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Interactions between N circular cylindrical inclusions in a piezoelectric matrix

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Summary. This paper studies the interactions between N randomly-distributed cylindrical inclusions in a piezoelectric matrix. The inclusions are assumed to be perfectly bounded to the matrix, which is subjected to an anti-plane shear stress and an in-plane electric field at infinity. Based on the complex variable method, the complex potentials in the matrix and inside the inclusions are first obtained in form of power series, and then approximate solutions for electroelastic fields are derived. Numerical examples are presented to discuss the influences of the inclusion array, inclusion size and inclusion properties on couple fields in the matrix and inclusions. Solutions for the case of an infinite piezoelectric matrix with N circular holes or an infinite elastic matrix containing N circular piezoelectric fibers can also be obtained as special cases of the present work. It is shown that the electroelastic field distribution in a piezoelectric material with multiple inclusions is significantly different from that in the case of a single inclusion.

1 Introduction

Studies on the inclusion problems in an infinite matrix are of theoretical and practical importance with increasingly wide application of composite materials [1]–[3]. In fact, much effort has been made on the subject since the pioneering work of Eshebly was published [4]. Even for the cases of multiple circular inclusions, it is difficult to cite all the works which were carried out in past decades. Recently, these related works have been extended to the case of an infinite piezoelectric matrix with circular inclusions. Pak [5] analyzed a circular piezoelectric inclusion embedded in an infinite piezoelectric matrix in the framework of linear piezoelectricity and obtained a closed-form solution for the case of a far-field antiplane mechanical load and a far-field inplane electrical load. Dunn and Wienecke [6] derived the electroelastic fields in and around inclusions and inhomogeneities in transversely isotropic piezoelectric solids using Eshelby's method. Xiao and Bai [7] investigated a circular piezoelectric fiber sensor embedded in a non-piezoelectric elastic material and obtained a closed-form solution for the stress field outside a circular piezoelectric inhomogeneity. Shen et al. [8] studied the interaction of a piezoelectric screw dislocation with a nonuniformly coated circular

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inclusion in an unbounded piezoelectric matrix subjected to remote anti-plane shear and electric fields. Deng and Meguid [9] considered the case of a partially debonded circular inclusion in piezoelectric materials. Gao and Noda [10] studied the anti-plane deformation of an arbitrarily-shaped inclusion embodied in an infinite piezoelectric material using Faber series expanding of complex potentials. For the cases of multi-inclusions, Ishihara and Noda [11] studied an anti-plane electroelastic problem of an infinite piezoelectric body with two circular piezoelectric inhomogeneities by introducing two complex potential functions and conformal mapping. Wu et al. [12] and Wang and Shen [13] derived the electro-elastic field of the infinite piezoelectric medium with two piezoelectric circular cylindrical inclusions under general loads based on the use of conformal mapping and the theorem of analytic continuation, respectively. Especially, Chao and Chang [14] addressed the problem of N interacting circular inclusions in anti-plane piezoelectricity based upon the complex variable theory and the method of successive approximations by extending their previous work on the problem of multiple inclusions in an elastic material [15]. The similar problem was solved for the case of multi-inclusions in plane magnetoelasticity [16]. More recently, Chen and Wu [17] proposed a null-field approach for piezoelectricity problems with arbitrary circular inclusions by using the separable expressions of fundamental solutions and Fourier series for boundary densities. In addition, Xu et al. [18] developed a rigorous analytical method for solving the problem of a doubly periodic parallelogrammic array of piezoelectric fibers in piezoelectric composites under anti-plane shear coupled with in-plane electrical load, based on the use of the doubly quasi-periodic Riemann boundary value problem theory integrated with the eigenstrain and eigen-electrical-field concepts.

In the present work we propose a straightforward and concise approach to analyze the problem of N interacting circular inclusions in anti-plane piezoelectricity based on complex variable theory. Mathematically, the key to the present problem is to solve a sub-problem taking place in the infinite matrix with holes, which is a multiply-connected region. The feature of the present work is to express the complex potential of the matrix in sum of those of N infinite regions having a single hole, respectively, and an infinite region with hole. Then, using the continuous conditions between the inclusions and the matrix produces a system of linear equations concerning unknown coefficients involved in the complex potentials. Once these equations are solved the potentials become known and thus all the fields can be determined. Below are the main contents of the work: following the Introduction, the considered problem and used assumptions are stated in Sect. 2, and then the general solutions of complex potentials are derived in Sect. 3. Presented in Sect. 4 are numerical examples to discuss the influences of inclusion array, inclusion size and inclusion properties on couple fields in the matrix and inclusions. Finally, this work is concluded in Sect. 5.

2 Statement of the problem

Consider an infinite piezoelectric matrix containing N cylindrical inclusions which are parallel to each other. The cross-section normal to the inclusions is shown in Fig. 1, where the regions occupied by the matrix and the cross-section of inclusions are denoted by M and I, respectively, and all the inclusions are assumed to be completely bounded to the matrix. In true piezoelectric composites, the matrix is purely elastic and the inclusions (fibers) are piezoelectric. However, in this work the matrix and inclusions are assumed to be piezoelectric to obtain a general solution, which can cover the solutions for special cases, for example, of an infinite elastic matrix with N piezoelectric fibers or circular holes, of an infinite piezoelectric matrix containing multiple inclusions or holes. In addition, the matrix and inclusions are assumed to have the same poling direction along the positive z axis and

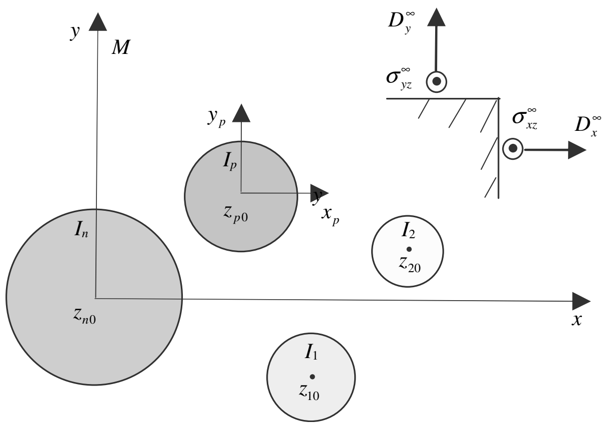

Fig. 1. N circular piezoelectric inclusions in an infinite matrix

the same isotropic plane in the x–y plane, as shown in Fig. 1, where the matrix is loaded by a uniform remote anti-plane shear and an in-plane electric loading.

In this case, the general solution for the generalized displacement function $\mathbf{u}$ and generalized stress function ϕ can be expressed as [19]

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}(z)}, \quad \mathbf{u} = (u_z, \varphi)^T, \tag{1.1}$$

$$\boldsymbol{\phi} = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}\mathbf{f}(z)}, \quad z = x + iy, \tag{1.2}$$

where u_z and φ are elastic displacement and electric potential, respectively; $\mathbf{f}(z)$ is a unknown complex vector; $\mathbf{A}$ and $\mathbf{B}$ stand for the material constant matrices defined as

$$\mathbf{A} = \mathbf{I}, \quad \mathbf{B} = i\mathbf{B}_0, \quad \mathbf{B}_0 = \begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}. \tag{2}$$

In Eq. (2), c_{44}, e_{15} and ε_{11} represent the elastic constants, the piezoelectric constants and the dielectric constants, respectively.

Once the complex potential $\mathbf{f}(z)$ is obtained based on given boundary conditions, all the fields, e.g., the components of stress σ, electric displacement D and electric field E, can be determined from

$$(\sigma_{zx}, D_x)^T = -\phi_{,2}, \quad (\sigma_{zy}, D_y)^T = \phi_{,1}, \quad u_{2,1} = -E_x, u_{2,2} = -E_y. \tag{3}$$

From Eq. (3) one has

$$\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}_1 = 2\mathbf{B}\mathbf{F}(z), \tag{4}$$

where

$$\boldsymbol{\sigma}_1 = (\sigma_{zx}, D_x)^T, \quad \boldsymbol{\sigma}_2 = (\sigma_{zy}, D_y)^T, \quad \mathbf{F}(z) = d\mathbf{f}(z)dz.$$

Thus, the key task is to find the complex potential vector $\mathbf{f}(z)$.

3 Complex potentials

In this case, the complex potential in the matrix has the form of

$$\mathbf{f}(z) = \mathbf{c}^\infty z + \mathbf{f}_0(z), \tag{5}$$

where c^∞ is a constant related to the loading condition at infinity, and $\mathbf{f}_0(z)$ is an unknown complex function that nulls at infinity, i.e., $\mathbf{f}_0(\infty) = 0$.

In general, $\mathbf{f}_0(z)$ can be expressed as

$$\mathbf{f}_0(z) = \sum_{n=1}^{N} \mathbf{f}_{n0}^{(0)}(z), \tag{6}$$

where $\mathbf{f}_{n0}^{(0)}(z)$ is an analytical function outside the inclusion l_n, and it can be expanded into the Laurent series as

$$\mathbf{f}_{n0}^{(0)}(z) = \sum_{j=1}^{\infty} \mathbf{a}_{nj} \left(\frac{z - z_{n0}}{r_n} \right)^{-j}, \tag{7}$$

where $\mathbf{a}_{nj}$ are unknown coefficients, and r_n is the radius of the n-th inclusion.

Inserting Eq. (7) into (6) gives

$$\mathbf{f}_0(z) = \sum_{n=1}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \left(\frac{z - z_{n0}}{r_n} \right)^{-j}. \tag{8}$$

On the other hand, inside any inclusion $l_p (p = 1, 2 \ldots N)$ the complex potential $\mathbf{f}_{p0}(z)$ can be expanded into the Taylor series as

$$\mathbf{f}_{p0}(z) = \sum_{j=1}^{\infty} \mathbf{b}_{pj} \left(\frac{z - z_{p0}}{r_p} \right)^{j}, \tag{9}$$

where $\mathbf{b}_{pj}$ are unknown coefficients.

Now, we move the origin of the global system $x{-}y$ into the point z_{p0}, that is, we make the following coordinate translation: $z - z_{p0} = z_p$. In the local coordinate system $x_p{-}y_p$, Eqs. (9) and (8) can be rewritten as

$$\mathbf{f}_{p0}(z_p) = \sum_{j=1}^{\infty} \mathbf{b}_{pj} \left(\frac{z_p}{r_p} \right)^{j}, \tag{10}$$

$$\mathbf{f}_0(z_p) = \sum_{j=1}^{\infty} \mathbf{a}_{pj} \left(\frac{z_p}{r_p} \right)^{-j} + \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \left(\frac{z_p + z_{p0} - z_{n0}}{r_n} \right)^{-j}. \tag{11}$$

In Eq. (11), the term $\left(\frac{z_p + z_{p0} - z_{n0}}{r_n} \right)^{-j}$ is a given function that is analytic outside the inclusion $l_n (n \neq p)$. This indicates that the function is also analytic inside the inclusion l_p, and thus it can be expanded into the Taylor series in the inclusion l_p, that is

$$\begin{aligned}
\left(\frac{r_n}{z_p + z_{p0} - z_{n0}} \right)^{j} &= \left(\frac{r_n}{z_{p0} - z_{n0}} \right)^{j} \left(1 + \frac{z_p}{z_{p0} - z_{n0}} \right)^{-j} \\
&= \left(\frac{r_n}{z_{p0} - z_{n0}} \right)^{j} \left(1 + \sum_{i=1}^{\infty} (-1)^i c_{ji} \left(\frac{z_p}{z_{p0} - z_{n0}} \right)^{i} \right) \\
&= \left(\frac{r_n}{z_{p0} - z_{n0}} \right)^{j} + \sum_{i=1}^{\infty} (-1)^i c_{ji} \left(\frac{r_n}{z_{p0} - z_{n0}} \right)^{j+i} \left(\frac{z_p}{r_n} \right)^{i} \\
&= \beta_{nj0}^{p} + \sum_{i=1}^{\infty} \beta_{nji}^{p} \left(\frac{z_p}{r_n} \right)^{i},
\end{aligned} \tag{12}$$

where

$$\beta_{nj0}^{p} = \left(\frac{R_n}{z_{p0} - z_{n0}}\right)^{j}, \quad \beta_{nji}^{p} = (-1)^{i} c_{ji} \left(\frac{R_n}{z_{p0} - z_{n0}}\right)^{j+i}, \quad c_{ji} = \frac{j(j+1)\ldots(j+i-1)}{i!}.$$

Substituting Eq. (12) into Eq. (11) produces

$$\mathbf{f}_0(z_p) = \sum_{i=1}^{\infty} \mathbf{a}_{pi} \left(\frac{z_p}{r_p}\right)^{-i} + \sum_{i=1}^{\infty} \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \beta_{nji}^{p} \left(\frac{z_p}{r_n}\right)^{i} + \text{const.} \tag{13}$$

On the interface between the matrix and the inclusion l_p, the continuous conditions are

$$\mathbf{u}_M = \mathbf{u}_p, \quad \boldsymbol{\phi}_M = \boldsymbol{\phi}_p, \quad (p = 1, 2 \ldots N). \tag{14}$$

Considering Eq. (1) together with (5), Eq. (14) can be reduced to

$$2\mathrm{Re}\lfloor \mathbf{A}\mathbf{c}^{\infty}(z_p + z_{p0}) \rfloor + 2\mathrm{Re}[\mathbf{A}\mathbf{f}_0(z_p)] = 2\mathrm{Re}[\mathbf{A}_p \mathbf{f}_{p0}(z_p)], \tag{15}$$

$$2\mathrm{Re}\lfloor \mathbf{B}\mathbf{c}^{\infty}(z_p + z_{p0}) \rfloor + 2\mathrm{Re}[\mathbf{B}\mathbf{f}_0(z_p)] = 2\mathrm{Re}[\mathbf{B}_p \mathbf{f}_{p0}(z_p)]. \tag{16}$$

Using the condition $z_p = r_p\,\sigma = r_p\,e^{i\theta}$ at the rim of the inclusion l_p, one has from Eqs. (15) and (16) that

$$2\mathrm{Re}\lfloor \mathbf{A}\mathbf{c}^{\infty}(R_p\sigma + z_{p0}) \rfloor + 2\mathrm{Re}[\mathbf{A}\mathbf{f}_0(\sigma)] = 2\mathrm{Re}[\mathbf{A}_p \mathbf{f}_{p0}(\sigma)], \tag{17}$$

$$2\mathrm{Re}\lfloor \mathbf{B}\mathbf{c}^{\infty}(R_p\sigma + z_{p0}) \rfloor + 2\mathrm{Re}[\mathbf{B}\mathbf{f}_0(\sigma)] = 2\mathrm{Re}[\mathbf{B}_p \mathbf{f}_{p0}(\sigma)], \tag{18}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}_p$ and $\mathbf{B}_p$ are the elastic matrices related to the matrix and the inclusion l_p, respectively, and

$$\mathbf{f}_{p0}(\sigma) = \sum_{i=1}^{\infty} \mathbf{b}_{pi} \sigma^{i}, \tag{19}$$

$$\mathbf{f}_0(\sigma) = \sum_{i=1}^{\infty} \mathbf{a}_{pi} \sigma^{-i} + \sum_{i=1}^{\infty} \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \beta_{nji}^{p} \left(\frac{r_p}{r_n}\right)^{i} \sigma^{i}. \tag{20}$$

Inserting Eqs. (19) and (20) into (17), and then equating the coefficients of σ^i $(i \geq 1)$ gives

$$\mathbf{A}\mathbf{c}^{\infty} r_p \delta_{1i} + \overline{\mathbf{A}}\overline{\mathbf{a}_{pi}} + \mathbf{A} \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \beta_{nji}^{p} \left(\frac{r_p}{r_n}\right)^{i} = \mathbf{A}_p \mathbf{b}_{pi}, \quad i = 1, 2 \ldots \infty;\ p = 1, 2 \ldots N, \tag{21}$$

where $\delta_{1i} = 1$ for $i = 1$ or $\delta_{1i} = 0$ for $i \neq 1$.

Similarly, we have from Eq. (18) that

$$\mathbf{B}\mathbf{c}^{\infty} r_p \delta_{1i} + \overline{\mathbf{B}}\overline{\mathbf{a}_{pi}} + \mathbf{B} \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \beta_{nji}^{p} \left(\frac{r_p}{r_n}\right)^{i} = \mathbf{B}_p \mathbf{b}_{pi}, \quad i = 1, 2 \ldots \infty;\ p = 1, 2 \ldots N. \tag{22}$$

If all the inclusions have the same radius, i.e., $r_n = a$ $(n = 1, 2 \ldots N)$, Eqs. (21) and (22) become

$$\mathbf{A}\mathbf{c}^{\infty} a \delta_{1i} + \overline{\mathbf{A}}\overline{\mathbf{a}_{pi}} + \mathbf{A} \sum_{\substack{n=1 \\ n \neq p}}^{N} \sum_{j=1}^{\infty} \mathbf{a}_{nj} \beta_{nji}^{p} = \mathbf{A}_p \mathbf{b}_{pi}, \quad i = 1, 2 \ldots \infty;\ p = 1, 2 \ldots N, \tag{23}$$

$$\mathbf{B}\mathbf{c}^{\infty}a\delta_{1i} + \overline{\mathbf{B}}\overline{\mathbf{a}_{pi}} + \mathbf{B}\sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\mathbf{a}_{nj}\beta_{nji}^{p} = \mathbf{B}_p\mathbf{b}_{pi}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N. \tag{24}$$

Using Eq. (2), Eqs. (23) and (24) can be rewritten as

$$\mathbf{b}_{pi} = \mathbf{c}^{\infty}R_p\delta_{1i} + \overline{\mathbf{a}_{pi}} + \sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\beta_{nji}^{p}\mathbf{a}_{nj}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N, \tag{25}$$

$$\mathbf{c}^{\infty}R_p\delta_{1i} - \overline{\mathbf{a}_{pi}} + \sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\beta_{nji}^{p}\mathbf{a}_{nj} = \mathbf{B}^{-1}\mathbf{B}_p\mathbf{b}_{pi}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N. \tag{26}$$

Inserting Eq. (25) into Eq. (26) one can obtain a set of linear equations only containing the unknown coefficients $\mathbf{a}_{pi}$ as

$$(\mathbf{I}+\mathbf{k}_p)\overline{\mathbf{a}_{pi}} - (\mathbf{I}-\mathbf{k}_p)\sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\beta_{nji}^{p}\mathbf{a}_{nj} = (\mathbf{I}-\mathbf{k}_p)\mathbf{c}^{\infty}R_p\delta_{1i}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N, \tag{27}$$

where $\mathbf{k}_p$ is a real matrix defined as

$$\mathbf{k}_p = \mathbf{B}^{-1}\mathbf{B}_p, \quad p = 1,2\ldots N. \tag{28}$$

For the case of N circular holes $\mathbf{k}_p = 0$, and Eq. (27) becomes

$$\overline{\mathbf{a}_{pi}} - \sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\beta_{nji}^{p}\mathbf{a}_{nj} = \mathbf{c}^{\infty}R_p\delta_{1i}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N. \tag{29}$$

For the case of N rigid conductive inclusions $\mathbf{k}_p \to \infty$ and (27) degenerates into

$$\overline{\mathbf{a}_{pi}} + \sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{\infty}\beta_{nji}^{p}\mathbf{a}_{nj} = -\mathbf{c}^{\infty}R_p\delta_{1i}, \quad i = 1,2\ldots\infty;\ p = 1,2\ldots N. \tag{30}$$

In general cases, one may take i and j up to the M-th terms, i.e., letting $i = j = M$ in Eq. (27) results in

$$(\mathbf{I}+\mathbf{k}_p)\overline{\mathbf{a}_{pi}} - (\mathbf{I}-\mathbf{k}_p)\sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{M}\beta_{nji}^{p}\mathbf{a}_{nj} = (\mathbf{I}-\mathbf{k}_p)\mathbf{c}^{\infty}R_p\delta_{1i}, \quad i = 1,2\ldots M;\ p = 1,2\ldots N. \tag{31}$$

Taking the conjugate of Eq. (31) leads to

$$(\mathbf{I}+\mathbf{k}_p)\mathbf{a}_{pi} - (\mathbf{I}-\mathbf{k}_p)\sum_{\substack{n=1\\n\neq p}}^{N}\sum_{j=1}^{M}\overline{\beta_{nji}^{p}}\overline{\mathbf{a}_{nj}} = (\mathbf{I}-\mathbf{k}_p)\overline{\mathbf{c}^{\infty}}R_p\delta_{1i}, \quad i = 1,2\ldots M;\ p = 1,2\ldots N. \tag{32}$$

Equations (31) and (32) constitute a set of $2N \times M$ linear equations concerning $2N \times M$ unknown coefficients $\mathbf{a}_{pi}$ and $\overline{\mathbf{a}_{pi}}$ $(p = 1,2,\ \ldots N;\ i = M)$. After $\mathbf{a}_{pi}$ is determined from Eqs. (31) and (32),

$\mathbf{b}_{pi}$ can be written out from Eq. (25). With $\mathbf{a}_{pi}$ and $\mathbf{b}_{pi}$, all the complex potentials both in the matrix and inclusions become unknown, and then all the field variables can be easily obtained by using Eq. (3).

Especially, for a single inclusion with a radius R and a centre location at $z_{p0} = 0$, Eqs. (25) and (27) reduce to

$$\mathbf{b}_{1p} = \mathbf{c}^{\infty}R + \overline{\mathbf{a}_{1p}}, \quad (\mathbf{I} + \mathbf{k}_p)\mathbf{a}_{1p} = (\mathbf{I} - \mathbf{k}_p)\overline{\mathbf{c}^{\infty}}R. \tag{33}$$

From Eqs. (33) we have

$$\mathbf{a}_{1p} = (\mathbf{B} + \mathbf{B}_p)^{-1}(\mathbf{B} - \mathbf{B}_p)\overline{\mathbf{c}^{\infty}}R, \quad \mathbf{b}_{1p} = \mathbf{M}\big(\boldsymbol{\sigma}_2^{\infty} + i\boldsymbol{\sigma}_1^{\infty}\big), \tag{34}$$

where

$$\mathbf{M} = 2\mathbf{B}_p(\mathbf{B} + \mathbf{B}_p)^{-1}, \quad \boldsymbol{\sigma}_2^{\infty} = \Big(\sigma_{zy}^{\infty}, D_y^{\infty}\Big)^T, \quad \boldsymbol{\sigma}_1^{\infty} = \big(\sigma_{zx}^{\infty}, D_x^{\infty}\big)^T, \quad \mathbf{F}(z) = d\mathbf{f}(z)dz.$$

Using Eqs. (4) and (33) we can establish the relationship between the field variables inside the inclusion and the applied ones as

$$\boldsymbol{\sigma}_2^{(p)} + i\boldsymbol{\sigma}_1^{(p)} = \mathbf{M}\big(\boldsymbol{\sigma}_2^{\infty} + i\boldsymbol{\sigma}_1^{\infty}\big), \tag{35}$$

that is

$$\begin{aligned} \sigma_{zy}^{(p)} &= M_{11}\sigma_{zy}^{\infty} + M_{12}D_y^{\infty}, \\ \sigma_{zx}^{(p)} &= M_{11}\sigma_{zx}^{\infty} + M_{12}D_x^{\infty}, \\ D_y^{(p)} &= M_{21}\sigma_{zy}^{\infty} + M_{22}D_y^{\infty}, \\ D_x^{(p)} &= M_{21}\sigma_{zx}^{\infty} + M_{22}D_x^{\infty}. \end{aligned} \tag{36}$$

Similarly, the electric field inside can be expressed as

$$E_y^{(p)} = -W_{21}\sigma_{zy}^{\infty} - W_{22}D_y^{\infty}, \quad E_x^{(p)} = -W_{21}\sigma_{zx}^{\infty} - W_{22}D_x^{\infty}, \tag{37}$$

where $\mathbf{W} = 2i(\mathbf{B}_p + \mathbf{B})^{-1}$.

For this case, the field variable in the matrix can be expressed as

$$\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2^{\infty} + i\boldsymbol{\sigma}_1^{\infty} - \mathbf{B}(\mathbf{B} + \mathbf{B}_p)^{-1}(\mathbf{B} - \mathbf{B}_p)\mathbf{B}^{-1}\big(\boldsymbol{\sigma}_2^{\infty} - i\boldsymbol{\sigma}_1^{\infty}\big)\frac{R^2}{z^2}. \tag{38}$$

4 Numerical examples

Choose two model materials with the following constants:

$$C_{44}^I = 3.52 \times 10^{10}\mathrm{Nm}^{-2}, \quad e_{15}^I = 17.00\,\mathrm{Cm}^{-2}, \quad \varepsilon_{11}^I = 1.51 \times 10^{-8}\mathrm{CVm}^{-1}$$

for all the inclusions, and

$$C_{44}^M = 2.56 \times 10^{10}\mathrm{Nm}^{-2}, \quad e_{15}^M = 13.44\,\mathrm{Cm}^{-2}, \quad \varepsilon_{11}^M = 6.00 \times 10^{-9}\mathrm{CVm}^{-1}$$

for the matrix.

In Fig. 2, the distribution of electric displacement along the interface at the side of the matrix with two same inclusions under combined mechanical load and electric field is shown. It is found that as the distance between two inclusions increases, the value of the electric displacement is close to the exact one, which is calculated from Eq. (38). In Figs. 3 and 4, stress and electric displacement are

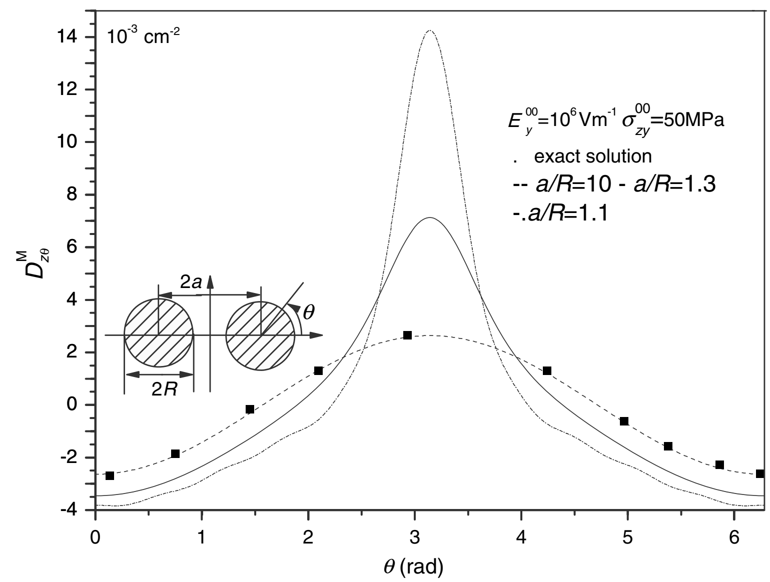

Fig. 2. Distribution of electric displacement along the interface on the side of the matrix for two inclusions with equal radius

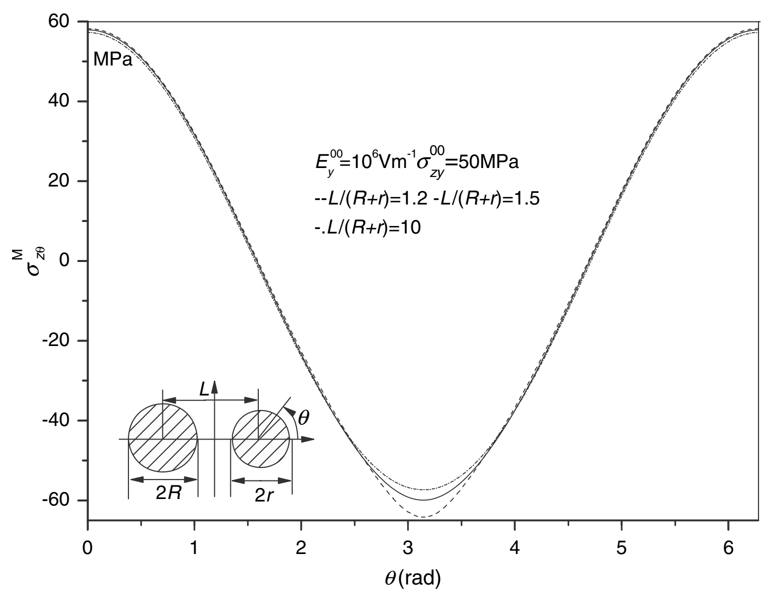

Fig. 3. Distribution of stress along the interface on the side of the matrix for two inclusions with unequal radius

plotted in the matrix with two inclusions with unequal radii, and it is shown that the applied electric load has less influence on the variation of stress concentration around the inclusions, and the size of inclusions has no remarkable effect on the distribution form of the field variables. For the case of three equal inclusions with triangle array, the maximum values of stress and electric displacement in the matrix have a small deviation compared with the case of two inclusions, as shown in Figs. 5

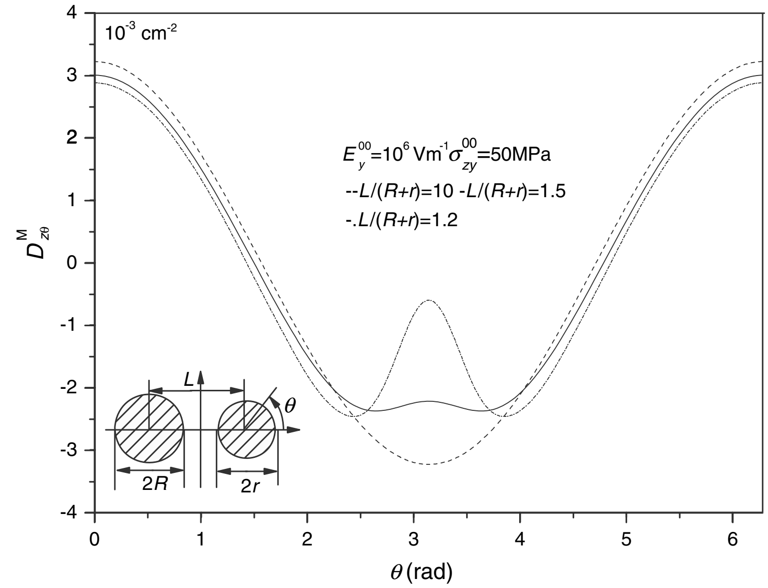

Fig. 4. Distribution of electric displacement along the interface on the side of the matrix for two inclusions with unequal radius

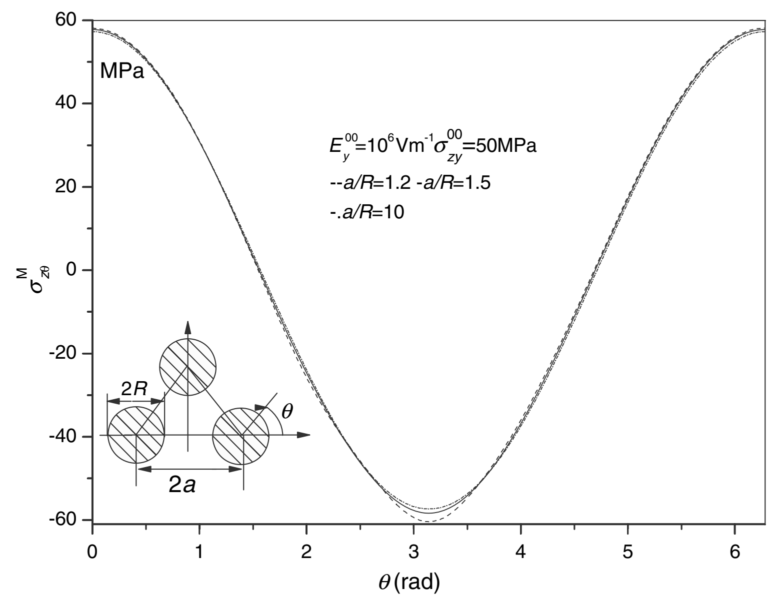

Fig. 5. Distribution of stress along the interface on the side of the matrix for three equal inclusions with triangle array

and 6. However, as the number of inclusions increases, e.g., for the case of four inclusions located in a square array as shown in Figs. 7 and 8, the changes of stress and displacement around the inclusions become aggravate.

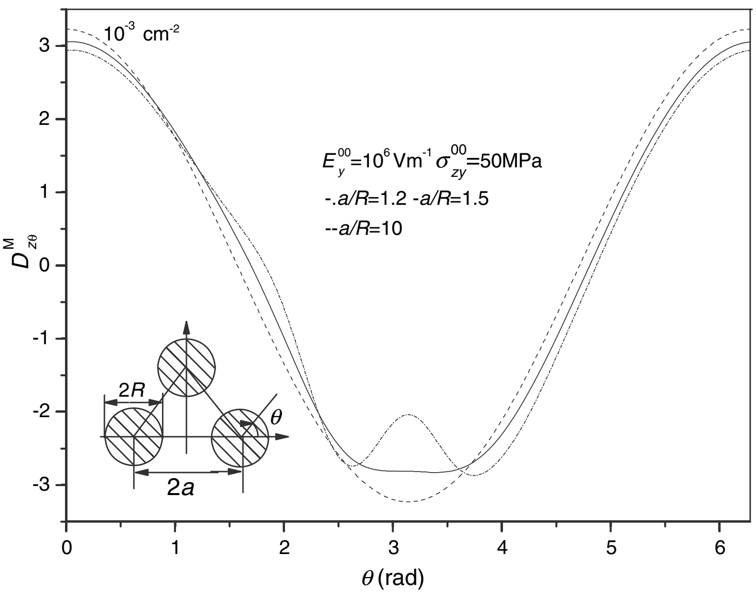

Fig. 6. Distribution of electric displacement along the interface on the side of the matrix for three equal inclusions with triangle array

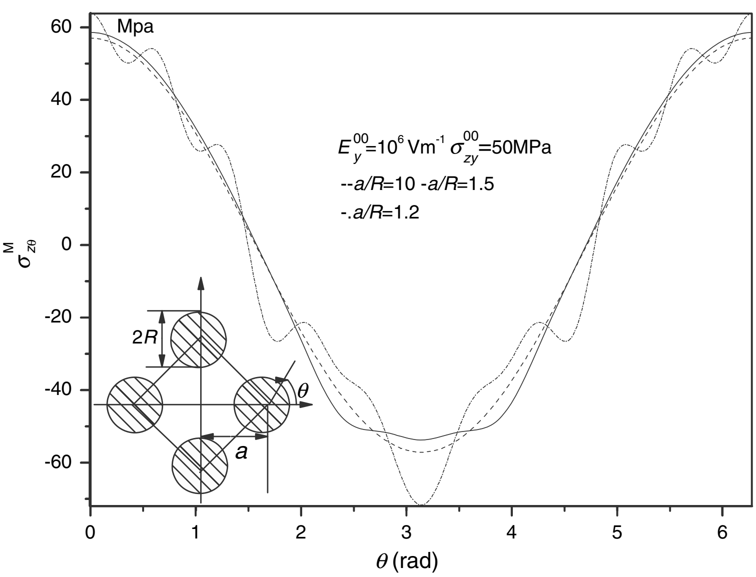

Fig. 7. Distribution of stress along the interface on the side of the matrix for four equal inclusions with square array

5 Conclusions

This work presents a simple method to study a multi-inclusion-matrix system based on power series expansion of complex potentials. The general solutions of complex potentials are derived when the matrix is uniformly loaded by an in-plane electric field combined with an anti-plane shear stress at infinity. Numerical calculations are conducted to discuss the influences of inclusion array, inclusion size and inclusion properties on couple fields in the matrix and inclusions. It is found that the size of

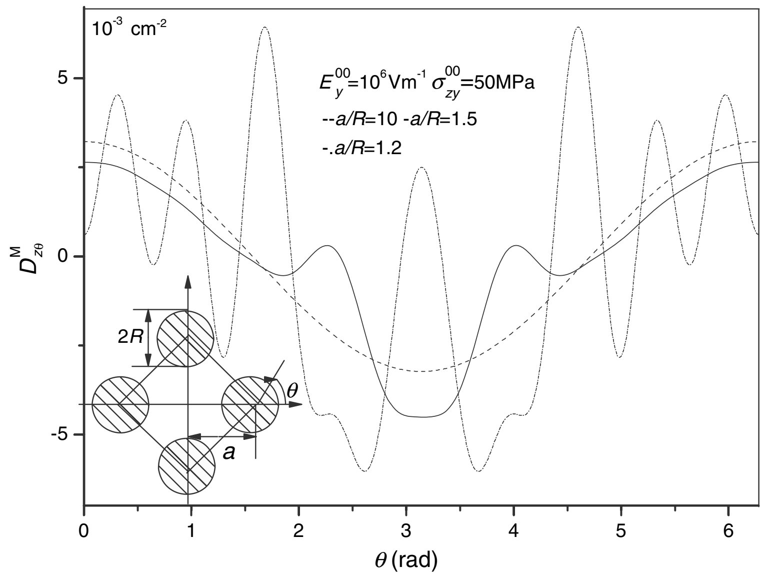

Fig. 8. Distribution of electric displacement along the interface on the side of the matrix for four equal inclusions with square array

inclusions has no remarkable effect on the distribution of field variables, but as the number of inclusions increases, the change of all field variables around the inclusions becomes very complicated, especially when the inclusions are close to each other. Finally, it should be noted that the proposed approach for solving a multiple-inclusion system can be extended to the case of generalized plane strain in an infinite region with N circular inclusions or holes. For the case of multiple elliptic inclusions or holes, the related problem can also be solved based on series expansion of complex potentials in form of Faber series. The interested readers may refer to the work of Kosmodamianskii [20].

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