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Shear deformation effect in nonlinear analysis of spatial composite beams subjected to variable axial loading by bem

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Summary. In this paper a boundary element method is developed for the nonlinear analysis of composite beams of arbitrary doubly symmetric constant cross section, taking into account the shear deformation effect. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. The beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, the axial displacement and to two stress functions and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. Numerical examples with great practical interest are worked out to illustrate the efficiency and the range of applications of the developed method. The influence of both the shear deformation effect and the variableness of the axial loading are remarkable.

1 Introduction

An important consideration in the analysis of the components of plane and space frames is the influence of the action of axial, lateral forces and end moments on the deformed shape of a beam. Lateral loads and end moments generate deflection that is further amplified by axial compression loading. On the other hand, composite beams have been increasingly used in recent years as structural members due to their high strength/stiffness properties for light weight materials. The extensive use of these structural elements requires an accurate analysis which is achieved taking into account that the axial force is nonlinearly coupled with the transverse deflections, avoiding in this way the inaccuracies arising from a linearized second-order analysis.

Over the past twenty years, many researchers have developed and validated various methods of performing a linearized second-order analyses on structures. Early efforts led to methods based on accounting for the aforementioned effect by using magnification factors applied to the results obtained from first-order analyses [1]–[3]. An example of such a method is the "B_1 and B_2 factor approach" provided in the AISC-LRFD specification [4]. Since the modifications used in this method are only applied to the moments of the columns and not of the beams, the results obtained from this method are often unsatisfactory especially for cases involving moderate to large deformations [2]. Consequently, due to the demand of more rigorous and accurate second-order analysis of structural components several research papers have been published including a non-linear incremental stiffness method [5], closed-form stiffness methods [6], [7], the analysis of non-linear effects by treating every element as a "beam-column" element [8], a design method for space frames using stability functions to capture second-order effects associated with P-δ and P-Δ effects [9], uniform formulae restricted to a single bar of a skeletal structure and to only a few loadings [10], the finite element method using linear and cubic shape functions [11] and a 3-D second-order plastic-hinge analysis accounting for material and geometric non-linearities [12], [13]. Recently, Katsikadelis and Tsiatas [14] presented a BEM-based method for the nonlinear analysis of homogeneous beams with variable stiffness. In all these studies shear deformation effect is ignored.

Moreover, Kim et al. presented a practical second-order inelastic static [15] and dynamic [16] analysis for 3-D steel frames, Machado and Cortinez [17] a geometrically non-linear beam theory for the lateral buckling problem of bisymmetric thin-walled composite simply supported or cantilever beams, taking into account shear deformation effects. Nevertheless, in all of the aforementioned research efforts the axial loading of the structural components is assumed to be constant. Finally, the boundary element method taking into account shear deformation effects has been employed only for first-order analyses [18]–[20].

In this paper a boundary element method is developed for the nonlinear analysis of composite beams of arbitrary doubly symmetric constant cross section, taking into account the shear deformation effect. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. The beam is subjected to an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, the axial displacement and to two stress functions and solved using the Analog Equation Method [21], a BEM based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows:

(i) The beam is subjected in an arbitrarily concentrated or distributed variable axial loading.
(ii) The beam is supported by the most general boundary conditions including elastic support or restrain.
(iii) The analysis is not restricted to a linearized second-order one but is a nonlinear one arising from the fact that the axial force is nonlinearly coupled with the transverse deflections (additional terms are taken into account).
(iv) Shear deformation effect is taken into account.
(v) The boundary conditions at the interfaces between different material regions have been considered.

(vi) The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko's [22] and Cowper's [23] definitions, for which several authors [24], [25] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [26], [27], for which these factors take negative values.
(vii) The effect of the material's Poisson ratio $\acute{\imath}$ is taken into account.
(viii) The proposed method employs a pure BEM approach (requiring only boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples with great practical interest are worked out to illustrate the efficiency and the range of applications of the developed method. The influence of both the shear deformation effect and the variableness of the axial loading are remarkable.

2 Statement of the problem

Consider a prismatic beam of length l with a doubly symmetric composite cross section of arbitrary shape, consisting of materials in contact, each of which can surround a finite number of inclusions, with modulus of elasticity E_j, shear modulus G_j and common Poisson's ratio v, occupying the regions Ω_j $(j = 1, 2, \ldots, K)$ of the y, z plane (Fig. 1). The materials of these regions are firmly bonded together and are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions Ω_j be denoted by Γ_j $(j = 1, 2, \ldots, K.)$. These boundary curves are piecewise smooth, i.e., they may have a finite number of corners. Without loss of generality, it may be assumed that the x-axis of the beam principal coordinate system is the line joining the centroids of the cross sections. The beam is subjected to an arbitrarily distributed axial loading p_x and to torsionless bending arising from arbitrarily distributed transverse loading p_y, p_z and bending moments m_y, m_z along the y- and z-axes, respectively (Fig. 1a).

According to the nonlinear theory of beams for moderate large deflections $((\partial u/\partial x)^2 \ll \partial u/\partial x,\ (\partial u/\partial x)(\partial u/\partial y) \ll (\partial v/\partial x) + (\partial u/\partial y), (\partial u/\partial x)(\partial u/\partial z) \ll (\partial w/\partial x) + (\partial u/\partial z))$ and assuming small rotations [28], [29], the angles of rotation of the cross-section in the x-z and x-y planes of the beam subjected to the aforementioned loading and taking into account shear deformation effect satisfy the following relations:

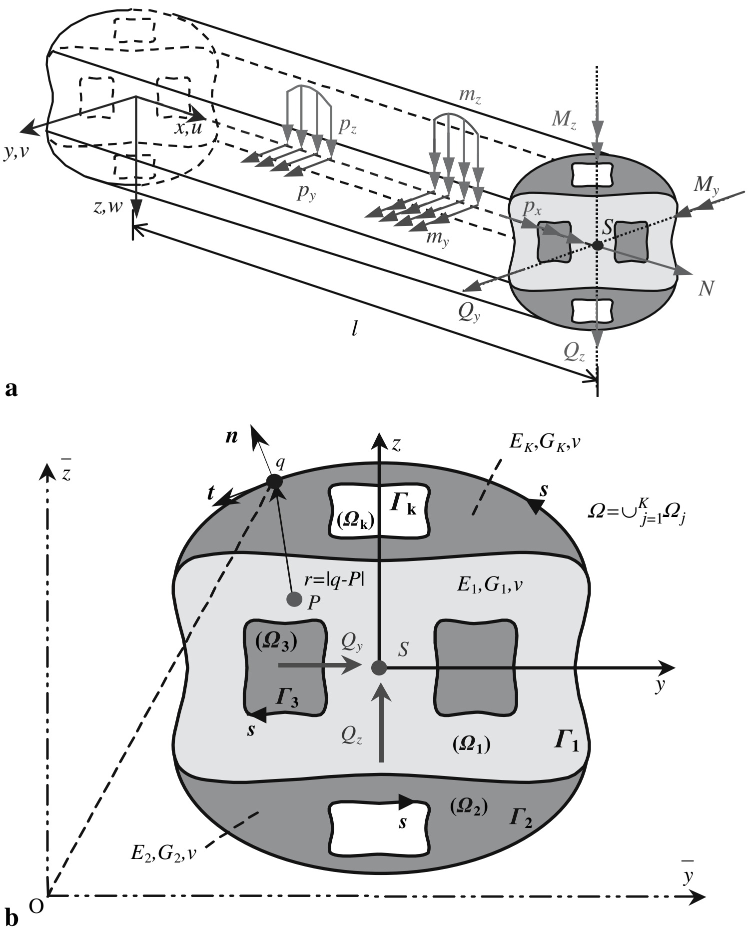
$$\cos\omega_y \approx 1, \quad \cos\omega_z \approx 1, \tag{1.1,2}$$

$$\sin\omega_y \approx \omega_y = -\frac{dw}{dx} = \theta_y - \gamma_z, \quad \sin\omega_z \approx \omega_z = -\frac{dv}{dx} = -\theta_z - \gamma_y, \tag{1.3,4}$$

where $w = w(x)$, $v = v(x)$ are the beam transverse displacements with respect to the z- and y-axes, respectively, γ_y, γ_z are the additional angles of rotation of the cross-section due to shear deformation (Fig. 2a) [28], [29], while the corresponding curvatures are given as

$$k_y = \frac{d\theta_y}{dx} = -\frac{d^2w}{dx^2} + \frac{d\gamma_z}{dx} = -\frac{d^2w}{dx^2} - \frac{p_z}{G_1 A_z}, \tag{2.1}$$

$$k_z = \frac{d\theta_z}{dx} = \frac{d^2v}{dx^2} - \frac{d\gamma_y}{dx} = \frac{d^2v}{dx^2} + \frac{p_y}{G_1 A_y}, \tag{2.2}$$



a

b

Fig. 1. Prismatic beam in torsionless bending (**a**) with an arbitrary composite doubly symmetric cross-section occupying the two dimensional region Ω (**b**)

where the first material is considered as reference material, G_1A_y, G_1A_z are shear rigidities of the Timoshenko's beam theory and

$$A_z = \kappa_z A = \frac{1}{a_z}A, \qquad A_y = \kappa_y A = \frac{1}{a_y}A \tag{3.1,2}$$

are the shear areas with respect to the y- and z-axes, respectively, with κ_y, κ_z the shear correction factors, a_y, a_z the shear deformation coefficients and A the composite cross section area given as

$$A = \sum_{j=1}^{K} \frac{G_j}{G_1} \int_{\Omega_j} d\Omega_j. \tag{4}$$

It is worth here noting that the reduction of Eq. (4) using the shear modulus G_1 of the first material could be achieved using any other material, considering it as reference material.

Referring to Fig. 2b [28], [29], the stress resultants R_x, R_z acting in the x- and z-directions, respectively, are related to the axial N and shear Q_z forces as

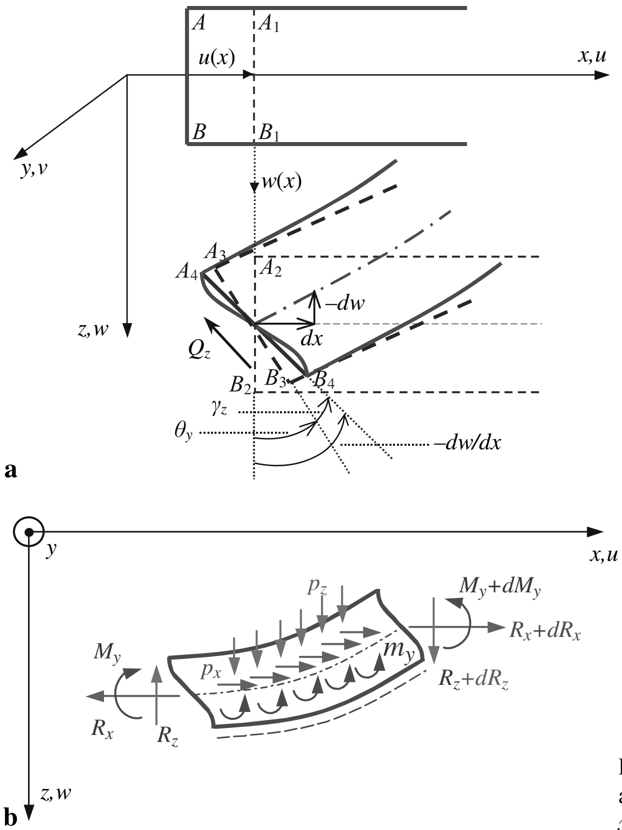


Fig. 2. Displacements (**a**) and forces (**b**) acting on the deformed element in the xz plane

$$R_x = N \cos \omega_y + Q_z \sin \omega_y, \tag{5.1}$$

$$R_z = Q_z \cos \omega_y - N \sin \omega_y, \tag{5.2}$$

which by virtue of Eqs. (1) become

$$R_x = N - Q_z \frac{dw}{dx}, \tag{6.1}$$

$$R_z = Q_z + N \frac{dw}{dx}. \tag{6.2}$$

The second term in the right hand side of Eq. (6.1), expresses the influence of the shear force Q_z on the horizontal stress resultant R_x. However, this term can be neglected since $Q_z w'$ is much smaller than N [28], [29] and thus Eq. (6.1) can be written as

$$R_x \approx N. \tag{7}$$

Similarly, the stress resultant R_y acting in the y-direction is related to the axial N and shear Q_y forces as

$$R_y = Q_y + N \frac{dv}{dx}. \tag{8}$$

The governing equation of the beam transverse displacement $w = w(x)$ will be derived by considering the equilibrium of the deformed element in the x-z plane. Thus, referring to Fig. 2b we obtain

$$\frac{dR_x}{dx} + p_x = 0, \tag{9.1}$$

$$\frac{dR_z}{dx} + p_z = 0, \tag{9.2}$$

$$\frac{dM_y}{dx} - Q_z + m_y = 0. \tag{9.3}$$

Substituting Eqs. (7), (6.2) into Eqs. (9.1,2), using Eq. (9.3) to eliminate Q_z, employing the well-known relation

$$M_y = E_1 I_y k_y, \tag{10}$$

where the moment of inertia of the composite cross section with respect to the y-axis is given as

$$I_y = \sum_{j=1}^{K} \frac{E_j}{E_1} \int_{\Omega_j} z^2 d\mathbf{\Omega}_j, \tag{11}$$

and utilizing Eq. (2.1) we obtain the expressions of the angle of rotation due to bending θ_y and the stress resultants M_y, R_z as

$$\theta_y = -\frac{dw}{dx} + \frac{1}{G_1 A_z}\left(-E_1 I_y \frac{d^3 w}{dx^3} - \frac{E_1 I_y}{G_1 A_z}\left(\frac{dp_z}{dx} + N\frac{d^3 w}{dx^3} - 2p_x \frac{d^2 w}{dx^2} - \frac{dp_x}{dx}\frac{dw}{dx}\right) + m_y\right), \tag{12}$$

$$M_y = -E_1 I_y \frac{d^2 w}{dx^2} - \frac{E_1 I_y}{G_1 A_z}\left(p_z + \frac{dN}{dx}\frac{dw}{dx} + N\frac{d^2 w}{dx^2}\right), \tag{13.1}$$

$$R_z = -E_1 I_y \frac{d^3 w}{dx^3} - \frac{E_1 I_y}{G_1 A_z}\left(\frac{dp_z}{dx} + N\frac{d^3 w}{dx^3} - 2p_x \frac{d^2 w}{dx^2} - \frac{dp_x}{dx}\frac{dw}{dx}\right) + m_y + N\frac{dw}{dx}. \tag{13.2}$$

and the governing differential equation as

$$E_1 I_y \left(1 + \frac{N}{G_1 A_z}\right)\frac{d^4 w}{dx^4} = p_z - p_x \frac{dw}{dx} + N\frac{d^2 w}{dx^2} + \frac{dm_y}{dx} - \frac{E_1 I_y}{G_1 A_z}\left(\frac{d^2 p_z}{dx^2} - 3p_x \frac{d^3 w}{dx^3} - 3\frac{dp_x}{dx}\frac{d^2 w}{dx^2} - \frac{d^2 p_x}{dx^2}\frac{dw}{dx}\right). \tag{14}$$

Moreover, the pertinent boundary conditions of the problem are given as

$$\alpha_1^z w(x) + \alpha_2^z R_z(x) = \alpha_3^z, \tag{15.1}$$

$$\beta_1^z \theta_y(x) + \beta_2^z M_y(x) = \beta_3^z \quad \text{at the beam ends } x = 0, l, \tag{15.2}$$

where α_i^z, β_i^z $(i = 1, 2, 3)$ are given constants, while the angle of rotation θ_y and the stress resultants M_y, R_z are given as

$$\theta_y = -\frac{E_1 I_y}{G_1 A_z}\left(1 + \frac{N}{G_1 A_z}\right)\frac{d^3 w}{dx^3} - \frac{dw}{dx}, \tag{16.1}$$

$$M_y = -E_1 I_y \left(1 + \frac{N}{G_1 A_z}\right)\frac{d^2 w}{dx^2}, \tag{16.2}$$

$$R_z = -E_1 I_y \left(1 + \frac{N}{G_1 A_z}\right)\frac{d^3 w}{dx^3} + N\frac{dw}{dx} \quad \text{at the beam ends } x = 0, l, \tag{16.3}$$

Equations (15) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restrain. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately the functions α_i^z and β_i^z (e.g., for a clamped edge it is $\alpha_1^z = \beta_1^z = 1,\ \alpha_2^z = \alpha_3^z = \beta_2^z = \beta_3^z = 0$).

Similarly, considering the beam in the x-y plane we obtain the boundary value problem of the beam transverse displacement $v = v(x)$ as

$$E_1 I_z\left(1+\frac{N}{G_1 A_y}\right)\frac{d^4 v}{dx^4} = p_y - p_x\frac{dv}{dx} + N\frac{d^2 v}{dx^2} - \frac{dm_z}{dx}$$

$$-\frac{E_1 I_z}{G_1 A_y}\left(\frac{d^2 p_y}{dx^2} - 3p_x\frac{d^3 v}{dx^3} - 3\frac{dp_x}{dx}\frac{d^2 v}{dx^2} - \frac{d^2 p_x}{dx^2}\frac{dv}{dx}\right) \quad \text{inside the beam,} \tag{17}$$

$$\alpha_1^y v(x) + \alpha_2^y R_y(x) = \alpha_3^y, \tag{18.1}$$

$$\beta_1^y \frac{dv(x)}{dx} + \beta_2^y M_z(x) = \beta_3^y \quad \text{at the beam ends } x = 0, l, \tag{18.2}$$

where $\alpha_i^y,\ \beta_i^y\ (i = 1, 2, 3)$ are given constants and the expressions of the angle of rotation θ_z and the stress resultants M_z, R_y inside the beam are given as

$$\theta_z = \frac{dv}{dx} - \frac{1}{G_1 A_y}\left(-E_1 I_z\frac{d^3 v}{dx^3} - \frac{E_1 I_z}{G_1 A_y}\left(\frac{dp_y}{dx} + N\frac{d^3 v}{dx^3} - 2p_x\frac{d^2 v}{dx^2} - \frac{dp_x}{dx}\frac{dv}{dx}\right) - m_y\right), \tag{19.1}$$

$$M_z = E_1 I_z\frac{d^2 v}{dx^2} + \frac{E_1 I_z}{G_1 A_y}\left(p_y + \frac{dN}{dx}\frac{dv}{dx} + N\frac{d^2 v}{dx^2}\right), \tag{19.2}$$

$$R_y = -E_1 I_z\frac{d^3 v}{dx^3} - \frac{E_1 I_z}{G_1 A_y}\left(\frac{dp_y}{dx} + N\frac{d^3 v}{dx^3} - 2p_x\frac{d^2 v}{dx^2} - \frac{dp_x}{dx}\frac{dv}{dx}\right) - m_y + N\frac{dv}{dx}, \tag{19.3}$$

and the moment of inertia of the composite cross section with respect to the z-axis is given as

$$I_z = \sum_{j=1}^{K}\frac{E_j}{E_1}\int_{\Omega_j} y^2 d\Omega_j. \tag{20}$$

In both of the aforementioned boundary value problems the axial force N inside the beam or at its boundary is given from the following relation [28], [29]:

$$N = E_1 A\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right], \tag{21}$$

where $u = u(x)$ is the bar axial displacement, which can be evaluated from the solution of the following boundary value problem:

$$E_1 A\left[\frac{d^2 u}{dx^2} + \frac{d^2 w}{dx^2}\frac{dw}{dx} + \frac{d^2 v}{dx^2}\frac{dv}{dx}\right] = -p_x \qquad \text{inside the beam,} \tag{22}$$

$$c_1 u(x) + c_2 N(x) = c_3 \qquad \text{at the beam ends } x = 0, l, \tag{23}$$

where $c_i\ (i = 1, 2, 3)$ are given constants.

The solution of the boundary value problems prescribed from Eqs. (14), (15.1,2) and (17), (18.1,2) presumes the evaluation of the shear deformation coefficients a_z, a_y corresponding to the principal centroidal system of axes Cyz. These coefficients are established equating the approximate formula of the shear strain energy per unit length [25],

$$U_{\text{appr.}} = \frac{a_y Q_y^2}{2AG_1} + \frac{a_z Q_z^2}{2AG_1}, \tag{24}$$

with the exact one given from

$$U_{\text{exact}} = \sum_{j=1}^{K} \frac{E_1}{E_j} \int_{\Omega_j} \frac{(\tau_{xz})_j^2 + (\tau_{xy})_j^2}{2G_1} d\Omega_j, \tag{25}$$

and are obtained as [30]

$$a_y = \frac{1}{\kappa_y} = \frac{A}{E_1 \Delta^2} \sum_{j=1}^{K} \int_{\Omega_j} E_j \Big((\nabla\Theta)_j - \boldsymbol{e} \Big) \cdot \Big((\nabla\Theta)_j - \boldsymbol{e} \Big) d\Omega_j, \tag{26.1}$$

$$a_z = \frac{1}{\kappa_z} = \frac{A}{E_1 \Delta^2} \sum_{j=1}^{K} \int_{\Omega_j} E_j \Big((\nabla\Phi)_j - \boldsymbol{d} \Big) \cdot \Big((\nabla\Phi)_j - \boldsymbol{d} \Big) d\Omega_j, \tag{26.2}$$

where $(\tau_{xz})_j, (\tau_{xy})_j$ are the transverse (direct) shear stress components, $(\nabla) \equiv i_{\boldsymbol{y}}(\partial/\partial y) + i_z(\partial/\partial z)$ is a symbolic vector with i_y, $i_{\boldsymbol{z}}$ the unit vectors along the y- and z-axes, respectively, Δ is given from

$$\Delta = 2(1+v) I_y I_z, \tag{27}$$

v is the Poisson ratio of the cross section material, $\boldsymbol{e}$ and $\boldsymbol{d}$ are vectors defined as

$$\boldsymbol{e} = e_y i_y + e_z i_z = \left(v I_y \frac{y^2 - z^2}{2} \right) i_y + (v I_y yz) i_z, \tag{28.1}$$

$$\boldsymbol{d} = d_y i_y + d_z i_z = (v I_z yz) i_y - \left(v I_z \frac{y^2 - z^2}{2} \right) i_z, \tag{28.2}$$

and $\Theta(y,z)$, $\Phi(y,z)$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [30]:

$$\nabla^2 \Theta_j = -2 I_y y \quad \text{in } \Omega_j \ (j = 1, 2, \ldots, K), \tag{29.1}$$

$$E_j \left(\frac{\partial \Theta}{\partial n} \right)_j - E_i \left(\frac{\partial \Theta}{\partial n} \right)_i = (E_j - E_i) \boldsymbol{n} \cdot \boldsymbol{e} \ \text{on} \ \Gamma_j \quad (j = 1, 2, \ldots, K), \tag{29.2}$$

$$\nabla^2 \Phi_j = -2 I_z z \qquad \text{in } \Omega_j \quad (j = 1, 2, \ldots, K), \tag{30.1}$$

$$E_j \left(\frac{\partial \Phi}{\partial n} \right)_j - E_i \left(\frac{\partial \Phi}{\partial n} \right)_i = (E_j - E_i) \boldsymbol{n} \cdot \boldsymbol{d} \qquad \text{on } \Gamma_j \quad (j = 1, 2, \ldots, K), \tag{30.2}$$

where E_i is the modulus of elasticity of the Ω_i region at the common part of the boundaries of Ω_j and Ω_i regions, or $E_i = 0$ at the free part of the boundary of Ω_j region, while $(\partial/\partial n)_j \equiv n_y(\partial/\partial y)_j + n_z(\partial/\partial z)_j$ denotes the directional derivative normal to the boundary Γ_j. The vector $\boldsymbol{n}$ normal to the boundary Γ_j is positive if it points to the exterior of the Ω_j region. It is worth here noting that the normal derivatives across the interior boundaries vary discontinuously. In the case of negligible shear deformations $a_z = a_y = 0$. The boundary conditions

(29.1) and (30.2) have been derived from the physical consideration that the traction vector in the direction of the normal vector $\boldsymbol{n}$ on the interfaces separating the j and i different materials are equal in magnitude and opposite in direction, while it vanishes on the free surface of the beam.

3 Integral representations – Numerical solution

According to the precedent statement of the problem, the nonlinear analysis of a beam including shear deformation reduces in establishing the transverse displacements $w = w(x)$, $v = v(x)$ having continuous derivatives up to the fourth order with respect to x, the axial displacement $u = u(x)$ having continuous derivatives up to the second order with respect to x and the stress functions $(\Phi(y,z))_j$ and $(\Theta(y,z))_j$ having continuous partial derivatives up to the second order with respect to y and z.

3 1 For the transverse displacements w, v

The numerical solution of the boundary value problems described by Eqs. (14), (15.1,2) and (17), (18.1,2) is similar. For this reason, in the following we will analyze the solution of the problem of Eqs. (14), (15.1,2) noting any alteration or addition for the problem of Eqs. (17), (18.1,2). Equation (14) is solved using the Analog Equation Method [21]. This method has been developed for the beam equation including axial forces by Katsikadelis and Tsiatas [31]. However, another formulation is presented in this investigation.

Let w be the sought solution of the boundary value problem described by Eqs. (14) and (15.1,2). Differentiating this function four times yields

$$\frac{d^4 w}{dx^4} = q_z(x). \tag{31}$$

Equation (31) indicates that the solution of the original problem can be obtained as the deflection of a beam with unit flexural rigidity and infinite shear rigidity subjected to a flexural fictitious load $q_z(x)$ under the same boundary conditions. The fictitious load is unknown. However, it can be established using BEM as described in the following.

The solution of Eq. (31) is given in integral form as

$$w(x) = \int_0^l q_z w^* dx - \left[w^* \frac{d^3 w}{dx^3} - \frac{dw^*}{dx}\frac{d^2 w}{dx^2} + \frac{d^2 w^*}{dx^2}\frac{dw}{dx} - \frac{d^3 w^*}{dx^3} w \right]_0^l, \tag{32}$$

where w^* is the fundamental solution, which is given as

$$w^* = \frac{1}{12} l^3 \left(2 + |\rho|^3 - 3|\rho|^2 \right) \tag{33}$$

with $\rho = r/l$, $r = x - \zeta$, x, ζ points of the beam, which is a particular singular solution of the equation

$$\frac{d^4 w^*}{dx^4} = \delta(x, \xi). \tag{34}$$

Employing Eq. (33) the integral representation (32) can be written as

$$w(x)=\int_0^l q_z\Lambda_4(r)dx-\left[\Lambda_4(r)\frac{d^3w}{dx^3}+\Lambda_3(r)\frac{d^2w}{dx^2}+\Lambda_2(r)\frac{dw}{dx}+\Lambda_1(r)w\right]_0^l, \tag{35}$$

where the kernels $\Lambda_i(r)$, $(i=1,2,3,4)$, are given as

$$\Lambda_1(r)=-\frac{1}{2}\operatorname{sgn}\rho \tag{36.1}$$

$$\Lambda_2(r)=-\frac{1}{2}l(1-|\rho|), \tag{36.2}$$

$$\Lambda_3(r)=-\frac{1}{4}l^2|\rho|(|\rho|-2)\operatorname{sgn}\rho, \tag{36.3}$$

$$\Lambda_4(r)=\frac{1}{12}l^3\left(2+|\rho|^3-3|\rho|^2\right). \tag{36.4}$$

Notice that in Eq. (35) for the line integral it is $r=x-\xi$, x,ξ points inside the beam, whereas for the remaining terms $r=x-\zeta$, x inside the beam, ζ at the beam ends $0,l$.

Differentiating Eq. (35) results in the integral representations of the derivatives of the deflection w as

$$\frac{dw(x)}{dx}=\int_0^l q_z\Lambda_3(r)dx-\left[\Lambda_3(r)\frac{d^3w}{dx^3}+\Lambda_2(r)\frac{d^2w}{dx^2}+\Lambda_1(r)\frac{dw}{dx}\right]_0^l, \tag{37.1}$$

$$\frac{d^2w(x)}{dx^2}=\int_0^l q_z\Lambda_2(r)dx-\left[\Lambda_2(r)\frac{d^3w}{dx^3}+\Lambda_1(r)\frac{d^2w}{dx^2}\right]_0^l, \tag{37.2}$$

$$\frac{d^3w(x)}{dx^3}=\int_0^l q_z\Lambda_1(r)dx-\left[\Lambda_1(r)\frac{d^3w}{dx^3}\right]_0^l. \tag{37.3}$$

The integral representations (35), (37.1) written for the beam ends $0,l$ together with the boundary conditions (15.1,2) can be employed to express the unknown boundary quantities w, w', w'' and w''' in terms of q_z. This is accomplished numerically. If L is the number of the nodal points along the beam axis, this procedure yields the following set of linear equations:

$$\begin{bmatrix}[E_{11}] & [E_{12}] & [E_{13}] & [E_{14}]\\ [0] & [E_{22}] & [E_{23}] & [0]\\ [E_{31}] & [E_{32}] & [E_{33}] & [E_{34}]\\ [0] & [E_{42}] & [E_{43}] & [E_{44}]\end{bmatrix}\begin{Bmatrix}\{w\}\\ \{w'\}\\ \{w''\}\\ \{w'''\}\end{Bmatrix}=\begin{Bmatrix}\{a_3^z\}\\ \{\beta_3^z\}\\ \{0\}\\ \{0\}\end{Bmatrix}+\begin{bmatrix}[0]\\ [0]\\ [F_3]\\ [F_4]\end{bmatrix}\{q_z\} \tag{38}$$

in which $[E_{22}]$, $[E_{23}]$, $[E_{1i}]$, $(i=1,2,3,4)$, are 2×2 matrices including the nodal values of the functions a_1^z, a_2^z, β_1^z, β_2^z of Eqs. (15.1,2) and $[E_{ij}]$, $(i=3,4,\ j=1,2,3,4)$ are square 2×2 known coefficient matrices resulting from the values of the kernels Λ_i at the beam ends; $\{a_3^z\}$, $\{\beta_3^z\}$ are 2×1 known column matrices including the boundary values of the functions a_3^z, β_3^z in Eqs. (15.1,2) and $[F_i]$ $(i=3,4)$ are $2\times L$ rectangular known matrices originating from the integration of the kernels on the axis of the beam. Finally, $\{w\}$, $\{w'\}$, $\{w''\}$ and $\{w'''\}$ are vectors including the two unknown nodal values of the respective boundary quantities and $\{q_z\}$ is a vector including the L unknown nodal values of the fictitious load.

The discretized counterpart of Eq. (35) when applied to all nodal points in the interior of the beam yields

$$\{W\} = [F]\{q_z\} - ([E_1]\{w\} + [E_2]\{w'\} + [E_3]\{w''\} + [E_4]\{w'''\}), \tag{39}$$

where $[F]$ is an $L \times L$ known matrix and $[E_i]$ $(i = 1, 2, 3, 4)$ are $L \times 2$ also known matrices. Elimination of the boundary quantities from Eq. (39) using Eq. (38) for homogeneous boundary conditions (15.1,2) ($a_3^z = \beta_3^z = 0$) yields

$$\{W\} = [B_z]\{q_z\}, \tag{40}$$

where $[B_z]$ is an $L \times L$ matrix.

Moreover, the discretized counterpart of Eqs. (37.1–3) when applied to all nodal points in the interior of the beam, after elimination of the boundary quantities using Eq. (38) yields

$$\{W'\} = [B_z']\{q_z\}, \quad \{W''\} = [B_z'']\{q_z\}, \quad \{W'''\} = [B_z''']\{q_z\}, \tag{41.1 - 3}$$

where $[B_z']$, $[B_z'']$, $[B_z''']$ are known $L \times L$ coefficient matrices. Note that Eqs. (40) and (41) are valid for homogeneous boundary conditions ($a_3^z = \beta_3^z = 0$). For non-homogeneous boundary conditions, an additive constant vector will appear in the right hand side of these equations.

Finally, applying Eq. (14) to the L nodal points in the interior of the beam we obtain the following linear system of equations with respect to q_z:

$$\left[[D_z''''] - [D_z'''][B_z'''] - [D_z''][B_z''] - [D_z'][B_z']\right]\{q_z\} = \{p_z\} + \left\{m_y'\right\} - [D_z]\{p_z''\}, \tag{42}$$

where $[D_z'''']$, $[D_z''']$, $[D_z'']$, $[D_z']$, $[D_z]$ are diagonal $L \times L$ matrices whose elements are given from

$$(D_z'''')_{ii} = E_1 I_y \left(1 + \frac{N_i}{G_1 A_z}\right), \tag{43.1}$$

$$(D_z''')_{ii} = \frac{3E_1 I_y}{G_1 A_z}(p_x)_i, \tag{43.2}$$

$$(D_z'')_{ii} = \frac{3E_1 I_y}{G_1 A_z}(p_x')_i + N_i, \tag{43.3}$$

$$(D_z')_{ii} = \frac{E_1 I_y}{G_1 A_z}(p_x'')_i - (p_x)_i, \tag{43.4}$$

$$(D_z)_{ii} = \frac{E_1 I_y}{G_1 A_z} \tag{43.5}$$

at the L nodal points in the interior of the beam; $\{q_z\}$, $\{p_z\}$, $\left\{m_y'\right\}$ and $\{p_z''\}$ are vectors with L elements including the values of the fictitious loading, the transverse loading, the first derivative of the bending moment distributed loading and the second derivative of the transverse loading at the L nodal points in the interior of the beam. The values of the quantities $(m_y)'$, $(p_z)''$, $(p_x)'$ and $(p_x)''$ result after approximating the corresponding derivatives with appropriate central, forward or backward finite differences.

Similarly for the transverse deflection $v = v(x)$, application of Eq. (17) to the L nodal points in the interior of the beam results the following linear system of equations with respect to q_y:

$$\left[\left[D_y''''\right] - \left[D_y'''\right]\left[B_y'''\right] - \left[D_y''\right]\left[B_y''\right] - \left[D_y'\right]\left[B_y'\right]\right]\{q_y\} = \{p_y\} - \{m_z'\} - [D_y]\left\{p_y''\right\}, \tag{44}$$

where $\left[B'_y\right]$, $\left[B''_y\right]$, $\left[B'''_y\right]$ are known $L \times L$ coefficient matrices similar to those mentioned before for the deflection w; $\left[D''''_y\right]$, $\left[D'''_y\right]$, $\left[D''_y\right]$, $\left[D'_y\right]$, $[D_y]$ are diagonal $L \times L$ matrices whose elements are given from

$$\left(D''''_y\right)_{ii} = E_1 I_z \left(1 + \frac{N_i}{G_1 A_y}\right), \tag{45.1}$$

$$\left(D'''_y\right)_{ii} = \frac{3E_1 I_z}{G_1 A_y} (p_x)_i, \tag{45.2}$$

$$\left(D''_y\right)_{ii} = \frac{3E_1 I_z}{G_1 A_y} \left(p'_x\right)_i + N_i, \tag{45.3}$$

$$\left(D'_y\right)_{ii} = \frac{E_1 I_z}{G_1 A_y} \left(p''_x\right)_i - (p_x)_i, \tag{45.4}$$

$$(D_y)_{ii} = \frac{E_1 I_z}{G_1 A_y} \tag{45.5}$$

at the L nodal points in the interior of the beam; $\{q_y\}$, $\{p_y\}$, $\{m'_z\}$ and $\left\{p''_y\right\}$ are vectors with L elements, similar with those mentioned before for the deflection w.

3.2 For the axial displacement u

Let u be the sought solution of the boundary value problem described by Eqs. (22) and (23). Differentiating this function two times yields

$$\frac{d^2 u}{dx^2} = q_x(x). \tag{46}$$

Equation (46) indicates that the solution of the original problem can be obtained as the axial displacement of a beam with unit axial rigidity subjected to a flexural fictitious load $q_x(x)$ under the same boundary conditions. The fictitious load is unknown.

The solution of Eq. (46) is given in integral form as

$$u(x) = \int_o^l u^* q_x dx + \left[u^* \frac{du}{dx} - \frac{du^*}{dx} u\right]_0^l, \tag{47}$$

where u^* is the fundamental solution, which is given as

$$u^* = \frac{1}{2}|r|, \tag{48}$$

with $r = x - \zeta$, x, ζ points of the beam.

Following the same procedure as in Sect. 3.1, the discretized counterpart of Eq. (47) and its first derivative with respect to x, when applied to all nodal points in the interior of the beam yields

$$\{U\} = [B_x]\{q_x\}, \tag{49.1}$$

$$\{U'\} = \left[B'_x\right]\{q_x\}, \tag{49.2}$$

where $[B_x]$, $[B'_x]$ are known matrices with dimensions $L \times L$, similar to those mentioned before for the deflection w and the following system of equations with respect to q_x, q_y and q_z is obtained:

$$[D''_x]\{q_x\} + [D''_x]\left[[B''_z]\{q_z\}\right]_{\text{dg.}}[B'_z]\{q_z\} + [D''_x]\left[\left[B''_y\right]\{q_y\}\right]_{\text{dg.}}\left[B'_y\right]\{q_y\} = -\{p_x\}, \tag{50}$$

where the symbol $[\,]_{\text{dg}}$. indicates a diagonal matrix with the elements of the included column matrix, $\{q_x\}$, $\{p_x\}$ are vectors with L elements, similar to those mentioned before for the deflection w, while the axial force at the neutral axis of the beam can be expressed as follows:

$$\{N\} = [D'_x][B'_x]\{q_x\} + [D'_x]\left[[B'_z]\{q_z\}\right]_{\text{dg.}}[B'_z]\{q_z\} + [D'_x]\left[[B'_z]\{q_z\}\right]_{\text{dg.}}[B'_z]\{q_z\}. \tag{51}$$

In Eqs. (50) and (51) $[D''_x]$, $[D'_x]$ are diagonal $L \times L$ matrices whose elements are given from

$$(D''_x)_{ii} = E_1 A, \tag{52.1}$$

$$(D'_x)_{ii} = \frac{E_1 A}{2} \tag{52.2}$$

at the L nodal points in the interior of the beam. Equations (42), (44), (50) and (51) constitute a nonlinear coupled system of equations with respect to q_x, q_y, q_z and N. The solution of this system is accomplished iteratively by employing the two term acceleration method [32], [33].

3.3 For the stress functions $(\Phi(y,z))_j$ and $(\Theta(y,z))_j$

The evaluation of the stress functions $(\Phi(y,z))_j$ and $(\Theta(y,z))_j$ is accomplished using BEM as presented in [30].

Moreover, since the torsionless bending problem of beams is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients Eqs. (26.1,2) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments of inertia and the cross section area can be written as

$$I_y = \frac{1}{E_1}\sum_{j=1}^{K}\int_{\Gamma_j}(E_j - E_i)\left(yz^2\cos\beta\right)ds, \tag{53.1}$$

$$I_z = \frac{1}{E_1}\sum_{j=1}^{K}\int_{\Gamma_j}(E_j - E_i)\left(zy^2\sin\beta\right)ds, \tag{53.2}$$

$$A = \frac{1}{2G_1}\sum_{j=1}^{K}\int_{\Gamma_j}(G_j - G_i)(y\cos\beta + z\sin\beta)ds, \tag{53.3}$$

while the shear deformation coefficients a_y and a_z are obtained from the relations

$$a_y = \frac{A}{E_1\Delta^2}\left((4v+2)I_y I_{\Theta y} + \frac{1}{4}v^2 I_y^2 I_{ed} - I_{\Theta e}\right), \tag{54.1}$$

$$a_z = \frac{A}{E_1 \Delta^2} \left((4v+2) I_z I_{\Phi z} + \frac{1}{4} v^2 I_z^2 I_{ed} - I_{\Phi d} \right), \tag{54.2}$$

where

$$I_{\Theta e} = \sum_{j=1}^{K} \int_{\Gamma_j} (E_j - E_i)(\Theta)_j (\boldsymbol{n} \cdot \boldsymbol{e}) ds, \tag{55.1}$$

$$I_{\Phi d} = \sum_{j=1}^{K} \int_{\Gamma_j} (E_j - E_i)(\Phi)_j (\boldsymbol{n} \cdot \boldsymbol{d}) ds, \tag{55.2}$$

$$I_{ed} = \sum_{j=1}^{K} \int_{\Gamma_j} (E_j - E_i) \left(y^4 z \sin\beta + z^4 y \cos\beta + \frac{2}{3} y^2 z^3 \sin\beta \right) ds, \tag{55.3}$$

$$I_{\Theta y} = \frac{1}{6} \sum_{j=1}^{K} \int_{\Gamma_j} (E_j - E_i) \left[-2 I_{yy} y^4 z \sin\beta + \left(3(\Theta)_j \cos\beta - y(\boldsymbol{n} \cdot \boldsymbol{e}) \right) y^2 \right] ds, \tag{55.4}$$

$$I_{\Phi z} = \frac{1}{6} \sum_{j=1}^{K} \int_{\Gamma_j} (E_j - E_i) \left[-2 I_{zz} z^4 y \cos\beta + \left(3(\Phi)_j \sin\beta - z(\boldsymbol{n} \cdot \boldsymbol{d}) \right) z^2 \right] ds. \tag{55.5}$$

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency and the range of applications of the developed method.

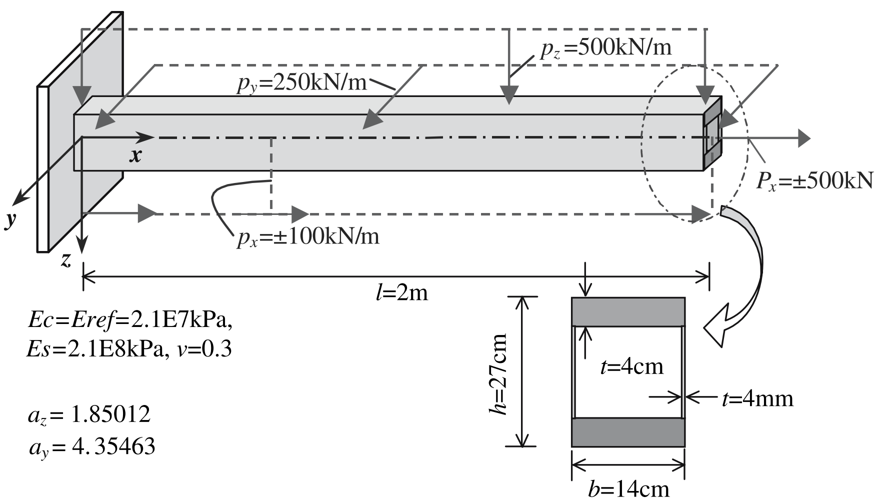


Fig. 3. Cantilever beam of example 1

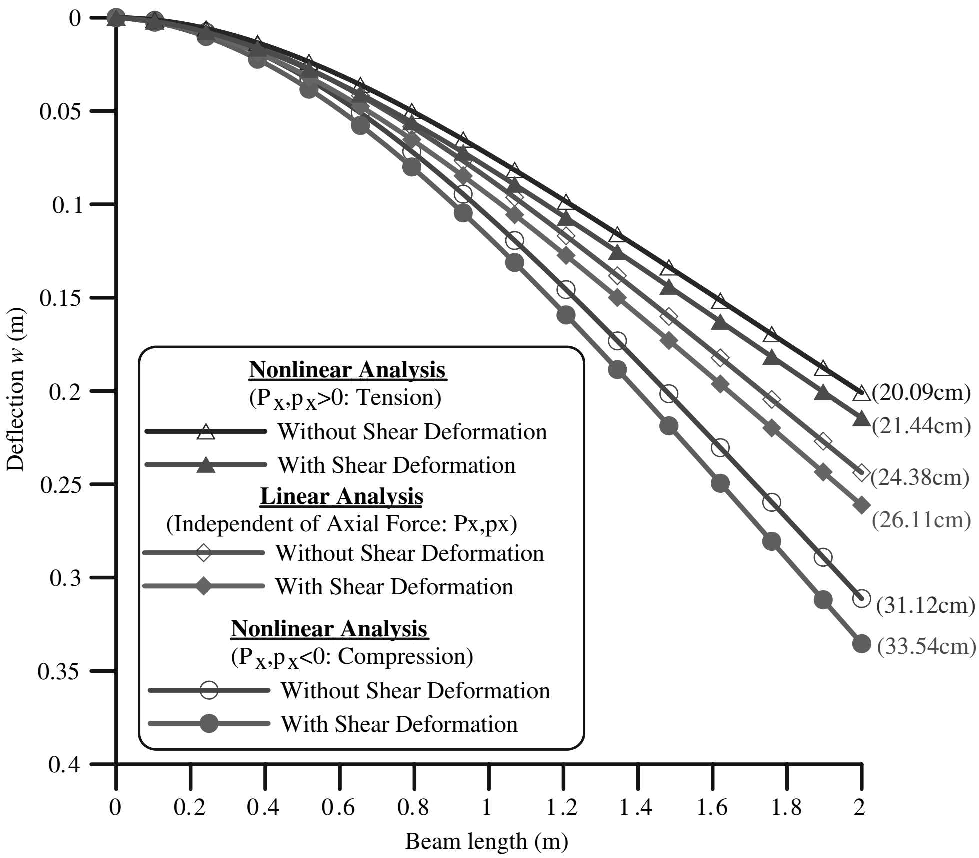


Fig. 4. Deflections w of the beam of example 1

Example 1

A cantilever beam of a hollow composite rectangular cross section $b \times h$ consisting of two concrete and two steel rectangular parts in contact, loaded axially (either tensionally or compressively) and transversely in both directions, as shown in Fig. 3 has been studied. In Figs. 4 and 5 the transverse deflections w, v and in Fig. 6 the axial displacement u along the beam axis are presented for both cases of tensile or compressive axial loading as compared with those obtained from a linear analysis taking into account or ignoring the shear deformation effect. As expected, from the first two figures it is easily verified that tensile axial loading decreases while compressive loading increases the resulting deflections. From the last figure it comes up that even for tensile axial loading the results of the nonlinear analysis show a remarkable negative beam axial displacement u coming from the intense transverse load. For this reason, in Table 1 the values of the axial displacements u along the beam are given for both cases of tensile or compressive axial loading. Moreover, from the aforementioned figures and table the increment of all the deflections and the axial displacement due to the influence of the shear deformation effect is remarkable.

Example 2

A cantilever beam having the composite cross section consisting of a HEB-300 totally encased in a circular matrix, loaded axially (either tensionally or compressively) with a parabolic variation of the third order of the axial loading p_x and uniformly transversely in

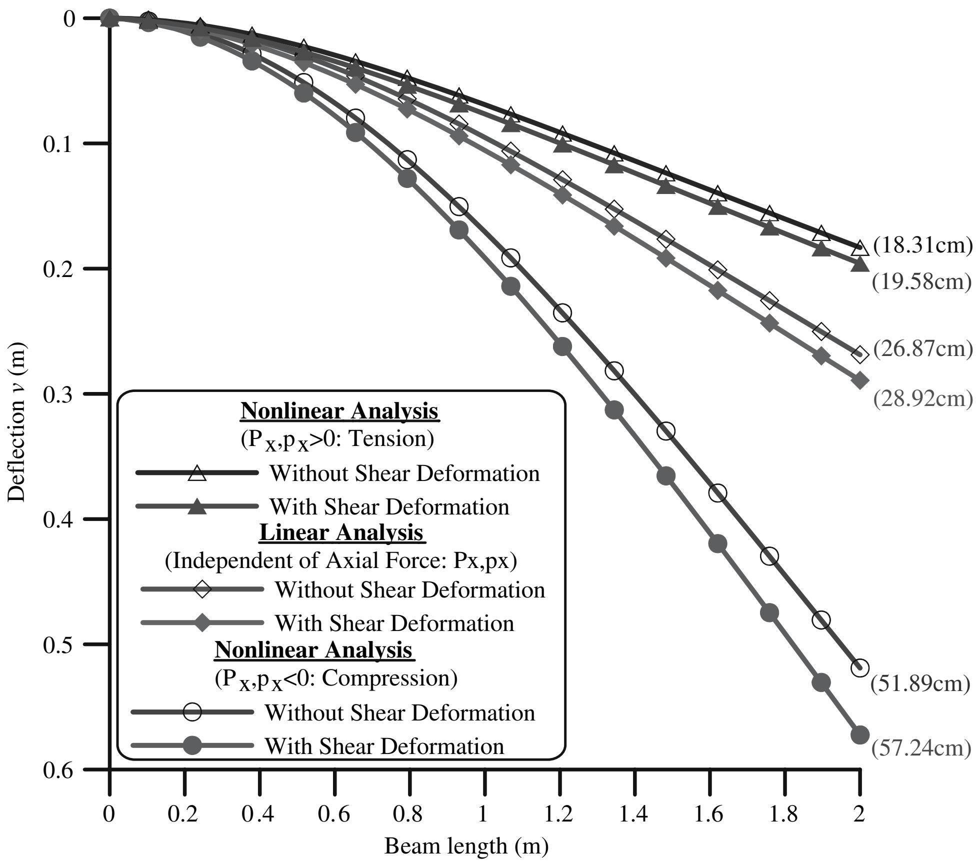


Fig. 5. Deflections v of the beam of example 1

both directions, as shown in Fig. 7, has been studied. In Tables 2 and 3 the transverse deflections w, v and in Fig. 8 the axial displacement u along the beam axis are presented for both cases of tensile or compressive axial loading as compared with those obtained from a linear analysis taking into account or ignoring shear deformation effect. From these tables and figure it comes up that for this solid cross section the resulting from the nonlinear analysis deflections decrement due to tensile axial loading, deflections increment due to compressive one and the axial displacement arising from the transverse loading are not so intense as in the case of the thin-walled cross section. Moreover, the increment of all the deflections and the axial displacement due to the influence of the shear deformation effect in this case could be ignored.

Example 3

A clamped beam having the composite cross section consisting of an I-section ($E_c = E_{ref} = 2.1 \times 10^7$ kN/m^2) totally filled ($E_s = 2.1 \times 10^6$ kN/m^2) so as to form a composite rectangular cross section, loaded transversely in both directions, as shown in Fig. 9, has been studied. In Figs. 10 and 11 and in Table 4 the transverse deflections w, v, respectively, along the beam axis are presented as compared with those obtained from a linear analysis taking into account or ignoring shear deformation effect. From the aforementioned figures and table the influence of the shear deformation effect is once again remarkable and should not be ignored in nonlinear analysis.

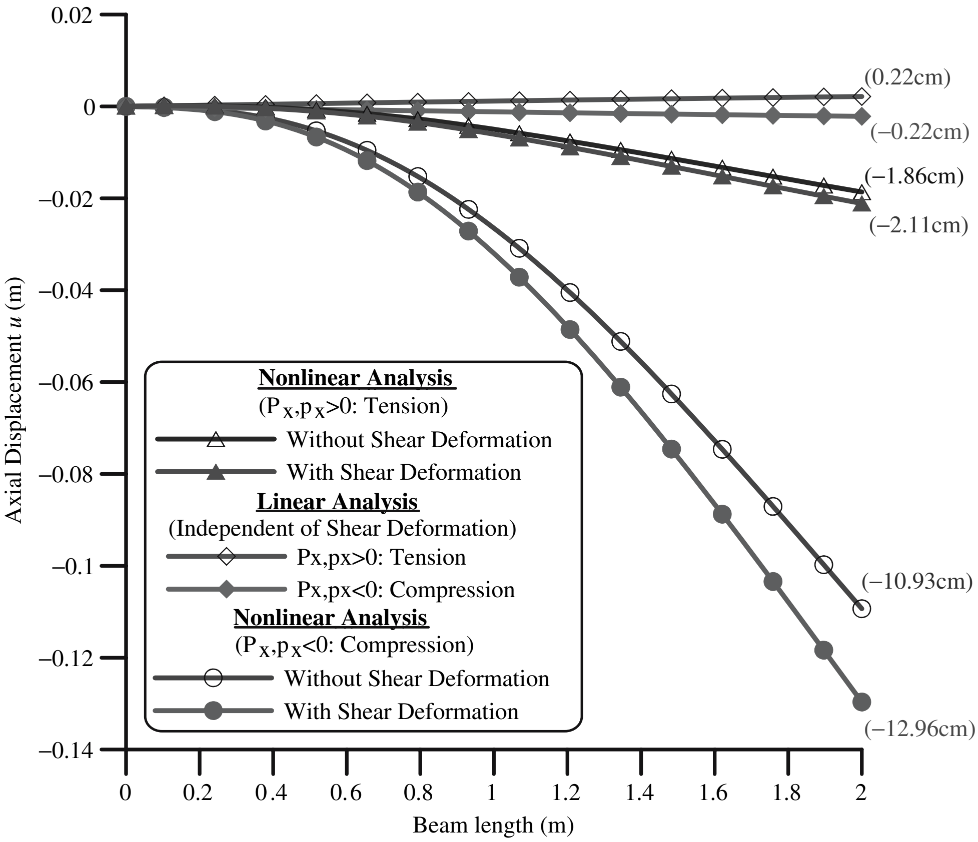


Fig. 6. Axial displacement u of the beam of example 1

Table 1. Axial displacements u (cm) along the beam of example 1

No.	x (cm)	Tension			Compression		
		Linear analysis	Nonlinear analysis		Linear analysis	Nonlinear nalysis	
		Independent of s.d.	Without s.d.	With s.d.	Independent of s.d.	Without s.d.	With s.d.
1	3.448	0.004	0.004	0.004	−0.004	−0.005	−0.005
2	31.034	0.038	0.004	−0.014	−0.038	−0.152	−0.199
3	65.517	0.079	−0.157	−0.214	−0.079	−0.955	−1.180
4	100.000	0.117	−0.496	−0.601	−0.117	−2.648	−3.197
5	134.483	0.153	−0.942	−1.097	−0.153	−5.115	−6.111
6	168.966	0.188	−1.426	−1.630	−0.188	−8.081	−9.605
7	200.000	0.216	−1.864	−2.111	−0.216	−10.935	−12.965

5 Concluding remarks

In this paper a boundary element method is developed for the nonlinear analysis of beams of arbitrary doubly symmetric composite constant cross section, taking into account the shear deformation effect. Five boundary value problems are formulated with respect to the transverse displacements, the axial displacement and to two stress functions and solved

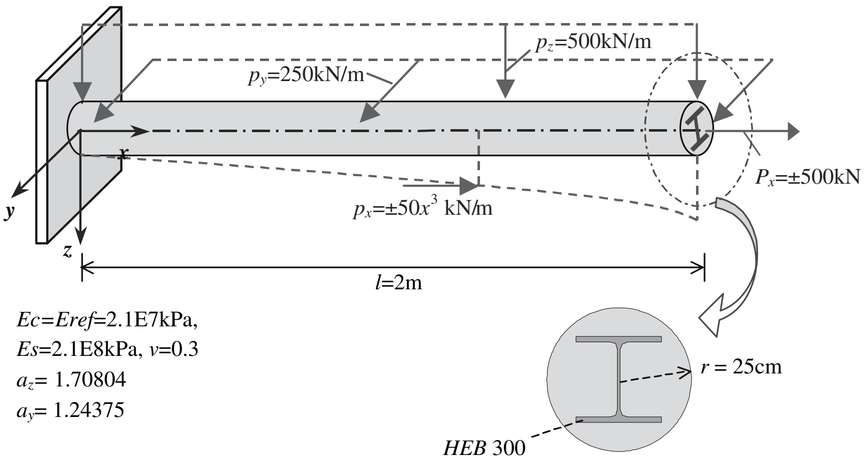


Fig. 7. Cantilever beam of example 2

Table 2. Transverse deflections w (mm) along the beam of example 2

No.	x (cm)	Linear analysis		Nonlinear analysis			
		Independent of axial loading P_x, p_x		Tension		Compression	
		Without s.d.	With s.d.	Without s.d.	With s.d.	Without s.d.	With s.d.
1	3.448	0.005	0.027	0.005	0.027	0.005	0.027
2	31.034	0.387	0.586	0.385	0.583	0.390	0.589
3	65.517	1.533	1.952	1.522	1.940	1.543	1.964
4	100.000	3.163	3.803	3.139	3.776	3.187	3.831
5	134.483	5.065	5.925	5.024	5.879	5.106	5.972
6	168.966	7.086	8.167	7.026	8.099	7.146	8.236
7	200.000	8.932	10.212	8.854	10.123	9.011	10.302

Table 3. Transverse deflections v (mm) along the beam of example 2

No.	x (cm)	Linear analysis		Nonlinear analysis			
		Independent of axial loading P_x, p_x		Tension		Compression	
		Without s.d.	With s.d.	Without s.d.	With s.d.	Without s.d.	With s.d.
1	3.448	0.004	0.012	0.004	0.012	0.004	0.012
2	31.034	0.269	0.341	0.267	0.339	0.271	0.344
3	65.517	1.064	1.217	1.054	1.206	1.075	1.228
4	100.000	2.197	2.431	2.174	2.406	2.221	2.455
5	134.483	3.517	3.832	3.479	3.789	3.558	3.874
6	168.966	4.922	5.315	4.864	5.253	4.981	5.378
7	200.000	6.204	6.669	6.129	6.589	6.281	6.752

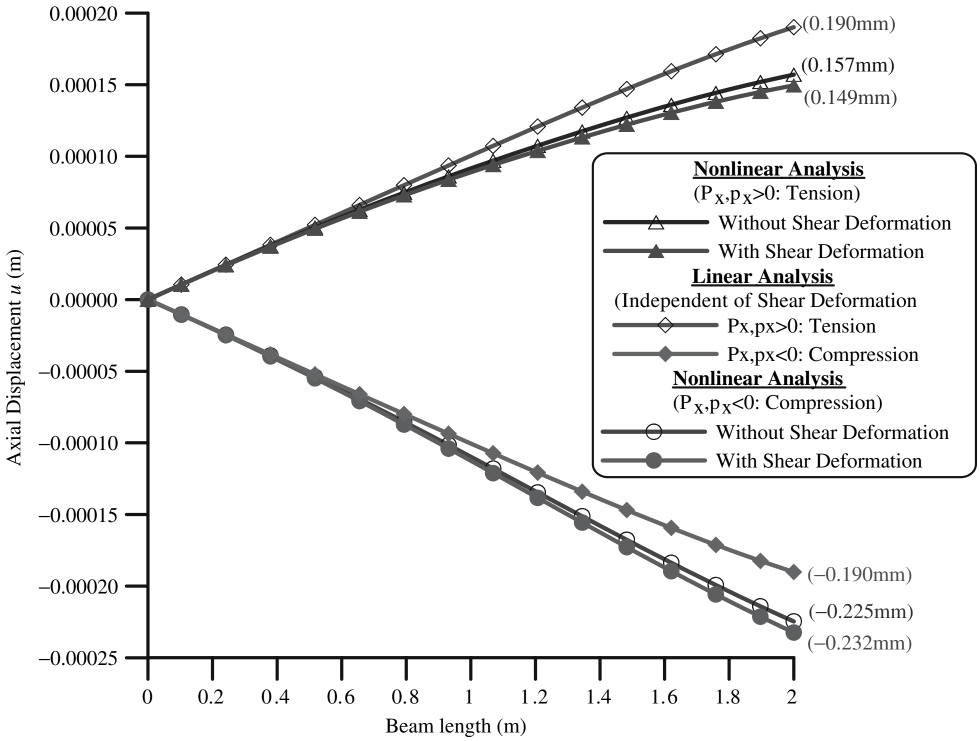


Fig. 8. Axial displacement u of the beam of example 2

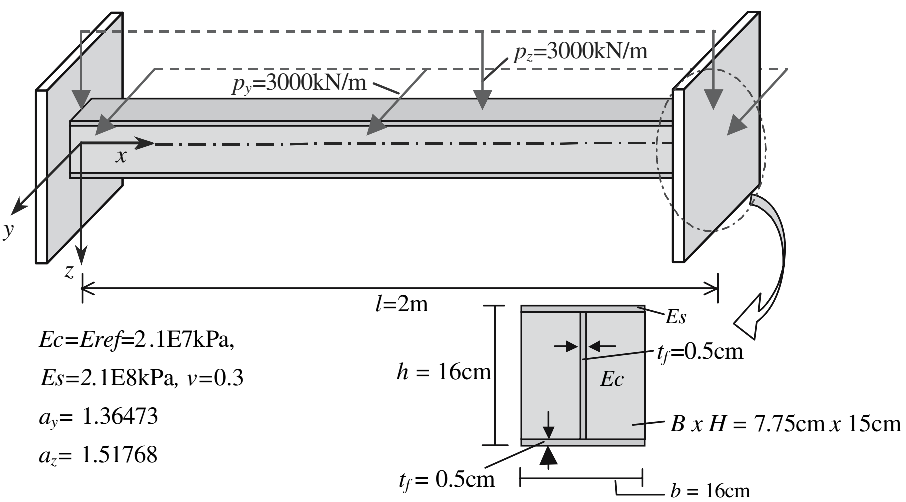


Fig. 9. Clamped beam of example 3

using the Analog Equation Method, a BEM based method. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The main conclusions that can be drawn from this investigation are

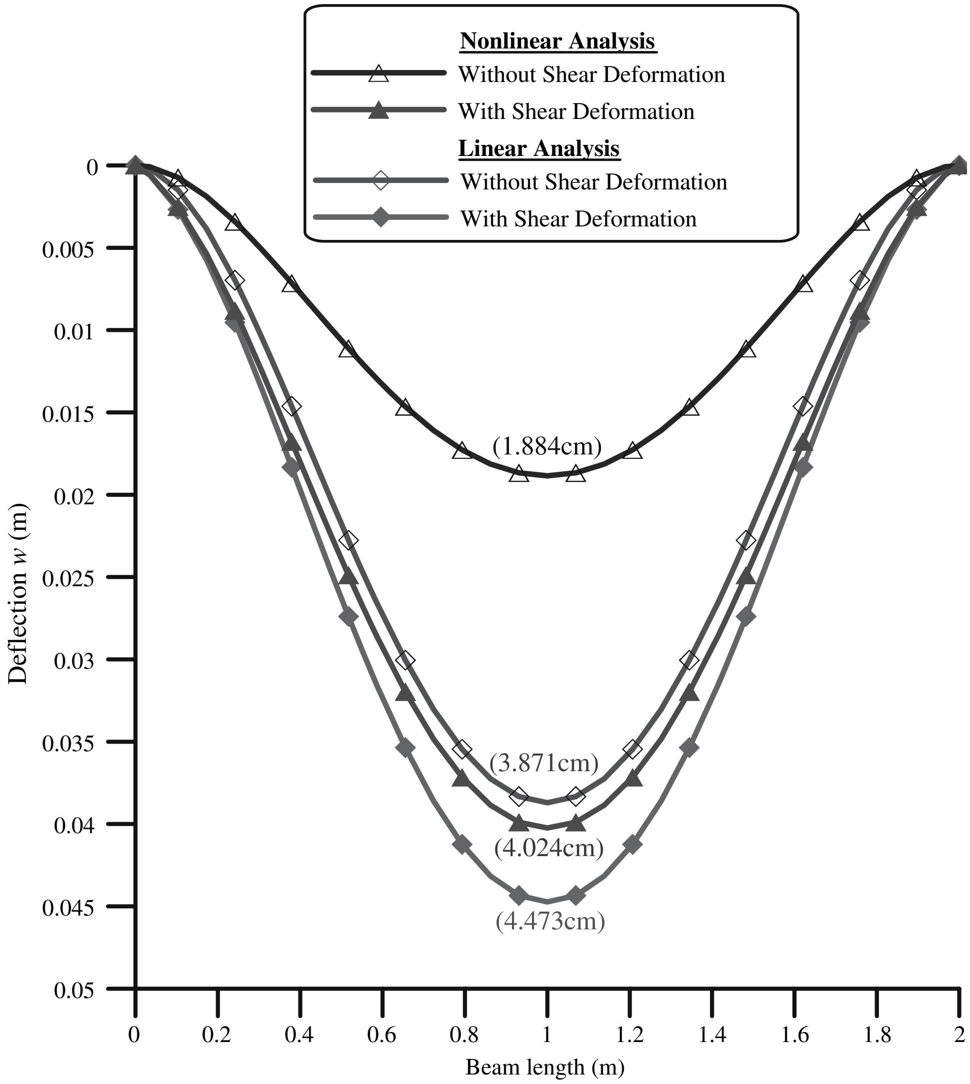


Fig. 10. Deflections w of the beam of example 3

(i) The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary doubly symmetric composite cross section.

(ii) The significant influence of geometrical nonlinear analysis especially in thin walled beam elements subjected in intense transverse loading is verified.

(iii) In some cases the discrepancy between the results of the linear and the nonlinear analysis demonstrates the significant influence of the axial loading.

(iv) The remarkable increment of all the deflections and the axial displacements due to the influence of the shear deformation effect demonstrates its significant influence in nonlinear analysis, especially in thin walled beam elements.

(v) The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

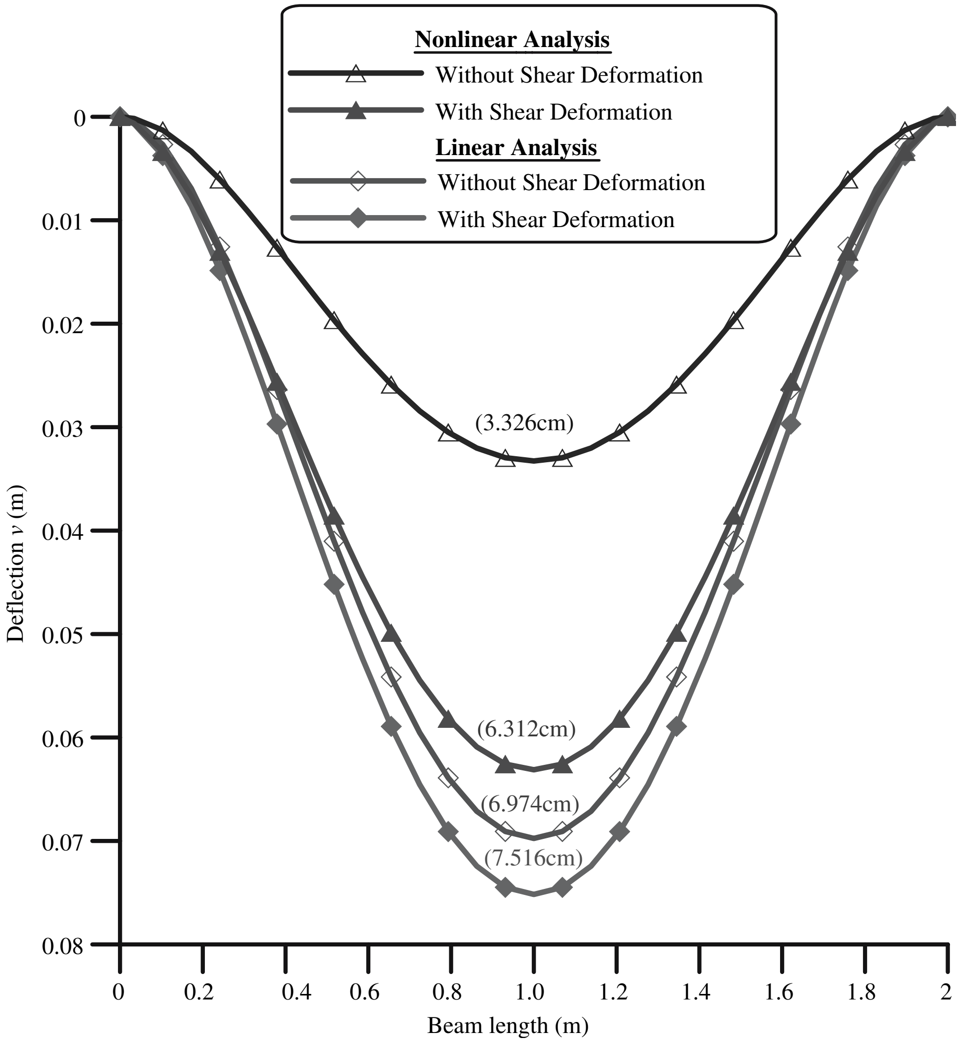


Fig. 11. Deflections v of the beam of example 3

Table 4. Transverse deflections w, v (cm) along the beam of example 3

No.	x (cm)	Linear Analysis				Nonlinear Analysis			
		Deflection w		Deflection v		Deflection w		Deflection v	
		Without s.d.	With s.d.	Without s.d.	With s.d.	Without s.d.	With s.d.	Without s.d.	With s.d.
1	3.448	0.017	0.058	0.032	0.068	0.008	0.056	0.016	0.064
2	31.034	1.064	1.381	1.917	2.202	0.52	1.269	0.922	1.931
3	65.517	3.005	3.536	5.414	5.892	1.464	3.196	2.587	4.986
4	100.000	3.871	4.473	6.974	7.516	1.884	4.024	3.326	6.312

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