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The refined theory of beams for a transversely isotropic body

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Summary. The refined theory of transversely isotropic beams is proposed on the basis of the classical elasticity theory. By using E-L solution and Lur'e method, the refined theory provides the solutions of transversely isotropic beams without ad hoc assumptions. Exact solutions, including a fourth-order part and a transcendental part, are obtained for beams with homogeneous boundary conditions, whereas approximate solutions are derived for beams under transverse surface loadings by dropping terms of high order. It is shown that the displacements and stresses of the beam can be represented by the angle of rotation and the deflection of the neutral surface. In this paper, separate discussions are given to the cases in which characteristic roots are distinct or equal to each other. To the authors' knowledge, the latter has not been covered in the literature. To illustrate the application of the beam theory developed, three examples are examined: a cantilever beam under end loading, a simply supported beam under uniform loading, and a cantilever beam under linear loading. Results show that the refined theory of transversely isotropic beams can be degenerated into that of isotropic beams by omitting anisotropic terms.

1 Introduction

Due to the wide use of anisotropic composite materials, the study of anisotropic elasticity becomes increasingly important. Transversely isotropic material is a noticeable kind of anisotropic material. Lekhnitskii-Hu-Nowacki (LHN) solution [1]–[3] and Elliott-Lodge (E-L) solution [4], [5] are well known general solutions for transversely isotropic body. Wang and Wang [6] pointed out that LHN solution and E-L solution are complete if the elastic region is z-convex (a z-convex domain is a domain which intersects each line, parallels to z-axis at an open interval or does not intersect the line) and characteristic roots s_0^2, s_1^2 and s_2^2 are distinct. Furthermore, Wang and Shi [7] gave a new form of E-L solution for $s_1^2 = s_2^2$, and proved the completeness of LHN solution and the new E-L solution in such cases that s_0^2, s_1^2 and s_2^2 are equal to each other.

Without employing ad hoc assumptions, Cheng [8] presented a method for the solution of three-dimensional elasticity equations, and with the method deduced directly a refined theory of the plate from Boussinesq-Galerkin solution and Lur'e method [9]. A parallel development of Cheng's theory has been obtained by Barrett and Ellis [10] for the isotropic plates under transverse surface loadings (only homogeneous cases are considered in the previous works). By using LHN solution, Wang [11], [12] extended it for the transversely isotropic body and obtained the refined theory of transversely isotropic plate problems and plane problems. Wang and Shi [13] derived a new thick plate theory by using Papkovich-Neuber solution and Lur'e method [9] without ad hoc assumptions, and derived shear theory of plates from the refined plate theory. Their work was also extended to transversely isotropic plates by Yin and Wang [14] through E-L solution. Xu and Wang [15] applied results [13] to the problem of a transversely isotropic piezoelectric plate, and derived approximate equations for the plate under transverse loadings.

As an extension of Cheng's theory, the refined theory was developed for the rectangular elastic beam problems [16]–[18]. Then the refined theory of beams in the coupling fields was investigated, for example, magnetoelastic beams [19], piezoelectric beams [20] and thermoelastic beams [21]. Moreover, the exact equations for the beam without transverse surface loadings and the approximate equations for the beam under transverse surface loadings were derived from the refined beam theory, respectively.

It is the purpose of this paper to extend our previous work [16]–[21] to transversely isotropic beams, and present a systematic method for the derivation of the refined theory of transversely isotropic beams. In this paper, discussions are given to both the case 1 $s_1^2 \neq s_2^2$ and the case 2 $s_1^2 = s_2^2$. To the authors' knowledge, the case 2, which can be directly reduced to isotropic beams, has not been studied in the literature.

The paper is organized as follows: In the next section, E-L solution of the plane stress problem is given in light of the work [6], [7]. In Sect. 3, by using E-L solution and Lur'e method, the refined theory of transversely isotropic beams is derived for both the case 1 and the case 2 without ad hoc assumptions. In virtue of the refined theory developed in Sect. 3, the exact equations are obtained for the beams with homogeneous boundary conditions in Sect. 4. It is shown that the equations can be decomposed into two governing differential equations: the fourth-order equation and the transcendental equation. Then in Sect. 5, the approximate equations are derived for the beams under transverse loadings. Finally, three examples are examined to illustrate the application of the theory proposed in this paper.

2 The general solution of transversely isotropic elasticity

For a narrow rectangular straight beam, the width in the y-direction is stress free. Therefore, it is plausible to set the components of stress $\sigma_y = \tau_{xy} = \tau_{yz} = 0$. This is a plane stress assumption. In a fixed rectangular coordinate system, let z-axis be perpendicular to the isotropic plane ($x - y$ plane) of the beam. We assume that the beam length in x-direction is l, the beam width in y-direction is 1, the beam height in z-direction is h, and $l \gg h \gg 1$. The constitutive equations of two-dimensional elasticity are:

$$\sigma_x = \bar{c}_{11} u_{x,x} + \bar{c}_{13} u_{z,z}, \quad \sigma_z = \bar{c}_{13} u_{x,x} + \bar{c}_{33} u_{z,z}, \quad \tau_{xz} = \bar{c}_{44}\big(u_{z,x} + u_{x,z}\big), \tag{1}$$

where σ_x, σ_z, τ_{xz}, u_x and u_z are the components of stress and displacement, respectively, $\bar{c}_{11}$, $\bar{c}_{13}$, $\bar{c}_{33}$ and $\bar{c}_{44}$ are the elastic stiffness constants, and the comma in the subscript denotes the partial derivative with respect to the spatial variable. For the plane stress problem, $\bar{c}_{11}$, $\bar{c}_{13}$, $\bar{c}_{33}$ and $\bar{c}_{44}$ in Eqs. (1) should be replaced by, respectively,

$$c_{11} = \bar{c}_{11} - \frac{\bar{c}_{12}^2}{\bar{c}_{11}}, \quad c_{13} = \bar{c}_{13} - \frac{\bar{c}_{12}\bar{c}_{13}}{\bar{c}_{11}}, \quad c_{33} = \bar{c}_{33} - \frac{\bar{c}_{13}^2}{\bar{c}_{11}}, \quad c_{44} = \bar{c}_{44}.$$

In the absence of body force, the equilibrium equations of elasticity plane stress problem denoted by u_x and u_z are expressed as

$$c_{11} u_{x,xx} + c_{44} u_{x,zz} + (c_{13} + c_{44}) u_{z,xz} = 0, \quad (c_{13} + c_{44}) u_{x,xz} + c_{44} u_{z,xx} + c_{33} u_{z,zz} = 0. \tag{2}$$

According to E-L solution for transversely isotropic elasticity [7], the components of displacements can be expressed in two cases as follows:

2.1 The case 1, $s_1^2 \neq s_2^2$

When $s_1^2 \neq s_2^2$, the solutions of Eqs. (2) can take the form

$$u_x = (\phi_1 + \phi_2)_{,x}, \quad u_z = (k_1\phi_1 + k_2\phi_2)_{,z}, \tag{3}$$

where the constants k_1 and k_2 satisfy

$$\frac{c_{11}}{c_{44} + (c_{13} + c_{44})k_i} = \frac{c_{13} + c_{44}(1 + k_i)}{c_{33}k_i} = s_i^2 \quad (i = 1, 2), \tag{4}$$

and s_1^2 and s_2^2 are two characteristic roots of the following quadratic algebra equation of s^2,

$$c_{33}c_{44}s^4 + \left(c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33}\right)s^2 + c_{11}c_{44} = 0. \tag{5}$$

We obtain the two roots s_1^2 and s_2^2 of the algebra equation (5) and assume that they are distinct. So the potential function ϕ_i satisfies the following equation:

$$\nabla_i^2\phi_i = 0 \quad (i = 1, 2) \tag{6}$$

with $\nabla_i^2 = \partial_z^2 + s_i^2\partial_x^2$.

According to Wang and Wang [6], it can be further proved that $k_1k_2 = 1$. Lekhnitskii [1] proved that the numbers s_1 and s_2 for any transversely isotropic body can be real or complex (with a real part different from zero), but can not be purely imaginary.

2.2 The case 2, $s_1^2 = s_2^2$

When $s_1^2 = s_2^2$, the solutions of Eqs. (2) are expressed in a different form,

$$u_x = (\varphi_1 + z\varphi_2)_{,x}, \quad u_z = (\varphi_1 + z\varphi_2)_{,z} - \lambda\varphi_2, \tag{7}$$

and the potential function φ_i satisfies the following equation:

$$\nabla_s^2\varphi_i = \partial_z^2\varphi_i + s^2\partial_x^2\varphi_i = 0, \tag{8}$$

where

$$\lambda = 2\frac{c_{13} + 2c_{44}}{c_{13} + c_{44}}, \quad \frac{c_{11}}{c_{13} + 2c_{44}} = \frac{c_{13} + 2c_{44}}{c_{33}} = s^2. \tag{9}$$

It is proved that E-L solution in different cases is complete for z-convex domains (like the domain considered here) [6], [7]. Thus the completeness of solutions (4) and (7) can be guaranteed in the case of rectangular straight beams which are of interest in this paper.

3 The refined theory of transversely isotropic beams

As we know, the problem of beams may be decomposed into two fundamental problems: the extension of a beam and the bending of a beam. In the case of bending of a beam, the beam is subjected only to anti-symmetrical loadings and edge conditions, thus only odd functions of z are required for u_x and even functions of z for u_z.

3.1 The case 1, $s_1^2 \neq s_2^2$

Based on Lur'e method [9] and with these requirements satisfied, treating Eqs. (6) as an ordinary differential equation in z with constant coefficients, one obtains the following symbolic solution of Eqs. (6):

$$\phi_i(x,z) = \frac{\sin(s_i z \partial_x)}{s_i \partial_x} g_i(x), \tag{10}$$

where g_i is unknown function of x yet to be determined. The operators $\sin(s_i z\partial_x)/s_i\partial_x$ and $\cos(s_i z \partial_x)$, which must be interpreted as representing series in powers of operators $(s_i z\partial_x)^2$, have the following symbolic expressions:

$$\begin{aligned} \frac{\sin(s_i z\partial_x)}{s_i\partial_x} &= z\left(1 - \frac{1}{3!}s_i^2 z^2 \partial_x^2 + \frac{1}{5!}s_i^4 z^4 \partial_x^4 - \cdots\right), \\ \cos(s_i z \partial_x) &= \left(1 - \frac{1}{2!}s_i^2 z^2 \partial_x^2 + \frac{1}{4!}s_i^4 z^4 \partial_x^4 - \cdots\right). \end{aligned} \tag{11}$$

Substituting Eq. (10) into Eq. (3), one obtains

$$u_x = \frac{\sin(s_1 z\partial_x)}{s_1} g_1 + \frac{\sin(s_2 z\partial_x)}{s_2} g_2, \quad u_z = k_1 \cos(s_1 z\partial_x) g_1 + k_2 \cos(s_2 z \partial_x) g_2. \tag{12}$$

From Eqs. (12), we can get the angle of rotation and the deflection of the neutral surface,

$$\psi = -u_{x,z}|_{z=0} = -g_{1,x} - g_{2,x}, \quad w = u_z|_{z=0} = k_1 g_1 + k_2 g_2. \tag{13}$$

In terms of Eqs. (13), it can be found to be

$$g_1' = \frac{k_2\psi + w'}{k_1 - k_2}, \quad g_2' = -\frac{k_1 \psi + w'}{k_1 - k_2}. \tag{14}$$

For the sake of simplicity, the differential symbol "′" denotes differentiation with respect to x. From Eqs. (12) and (14), the final expressions for the displacements are

$$\begin{aligned} (k_1 - k_2)u_x &= \left[k_2 \frac{\sin(s_1 z\partial_x)}{s_1} - k_1 \frac{\sin(s_2 z \partial_x)}{s_2}\right] \frac{\psi}{\partial_x} + \left[\frac{\sin(s_1 z\partial_x)}{s_1} - \frac{\sin(s_2 z\partial_x)}{s_2}\right] w, \\ (k_1 - k_2)u_z &= k_1 k_2 [\cos(s_1 z\partial_x) - \cos(s_2 z\partial_x)] \frac{\psi}{\partial_x} + [k_1 \cos(s_1 z \partial_x) - k_2 \cos(s_2 z\partial_x)] w. \end{aligned} \tag{15}$$

Using Hooke's law, from Eqs. (15) the components of stress can be indicated as

$$\begin{aligned} \frac{k_1 - k_2}{c_{44}} \sigma_x &= \left[k_2(1+k_1)s_1^2 \frac{\sin(s_1 z\partial_x)}{s_1} - k_1(1+k_2)s_2^2 \frac{\sin(s_2 z \partial_x)}{s_2}\right]\psi \\ &\quad + \left[(1+k_1)s_1^2 \frac{\sin(s_1 z\partial_x)}{s_1} - (1+k_2)s_2^2 \frac{\sin(s_2 z\partial_x)}{s_2}\right] w', \\ \frac{k_1 - k_2}{c_{44}} \sigma_z &= -\left[k_2(1+k_1) \frac{\sin(s_1 z\partial_x)}{s_1} - k_1(1+k_2) \frac{\sin(s_2 z\partial_x)}{s_2}\right]\psi \\ &\quad - \left[(1+k_1)\frac{\sin(s_1 z\partial_x)}{s_1} - (1+k_2)\frac{\sin(s_2 z\partial_x)}{s_2}\right] w', \\ \frac{k_1 - k_2}{c_{44}} \tau_{xz} &= [k_2(1+k_1)\cos(s_1 z\partial_x) - k_1(1+k_2)\cos(s_2 z\partial_x)]\psi \\ &\quad + [(1+k_1)\cos(s_1 z\partial_x) - (1+k_2)\cos(s_2 z \partial_x)] w', \end{aligned} \tag{16}$$

3.2 *The case 2, $s_1^2 = s_2^2$*

After the same manipulation as in case 1, the components of displacement and stress in terms of the angle of rotation and the deflection of the neutral surface can be expressed as

$$\begin{aligned} u_x &= -\frac{\sin(sz\partial_x)}{s}\frac{\psi}{\partial_x} + \frac{1}{\lambda}\left[z\partial_x\cos(sz\partial_x) - \frac{\sin(sz\partial_x)}{s}\right]f, \\ u_z &= \cos(sz\partial_x)w - \frac{s^2}{\lambda}z\partial_x\frac{\sin(sz\partial_x)}{s}f, \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\sigma_x}{c_{44}} &= -s^2\frac{\sin(sz\partial_x)}{s}(\psi - w') + \frac{2s^2}{\lambda}\left[\frac{\sin(sz\partial_x)}{s} + z\partial_x\cos(sz\partial_x)\right]f', \\ \frac{\sigma_z}{c_{44}} &= \frac{\sin(sz\partial_x)}{s}(\psi - w') + \frac{2}{\lambda}\left[\frac{\sin(sz\partial_x)}{s} - z\partial_x\cos(sz\partial_x)\right]f', \\ \frac{\tau_{xz}}{c_{44}} &= -\cos(sz\partial_x)(\psi - w') - \frac{2s^2}{\lambda}z\partial_x\frac{\sin(sz\partial_x)}{s}f', \end{aligned} \tag{18}$$

where

$$f' = -\psi - w'. \tag{19}$$

4 Exact equations: no transverse surface loadings

Making use of the refined theory of transversely isotropic beams obtained in the previous section, we will investigate the rectangular straight beam with homogeneous boundary conditions on the upper and lower surfaces in this section. Namely, the following boundary conditions are prescribed:

$$\tau_{xz} = 0, \quad \sigma_z = 0 \quad \text{at} \quad z = \pm h/2. \tag{20}$$

Corresponding to the discussion in the previous section, analysis will be given separately to two cases.

4.1 *The case 1, $s_1^2 \neq s_2^2$*

Substituting the stress expressions in Eqs. (16) into the boundary conditions (20) of beams, we get the following equations:

$$\begin{aligned} &-[k_2(1+k_1)CS_1 - k_1(1+k_2)CS_2]\psi - [(1+k_1)CS_1 - (1+k_2)CS_2]w' = 0, \\ &[k_2(1+k_1)SN_1 - k_1(1+k_2)SN_2]\psi + [(1+k_1)SN_1 - (1+k_2)SN_2]w' = 0. \end{aligned} \tag{21}$$

The differential operators SN_i and CS_i are defined by

$$SN_i = \sin\left(\frac{s_i h\partial_x}{2}\right)\Big/ s_i, \quad CS_i = \cos\left(\frac{s_i h\partial_x}{2}\right).$$

Equations (21) are differential equations with respect to unknown deflection and angle of rotation of the neutral surface. Let L_0 be the determinant of the 2×2 operator coefficient matrix of Eqs. (21), there is

$$L_0 = -(k_1 - k_2)(1+k_1)(1+k_2)(SN_1CS_2 - SN_2CS_1)\partial_x, \tag{22}$$

and $L_{ij}\ (i,j=1,2)$ be the co-factors of the matrix. The solutions of Eqs. (21) are

$$\begin{bmatrix} \psi \\ w \end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{23}$$

and ξ_i satisfies

$$L_0\xi_i = 0 \quad (i=1,2). \tag{24}$$

In Appendix B of Ref. [19], it is proved that the solutions of Eqs. (24) can be decomposed into two parts which are governed by a fourth-order equation and a transcendental equation, respectively. That is, ξ_i can be rewritten as

$$\xi_i = \xi_i^{(1)} + \xi_i^{(2)}, \tag{25}$$

where the superscripts "$^{(1)}$" and "$^{(2)}$" indicate the fourth-order part and the transcendental part, respectively, and $\xi_i^{(1)}$ and $\xi_i^{(2)}$ have to satisfy the following governing differential equations of the beam problem, respectively,

$$\partial_x^4 \xi_i^{(1)} = 0, \quad (SN_1CS_2 - SN_2CS_1)\frac{\xi_i^{(2)}}{\partial_x^3} = 0, \tag{26}$$

then the angle of rotation and the deflection of the neutral surface can also be decomposed into two parts, respectively,

$$\psi = \psi^{(1)} + \psi^{(2)}, \quad w = w^{(1)} + w^{(2)}. \tag{27}$$

The solutions corresponding to these two parts will be given separately in the following discussions.

4.1.1 The fourth-order equation

$\xi_i^{(1)}$ satisfies the following fourth-order equation:

$$\partial_x^4 \xi_i^{(1)} = 0. \tag{28}$$

From Eqs. (23), the corresponding solutions of $\psi^{(1)}$ and $w^{(1)}$ have the form

$$\begin{bmatrix} \psi^{(1)} \\ w^{(1)} \end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix}. \tag{29}$$

By using Eqs. (28) and (29), we can get

$$\psi^{(1)} = \left[1 + \frac{(1+k_1)(1+k_2)\left(s_2^2 - s_1^2\right)}{8(k_1-k_2)} h^2 \partial_x^2\right] \partial_x w^{(1)}, \tag{30}$$

where

$$\partial_x^4 w^{(1)} = 0, \tag{31}$$

and from Eqs. (16), the normal stress and shear stress can be found to be

$$\begin{aligned} \sigma_x^{(1)} &= -\frac{c_{44}(1+k_1)(1+k_2)\left(s_2^2-s_1^2\right)}{k_1-k_2} z\left(w^{(1)}\right)'', \quad \sigma_z^{(1)} = 0, \\ \tau_{xz}^{(1)} &= -\frac{c_{44}(1+k_1)(1+k_2)\left(s_2^2-s_1^2\right)}{2(k_1-k_2)} \left(\frac{h^2}{4} - z^2\right)\left(w^{(1)}\right)'''. \end{aligned} \tag{32}$$

By the same arguments made in Cheng [8], Eqs. (32) constitute a first-order refined theory of transversely isotropic beams in the case 1 with the differential governing equation (31), which can satisfy two edge conditions along the boundary of beams and coincide with the corresponding expressions of classical elasticity.

4.1.2 The transcendental equation

$\xi_i^{(2)}$ satisfies the transcendental equation

$$(SN_1CS_2 - SN_2CS_1)\frac{\xi_i^{(2)}}{\partial_x^3} = 0, \tag{33}$$

and the related solutions of $\psi^{(2)}$ and $w^{(2)}$ become

$$\begin{bmatrix}\psi^{(2)}\\ w^{(2)}\end{bmatrix} = \begin{bmatrix}L_{22} & -L_{12}\\ -L_{21} & L_{11}\end{bmatrix}\begin{bmatrix}\xi_1^{(2)}\\ \xi_2^{(2)}\end{bmatrix}. \tag{34}$$

Substituting Eqs. (34) into the stress expressions (16) gives

$$\sigma_x^{(2)} = \frac{\partial^4\Phi}{\partial x^2\partial z^2}, \quad \sigma_z^{(2)} = \frac{\partial^4\Phi}{\partial x^4}, \quad \tau_{xz}^{(2)} = -\frac{\partial^4\Phi}{\partial x^3\partial z}, \tag{35}$$

where the function $\Phi(x,z)$ has the form

$$\begin{aligned}\frac{\Phi}{c_{44}(1+k_1)(1+k_2)} = &-\left[\sin\left(\frac{s_1h\partial_x}{2}\right)\frac{\sin(s_2z\partial_x)}{s_1s_2} - \sin\left(\frac{s_2h\partial_x}{2}\right)\frac{\sin(s_1z\partial_x)}{s_1s_2}\right]\frac{1}{\partial_x^3}\xi_1^{(2)}\\ &+\left[\cos\left(\frac{s_1h\partial_x}{2}\right)\frac{\sin(s_2z\partial_x)}{s_2} - \cos\left(\frac{s_2h\partial_x}{2}\right)\frac{\sin(s_1z\partial_x)}{s_1}\right]\frac{1}{\partial_x^3}\xi_2^{(2)},\end{aligned} \tag{36}$$

and Φ satisfies the following equations:

$$\nabla_1^2\nabla_2^2\Phi = 0, \tag{37}$$

$$\Phi = 0, \quad \partial\Phi/\partial z = 0 \quad \text{at} \quad z = \pm h/2. \tag{38}$$

Equation (35) satisfy two edge conditions along the boundary of beams, and yet satisfy exactly all the fundamental equations in theory of transversely isotropic elasticity. Combining the fourth-order solution of Eqs. (32) and the transcendental solution of Eqs. (35), we arrive at a second-order refined theory for the bending transversely isotropic elastic beams with the two differential governing equations (31) and (37). It is important to note that the equilibrium equations (2) are satisfied by any solution of the refined elastic beam theory.

4.2 The case 2, $s_1^2 = s_2^2$

Likewise, we apply boundary conditions (20) to Eqs. (18), and then obtain the differential equations with respect to unknown deflection and angle of rotation of the neutral surface:

$$\begin{aligned}&\left(-CS + \frac{1}{\lambda}s^2h\partial_xSN\right)\psi + \left(CS + \frac{1}{\lambda}s^2h\partial_xSN\right)w' = 0,\\ &\left(\frac{\lambda-2}{\lambda}SN + \frac{1}{\lambda}h\partial_xCS\right)\psi + \left(-\frac{\lambda+2}{\lambda}SN + \frac{1}{\lambda}h\partial_xCS\right)w' = 0,\end{aligned} \tag{39}$$

where the differential operators SN and CS have the form

$$SN = \sin\left(\frac{sh\partial_x}{2}\right)\Big/s, \quad CS = \cos\left(\frac{sh\partial_x}{2}\right). \tag{40}$$

It is not difficult to write out the determinant of the operator coefficient matrix of Eqs. (39),

$$L_0 = -\frac{2hc_{44}}{\lambda}\left\{\left[1-\frac{\sin(sh\partial_x)}{sh\partial_x}\right]\frac{1}{\partial_x^2}\right\}\partial_x^4. \tag{41}$$

After going through the similar procedure, one can see that Eqs. (24), (25) and (27) still hold in the case 2. But the transcendental equation takes a different form from Eqs. (26) which may be changed into

$$\partial_x^4\xi_i^{(1)} = 0, \quad \left[1-\frac{\sin(sh\partial_x)}{sh\partial_x}\right]\frac{1}{\partial_x^2}\xi_i^{(2)} = 0. \tag{42}$$

Similarly, the solutions corresponding to these two equations will be presented separately as follows.

4.2.1 The fourth-order equation

$\xi_i^{(1)}$ satisfies the following fourth-order equation:

$$\partial_x^4\xi_i^{(1)} = 0. \tag{43}$$

After the same manipulation as in case 1, we can obtain ψ and w which satisfy the following equations:

$$\psi^{(1)} = \left(1+\frac{1}{\lambda}s^2h^2\partial_x^2\right)\partial_x w^{(1)}, \quad \partial_x^4 w^{(1)} = 0. \tag{44}$$

From Eqs. (18), the normal stress and shear stress can be found to be

$$\sigma_x^{(1)} = -\frac{8c_{44}s^2}{\lambda}z\left(w^{(1)}\right)'', \quad \sigma_z^{(1)} = 0, \quad \tau_{xz}^{(1)} = -\frac{c_{44}s^2}{4\lambda}\left(\frac{h^2}{4}-z^2\right)\left(w^{(1)}\right)'''. \tag{45}$$

4.2.2 The transcendental equation

$\xi_i^{(2)}$ satisfies the following transcendental equation:

$$\left[1-\frac{\sin(sh\partial_x)}{sh\partial_x}\right]\frac{1}{\partial_x^2}\xi_i^{(2)} = 0. \tag{46}$$

Repeating the procedure in the case 1 leads to the expressions of the normal stress and shear stress

$$\sigma_x^{(2)} = \frac{\partial^4\Phi}{\partial x^2\partial z^2}, \quad \sigma_z^{(2)} = \frac{\partial^4\Phi}{\partial x^4}, \quad \tau_{xz}^{(2)} = -\frac{\partial^4\Phi}{\partial x^3\partial z}, \tag{47}$$

where $\Phi(x,z)$ is given as

$$\begin{aligned}-\frac{\lambda}{2c_{44}}\Phi = &\left[-\frac{h}{s}\cos\frac{sh\partial_x}{2}\sin(sz\partial_x)+\frac{2z}{s}\sin\frac{sh\partial_x}{2}\cos(sz\partial_x)\right]\frac{\xi_1^{(2)}}{\partial_x^2}\\ &+\left[h\sin\frac{sh\partial_x}{2}\sin(sz\partial_x)-\frac{2}{s\partial_x}\cos\frac{sh\partial_x}{2}\sin(sz\partial_x)+2z\cos\frac{sh\partial_x}{2}\cos(sz\partial_x)\right]\frac{\xi_2^{(2)}}{\partial_x^2},\end{aligned} \tag{48}$$

and Φ satisfies the following equations:

$$\nabla_s^2\nabla_s^2\Phi = 0, \tag{49}$$

$$\Phi = 0, \quad \partial\Phi/\partial z = 0 \quad \text{at} \quad z = \pm h/2. \tag{50}$$

Combining the fourth-order solution of Eqs. (45) and the transcendental solution of Eqs. (47), we arrive at a second-order refined theory for the bending transversely isotropic elastic beams with the two differential governing equations (44) and (49) in case 2.

As a special case, an isotropic elastic beam is used to verify the correctness of the results obtained in this subsection. For an isotropic elastic beam, it is known that

$$c_{11} = c_{33} = \frac{E}{1-\nu^2}, \quad c_{13} = \frac{\nu E}{1-\nu^2}, \quad c_{44} = \frac{E}{2(1+\nu)}, \quad s^2 = 1, \quad \lambda = \frac{4}{1+\nu}. \tag{51}$$

where E and ν are the Young's modulus and Poisson's ratio of isotropic materials, respectively. It can be shown that all results of the second-order refined theory of transversely isotropic beams given in this subsection can reduce to the corresponding results for isotropic beams obtained in [17].

5 Approximate equations: transverse surface loadings

In this section a rectangular straight beam with transverse surface loadings is considered. The boundary conditions are given as

$$\tau_{xz} = 0, \quad \sigma_z = \pm q/2 \quad \text{at} \quad z = \pm h/2. \tag{52}$$

5.1 The case 1, $s_1^2 \neq s_2^2$

Substituting the stress expressions in Eqs. (16) into the boundary conditions (52) of the beam, we get the following nonhomogeneous matrix equation:

$$\begin{aligned} &-[k_2(1+k_1)CS_1 - k_1(1+k_2)CS_2]\psi - [(1+k_1)CS_1 - (1+k_2)CS_2]w' = 0, \\ &-[k_2(1+k_1)SN_1 - k_1(1+k_2)SN_2]\psi - [(1+k_1)SN_1 - (1+k_2)SN_2]w' = \frac{k_1-k_2}{2c_{44}}q. \end{aligned} \tag{53}$$

Simplifying Eqs. (53) leads to

$$\begin{aligned} &L_0 w = -[k_2(1+k_1)CS_1 - k_1(1+k_2)CS_2]\frac{k_1-k_2}{2c_{44}}q, \\ &\frac{L_0}{\partial_x}\psi = -[(1+k_1)CS_1 - (1+k_2)CS_2]\frac{k_1-k_2}{2c_{44}}q. \end{aligned} \tag{54}$$

Equations (54) are the exact governing equation for w and ψ in the beam problem subjected to the transverse surface loadings. Since these equations are of infinite order, however, they are not applicable in most cases. Using Taylor series of the trigonometric functions in Eqs. (11) and then dropping all the terms associated with h^4 or the higher orders, we arrive at the following equations

$$Dw'''' = \left[1 - \frac{5\alpha+\gamma}{40(k_1-k_2)}h^2\partial_x^2\right]q, \quad D\psi''' = \left[1 + \frac{5\beta-\gamma}{40(k_1-k_2)}h^2\partial_x^2\right]q, \tag{55}$$

where

$$\alpha = k_1(1+k_2)s_2^2 - k_2(1+k_1)s_1^2, \quad \beta = (1+k_2)s_2^2 - (1+k_1)s_1^2, \quad \gamma = (k_2-k_1)\left(s_1^2+s_2^2\right),$$

and

$$D = \frac{h^3}{12}\frac{c_{44}(1+k_1)(1+k_2)\left(s_2^2-s_1^2\right)}{k_1-k_2}$$

is the flexural rigidity of transversely isotropic beams in case 1. Equations (55) form the basic equations of an approximate first-order theory for the bending problem of the beam under the transverse loadings, in which boundary conditions can be prescribed at both upper and lower surfaces. It should be pointed out that Eqs. (16) in the previous section are still valid.

5.2 The case 2, $s_1^2 = s_2^2$

In the same way, applying the stress expressions in Eqs. (18) into the boundary conditions (52) and simplifying the results, one gets

$$L_0 w = \left(-CS + \frac{1}{\lambda} s^2 h \partial_x SN\right)\frac{q}{2}, \quad \frac{L_0}{\partial_x}\psi = \left(CS + \frac{1}{\lambda} s^2 h \partial_x SN\right)\frac{q}{2}. \tag{56}$$

Using Taylor series of the trigonometric functions and then truncating all the terms associated with h^4 or the higher orders, we arrive at the following equations:

$$Dw'''' = \left(1 - \frac{20+3\lambda}{40\lambda} s^2 h^2 \partial_x^2\right) q, \quad D\psi''' = \left(1 + \frac{20-3\lambda}{40\lambda} s^2 h^2 \partial_x^2\right) q, \tag{57}$$

where

$$D = \frac{2c_{44} s^2 h^3}{3\lambda}$$

is the flexural rigidity of transversely isotropic beams in the case 2. Likewise, the solutions (57) can be safely reduced to the results for isotropic elastic beams obtained in [17] in terms of Eqs. (51).

6 Several examples

To illustrate the application of the theory developed in the previous sections, three examples are considered: a cantilever beam with a transverse concentrated loading applied at the free end, a simply supported beam with a constant transverse distributed loading, and a cantilever beam with a linear transverse distributed loading. Analysis on these three examples will be given separately to two cases. It should be noted that the same examples for isotropic elastic beams have been discussed by Gao and Wang [16].

6.1 The case 1, $s_1^2 \neq s_2^2$

By dropping all the terms associated with h^4 or the higher orders in Eqs. (53), we arrive at the following equations:

$$\psi - w' - \frac{h^2}{8(k_1 - k_2)}(\alpha\psi'' + \beta w''') = 0, \quad \psi' - w'' = \frac{3q}{2c_{44}h}. \tag{58}$$

According to the stress expressions in Eqs. (16), assuming q is a linear function and omitting all the terms associated with h^4 or the higher orders, one can obtain the expressions of the moment and shear force

$$M_x = -\frac{c_{44}h^3}{60(k_1 - k_2)}[(5\alpha + \gamma)\psi' + (5\beta - \gamma)w''], \quad Q_x = -\frac{2}{3}c_{44}h(\psi - w'). \tag{59}$$

6.1.1 The end loaded cantilever beam

Considering a cantilever beam of uniform cross-section loaded by a transverse shear force of magnitude Q_0 at $x = 0$ and clamped at $x = l$. For the present theory, the boundary conditions are

$$(5\alpha+\gamma)\psi'(0) + (5\beta-\gamma)w''(0) = 0, \quad \psi(0) - w'(0) = -\frac{3Q_0}{2c_{44}h}, \quad w(l) = \psi(l) = 0. \tag{60}$$

From Eqs. (58) and (60), the solution for the deflection of the neutral surface is obtained as

$$w = \frac{2(k_1-k_2)Q_0l^3}{(\alpha+\beta)c_{44}h^3}\left(-\frac{x^3}{l^3} + 3\frac{x}{l} - 2\right) - \frac{3Q_0 l}{2c_{44}h}\left(1-\frac{x}{l}\right). \tag{61}$$

6.1.2 The uniformly loaded and simply supported beam

The second example is a beam of uniform cross-section which is simply supported at $x = \pm l$ and which carries a uniformly distributed load of intensity $q = q_0$. The boundary conditions for the present theory are

$$(5\alpha+\gamma)\psi'(\pm l) + (5\beta-\gamma)w''(\pm l) = 0, \quad w(\pm l) = 0. \tag{62}$$

From Eqs. (58) and (62), we obtain the solution for the deflection of the neutral surface

$$w(x) = \frac{(k_1-k_2)q_0l^4}{2(\alpha+\beta)c_{44}h^3}\left(\frac{x^4}{l^4} - 6\frac{x^2}{l^2} + 5\right) + \frac{3(5\alpha+\gamma)q_0l^2}{20(\alpha+\beta)c_{44}h}\left(1-\frac{x^2}{l^2}\right). \tag{63}$$

6.1.3 The linearly loaded cantilever beam

As a third example, a uniform cantilever beam clamped at $x = l$ is considered. The beam is subjected to a linearly distributed load $q(x) = q_0x$, where q_0 is a constant. For the present theory, the boundary conditions are

$$(5\alpha+\gamma)\psi'(0) + (5\beta-\gamma)w''(0) = 0, \quad \psi(0) - w'(0) = 0, \quad w(l) = \psi(l) = 0. \tag{64}$$

From Eqs. (58) and (64), we have

$$w(x) = \frac{(k_1-k_2)q_0l^5}{10(\alpha+\beta)c_{44}h^3}\left(\frac{x^5}{l^5} - 5\frac{x}{l} + 4\right) + \frac{q_0l^3}{4(\alpha+\beta)c_{44}h}\left[\alpha\left(1-\frac{x^3}{l^3}\right) + 3\beta\left(1-\frac{x}{l}\right)\right]. \tag{65}$$

For the three examples, comparing with the results for isotropic beams [16], we find that the solutions for the deflection of the neutral surface have almost the same conformation for both isotropic and transversely isotropic beams.

6.2 The case 2, $s_1^2 = s_2^2$

After the same manipulation as the case 1, we arrive at the following equations:

$$\psi - w' - \frac{1}{8}s^2h^2\left(\frac{4+\lambda}{\lambda}\psi'' + \frac{4-\lambda}{\lambda}w'''\right) = 0, \quad \psi' - w'' = \frac{3q}{2c_{44}h}. \tag{66}$$

According to the stress expressions in Eqs. (18), omitting all the terms associated with h^4 or the higher orders, we obtain the following expressions of the moment and shear force for the present case:

$$M_x = -\frac{c_{44}s^2h^3}{60\lambda}[(20+3\lambda)\psi' + (20-3\lambda)w''], \quad Q_x = -\frac{2}{3}c_{44}h(\psi - w'). \tag{67}$$

For the sake of simplicity, results for the three examples are presented in the following without any detailed derivation.

6.2.1 The end loaded cantilever beam

$$w = \frac{\lambda Q_0 l^3}{4c_{44}s^2h^3}\left(-\frac{x^3}{l^3} + 3\frac{x}{l} - 2\right) - \frac{3Q_0 l}{2c_{44}h}\left(1 - \frac{x}{l}\right). \tag{68}$$

6.2.2 The uniformly loaded and simply supported beam

$$w(x) = \frac{\lambda q_0 l^4}{16c_{44}s^2h^3}\left(\frac{x^4}{l^4} - 6\frac{x^2}{l^2} + 5\right) + \frac{3(20+3\lambda)q_0 l^2}{160c_{44}h}\left(1 - \frac{x^2}{l^2}\right). \tag{69}$$

6.2.3 The linearly loaded cantilever beam

$$w(x) = \frac{\lambda q_0 l^5}{80c_{44}s^2h^3}\left(\frac{x^5}{l^5} - 5\frac{x}{l} + 4\right) + \frac{q_0 l^3}{32c_{44}h}\left[(4+\lambda)\left(1 - \frac{x^3}{l^3}\right) + 3(4-\lambda)\left(1 - \frac{x}{l}\right)\right]. \tag{70}$$

Once again the results described above reduce to the corresponding results for isotropic beams [16]. These results also indicate that the deflection of the neutral surface for transversely isotropic beams can be obtained from the corresponding part for isotropic beams by replacing the related elastic stiffness constants through Eqs. (51).

7 Conclusion

In this paper, the refined theory for rectangular straight transversely isotropic beams has been deduced systematically and directly from elasticity theory. Based on E-L solution and Lur'e method, the refined theory yields the solutions for transversely isotropic beams without ad hoc assumptions. Discussions are given separately to the two cases $s_1^2 \neq s_2^2$ and $s_1^2 = s_2^2$, since they are provided with different types of general solutions [6], [7].

On the basis of the refined theory developed in the present paper, solutions are obtained for transversely isotropic beams with homogeneous boundary conditions and transverse surface loadings, respectively. For the beams with homogeneous boundary conditions, the refined theory provides exact solutions which satisfy all of the governing equations. Employing the results in [17], we decompose the exact solutions into two parts: the fourth-order equation and the transcendental equation. However, for the beams under transverse loadings, only approximate solutions are reached, by truncating those terms associated with h^4 or the higher orders. It is further shown that the refined theory of the transversely isotropic beams in this paper can be reduced to the corresponding results for isotropic beams obtained in [17].

In the purpose of illustrating the application of the refined beam theory developed in this paper, three examples are considered: a cantilever beam with a transverse concentrated load applied at the free end, a simply supported beam with a constant transverse distributed load, and a cantilever beam with a linear transverse distributed load. Results for these three examples can be safely reduced to the corresponding results for isotropic beams [17]. Hence, the results obtained here are considered reliable as a basis for more general applications.

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