Acta Mech 200, 247–261 (2008)
DOI 10.1007/s00707-008-0002-1

Xiaoshan Cao · Feng Jin · Zikun Wang

On dispersion relations of Rayleigh waves in a functionally graded piezoelectric material (FGPM) half-space

Received: 9 May 2007 / Revised: 16 October 2007 / Published online: 22 May 2008

Abstract For propagation of Rayleigh surface waves in a transversely isotropic graded piezoelectric half-space with material properties varying continuously along depth direction, the Wentzel–Kramers–Brillouin (WKB) technique is employed for the asymptotic analytical derivations. The phase velocity equations for both the electrically open and shorted cases at the free surface are obtained. Influences of piezoelectric material parameters graded variations on Rayleigh wave dispersion relations, particles' displacements magnitude and corresponding decay properties are discussed. Results obtained indicate that coupled Rayleigh waves can propagate at the surface of the graded piezoelectric half-space, and their dispersion relations and the particles displacements ellipticity at the free surface are dependent upon the graded variation tendency of the material parameters. By the Rayleigh surface waves phase velocities relative changing values combined with the relationship between the wave number and the material graded coefficient, a theoretical foundation can be provided for the graded material characterization by experimental measurement.

1 Introduction

Early in the 1980s, a new-type material called functionally graded material (FGM) was proposed to solve problems in the thermal-protection systems of aerospace structures. Since then, FGM has attracted interest of researchers from many engineering fields. Nowadays, it is known that FGM can be used not only in thermal-protection systems but also in electronics and many other fields. The results obtained for the FGM structures lead us to consider that FGM may be applicable to surface acoustic wave (SAW) devices on condition that functionally graded piezoelectric materials (FGPMs) can be properly manufactured, as known from recent techniques for fabricating FGPMs [1].

It is known that surface acoustic waves have been applied successfully in electronics for signal processing. With the development of material technology, FGPMs can be manufactured and also can be used in surface wave devices to improve their efficiency and other features. The demands from the ultrasonic technology and nondestructive evaluation (NDE) fields make the research of wave propagation behaviors and characteristics in FGPMs to be a topic of practical importance [2–8].

To study the wave propagation behavior in inhomogeneous media with material parameters varying continuously, analytical solutions can be obtained only for some special cases due to the complexity of the governing equations. Some numerical solutions undertaken are to divide an inhomogeneous medium into a multi-layer model, and within each layer a material parameters homogeneous assumption is adopted for the analysis by virtue of finite element method [2–6]. Some reports on the asymptotic analysis of wave propagation in inhomogeneous media also can be found: Love wave propagation in a functionally graded piezoelectric layer/substrate

X. Cao · F. Jin (✉) · Z. Wang
Department of Engineering Mechanics/MOE Key Laboratory for Strength and Vibration, Xi'an Jiaotong University, 710049 Xi'an, People's Republic of China
E-mail: jinfengzhao@263.net

system [9], Rayleigh surface waves propagating along curved surfaces [10]. In reference [11], dispersion relations of a Rayleigh wave in a functionally graded piezoelectric half-space are studied by the perturbation technique.

In the present contribution, a Rayleigh surface wave in a transversely isotropic functionally graded piezoelectric half-space with material parameters varying continuously along the depth direction is taken into account. The coupled second-order governing equations with three variables are first converted into a sixth-order equation with variable coefficients containing one variable, and then the WKB technique is employed for the wave functions solutions for both the electrically open and shorted cases. For the case of material parameters varying exponentially along the depth direction, influences of each piezoelectric parameter graded variation on Rayleigh wave dispersion relations, magnitudes and decay properties of particles' displacements at the free surfaces are investigated. A theoretical foundation is also provided for the characterization of the material graded coefficient by experimental measurement of Rayleigh waves.

2 Statement of the problem

Rayleigh surface wave propagation behavior at the surface of a transversely isotropic functionally graded piezoelectric material (FGPM) half-space, as shown in Fig. 1, is taken into account. The surface of the FGPM half-space is traction free and the poling direction of the piezoelectricity is along the z-axis. The material coefficients, say, elastic coefficients $c_{ijkl}(z)$, piezoelectric coefficients $e_{kij}(z)$ and dielectric coefficients $\varepsilon_{jk}(z)$ vary continuously along the depth direction, namely, they are the functions of the z-axis.

The constitutive equations of the piezoelectric medium can be expressed in the following forms:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} S_{kl} - e_{kij} E_k, \\ D_j &= e_{jkl} S_{kl} + \varepsilon_{jk} E_k, \end{aligned} \tag{1}$$

in which σ_{ij}, S_{kl}, D_j , and E_k are the stress, strain, electrical displacement, and electrical intensity components, respectively.

The motion equation and the electrical displacement equilibrium equations possess the following expressions:

$$\begin{aligned} \sigma_{ij,j} &= \rho \ddot{u}_i, \\ D_{i,i} &= 0, \end{aligned} \tag{2}$$

where ρ is the mass density of the piezoelectric medium, u_i the mechanical displacement components in the ith direction, a comma followed by the subscript i indicates space differentiation with respect to the corresponding coordinate x_i and the dot "•" denotes time differentiation. Also, the repeated index in the subscript implies summation with respect to that index.

The relationship between the mechanical displacement and the strain components is

$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{3}$$

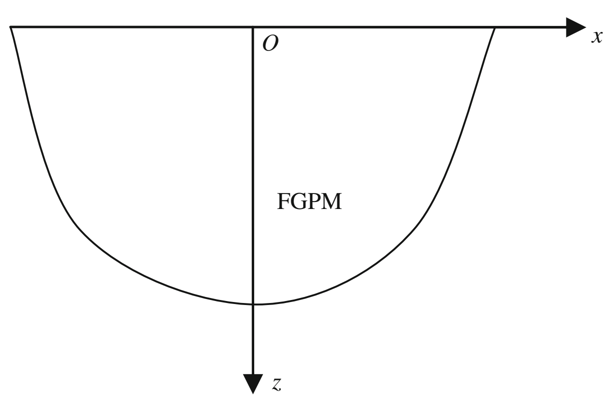

Fig. 1 The functionally graded piezoelectric material half-space and coordinate system

According to the quasi-static Maxwell's equation, the relationship between the electrical intensity and the electrical potential is

$$E_i = -\frac{\partial \varphi}{\partial x_i}. \tag{4}$$

To investigate the propagation behavior of the coupled Rayleigh wave in the above FGPM half-space, it is assumed that the wave is propagating in the positive direction of the x-axis, so the motion mode of Rayleigh wave is:

$$u = u(x, z, t), \quad v = 0, \quad w = w(x, z, t), \quad \varphi = \varphi(x, z, t). \tag{5}$$

Substitution of Eq. (5) into Eqs. (1), (3), and (4) together with Eq. (2) leads to the following governing equations for Rayleigh wave propagation:

$$\begin{aligned}
&c_{11}\frac{\partial^2 u}{\partial x^2} + c_{13}\frac{\partial^2 w}{\partial x \partial z} + e_{31}\frac{\partial^2 \varphi}{\partial x \partial z} + c_{44}\left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial z^2}\right) + e_{15}\frac{\partial^2 \varphi}{\partial x \partial z} + c'_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + e'_{15}\frac{\partial \varphi}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2},\\
&c_{33}\frac{\partial^2 w}{\partial z^2} + c_{13}\frac{\partial^2 u}{\partial x \partial z} + e_{33}\frac{\partial^2 \varphi}{\partial z^2} + c_{44}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}\right) + e_{15}\frac{\partial^2 \varphi}{\partial x^2} + c'_{13}\frac{\partial u}{\partial x} + c'_{33}\frac{\partial w}{\partial z} + e'_{33}\frac{\partial \varphi}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2},\\
&e_{15}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}\right) - \varepsilon_{11}\frac{\partial^2 \varphi}{\partial x^2} + e_{31}\frac{\partial^2 u}{\partial x \partial z} + e_{33}\frac{\partial^2 w}{\partial z^2} - \varepsilon_{33}\frac{\partial^2 \varphi}{\partial z^2} + e'_{31}\frac{\partial u}{\partial x} + e'_{33}\frac{\partial w}{\partial z} - \varepsilon'_{33}\frac{\partial \varphi}{\partial z} = 0,
\end{aligned} \tag{6}$$

in which superscript " $'$ " indicate space differentiation with respect to coordinate z.

The electrical potential $\varphi_1(x, z, t)$ in the air above the surface of the half-space ($z < 0$) should satisfy the following equation:

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = 0. \tag{7}$$

The electrical boundary conditions at the free surface can be described as:

For electrically open case:

$$\begin{aligned}
&\sigma_z(0, x) = 0,\\
&\tau_{xz}(0, x) = 0,\\
&\varphi(0, x) = \varphi_1(0, x),\\
&D\,(0, x) = D_1(0, x).
\end{aligned} \tag{8}$$

For electrically shorted case:

$$\begin{aligned}
&\sigma_z(0, x) = 0,\\
&\tau_{xz}(0, x) = 0,\\
&\varphi(0, x) = 0.
\end{aligned} \tag{9}$$

Up to this stage, the problem of Rayleigh wave propagation in the FGPM half-space becomes the solution of the governing differential Eqs. (6)–(7) under boundary conditions (8)–(9).

3 Solutions of the problem

The solutions of Eq. (6) can be assumed to possess the following forms [12]:

$$\begin{aligned}
u(x, z, t) &= Z_u \exp[ik(x - ct)],\\
w(x, z, t) &= Z_w \exp[ik(x - ct)],\\
\varphi(x, z, t) &= Z_\varphi \exp[ik(x - ct)],
\end{aligned} \tag{10}$$

where k is the wave number, c the phase velocity and Z_u, Z_w, Z_φ are undetermined functions with respect to the z-axis.

Substitution of Eq. (10) into Eq. (6) provides:

$$
\begin{aligned}
&c_{44}Z_u'' + c_{44}'Z_u' + \left(\rho c^2 - c_{11}\right)k^2 Z_u + \left[(c_{13}+c_{44})\,ikZ_w' + (e_{31}+e_{15})\,ikZ_\varphi' + c_{44}'ikZ_w + e_{15}'ikZ_\varphi\right] = 0,\\
&c_{33}Z_w'' + c_{33}'Z_w' + \left(\rho c^2 - c_{44}\right)k^2 Z_w + \left[(c_{13}+c_{44})\,ikZ_u' + e_{33}Z_\varphi'' - e_{15}k^2 Z_\varphi + c_{13}'ikZ_u + e_{33}'Z_\varphi'\right] = 0,\\
&\varepsilon_{33}Z_\varphi'' + \varepsilon_{33}'Z_\varphi' - \varepsilon_{11}k^2 Z_\varphi + \left[e_{15}k^2 Z_w - (e_{31}+e_{15})\,ikZ_u' - e_{33}Z_w'' - e_{31}'ikZ_u - e_{33}'Z_w'\right] = 0,
\end{aligned}
\tag{11}
$$

Equations (11) are second-order equations with variable coefficients with respect to Z_u, Z_w, and Z_φ. Usually, it is difficult to solve the equations directly. In the present contribution, Eq. (11) can be converted into a sixth-order differential equation with variable coefficients containing only one undetermined function, and an asymptotic analytical solution can be obtained by virtue of WKB technique.

Considering the fact that Eqs. (11) can be regarded as linear equations with respect to Z_φ'', Z_φ', Z_φ , we can obtain:

$$
\begin{aligned}
Z_\varphi &= F_0\left(Z_u'', Z_u', Z_u, Z_w'', Z_w', Z_w\right),\\
Z_\varphi' &= F_1\left(Z_u'', Z_u', Z_u, Z_w'', Z_w', Z_w\right),\\
Z_\varphi'' &= F_2\left(Z_u'', Z_u', Z_u, Z_w'', Z_w', Z_w\right).
\end{aligned}
\tag{12}
$$

Due to the fact that $F_0' = F_1$, $F_1' = F_2$, the following two equations can be obtained with respect to Z_u''', Z_u'', Z_u', Z_u, Z_w''', Z_w'', Z_w', Z_w:

$$
\begin{aligned}
&Z_u''' + a_1Z_u'' + a_2Z_u' + a_3Z_u + a_4Z_w''' + a_5Z_w'' + a_6Z_w' + a_7Z_w = 0,\\
&Z_w''' + b_1Z_w'' + b_2Z_w' + b_3Z_w + b_4Z_u''' + b_5Z_u'' + b_6Z_u' + b_7Z_u = 0.
\end{aligned}
\tag{13}
$$

Expressions $a_i, b_i\,(i = 1 \sim 7)$ are very complicated, but on condition that the material parameters are varying slowly and for high-frequency short waves (i.e., wave number $k \gg 1$), the high order terms containing $1/k^2$ in a_i, b_i can be omitted, thus Eq. (12) can be rewritten as:

$$
\begin{aligned}
G_1 &= Z_u''' + a_{10}Z_u'' + a_{20}k^2Z_u' + a_{30}k^2Z_u + a_{40}/kZ_w''' + a_{50}kZ_w'' + a_{60}kZ_w' + a_{70}k^3Z_w = 0,\\
G_2 &= Z_w''' + b_{10}Z_w'' + b_{20}k^2Z_w' + b_{30}k^2Z_w + b_{40}/kZ_u''' + b_{50}kZ_u'' + b_{60}kZ_u' + b_{70}k^3Z_u = 0
\end{aligned}
\tag{14}
$$

in which a_{i0} $(i = 1 \sim 7)$ are independent of the wave number k.

Undetermined coefficients $l_{i0}(i = 1 \sim 7)$ are introduced into the analysis to eliminate Z_w and its corresponding first to third differentials, and the summation is:

$$
H_0 = G_1''' + l_{10}G_1'' + l_{20}k^2G_1' + l_{30}k^2G_1 + l_{40}/kG_2''' + l_{50}kG_2'' + l_{60}kG_2' + l_{70}k^3G_2. \tag{15}
$$

Let the coefficients of Z_w and its first to sixth-order differentials in the above equation be zero, from $H_0 = 0$ and the high order terms containing $1/k^2$ are omitted also, an equation with respect to Z_u can be obtained as follows:

$$
Z_u^{(6)} + A_5Z_u^{(5)} + A_4k^2Z_u^{(4)} + A_3k^2Z_u''' + A_2k^4Z_u'' + A_1k^4Z_u' + A_0k^6Z_u = 0 \tag{16}
$$

in which

$$
\begin{aligned}
&A_5 = a_{10} + l_{10} + l_{40}b_{50} + l_{50}b_{40}, \quad A_4 = a_{20} + l_{20} + l_{50}b_{50}, \quad A_0 = l_{70}b_{70},\\
&A_3 = 3a_{20}' + a_{30} + l_1a_{20} + l_{20}a_{10} + l_{30} + l_{40}b_{70} + l_{50}\left(3b_{50}' + b_{60}\right) + l_{60}b_{50} + l_{70}b_{40},\\
&A_2 = l_{20}a_{20} + l_{50}b_{70} + l_{70}b_{50}, \quad A_1 = l_{20}\left(a_{20}' + a_{30}\right) + l_{30}a_{20} + 2l_{50}b_{70}' + l_{60}b_{70} + l_{70}b_{60}
\end{aligned}
$$

and

$$
\begin{aligned}
&l_{40} = -a_{40}, \quad l_{50} = -a_{50}, \quad l_{70} = -a_{70}, \quad l_{20} = b_{20}, \quad l_{30} = -\left(b_{20}a_{70}' - a_{70}b_{30}\right)/a_{70},\\
&l_{60} = \frac{a_{70}a_{50}\left(a_{60}b_{20} + b_{20}a_{50}' + 3a_{50}a_{70}' - 2a_{50}^2b_{20}'\right) - a_{70}^2\left(a_{60} + 3a_{50}'\right) - a_{50}^2b_{20}a_{70}'}{a_{70}\left(a_{70} - a_{50}b_{20}\right)},\\
&l_{10} = \frac{a_{70}\left(2a_{50}'b_{20} + 2a_{50}b_{20}' - 3a_{70}' - a_{50}b_{20}b_{10}\right) + a_{70}^2b_{10} + a_{50}b_{20}a_{70}'}{a_{70}\left(a_{70} - a_{50}b_{20}\right)}.
\end{aligned}
$$

Equation (16) is a sixth-order differential equation with variable coefficients. It can be proved that the coefficients A_i are independent of the wave number k. Now the WKB [13] technique is employed for the asymptotic analytical solution by converting Eq. (16) into a nonlinear equation.

By introducing $Z_u = \exp\left(\int \psi(z)dz\right)$, Eq. (16) can be transformed into the following form:

$$\begin{aligned}&\left[\psi^6 + 15\psi^4\psi' + 20\psi^3\psi'' + 45\psi^2\psi'^2 + 15\psi^2\psi''' + 60\psi\psi'\psi'' + 5\psi''^2 + 15\psi'\psi''' + 6\psi\psi^{(4)} + \psi^{(5)}\right]\\&+ A_5\left[\psi^5 + 10\psi^3\psi' + 10\psi^2\psi'' + 15\psi\psi'^2 + 5\psi\psi''' + 10\psi'\psi'' + \psi^{(4)}\right] + A_4\left(\psi^4 + 6\psi^2\psi' + 4\psi\psi''\right.\\&\left.+ 3\psi'^2 + \psi'''\right)k^2 + A_3\left(\psi^3 + 3\psi\psi' + \psi''\right)k^2 + A_2\left(\psi^2 + \psi'\right)k^4 + A_1\psi k^4 + A_0k^6 = 0.\end{aligned} \tag{17}$$

To solve Eq. (17), we seek an expansion of $\psi(z)$ in inverse powers of k, that is to say, we can write

$$\psi(z) = \psi_0(z)k + \psi_1(z) + \psi_2(z)k^{-1} + \cdots \tag{18}$$

Substituting Eq. (18) into (17) and equating the coefficients of each power of k to zero, we can get an infinite number of equations:

$$\psi_0^6 + A_4\psi_0^4 + A_2\psi_0^2 + A_0 = 0, \tag{19.1}$$

$$6\psi_0^5\psi_1 + 15\psi_0^4\psi_0' + A_5\psi_0^5 + A_4\left(4\psi_0^3\psi_1 + 6\psi_0^2\psi_0'\right) + A_3\psi_0^3 + A_2\left(2\psi_0\psi_1 + \psi_0'\right) + A_1\psi_0 = 0 \tag{19.2}$$

......

Six roots for Eq. (19.1) can be obtained, considering the decay property of the Rayleigh wave, we use $\psi_{01}, \psi_{02}, \psi_{03}$ to denote three roots whose real part are less than zero. From (19.2), we can get roots $\psi_{11}, \psi_{12}, \psi_{13}$ corresponding to $\psi_{01}, \psi_{02}, \psi_{03}$, and meanwhile $\psi_{11}, \psi_{12}, \psi_{13}$ should satisfy the following condition:

$$\psi_{1i} = -\frac{15\psi_{0i}^4\psi_{0i}' + A_5\psi_{0i}^5 + A_4 6\psi_{0i}^2\psi_{0i}' + A_3\psi_{0i}^3 + A_2\psi_{0i}' + A_1\psi_{0i}}{6\psi_{0i}^5 + 4A_4\psi_{0i}^3 + 2A_2\psi_{0i}} \quad (i = 1 \sim 3). \tag{20}$$

Thus Z_u can be expressed as:

$$\begin{aligned}Z_u = c_1 \exp\left(k\int \psi_{01}dz + \int \psi_{11}dz\right) + c_2 \exp\left(k\int \psi_{02}dz + \int \psi_{12}dz\right)\\+ c_3 \exp\left(k\int \psi_{03}dz + \int \psi_{13}dz\right)\end{aligned} \tag{21}$$

in which c_1, c_2, c_3 are undetermined constants.

Similarly, to obtain an equation only containing Z_w and its corresponding differentials, undetermined coefficients $m_i (i = 1 \sim 5)$ are introduced into the analysis to construct the summation:

$$H_1 = G_1'' + m_1G_1' + m_2G_1 + m_3G_2'' + m_4G_2' + m_5G_2. \tag{22}$$

Let the coefficients of Z_w and its corresponding first to fifth-order differentials be zero, then from $H_1 = 0$ and the high order terms containing $1/k^2$ omitting also, the following equation can be obtained:

$$k^5Z_w = B_5Z_u^{(5)} + B_4Z_u^{(4)} + B_3k^2Z_u''' + B_2k^2Z_u'' + B_1k^4Z_u' + B_0k^4Z_u. \tag{23}$$

It can be assumed that $Z_w = g(z)Z_u$, in which $g(z) = g_0(z)k + g_1(z) + g_2(z)k^{-1} + \cdots$. Substituting this equation into (23) and equating the coefficients of each power of k to zero, we can get the following equations:

$$g_0(z) = 0, \tag{24.1}$$

$$g_1(z) = B_5\psi_0^5 + B_3\psi_0^3 + B_1\psi_0. \tag{24.2}$$

From these two equations, we can obtain three roots $g_{11}(z)$, $g_{12}(z)$, and $g_{13}(z)$ corresponding to ψ_{01}, ψ_{02}, and ψ_{03}. Thus the solution of Z_w can be expressed as:

$$\begin{aligned} Z_w &= c_1 g_{11}(z)\exp\left(k\int \psi_{01}dz + \int \psi_{11}dz\right) + c_2 g_{12}(z)\exp\left(k\int \psi_{02}dz + \int \psi_{12}dz\right) \\ &\quad + c_3 g_{13}(z)\exp\left(k\int \psi_{03}dz + \int \psi_{13}dz\right). \end{aligned} \tag{25}$$

Equation (11) also can be expressed in the following form:

$$C_0 Z_\varphi k^3 = C_1 Z_u'' + C_2 k Z_w'' + C_3 k^2 Z_u' + C_4 k Z_w' + C_5 Z_u k^2 + C_6 k^3 Z_w, \tag{26}$$

The solutions of Z_u (Eq. (21)) and Z_w (Eq. (25)) are substituted into Eq. (26) and the high order terms are omitted, hence the solution of Z_φ can be expressed as follows:

$$\begin{aligned} Z_\varphi &= c_1 h_{11}(z)\exp\left(k\int \psi_{01}dz + \int \psi_{11}dz\right) + c_2 h_{12}(z)\exp\left(k\int \psi_{02}dz + \int \psi_{12}dz\right) \\ &\quad + c_3 h_{13}(z)\exp\left(k\int \psi_{03}dz + \int \psi_{13}dz\right) \end{aligned} \tag{27}$$

in which h_{1i} satisfy:

$$h_{1i} = \left(C_2 g_{1i}\psi_{0i}^2 + C_3\psi_{0i} + C_6 g_{1i}\right)/C_0.$$

As to Eq. (7), the solution of φ_1 can be expressed as follows:

$$\varphi_1(x, z, t) = Z_{\varphi 1}\exp(kx - i\omega t). \tag{28}$$

Substituting Eq. (28) into Eq. (7), and considering the attenuation condition $z \to -\infty$, $\varphi_1 \to 0$, we can obtain the solution of $Z_{\varphi 1}$ as follows:

$$Z_{\varphi 1} = c_4 e^{kz}, \tag{29}$$

where c_4 is an undetermined constant.

The electrically shorted boundary conditions described in Eq. (8) also can be expressed as:

$$\begin{aligned} & c_{13} Z_u ik + c_{33} dZ_w/dz + e_{33} dZ_\varphi/dz = 0, \\ & c_{44}\left(Z_w ik + dZ_u/dz\right) + e_{15} Z_\varphi ik = 0, \\ & Z_\varphi = 0. \end{aligned} \tag{30}$$

Substituting Eqs. (21), (25) and (27) into Eq. (30), equations with respect to c_1, c_2, and c_3 can be obtained, from the sufficient and necessary condition of the non-trivial solution to exist, the determinant of the coefficient matrix has to vanish, which leads to the following dispersion relation for the Rayleigh wave:

$$\left|P_{ij}\right| = 0, \tag{31}$$

where

$$\begin{aligned} P_{11} &= c_{13}ik + c_{33}\left[g_{11}' + g_{11}(k\psi_{01} + \psi_{11})\right] + e_{33}\left[h_{11}' + h_{11}(k\psi_{01} + \psi_{11})\right]\Big|_{z=0}, \\ P_{12} &= c_{13}ik + c_{33}\left[g_{12}' + g_{12}(k\psi_{02} + \psi_{12})\right] + e_{33}\left[h_{12}' + h_{12}(k\psi_{02} + \psi_{12})\right]\Big|_{z=0}, \\ P_{13} &= c_{13}ik + c_{33}\left[g_{13}' + g_{13}(k\psi_{03} + \psi_{13})\right] + e_{33}\left[h_{13}' + h_{13}(k\psi_{03} + \psi_{13})\right]\Big|_{z=0}, \\ P_{21} &= ikg_{11} + k\psi_{01} + \psi_{11}|_{z=0}, \quad P_{22} = ikg_{12} + k\psi_{02} + \psi_{12}|_{z=0}, \quad P_{23} = ikg_{13} + k\psi_{\ 03} + \psi_{13}|_{z=0}, \\ P_{31} &= h_{11}|_{z=0}, \quad P_{32} = h_{12}|_{z=0}, \quad P_{33} = h_{13}|_{z=0}. \end{aligned}$$

The electrically open boundary condition described in Eq. (9) also can be expressed as:

$$\begin{aligned}
&c_{13}Z_u ik + c_{33}dZ_w/dz + e_{33}dZ_\varphi/dz = 0,\\
&c_{44}\left(Z_w ik + dZ_u/dz\right) + e_{15}Z_\varphi ik = 0,\\
&Z_\varphi = Z_{\varphi 1},\\
&e_{31}Z_u ik + e_{33}dZ_w/dz - \varepsilon_{33}dZ_\varphi/dz = -\varepsilon_0 dZ_{\varphi 1}/dz.
\end{aligned} \tag{32}$$

Similarly, the dispersion relation of the Rayleigh wave for the electrically open case can be obtained as:

$$\left|Q_{ij}\right| = 0, \tag{33}$$

where

$$\begin{aligned}
&Q_{11} = P_{11}, \quad Q_{12} = P_{12}, \quad Q_{13} = P_{13}, \quad Q_{14} = 0,\\
&Q_{21} = P_{21}, \quad Q_{22} = P_{22}, \quad Q_{23} = P_{23}, \quad Q_{14} = e_{15}ik/c_{44}\big|_{z=0},\\
&Q_{31} = P_{31}, \quad Q_{32} = P_{32}, \quad Q_{33} = P_{33}, \quad Q_{34} = -1,\\
&Q_{41} = e_{31}ik + e_{33}\left[g'_{11} + g_{11}\left(k\psi_{01} + \psi_{11}\right)\right] - \varepsilon_{33}\left[h'_{11} + h_{11}\left(k\psi_{01} + \psi_{11}\right)\right]\big|_{z=0},\\
&Q_{42} = e_{31}ik + e_{33}\left[g'_{12} + g_{12}\left(k\psi_{02} + \psi_{12}\right)\right] - \varepsilon_{33}\left[h'_{12} + h_{12}\left(k\psi_{02} + \psi_{12}\right)\right]\big|_{z=0},\\
&Q_{43} = e_{31}ik + e_{33}\left[g'_{13} + g_{11}\left(k\psi_{03} + \psi_{13}\right)\right] - \varepsilon_{33}\left[h'_{13} + h_{13}\left(k\psi_{03} + \psi_{13}\right)\right]\big|_{z=0},\\
&Q_{44} = \varepsilon_0 k.
\end{aligned}$$

4 Numerical results

Now the numerical analysis results will be given to show the basic properties of Rayleigh surface waves propagating at the free surface of an FGPM half-space. The FGPM parameters are chosen to possess the exponential patterns for the calculation of dispersion relations, for both electrically open and shorted cases.

FGPM parameters are assumed to possess the following form [14]:

$$f(z) = f_\infty + \left(f_0 - f_\infty\right)\exp\left(-\alpha z\right)$$

in which α is the graded coefficient, f_0 and f_∞ stand for the material parameters at $z = 0$ and $z \to \infty$, respectively. In the following calculation, the values of parameters f_0 are chosen to be [15]:

$$\begin{aligned}
&c_{11} = 135\ \text{GPa}, \quad c_{13} = 67.9\ \text{GPa}, \quad c_{44} = 22.2\ \text{GPa}, \quad c_{33} = 113\ \text{GPa}, \quad c_{12} = 68.1\ \text{GPa},\\
&\quad \rho = 7.5 \times 10^3\ kg/m^3,\\
&e_{15} = 9.8\ C/m^2, \quad e_{33} = 9.0\ C/m^2, \quad e_{31} = -1.9\ C/m^2, \quad \varepsilon_{11} = 990\varepsilon_0, \quad \varepsilon_{33} = 450\varepsilon_0,\\
&\quad \varepsilon_0 = 8.854 \times 10^{-12}\ F/m
\end{aligned}$$

in which ε_0 is the dielectric constant in the air.

In the following numerical analysis, a relative variation coefficient of material parameters is introduced:

$$\delta = \left(f_\infty - f_0\right)/f_0.$$

4.1 Influence of material inhomogeneous character on dispersion relations

To reveal the influence of the inhomogeneous character of the FGPMs on the dispersion relations, a simplified scheme is adopted: only one of the material parameters is varied and other parameters keep constants. Influences of variations of different material parameters on the dispersion relations are shown in Figs. 2–10. In the numerical analysis, the relative variation of material parameters is chosen to be $\delta = \pm 0.1$, and the graded coefficient α is chosen to be 1, 3 and 5. In these figures, c is the phase velocity, k is the wave number, and c_{Rs} and c_{Ro} are phase velocities of the Rayleigh wave in transversely isotropic PZT-1 half-space for electrically shorted and open cases, respectively.

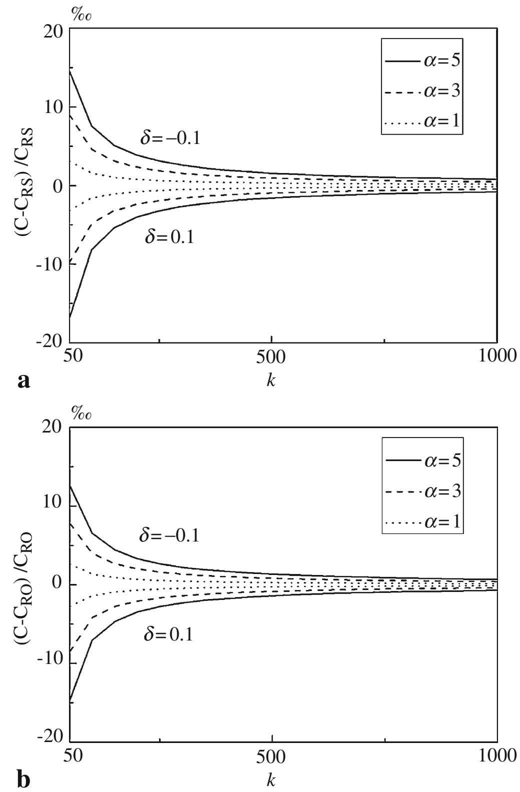

Fig. 2 Influence of different c_{11} on dispersion relations; **a** electrically shorted; **b** electrically open

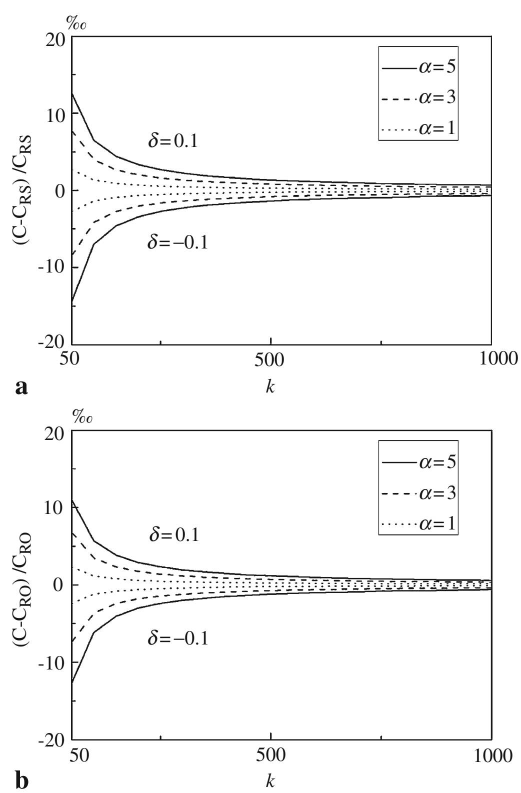

Fig. 3 Influence of different c_{13} on dispersion relations; **a** electrically shorted; **b** electrically open

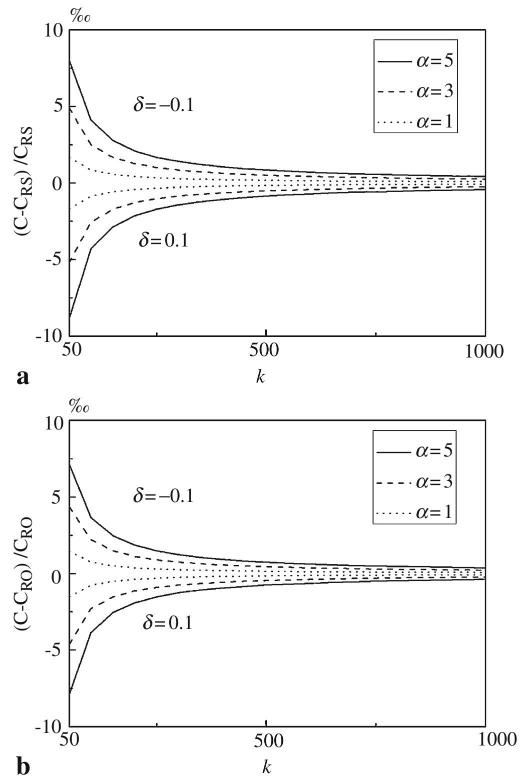

Fig. 4 Influence of different c_{33} on dispersion relations; **a** electrically shorted; **b** electrically open

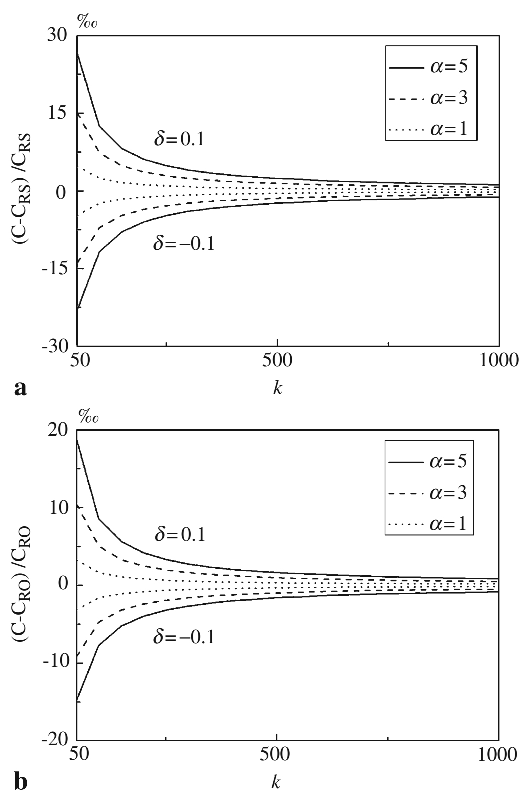

Fig. 5 Influence of different c_{44} on dispersion relations; **a** electrically shorted; **b** electrically open

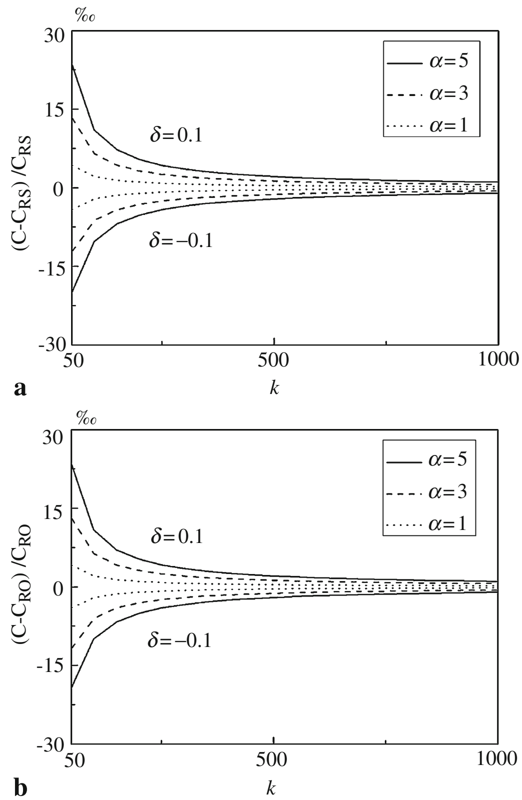

Fig. 6 Influence of different e_{15} on dispersion relations; **a** electrically shorted; **b** electrically open

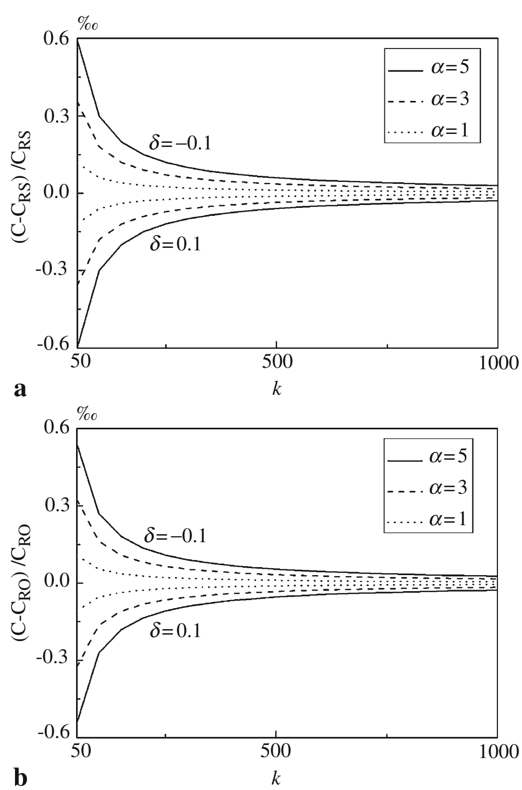

Fig. 7 Influence of different e_{31} on dispersion relations; **a** electrically shorted; **b** electrically open

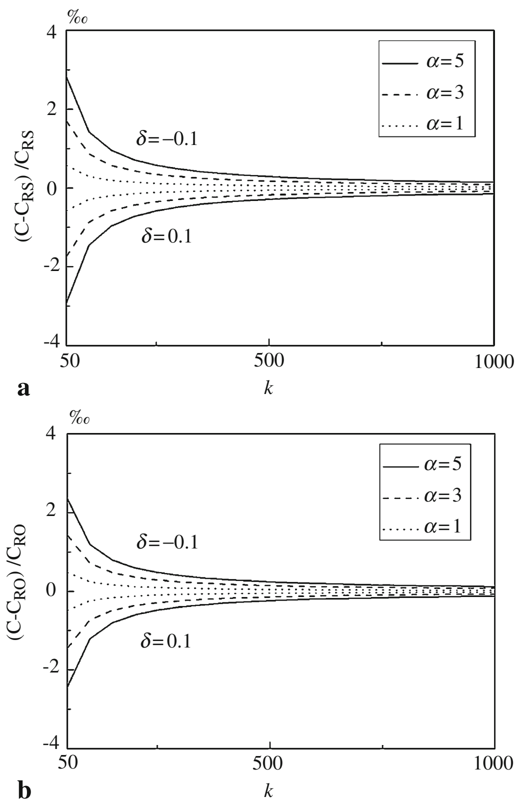

Fig. 8 Influence of different e_{33} on dispersion relations; **a** electrically shorted; **b** electrically open

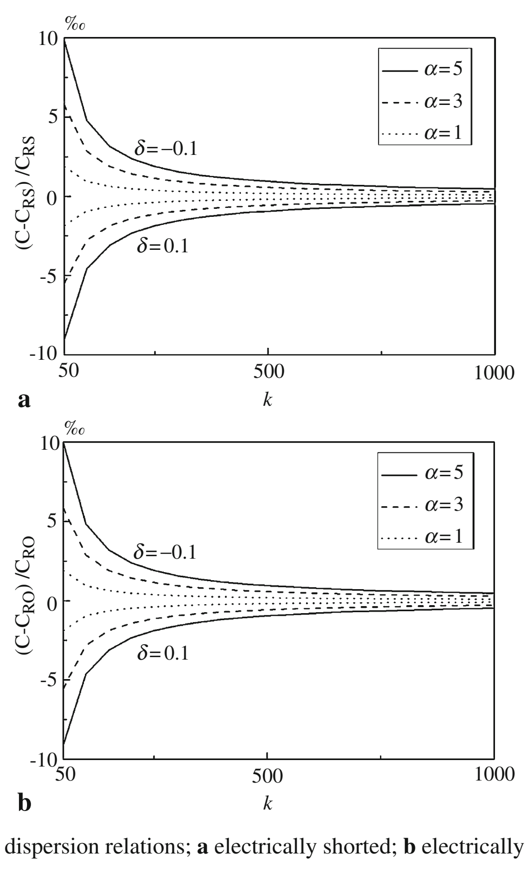

Fig. 9 Influence of different ε_{11} on dispersion relations; **a** electrically shorted; **b** electrically open

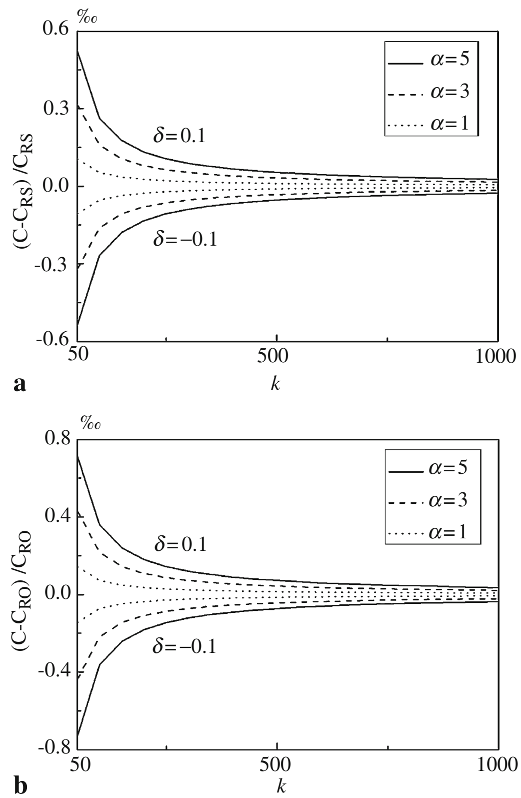

Fig. 10 Influence of different ε_{33} on dispersion relations; **a** electrically shorted; **b** electrically open

The above results indicate that, whether for the case of material parameters increasing along the depth direction ($\delta = 0.1$), or for the case of material parameters decreasing along the depth direction ($\delta = -0.1$), coupled Rayleigh surface waves can propagate at the free surface of the FGPM half-space. For both the electrically open and shorted cases, the influence of the graded coefficient α on the wave phase velocity shows a similar tendency. With the increase of α, the variation of the phase velocity also increases, but such kind of the influence is only obvious for the wave number less than 500. To further investigate the dispersion properties, the group velocity of the wave propagation, which is defined as $c_g = c + kdc/dk$, is introduced in our analysis. It can be found that, when the parameters c_{13}, c_{44}, e_{15}, and ε_{33} increase with the depth, or when the parameters c_{11}, c_{33}, e_{31}, e_{33}, and ε_{11} decrease with the depth, there exist $dc/dk < 0$, which means $c_g < c$, the Rayleigh wave is normal dispersion; on the other hand, when the parameters c_{13}, c_{44}, e_{15}, and ε_{33} decrease with the depth, or when the parameters c_{11}, c_{33}, e_{31}, e_{33} , and ε_{11} increase with the depth, there exist $dc/dk > 0$, which means $c_g > c$, the Rayleigh wave is anomalous dispersion.

From Figs. 2– 10, it also can be learned that for the same variation of material parameters variations of the elastic parameters c_{13}, c_{44}, c_{11}, and c_{33}, piezoelectric parameter e_{15} and dielectric ε_{11} significantly affect the phase velocity, the variation of piezoelectric parameter e_{33} affects the phase velocity little less, and variations of the piezoelectric parameter e_{31} and the dielectric parameter ε_{33} do almost not affect the phase velocity.

4.2 Influence of graded variation on the displacement components

The criterion for Rayleigh waves is that the mechanical displacement decays exponentially with the distance from the free surface. For the case of only the elastic parameter c_{44} variation, variations of the displacement components with depth for the electrically shorted case are shown in Fig. 11 by choosing the relative variation $\delta = \pm 0.1$, graded coefficient $\alpha = 5$ and wave number $k = 100$. It can be found that the displacements' decay character is similar to that in an isotropic medium. As to the influences of other parameters' variation on the mechanical displacement components, similar conclusions can be obtained.

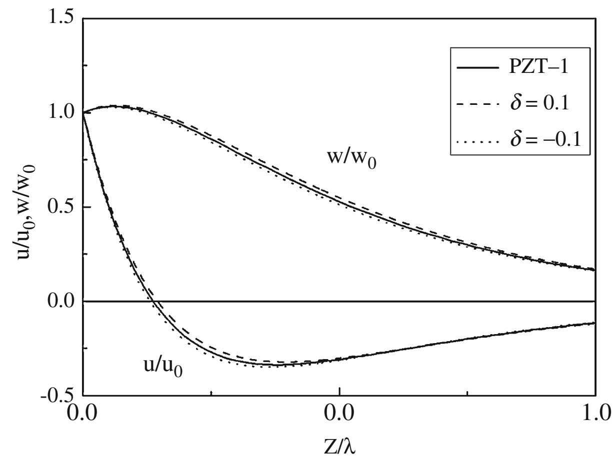

Fig. 11 Influence of variation of c_{44} on mechanical displacement amplitudes

Table 1 Ellipticity of particles' trajectories for different material parameters variation

δ	$k = 100, \alpha = 5$	c_{33}	e_{33}	c_{11}	e_{15}	c_{13}	c_{44}	e_{31}	ε_{11}	ε_{33}
0.1	Electrically shorted	1.928	1.907	1.957	1.835	1.854	1.818	1.902	1.922	1.903
	Electrically open	1.626	1.608	1.647	1.554	1.569	1.543	1.604	1.621	1.605
−0.1	Electrically shorted	1.873	1.893	1.844	1.956	1.947	1.973	1.899	1.876	1.898
	Electrically open	1.581	1.589	1.562	1.646	1.639	1.657	1.602	1.584	1.602

Table 2 Phase velocity variations (‰) for variation of material parameters

δ	$k/\alpha = 50$	c_{33}	e_{33}	c_{11}	e_{15}	c_{13}	c_{44}	e_{31}	ε_{11}	ε_{33}
0.1	Electrically shorted	−1.70	−0.58	−3.19	4.29	2.66	4.85	−0.12	−1.86	0.11
	Electrically open	−1.52	−0.48	−2.76	4.19	2.32	4.67	−0.11	−1.88	0.14
−0.1	Electrically shorted	1.67	0.57	3.10	−4.20	−2.73	−4.77	0.12	1.89	−0.11
	Electrically open	1.49	0.48	2.68	−4.05	−2.39	−4.53	0.11	1.91	−0.15

When Rayleigh surface wave propagate, the trajectories of the particles are ellipses. For the coordinate axes of Fig. 1 the motion is counter-clockwise at the free surface. To discuss the influence of graded variation on the displacement components, denote the ellipticity of particles' trajectories by $\gamma = |w_0|/|u_0|$, in which $|w_0|$ and $|u_0|$ are the particles' displacement amplitudes at the free surface ($z = 0$). Ellipticity of particles' trajectories for different material parameters variation is shown in Table 1, in which the wave number $k = 100$ and the graded coefficient $\alpha = 5$. For an isotropic piezoelectric medium, the ellipticity is 1.900 for the electrically open case and 1.603 for the electrically shorted case, respectively.

When parameters c_{13}, c_{44}, e_{15} , and ε_{33} increase with the depth, or $c_{11}, c_{33}, e_{31}, e_{33}$, and ε_{11} decrease with the depth, the ellipticity of the particles' motion trajectories increases also. On the contrary, decreasing of parameters c_{13}, c_{44}, e_{15} , and ε_{33} with the depth or the increasing of parameters $c_{11}, c_{33}, e_{31}, e_{33}$, and ε_{11} with the depth lead to the decrease of ellipticity.

4.3 Determination of FGPMs graded coefficient α

The analysis indicates that for the case of a single material parameter variation the wave number of the Rayleigh wave is proportional to the material graded parameter for the same phase velocities. The quantitative relations for the material parameters' variation with phase velocity relative variations are shown in Table 2 ($k/\alpha = 50$).

Table 2 indicates that on condition that the variation of a material parameter is given, it can be obtained the phase velocity and the corresponding phase velocity relative value by numerical calculation for a special value of k/α. By experimental measurement, the frequency of the Rayleigh surface wave can be got at this

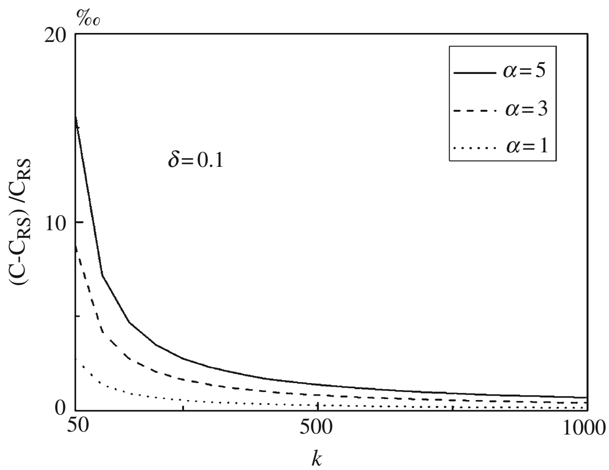

Fig. 12 Rayleigh wave dispersion curves for the elastic parameters' relative variation value $\delta_1 = 0.1$ if the electrical parameters keep constants

phase velocity and then the wave number k is known, and the material graded coefficient α can be determined accordingly.

The method can be extended to the case of variations of all the material parameters. On condition of the slowly varying assumption, for the case of the variation of all the elastic parameters $\delta_1 = 0.1$ and electrical parameters keeping constants, the dispersion relation of the Rayleigh waves is shown in Fig. 12 for the electrically shorted case. At the same value of k/α, the relative changing value of the phase velocity is the same. For example, at the value of $k/\alpha = 50$, the phase velocity relative changing value is $2.76^0/_{00}$.

Therefore, it is possible to determine the graded coefficient α of FGPMs by means of the experimental measurement of Rayleigh surface waves.

5 Conclusions

Dispersion characters of Rayleigh surface waves propagating along the free surface of a FGPM half-space are investigated by virtue of the WKB technique. The following conclusions can be drawn:

Coupled Rayleigh surface waves can propagate at the surface of a functionally graded piezoelectric material (FGPM) half-space. The dispersion relations of the Rayleigh wave and the particles' displacements at the free surface are dependent upon the graded variation tendency of the FGPM parameters. By the Rayleigh surface waves phase velocities relative changing values combined with the relationship between the wave number and the material graded coefficient, a theoretical foundation can be provided for the graded material characterization by experimental measurement.

Ackowledgments The authors gratefully acknowledge the financial support of the work by the National Basic Research Program of China through Grant No. 2006CB601202, the National Natural Science Foundation of China through Grant No.10632060, the National 111 Project of China through Grant No. B06024 and the specialized Research Fund for the Doctoral Program of Higher Education of China through Grant No. 20070698064.

References

1. Kawai, T., Miyazaki, S., Araragi, M.: A new method for forming a piezoelectric FGM using a dual dispenser system, In: Yamanouchi, et al. (ed.) Proceedings of the First International Symposium on Functionally Gradient Materials, pp. 191–196, Sendai (1990)
2. Chakraborty, A., Gopalakrishnan, S.: Wave propagation in inhomogeneous layered media: solution of forward and inverse problems. Acta. Mech. **169**, 153–185 (2004)
3. Liu, G.R., Tani, J.: SH waves in functionally gradient piezoelectric plates. Trans. ASME J. Appl. Mech. **116**, 440–448 (1994)
4. Liu, G.R., Tani, J., Ohyoshi, T.: Lamb waves in a functionally gradient material plates and its transient response. Part 1: theory; Part 2: calculation result. Trans. Jpn. Soc. Mech. Eng. **57**A, 131–142 (1991)

5. Han, X., Liu, G.R., Lam, K.Y.: A quadratic layer element for analyzing stress waves in functionally graded materials and its application for material characterization. J. Sound. Vibr. **236**, 307–321 (2000)
6. Liu, G.R., Han, X., Lam, K.Y.: An integration technique for evaluating confluent hypergeometric functions and its application to functionally graded materials. Comput. Struct. **79**, 1039–1047 (2001)
7. Wang, Q., Varadan, V.K.: Wave propagation in piezoelectric coupled plates by use of interdigital transducer Part 1. Dispersion characteristics. Int. J. Solids. Struct **39**, 1119–1130 (2002)
8. Li, X.Y., Wang, Z.K., Huang, S.H.: Love waves in functionally graded piezoelectric materials. Int. J. Solids. Struct. **41**, 7309–7328 (2004)
9. Liu, J., Wang, Z.: The propagation behavior of Love waves in a functionally graded layered piezoelectric structure. Smart. Mater. Struct. **14**, 137–146 (2005)
10. Jin, F., Wang, Z., Kishimoto, K.: Basic properties of Rayleigh surface wave propagation along curved surfaces. Int. J. Eng. Sci. **43**, 250–261 (2005)
11. Liu, J., Wang, Z.: Study on the propagation of Rayleigh surface waves in a graded half-space. Chin. J. Appl. Mech. **21**, 106–109 (2004)
12. Rose, J.L.: Ultrasonic waves in solid media. Cambridge University Press, Cambridge (1999)
13. Bender, C.M., Orszag, S.A.: Advanced mathematical methods for scientists and engineers. McGraw-Hill, New York (1978)
14. Koizumi, M.: The concept of FGM, ceramic transaction. Funct. Gradient. Mater. **34**, 3–10 (1993)
15. Jin, F., Wang, Z., Wang, T.: The Bleustein-Gulyaev (B-G) wave in a piezoelectric layered half-space. Int. J. Eng. Sci. **39**, 1271–1285 (2001)