of the response variance only when a is relatively near unity." Indeed, for $a = 1$ the error equals zero. For $a = 2$, the error constitutes $\eta = 5.7\%$; for $a = 3$, the error equals $\eta = 14.6\%$; for $a = 4$, the error is $\eta = 22.8\%$; for $a = 5$, the error reaches $\eta = 29.9\%$. It appears to be of interest to investigate this oscillator via the modified Anh and Di Paola [3] approach.

6 Application of Anh and Di Paola approach to the Lutes–Sarkani oscillator

For simplicity we will limit ourselves by considering the case when a is a positive integer. We intend to replace the power oscillator containing a nonlinear term $|X(t)|^a \operatorname{sgn}[X(t)]$ by a linear oscillator with the term $k_{eq}\, X(t) = k_{eq}\, |X(t)| \operatorname{sgn}[X(t)]$, the difference of powers of $|X(t)|$ being $a - 1$. During the "regulation" procedure by Anh and Di Paola [3] we are recommended to "increase" the nonlinearity, i.e. power a by original power plus the increment $a - 1$, i.e. to use the new regulation power of $a + (a - 1) = 2a - 1$. Hence the procedure can be represented schematically as follows:

$$k\,|X(t)|^a \to k_1\,|X(t)|^{2a-1} \to k_2\,|X(t)|^a \to k_{eq}X(t). \tag{58}$$

We form a difference $k\,|X(t)|^a - k_1\,|X(t)|^{2a-1}$ and demand the statistical orthogonality of this expression to $|X(t)|^{2a-1}$,

$$E\left\{\left(k\,|X(t)|^a - k_1\,|X(t)|^{2a-1}\right)|X(t)|^{2a-1}\right\} = 0, \tag{59}$$

which leads to

$$k_1 = k\frac{E[|X(t)|^{3a-1}]}{E[|X(t)|^{4a-2}]}. \tag{60}$$

The general expression for $E[|X(t)|^a]$ is:

$$E\left[\,|X(t)|^a\right] = \sigma_X^a \int\limits_{-\infty}^{\infty} \frac{|\xi|^a}{\sqrt{2\pi}} \exp\left(-\xi^2/2\right)\, d\xi. \tag{61}$$

We obtain the coefficient of regulation k_1 as follows:

$$k_1 = k\,\sigma_X^{1-a}\frac{\int_{-\infty}^{\infty}|\xi|^{3a-1}\exp\left(-\xi^2/2\right)\,d\xi}{\int_{-\infty}^{\infty}|\xi|^{4a-2}\exp\left(-\xi^2/2\right)\,d\xi}. \tag{62}$$

Proceeding in perfect analogy we obtain the following coefficient of regulation:

$$k_2 = k\,\frac{\left(\int_{-\infty}^{\infty}|\xi|^{3a-1}\exp\left(-\xi^2/2\right)\,d\xi\right)^2}{\int_{-\infty}^{\infty}|\xi|^{4a-2}\exp\left(-\xi^2/2\right)\,d\xi\;\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi} \tag{63}$$

as well as k_{eq},

$$k_{eq} = k\frac{\sigma_X^{a+1}}{E[X^2(t)]}\,\frac{1}{\sqrt{2\pi}}\,\frac{\left(\int_{-\infty}^{\infty}|\xi|^{3a-1}\exp\left(-\xi^2/2\right)\,d\xi\right)^2\int_{-\infty}^{\infty}|\xi|^{a+1}\exp\left(-\xi^2/2\right)\,d\xi}{\int_{-\infty}^{\infty}|\xi|^{4a-2}\exp\left(-\xi^2/2\right)\,d\xi\;\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi} \tag{64}$$

$$= \frac{k\;E\left[X^2(t)\right]^{\frac{a-1}{2}}}{R} \tag{65}$$

where

$$R = \sqrt{2\pi}\,\frac{\int_{-\infty}^{\infty}|\xi|^{4a-2}\exp\left(-\xi^2/2\right)\,d\xi\;\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi}{\left(\int_{-\infty}^{\infty}|\xi|^{3a-1}\exp\left(-\xi^2/2\right)\,d\xi\right)^2\int_{-\infty}^{\infty}|\xi|^{a+1}\exp\left(-\xi^2/2\right)\,d\xi}. \tag{66}$$

The equation of motion becomes

$$\dot{X}(t) + \frac{k\;E\left[X^2(t)\right]^{\frac{a-1}{2}}}{R}\,X(t) = F(t). \tag{67}$$

We deduce the mean-square displacement of the latter oscillator,

$$E\left[X^2_{\text{regulated}}(t)\right]_I = \left(\frac{\pi\;S_0}{k}\right)^{\frac{2}{a+1}} R^{\frac{2}{a+1}}. \tag{68}$$

In order to express R as a function of Gamma-functions, note that

$$\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi = 2\int_{0}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi. \tag{69}$$

Then we make a change in the variable $\eta = \xi^2/2$, to get

$$\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi = 2^{a+\frac{1}{2}}\int_{0}^{\infty}\eta^{a-\frac{1}{2}}\exp(-\eta)\,d\eta. \tag{70}$$

According to the definition of the Gamma-function,

$$\Gamma(z) = \int_{0}^{\infty}t^{z-1}\exp(-t)\,dt, \tag{71}$$

we finally obtain

$$\int_{-\infty}^{\infty}|\xi|^{2a}\exp\left(-\xi^2/2\right)\,d\xi = 2^{a+\frac{1}{2}}\Gamma\left(a+\frac{1}{2}\right). \tag{72}$$

By applying this process to the other integrals, we get:

$$E\left[X^2_{\text{regulated}}(t)\right]_I = \left(\frac{\pi\;S_0}{k}\right)^{\frac{2}{a+1}}\left[\sqrt{\pi}\,2^{-\frac{a+1}{2}}\frac{\Gamma\left(\frac{4a-1}{2}\right)\;\Gamma\left(\frac{2a+1}{2}\right)}{\Gamma\left(\frac{3a}{2}\right)^2\;\Gamma\left(\frac{a+2}{2}\right)}\right]^{\frac{2}{a+1}}. \tag{73}$$

Table 1 below presents the percentagewise error due to the approximate nature of the solutions (with both conventional linearization and RGEL method) in comparison to the exact solution provided by Eq. (55), for different integer values of a.

We can observe that there is an important improvement in the performance of the stochastic linearization when we utilize the RGEL method. Namely, whereas for $a = 2$, the classical linearization is in error of about 5.69%, the regulated linearization has an error which is over 7 times less, namely 0.77%. For large value of a namely, $a = 5$, the regulated linearization has about half the error of that classical linearization namely 15% versus 29.9%. For even larger values of a, the error is much less than that in the classical scheme but still quite large: for $a = 7$, the Anh and Di Paola approach leads to 25.8% error, whereas the classical approach is associated with an error of about 41%. Still, the regulation reduces the error in this case by about 15%.

7 Extension of Anh and Di Paola method: two-step regulation

A natural question arises: What is the effect of additional steps in regulation? Anh and Di Paola [3] considered only a single-step regulation. Here, the two step regulation is performed, as illustrated schematically below:

$$kX^a(t) \to k_1 X^{2a-1}(t) \to k_2 X^{3a-2}(t) \to k_3 X^{2a-1}(t) \to k_4 X^a(t) \to k_{eq,II} X(t). \tag{74}$$

Proceeding in perfect analogy with a single step procedure, we get

$$k_1 = k\ \sigma_X^{1-a} \frac{\int_{-\infty}^{\infty} |\xi|^{3a-1} \exp\left(-\xi^2/2\right)\ d\xi}{\int_{-\infty}^{\infty} |\xi|^{4a-1} \exp\left(-\xi^2/2\right)\ d\xi}, \tag{75}$$

$$k_2 = k_1\ \sigma_X^{1-a} \frac{\int_{-\infty}^{\infty} |\xi|^{5a-3} \exp\left(-\xi^2/2\right)\ d\xi}{\int_{-\infty}^{\infty} |\xi|^{6a-4} \exp\left(-\xi^2/2\right)\ d\xi}, \tag{76}$$

$$k_3 = k_2\ \sigma_X^{a-1} \frac{\int_{-\infty}^{\infty} |\xi|^{5a-3} \exp\left(-\xi^2/2\right)\ d\xi}{\int_{-\infty}^{\infty} |\xi|^{4a-2} \exp\left(-\xi^2/2\right)\ d\xi}, \tag{77}$$

$$k_4 = k_3\ \sigma_X^{a-1} \frac{\int_{-\infty}^{\infty} |\xi|^{3a-1} \exp\left(-\xi^2/2\right)\ d\xi}{\int_{-\infty}^{\infty} |\xi|^{2a} \exp\left(-\xi^2/2\right)\ d\xi}, \tag{78}$$

$$k_{eq,II} = k_4\ \sigma_X^{a+1} \frac{\int_{-\infty}^{\infty} |\xi|^{a+1} \exp\left(-\xi^2/2\right) d\xi}{\sqrt{2\pi}\ E[X^2(t)]}. \tag{79}$$

After expressing $k_{eq,II}$ via k, the initial equation of motion is replaced by

$$\dot{X}(t) + \frac{k\ E\left[X^2(t)\right]^{\frac{a-1}{2}}}{Q} X(t) = F(t) \tag{80}$$

where

$$Q = \sqrt{2\pi} \frac{\left(\int_{-\infty}^{\infty} |\xi|^{4a-2} \exp\left(-\xi^2/2\right)\ d\xi\right)^2 \int_{-\infty}^{\infty} |\xi|^{6a-4} \exp\left(-\xi^2/2\right)\ d\xi \int_{-\infty}^{\infty} |\xi|^{2a} \exp\left(-\xi^2/2\right)\ d\xi}{\left(\int_{-\infty}^{\infty} |\xi|^{5a-3} \exp\left(-\xi^2/2\right)\ d\xi\right)^2 \left(\int_{-\infty}^{\infty} |\xi|^{3a-1} \exp\left(-\xi^2/2\right)\ d\xi\right)^2 \int_{-\infty}^{\infty} |\xi|^{a+1} \exp\left(-\xi^2/2\right)\ d\xi}. \tag{81}$$

We arrive at the mean-square displacement

$$E\left[X^2_{\text{regulated}}(t)\right]_{II} = \left(\frac{\pi\ S_0}{k}\right)^{\frac{2}{a+1}} Q^{\frac{2}{a+1}} \tag{82}$$

or, via the Gamma-functions,

$$E[X^2_{\text{regulated}}(t)]_{II} = \left(\frac{\pi\ S_0}{k}\right)^{\frac{2}{a+1}} \left[\sqrt{\pi} 2^{-\frac{a+1}{2}} \frac{\Gamma\left(\frac{6a-3}{2}\right)\ \Gamma\left(\frac{2a+1}{2}\right)\ \Gamma\left(\frac{4a-1}{2}\right)^2}{\Gamma\left(\frac{5a-2}{2}\right)^2\ \Gamma\left(\frac{3a}{2}\right)^2\ \Gamma\left(\frac{a+2}{2}\right)}\right]^{\frac{2}{a+1}}. \tag{83}$$

The Roman subscript II indicates that the result is obtained in the second step of the regulation process. Table 2 presents a comparison of the two-step procedure with the exact solution on one hand, a single-step procedure, and classical stochastic linearization.

We note that two-step regulation provides an additional improvement in comparison to the single-step regulation; for the moderate value of $a = 4$, the classical stochastic linearization is associated with the error of about 23%; the single-step regulation results in an error of about 9% whereas the two-step regulation leads to the error of 1.23%. Thus, the error in two-step regulation is about 18 times less than in the classical scheme, and about 7 times less than in a single-step regulation. For larger values of a, though still much better than the classical single-step regulation linearization, the two-step regulation reduces the error by about additional 10% in comparison with a single-step regulation: The classical linearization yields about 41% for $a = 7$; the single-step regulation leads to 26% whereas the two-step regulation results in 14.6% of error. We should note that the two-step regulation turns out to be the optimal one for the Lutes and Sarkani oscillator, since it turns that the three-step regulation yields larger errors out than the two-step version. The optimum number of steps needed should be established for each oscillator at hand.

Table 2 Error incurred by using an extended two-step regulation

a	$\sigma^2_{X,\text{exact}}$	$E[X^2_{\text{regulated}}(t)]_I$	Error, %	$E[X^2_{\text{regulated}}(t)]_{II}$	Error, %
1	1	1.0000	0	1	0
2	0.7765	0.7824	0.7713	0.8205	5.6693
3	0.6760	0.6547	3.1546	0.7117	5.2820
4	0.6175	0.5620	8.9861	0.6251	1.2229
5	0.5786	0.4917	15.0206	0.5554	4.0131
6	0.5505	0.4367	20.6846	0.4988	9.4038
7	0.5291	0.3925	25.8224	0.4521	14.5548

8 Conclusion

In this study, authors first apply the Anh and Di Paola approach to the Atalik and Utku oscillator with dramatic reduction of the error in comparison to the classical stochastic linearization (3% vs. 14%). Then authors next extend the Anh and Di Paola approach to include the two-step regulation, which turns out to be the optimal one for the Lutes and Sarkani oscillator, providing considerable reduction in the error. It appears that the method has a large potential and it ought to be explored for wider classes of oscillators.

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