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SH-waves at a corrugated interface between a dry sandy half-space and an anisotropic elastic half-space

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Summary. A problem of reflection and transmission of a plane *SH*-wave incident at a corrugated interface between a dry sandy half-space and an anisotropic elastic half-space is investigated. Rayleigh's method of approximation is applied to derive the reflection and transmission coefficients for the first and second order approximation of the corrugation. The expressions for reflection and transmission coefficients for the first order approximation of the corrugation are obtained in closed form for a special type of interface, and the energy partition relation is derived. It is found that these coefficients are proportional to the amplitude of corrugation and are functions of elastic properties of the half-spaces and also of the angle of incidence. Numerical examples illustrating the effects of the sandiness, the anisotropy, the corrugation of the interface, the frequency, and the angle of the incidence on the coefficients are presented.

1 Introduction

The theory of elastic waves finds numerous applications in seismology and geophysics. Seismic signals are applied to investigate the internal structure of the Earth, and they are used in the exploration of valuable materials, e.g., oils, water, minerals etc. The mathematical analysis of seismic waves is mainly devoted to the study of propagation, reflection/refraction and diffraction problems.

Irregularities, such as mountains, basins, mountain roots and salt and ore deposits, affect the reflection and transmission of seismic waves. The interior boundaries of the Earth's media are not planar but are of undulated nature. The roughness or undulation of interfaces do affect the energy partition between the reflected and transmitted waves. The problem of reflection and transmission of waves at irregular interfaces was studied by Asano [1]–[3], Abubakar [4]–[6], Dunkin and Eringen [7], Salvin and Wolf [8], Sumner and Deresiewicz [9], Gupta [10]–[12], Zhang and Shinozuka [13], Tomar and Saini [14], Tomar and Kaur [15] and Kaur et al. [16] among others.

Earth is very complex in nature and contains various types of rocks and materials with amazing characteristics such as anisotropy, heterogeneity, sandiness etc. The effect of anisotropy on reflection and transmission of elastic waves at a plane interface between two elastic media was studied by Musgrave [17], Henneke [18], Daley and Hron [19], Saini and Singh [20], Keith and Crampin [21], Rokhlin et al. [22], Mandal [23], Ruger [24], among others.

Chakraborty and Chandra [25] studied the reflection and refraction of plane SH-waves at a plane interface between a dry sandy layer and sedimentary rock (anisotropic of transversely isotropic type).

In this paper, we analyze the problem of reflection and transmission of SH-waves at a corrugated interface between a dry sandy half-space and an anisotropic elastic half-space. Rayleigh's method of approximation [26], [27, pp. 204–205] is applied to derive the reflection and transmission coefficients for the first and the second order approximation of the corrugation, and the energy partition relation is also derived. It is found that the sandiness and the anisotropic characteristics of the half-spaces, the corrugation of the interface, the frequency and the angle of incidence have significant effect on the reflection and transmission coefficients.

2 Formulation of the problem and its solution

Let $z = \zeta(x)$ be the equation of the corrugated interface separating a dry sandy half-space denoted by H_1 $[-\infty < z \leq \zeta(x)]$ and an anisotropic elastic half-space denoted by H_2 $[\zeta(x) \leq z < \infty]$. In the equation of the corrugated interface, ζ is a periodic function of x, independent of y and whose mean value is zero. The geometry of the problem is shown in Fig. 1.

The Fourier series representation of the function $\zeta(x)$ is

$$\zeta(x) = \sum_{n=1}^{\infty} [\zeta_n e^{\imath n k^* x} + \zeta_{-n} e^{-\imath n k^* x}], \tag{1}$$

where ζ_n and ζ_{-n} are Fourier expansion coefficients, n is the series expansion order, the wavelength of corrugation is $2\pi/k^*$, and $\imath = \sqrt{-1}$. Introducing constants c_1, c_n and s_n

$$\zeta_1 = \zeta_{-1} = \frac{c_1}{2}, \quad \zeta_n = \frac{c_n - \imath s_n}{2}, \quad \zeta_{-n} = \frac{c_n + \imath S_n}{2}, \quad n = 2, 3, 4, \ldots$$

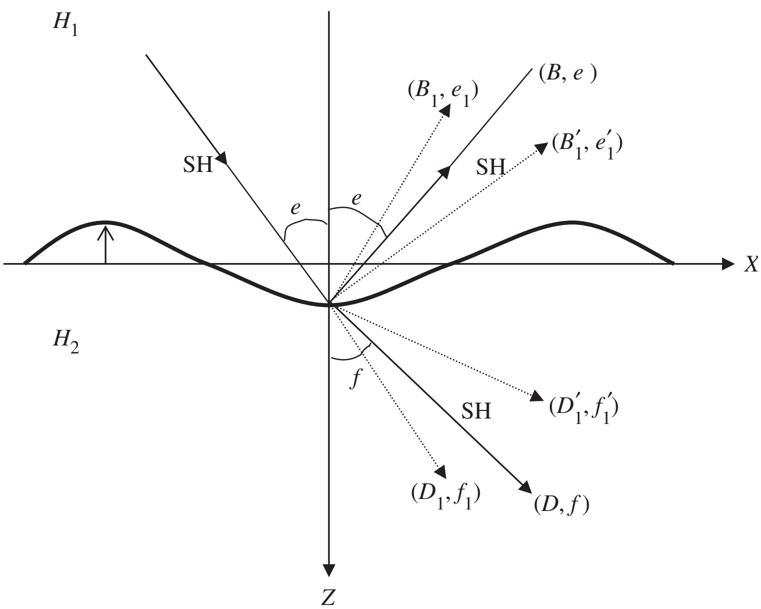

Fig. 1. Geometry of the problem

we obtain

$$\zeta = c_1 \cos k^* x + \sum_{n=2}^{\infty} [c_n \cos nk^* x \, + \, s_n \sin nk^* x]. \tag{2}$$

A stress − strain relation for real earth material such as sand/soil was given by Weiskopff [28]. According to his theory, in an idealized soil the resistance to shear is much smaller than that in a solid because of the slipping of granules on each other, and the resulting shearing deflection is thus much greater. For such materials, the relation $E/\mu = 2(1+\sigma)$ valid for a purely elastic solid, where E is Young's modulus, μ is the modulus of rigidity and σ is Poission's ratio, may be modified as follows:

$$\frac{E}{\mu} = 2\eta(1+\sigma).$$

Here, $\eta > 1$ corresponds to sandy materials and $\eta = 1$ corresponds to an elastic solid. Chakraborty and Chandra [25] applied Weiskopff's theory to investigate the problem of reflection and transmission of plane SH-waves at the boundary of a dry sandy layer and an anisotropic elastic medium.

Neglecting body forces, the equation of motion for the plane SH-wave propagating in a sandy elastic medium H_1 is [29]:

$$\frac{\mu_1}{\eta}\left[\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial z^2}\right] = \rho_1 \frac{\partial^2 V_1}{\partial t^2}, \tag{3}$$

where V_1 is the y-component of the displacement vector, μ_1, ρ_1 and η are, respectively, the rigidity, the density and the sandiness of the medium. The shear wave velocity in H_1 is $\beta_1 = \sqrt{\frac{\mu_1}{\eta\rho_1}}$. Using the method of separation of variables, the time harmonic solution of Eq. (3) for the SH-wave propagating in the positive direction of the x-axis is

$$V_1 = [A_0 e^{-sz} + B_0 e^{sz}] e^{i(\omega t - k_1 x)},$$

where A_0 and B_0 are constants, ω is the angular frequency, $k_1 (= \frac{\omega \sin e}{\beta_1})$ is the horizontal component of the wave number [10], e being the angle of incidence, and

$$s = \sqrt{k_1^2 - \frac{\omega^2}{\beta_1^2}}. \tag{4}$$

The equation of motion for the plane SH-wave in a transversely isotropic medium H_2, in the absence of body forces, is

$$N \frac{\partial^2 V_2}{\partial x^2} + L \frac{\partial^2 V_2}{\partial z^2} = \rho_2 \frac{\partial^2 V_2}{\partial t^2}, \tag{5}$$

where V_2 is the y-component of the displacement vector, and N, L and ρ_2 are, respectively, the elastic constants and the mass density. The shear waves velocities in H_2 along the x- and z-axes are $\beta_2 = \sqrt{\frac{N}{\rho_2}}$ and $\beta_2' = \sqrt{\frac{L}{\rho_2}}$, respectively. The time harmonic solution of Eq. (5) for the SH-wave propagating in the positive direction of the x-axis is

$$V_2 = [C_0 e^{-qz} + D_0 e^{qz}] e^{i(\omega t - k_2 x)},$$

where C_0 and D_0 are constants, k_2 is the wavenumber defined by the law of refraction $k_2 : k_1 = \sin e : \sin f, f$ being the angle of refraction and

$$q = \sqrt{\frac{N}{L}\left(k_2^2 - \frac{\omega^2}{\beta_2^2}\right)}. \tag{6}$$

Let us assume that a ray of plane SH wave of unit amplitude propagating in the upper half space H_1 be incident at the corrugated interface $z = \zeta$ making an angle e with the z-axis. Due to corrugation of the interface, the reflection and refraction phenomena will be affected, and the incident SH-wave will give rise to (i) a regularly reflected and a regularly refracted wave at angles e and f with the z-axis in the upper and lower half-spaces H_1 and H_2, respectively, (ii) a spectrum of n^{th} order of irregularly reflected and irregularly refracted waves at angles e_n and f_n in the left side of regularly reflected and regularly refracted waves, respectively, and (iii) a similar spectrum of irregularly reflected and irregularly refracted waves at angles e'_n and f'_n in the right side of regularly reflected and regularly refracted waves, respectively, at the corrugated interface. The angle of refraction f is related to the incidence angle e through Snell's law

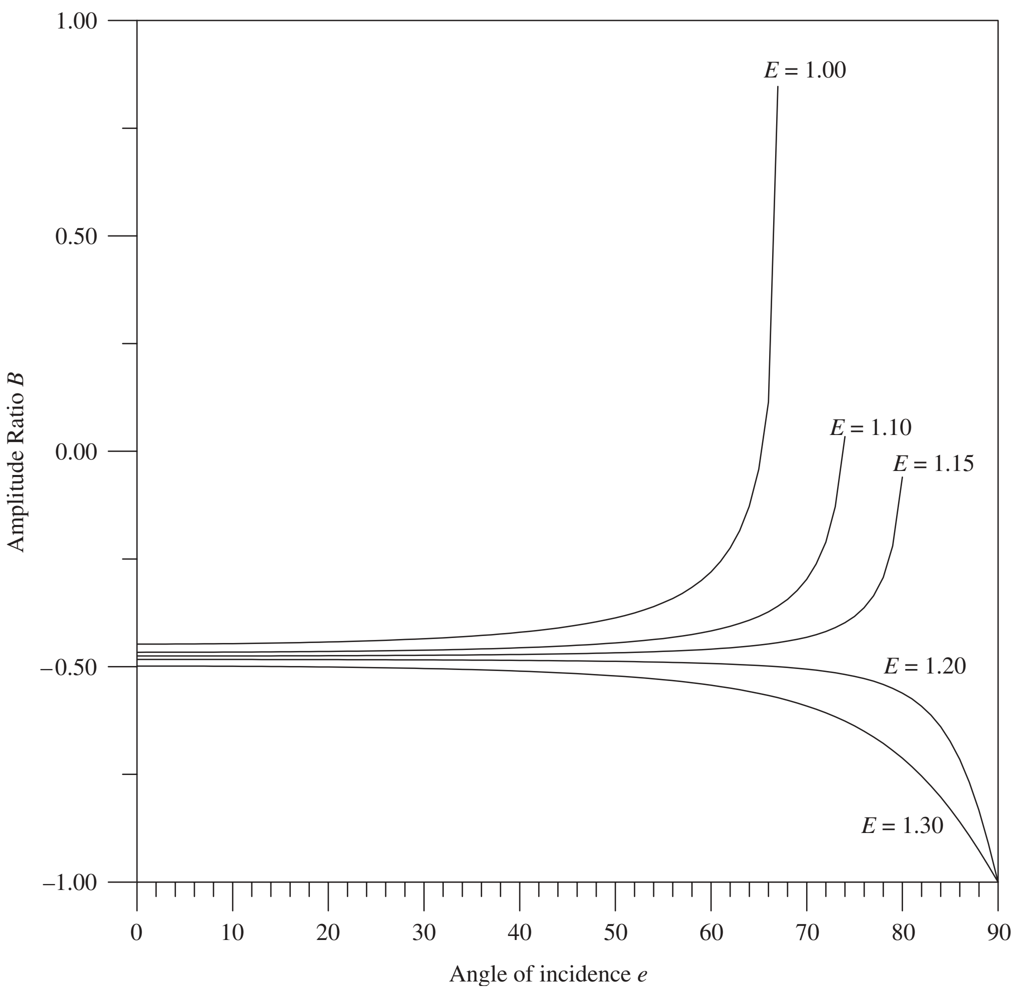

Fig. 2. Variation of the amplitude ratio B with the angle of incidence e for different values of η, where $\eta = E$

$$\frac{\sin e}{\beta_1} = \frac{\sin f}{\beta_2}. \tag{7}$$

The angles e_n, e'_n, f_n and f'_n are given by the following Spectrum theorem [1]:

$$\sin e_n - \sin e = \frac{nk^* \beta_1}{\omega}, \quad \sin e'_n - \sin e = -\frac{nk^* \beta_1}{\omega},$$
$$\sin f_n - \sin f = \frac{nk^* \beta_2}{\omega}, \quad \sin f'_n - \sin f = -\frac{nk^* \beta_2}{\omega}. \tag{8}$$

The total displacement in the half-space H_1 is then given by the sum of incident, regularly reflected and irregularly reflected waves

$$V_1 = \left[e^{-sz} + Be^{sz} + \sum_{n=1}^{\infty} B_n e^{s_n z} e^{-\imath nk^* x} + \sum_{n=1}^{\infty} B'_n e^{s'_n z} e^{\imath nk^* x} \right] e^{\imath\omega(t - \frac{x \sin e}{\beta_1})}, \tag{9}$$

where $s_n = \frac{\imath\omega \cos e_n}{\beta_1}$ and $s'_n = \frac{\imath\omega \cos e'_n}{\beta_1}$.

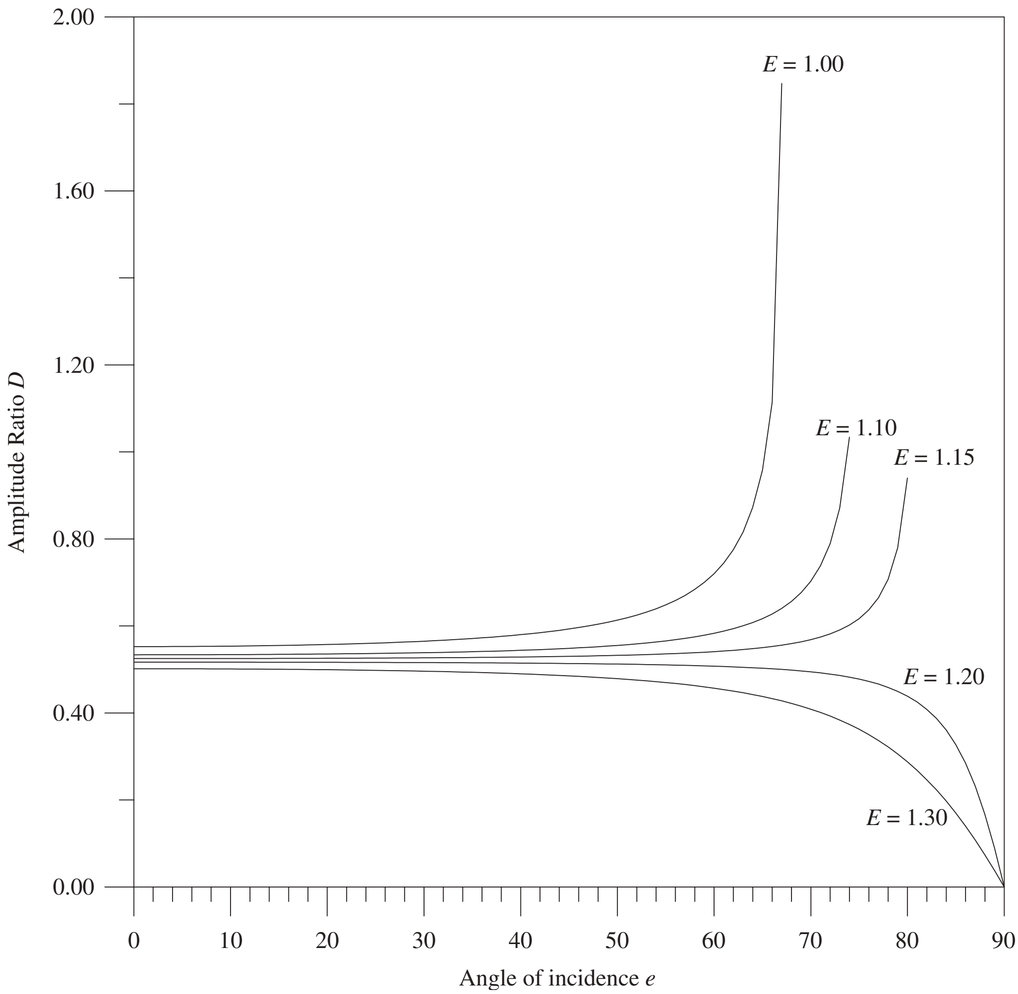

Fig. 3. Variation of the amplitude ratio D with the angle of incidence e for different values of η, where $\eta = E$

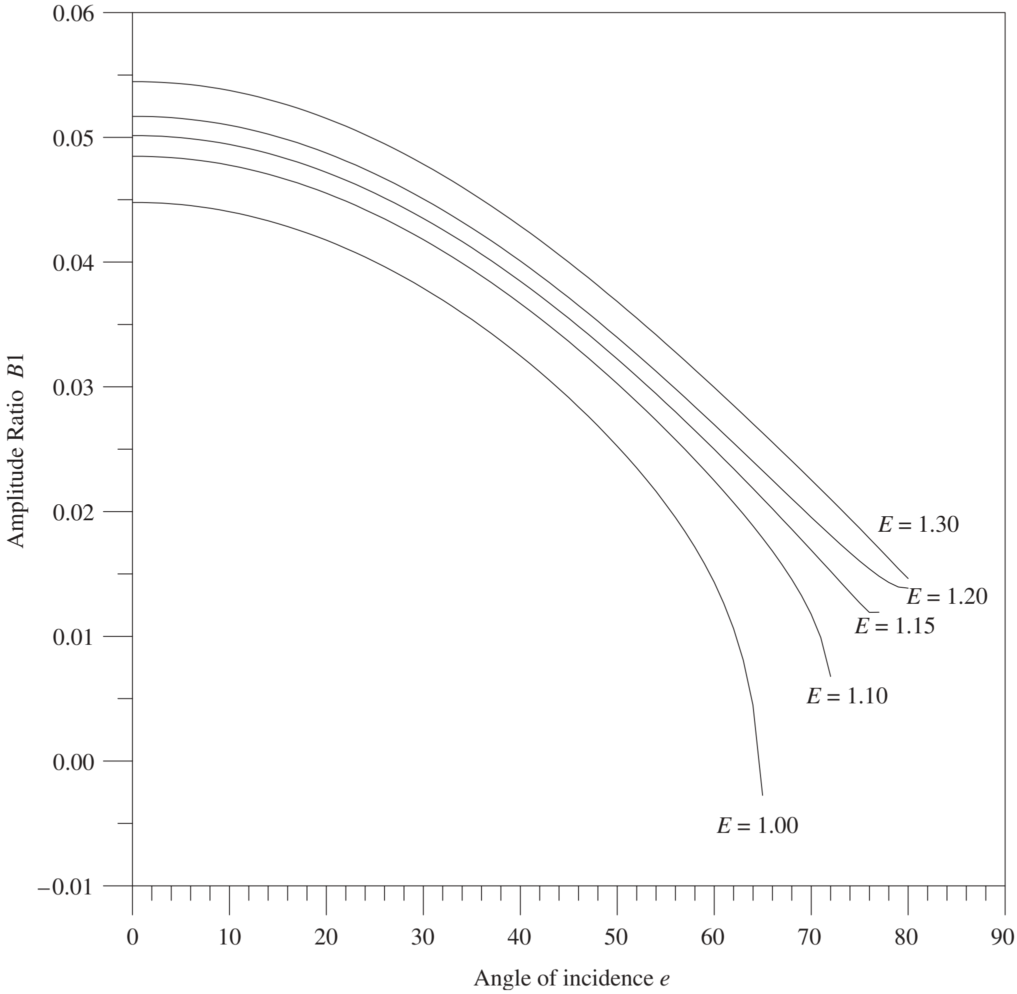

Fig. 4. Variation of the amplitude ratio $B1$ with the angle of incidence e for different values of η, where $\eta = E$

Similarly, the total displacement V_2 in the half-space H_2 is the sum of regularly refracted and irregularly refracted waves,

$$V_2 = \left[De^{-qz} + \sum_{n=1}^{\infty} D_n e^{-q_n z} e^{-\imath n k^* x} + \sum_{n=1}^{\infty} D'_n e^{-q'_n z} e^{\imath n k^* x}\right] e^{\imath\omega\left(t - \frac{x \sin f}{\beta_2}\right)}, \tag{10}$$

where $q_n = \imath\sqrt{\frac{N}{L}}\frac{\omega}{\beta_2}\cos f_n$ and $q'_n = \imath\sqrt{\frac{N}{L}}\frac{\omega}{\beta_2}\cos f'_n$.

The constants B, D, B_n, D_n, B'_n and D'_n are determined from the boundary conditions at the interface.

3 Boundary conditions

The boundary conditions at the corrugated interface $z = \zeta$ ensure the continuity of displacement and traction, that is,

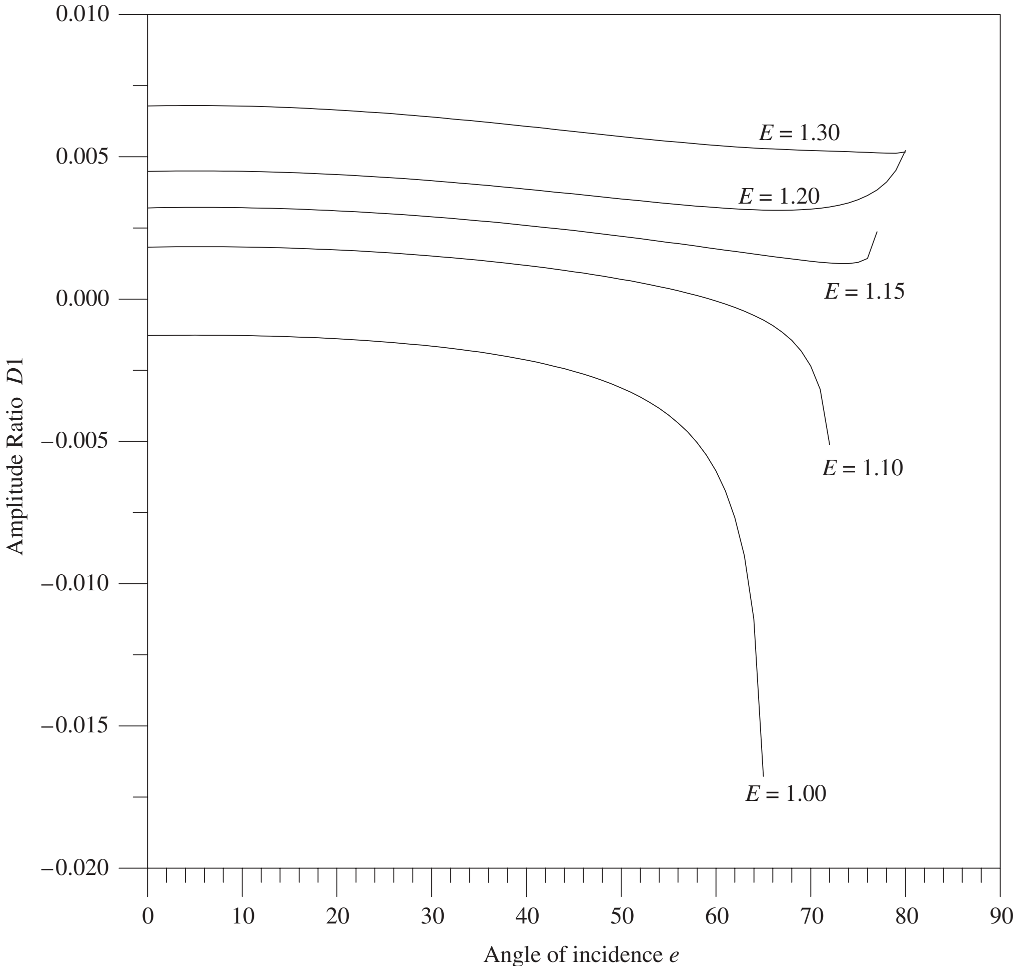

Fig. 5. Variation of the amplitude ratio $D1$ with the angle of incidence e for different values of η, where $\eta = E$

(I) $[V_1]_{H_1} = [V_2]_{H_2}$,
(II) $[\tau_v]_{H_1} = [\tau_v]_{H_2}$,

where τ_v denotes the normal stress to the interface. The boundary condition (II) can be written as

$$\frac{\mu_1}{\eta}\left[\frac{\partial V_1}{\partial z} - \frac{\partial V_1}{\partial x}\zeta'\right] = \left[L\frac{\partial V_2}{\partial z} - N\frac{\partial V_2}{\partial x}\zeta'\right], \tag{11}$$

where ζ' is the derivative of ζ with respect to x. Substituting Eqs. (9) and (10) in the above boundary conditions, we obtain

$$\begin{aligned} &e^{-s\zeta} + B\,e^{s\zeta} + \sum_{n=1}^{\infty} B_n e^{s_n\zeta}\, e^{-\imath mk^*x} + \sum_{n=1}^{\infty} B'_n\, e^{s'_n\zeta}\, e^{\imath mk^*x} \\ &\quad = De^{-q\zeta} + \sum_{n=1}^{\infty} D_n e^{-q_n\zeta} e^{-\imath mk^*x} + \sum_{n=1}^{\infty} D'_n e^{-q'_n\zeta} e^{\imath mk^*x}, \end{aligned} \tag{12}$$

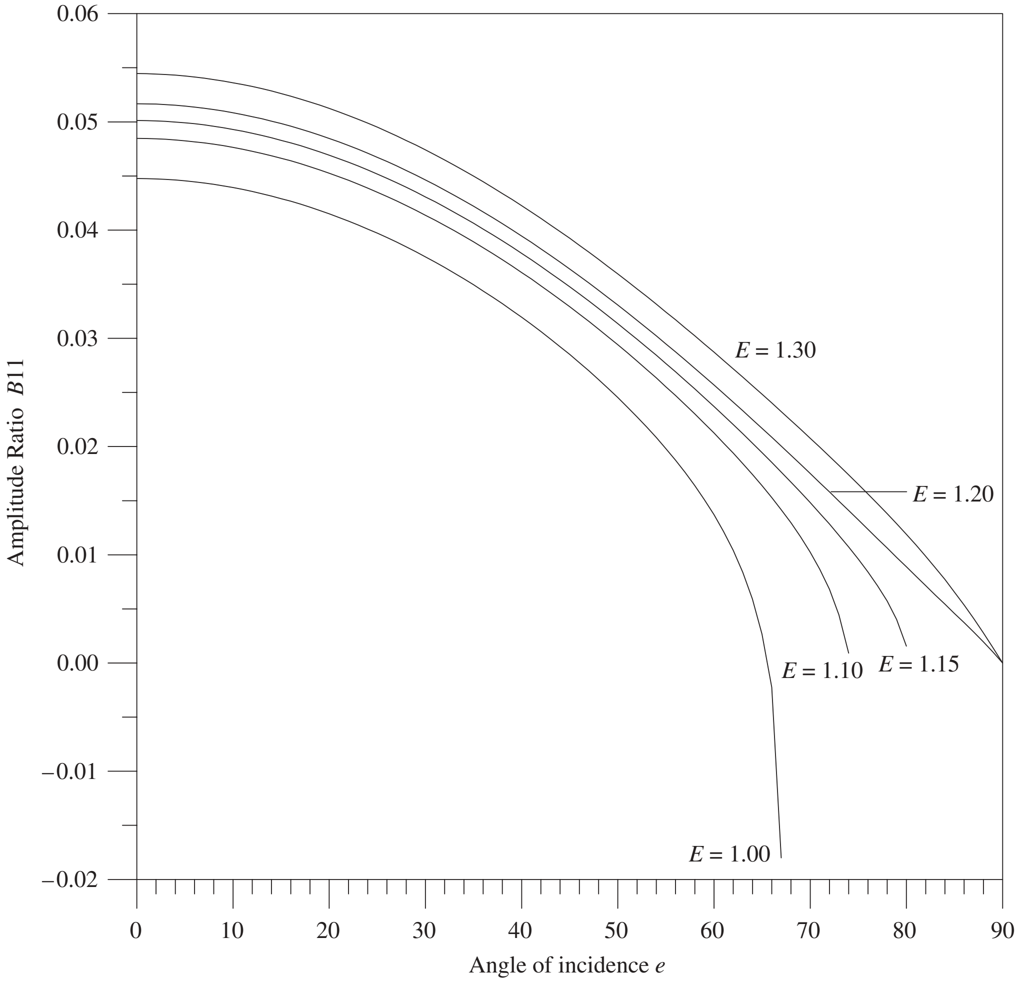

Fig. 6. Variation of the amplitude ratio $B11$ with the angle of incidence e for different values of η, where $\eta = E$

and

$$\begin{aligned}
&\frac{\mu_1}{\eta}\Bigg[\left(-s+\frac{\imath\omega\sin e}{\beta_1}\zeta'\right)e^{-s\zeta}+B\left(s+\frac{\imath\omega\sin e}{\beta_1}\zeta'\right)e^{s\zeta}+\sum_{n=1}^{\infty}B_n e^{-\imath nk^*x}\left\{s_n+\imath\left(\frac{\omega\sin e}{\beta_1}+nk^*\right)\zeta'\right\}e^{s_n\zeta}\\
&\qquad+\sum_{n=1}^{\infty}B'_n e^{\imath nk^*x}\left\{s'_n+\imath\left(\frac{\omega\sin e}{\beta_1}-nk^*\right)\zeta'\right\}e^{s'_n\zeta}\Bigg]\\
&\quad=L\Bigg[D\left(-q+\frac{\imath N\omega\sin f}{L\beta_2}\zeta'\right)e^{-q\zeta}+\sum_{n=1}^{\infty}D_n e^{-\imath nk^*x}\left\{-q_n+\frac{\imath N}{L}\left(\frac{\omega\sin f}{\beta_2}+nk^*\right)\zeta'\right\}e^{-q_n\zeta}\\
&\qquad+\sum_{n=1}^{\infty}D'_n e^{\imath nk^*x}\left\{-q'_n+\frac{\imath N}{L}\left(\frac{\omega\sin f}{\beta_2}-nk^*\right)\zeta'\right\}e^{-q'_n\zeta}\Bigg].
\end{aligned}\tag{13}$$

From Eqs. (12) and (13), the reflection and transmission coefficients of n^{th} order of approximation of the corrugated interface can be determined.

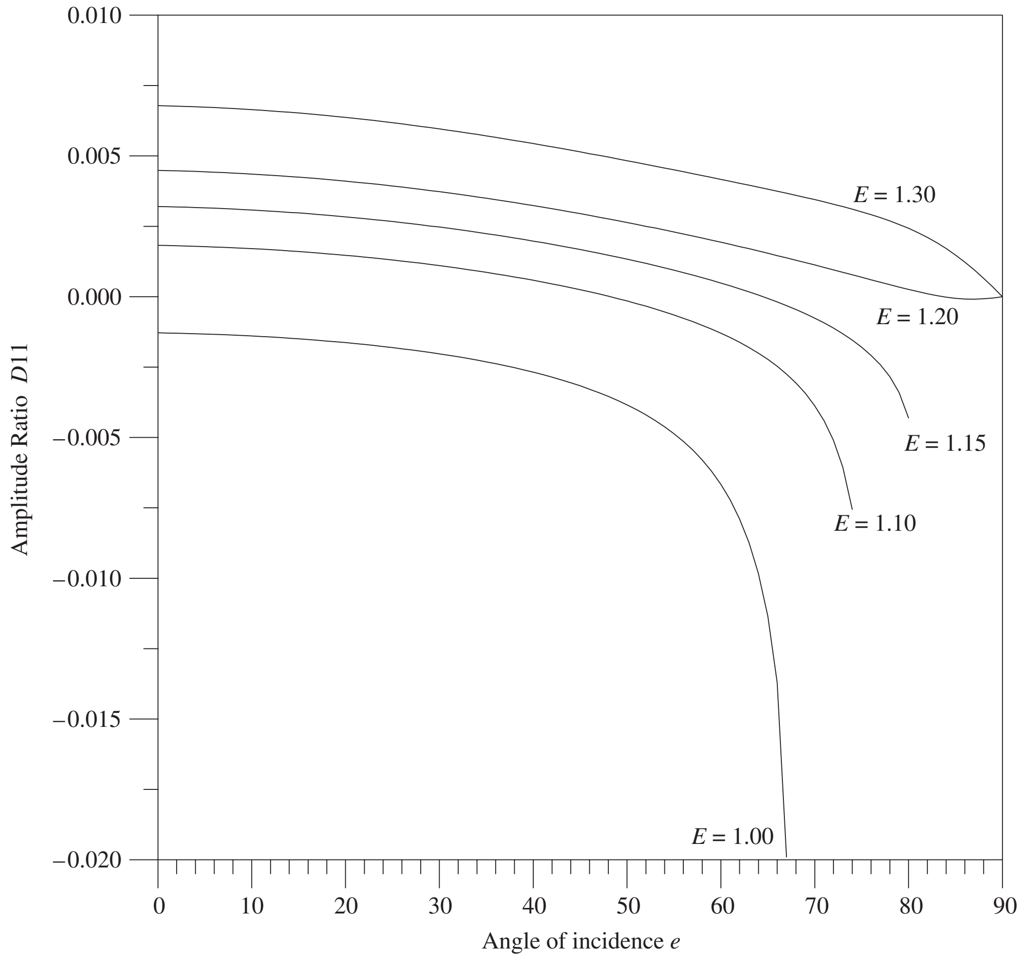

Fig. 7. Variation of the amplitude ratio $D11$ with the angle of incidence e for different values of η, where $\eta = E$

4 Solution of the first order approximation

We assume that the amplitude and the slope of corrugation of the interface $z = \zeta(x)$ is so small that higher powers of ζ can be neglected. The exponential functions involving ζ can then be approximated as

$$e^{\pm s\zeta} \simeq 1 \pm s\zeta. \tag{14}$$

In view of Eq. (14), the first order approximation for the coefficients B and D can be obtained from Eqs. (12) and (13) by collecting the terms independent of x and ζ to both sides:

$$1 + B = D, \tag{15}$$

$$s\mu_1(1 - B) = q\eta LD. \tag{16}$$

These equations provide the values of reflection coefficient B and transmission coefficient D at a plane interface between sandy and anisotropic elastic half-spaces. Solving Eqs. (12) and (13) for B and D, we obtain

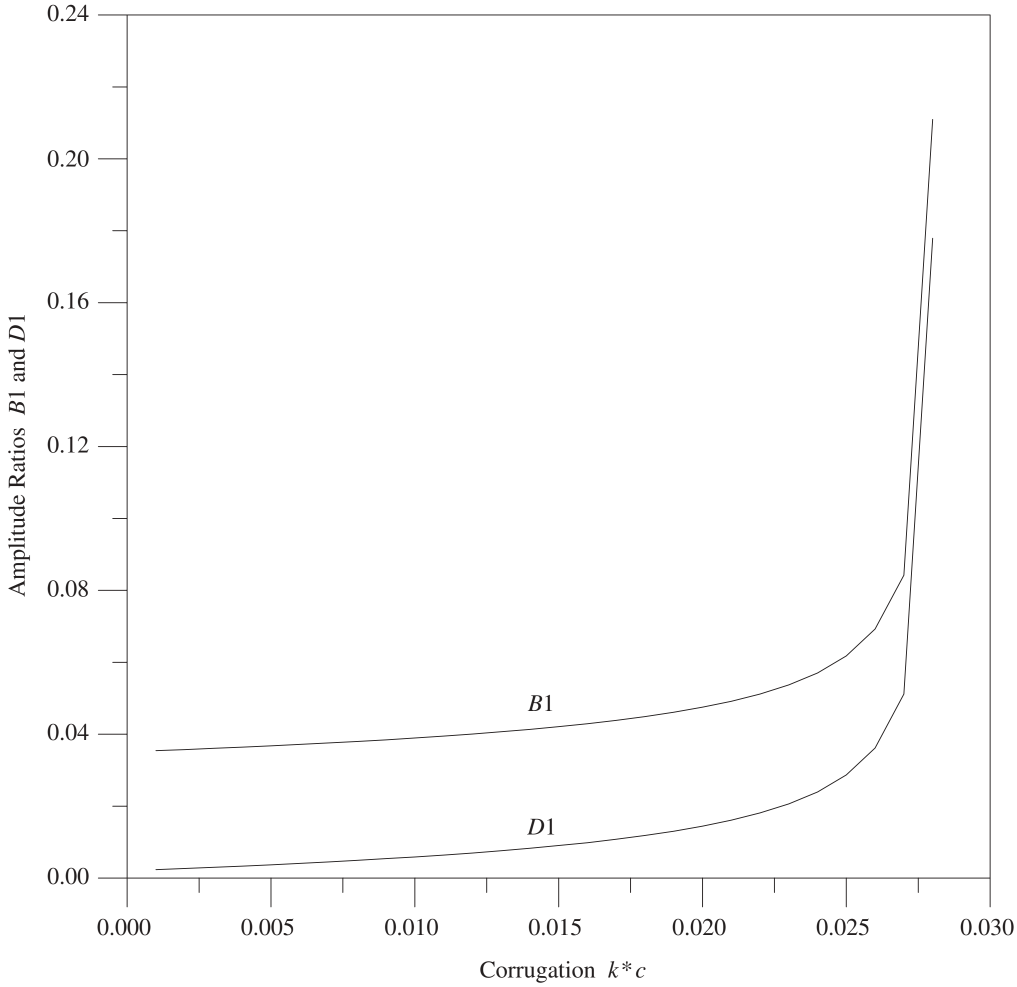

Fig. 8. Variation of the amplitude ratios $B1$ and $D1$ with the corrugation k^*c

$$B = \frac{\mu_1 s - L\eta q}{\mu_1 s + L\eta q}, \quad D = \frac{2\mu_1 s}{\mu_1 s + L\eta q}. \tag{17}$$

To find the solutions of the first order approximation for the coefficients B_n and D_n, we collect the coefficients of $e^{-\imath nk^*x}$ at both sides of Eqs. (12) and (13),

$$B_n - D_n = [(1 - B)s - qD]\zeta_{-n}, \tag{18}$$

$$\mu_1 s_n B_n + L\eta q_n D_n = \left[-\mu_1\left(s^2 + \frac{nk^*\omega \sin e}{\beta_1}\right)(1 + B) + L\eta\left(q^2 + \frac{nk^* N\omega \sin f}{L\beta_2}\right)D\right]\zeta_{-n}. \tag{19}$$

Equating the coefficients of $e^{\imath nk^*x}$, we obtain the first order approximation for the coefficients B'_n and D'_n,

$$B'_n - D'_n = [(1 - B)s - qD]\zeta_n, \tag{20}$$

$$\mu_1 s'_n B'_n + L\eta q'_n D'_n = \left[-\mu_1\left(s^2 - \frac{nk^*\omega \sin e}{\beta_1}\right)(1 + B) + L\eta\left(q^2 - \frac{nk^* N\omega \sin f}{L\beta_2}\right)D\right]\zeta_n. \tag{21}$$

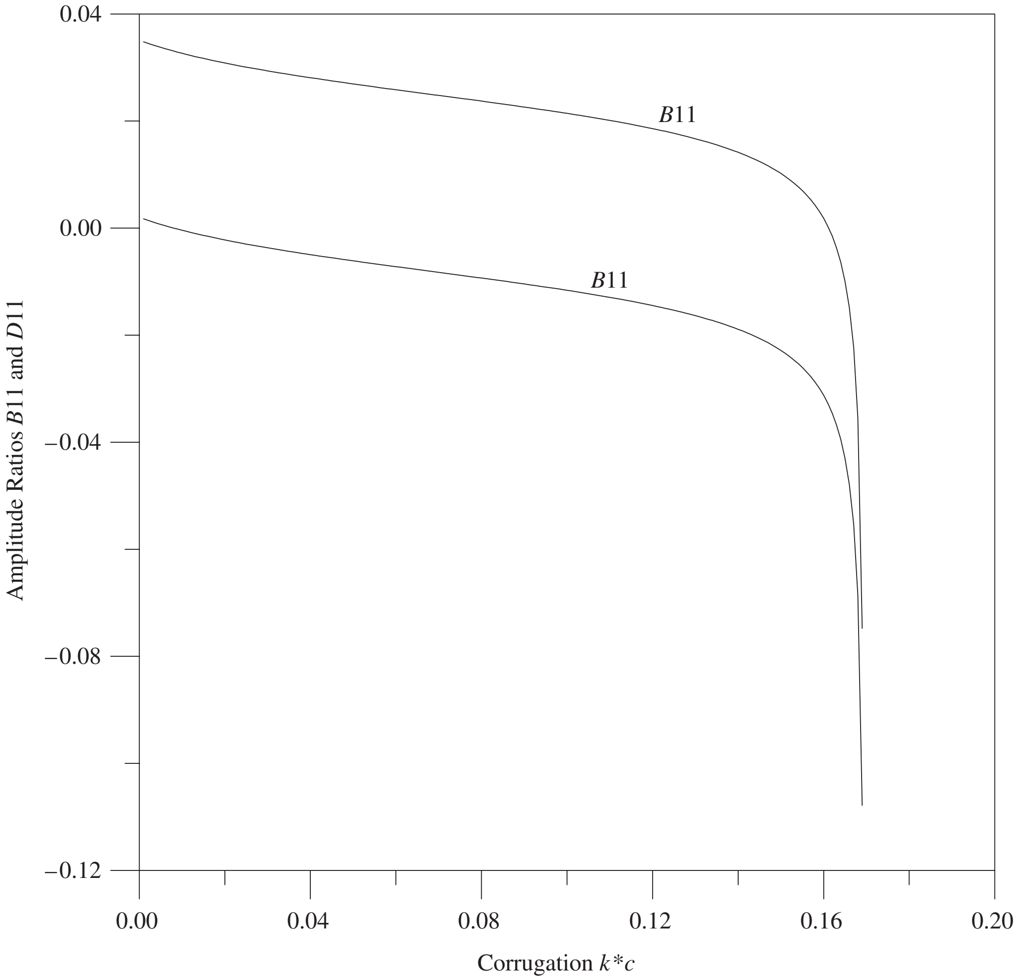

Fig. 9. Variation of the amplitude ratios $B11$ and $D11$ with the corrugation k^*c

Equations (18)–(21) provide the values of B_n, D_n, B'_n and D'_n

$$B_n = \frac{\Delta_{B_n}}{\Delta_n}, \quad D_n = \frac{\Delta_{Dn}}{\Delta_n}, \quad B'_n = \frac{\Delta_{B'_n}}{\Delta'_n}, \quad D'_n = \frac{\Delta_{D'_n}}{\Delta'_n}, \tag{22}$$

where

$$\Delta_{B_n} = \left[-\mu_1(1+B)\left(s^2 + \frac{nk^*\omega \sin e}{\beta_1}\right) + (1-B)L\eta s\, q_n + \eta LD\left(q^2 - qq_n + \frac{nk^*N\omega \sin f}{L\beta_2}\right)\right]\zeta_{-n},$$

$$\Delta_{D_n} = \left[-\mu_1(1+B)\left(s^2 - qs_n + \frac{nk^*\omega \sin e}{\beta_1}\right) - \mu_1(1-B)ss_n + \eta LD\left(q^2 + \frac{nk^*N\omega \sin f}{L\beta_2}\right)\right]\zeta_{-n},$$

$$\Delta_{B'_n} = \left[-\mu_1(1+B)\left(s^2 - \frac{nk^*\omega \sin e}{\beta_1}\right) + (1-B)L\eta s\, q'_n + \eta LD\left(q^2 - qq'_n - \frac{nk^*N\omega \sin f}{L\beta_2}\right)\right]\zeta_n,$$

$$\Delta_{D'_n} = \left[-\mu_1(1+B)\left(s^2 - qs'_n - \frac{nk^*\omega \sin e}{\beta_1}\right) - \mu_1(1-B)ss'_n + \eta LD\left(q^2 - \frac{nk^*N\omega \sin f}{L\beta_2}\right)\right]\zeta_n,$$

$$\Delta_n = \mu_1 s_n + L\eta q_n, \quad \Delta'_n = \mu_1 s'_n + L\eta q'_n.$$

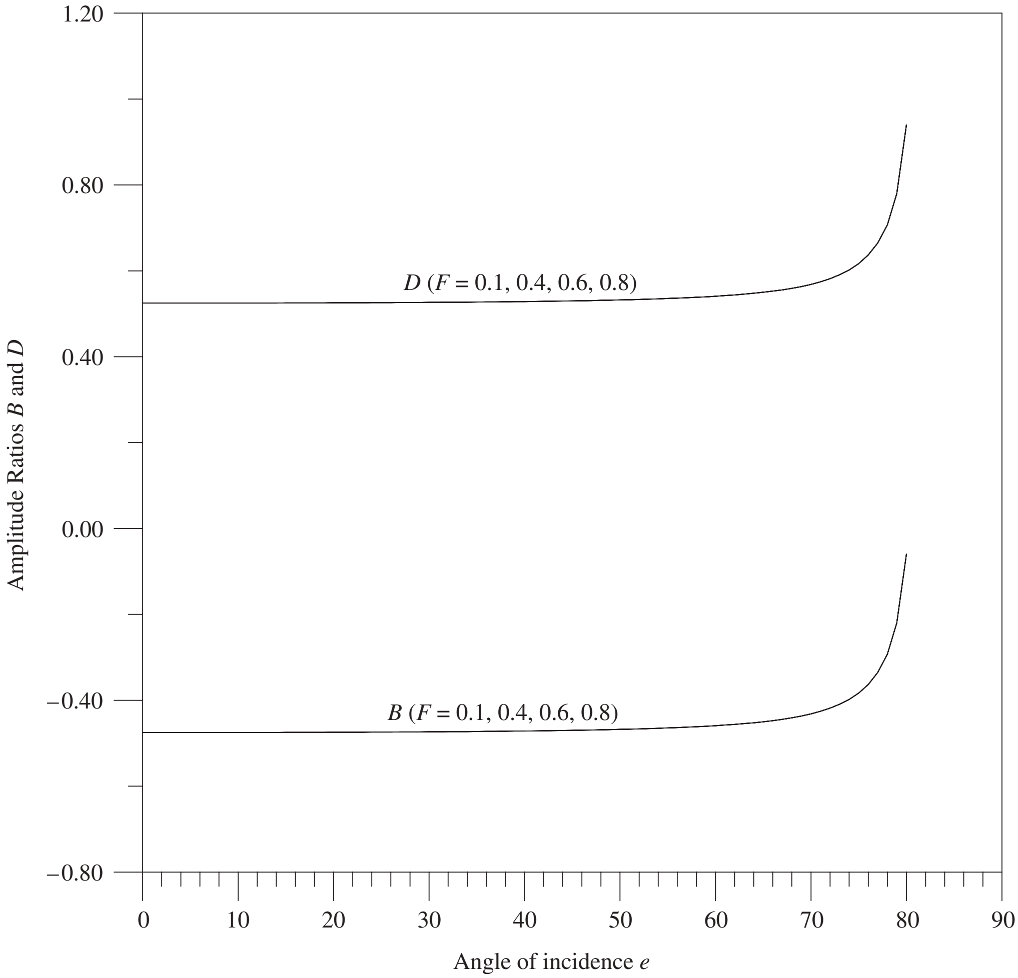

Fig. 10. Variation of the amplitude ratios B and D with the angle of incidence when the frequency $\omega\, c/\beta_1 = 0.1, 0.4, 0.6, 0.8$, where $F = \omega\, c/\beta_1$

The values of B and D appearing in the above expressions are given by Eq. (17). Here B_n, B'_n and D_n, D'_n are the reflection and transmission coefficients, respectively, for the first order approximation of the corrugation.

5 Solution of the second order approximation

If the terms of higher order other than ζ^2 are neglected, then

$$e^{\pm q\zeta} \simeq 1 \pm q\zeta + \frac{q^2\zeta^2}{2}, \text{ etc.} \tag{23}$$

To find the solution of the second order approximation for B, D, B_n, D_n, B'_n and D'_n, we collect the terms independent of x, the coefficients of $e^{-\imath m k^* x}$ and the coefficients of $e^{\imath m k^* x}$ at both sides of Eqs. (12) and (13), after inserting Eq. (23), so that

$$(1+B)(1+s^2\zeta_n\zeta_{-n}) + s_n B_n \zeta_n + s'_n B'_n \zeta_{-n} = D(1+q^2\zeta_n\zeta_{-n}) - q_n D_n \zeta_n - q'_n D'_n \zeta_{-n}, \tag{24}$$

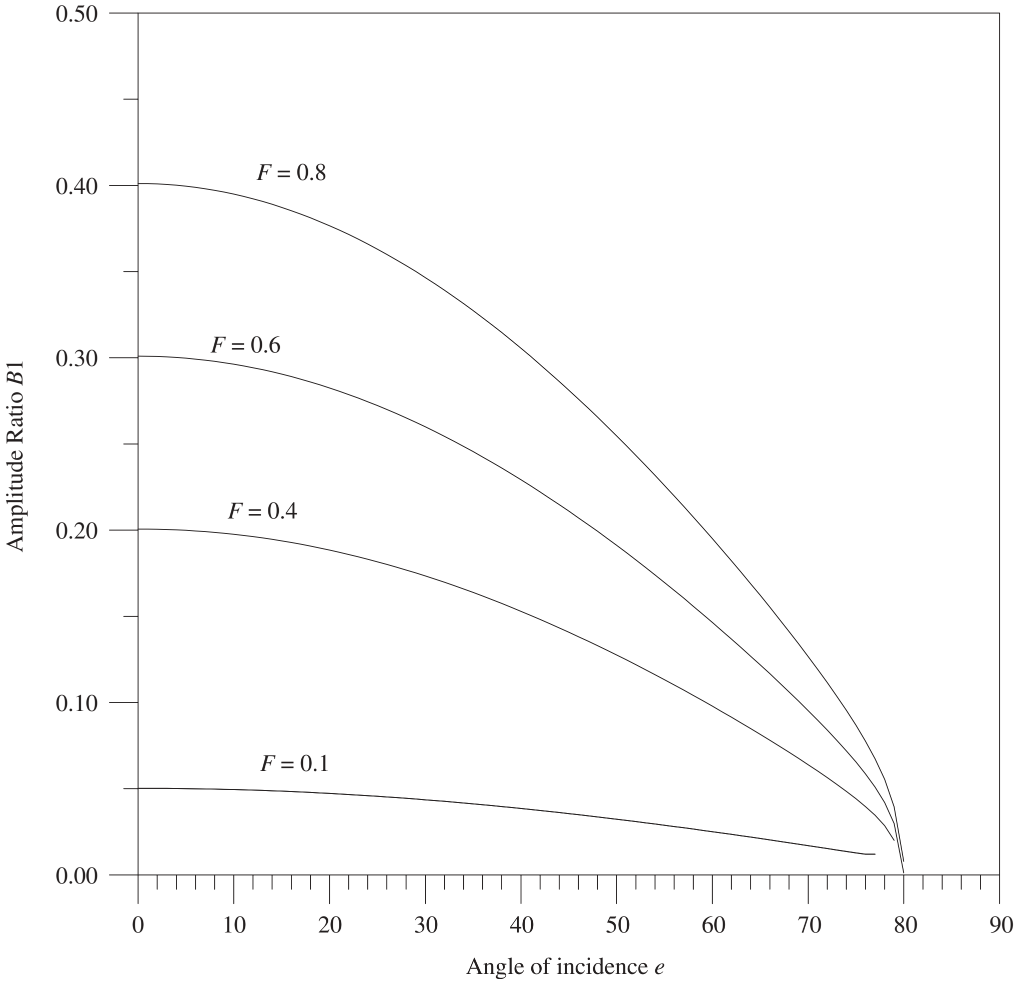

Fig. 11. Variation of the amplitude ratio $B1$ with the angle of incidence when the frequency $\omega\, c/\beta_1 = 0.1, 0.4, 0.6, 0.8$, where $F = \omega\, c/\beta_1$

$$s(1-B)[1+s^2\zeta_n\zeta_{-n}] - B_n\zeta_n\left[s_n^2 - nk^*(k+nk^*)\left(1+\frac{s_n^2}{2}\zeta_n\zeta_{-n}\right)\right] - B'_n\zeta_{-n}\left[s_n'^2 + nk^*(k-nk^*)\right.$$
$$\left.\times\left(1+\frac{s_n'^2}{2}\zeta_n\zeta_{-n}\right)\right] = \frac{L\eta}{\mu_1}\left[Dq(1+q^2\zeta_n\zeta_{-n}) - D_n\zeta_n\left\{q_n^2 - \frac{N}{L}nk^*(k+nk^*)\right.\right.$$
$$\left.\left.\times\left(1+\frac{q_n^2}{2}\zeta_n\zeta_{-n}\right)\right\} - D'_n\zeta_{-n}\left\{q_n'^2 - \frac{N}{L}nk^*(k-nk^*)\left(1+\frac{q_n'^2}{2}\zeta_n\zeta_{-n}\right)\right\}\right], \tag{25}$$

$$s(1-B)\zeta_{-n} - B_n(1+s_n^2\zeta_n\zeta_{-n}) - B'_n\frac{s_n'^2}{2}\zeta_{-n}^2 = Dq\zeta_{-n} - D_n(1+q_n^2\zeta_n\zeta_{-n}) - D'_nq_n'^2\zeta_{-n}^2, \tag{26}$$

$$(1+B)(s^2+knk^*)\zeta_{-n} - \frac{s^2}{2}(1-B)knk^*\zeta_n\zeta_{-n}^2 + B_ns_n(1+s_n^2\zeta_n\zeta_{-n}) + B'_ns'_n(1+s_n'^2\zeta_n\zeta_{-n})$$
$$= \frac{L\eta}{\mu_1}\left[D\zeta_{-n}\left\{q^2+\frac{N}{L}knk^*\left(1+\frac{q^2}{2}\zeta_n\zeta_{-n}\right)\right\} - D_nq_n(1+q_n^2\zeta_n\zeta_{-n}) - D'_nq'_n(1+q_n'^2\zeta_n\zeta_{-n})\right], \tag{27}$$

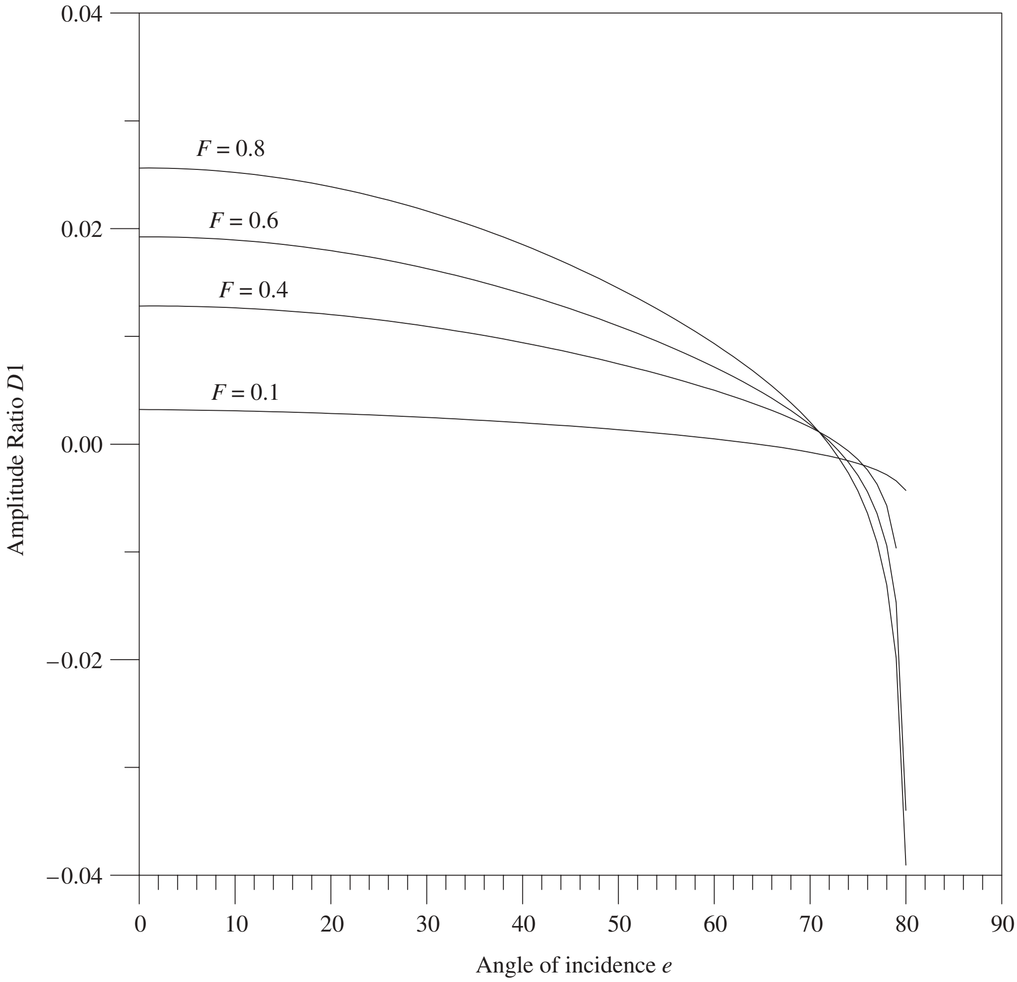

Fig. 12. Variation of the amplitude ratio $B1$ with the angle of incidence when the frequency $\omega\, c/\beta_1 = 0.1, 0.4, 0.6, 0.8$, where $F = \omega\, c/\beta_1$

$$s(1-B)\zeta_n - B_n \frac{s_n^2}{2}\zeta_n^2 - B'_n(1+s'^2_n\zeta_n\zeta_{-n}) = Dq\zeta_n - D_n\frac{q_n^2}{2}\zeta_n^2 - D'_n(1+q'^2_n\zeta_n\zeta_{-n}) \tag{28}$$

$$\begin{aligned}&(1+B)(s^2-knk^*)\zeta_n + \frac{s^2}{2}(1-B)knk^*\zeta_{-n}\zeta_n^2 + B_n s_n(1+s_n^2\zeta_n\zeta_{-n}) + B'_n s'_n(1+s'^2_n\zeta_n\zeta_{-n})\\ &\quad = \frac{L\eta}{\mu_1}\left[D\zeta_n\left\{q^2 - \frac{N}{L}knk^*\left(1+\frac{q^2}{2}\zeta_n\zeta_{-n}\right)\right\} - D_n q_n(1+q_n^2\zeta_n\zeta_{-n}) - D'_n q'_n(1+q'^2_n\zeta_n\zeta_{-n})\right],\\ &\quad \text{where } k = k_1\sin e = k_2 \sin f\end{aligned} \tag{29}$$

Equations (24)–(29) enable one to calculate the reflection and transmission coefficients for the second order approximation of the corrugation.

6 Special case

For $\zeta_n = \zeta_{-n} = 0, (n \neq 1)$ and $\zeta_1 = \zeta_{-1} = c/2$, the interface is given by $z = c\cos k^* x$, where c is the amplitude of the corrugation. From Eq. (22), we obtain the formulae for B_1, D_1, B'_1 and D'_1 for the first order approximation of the corrugation,

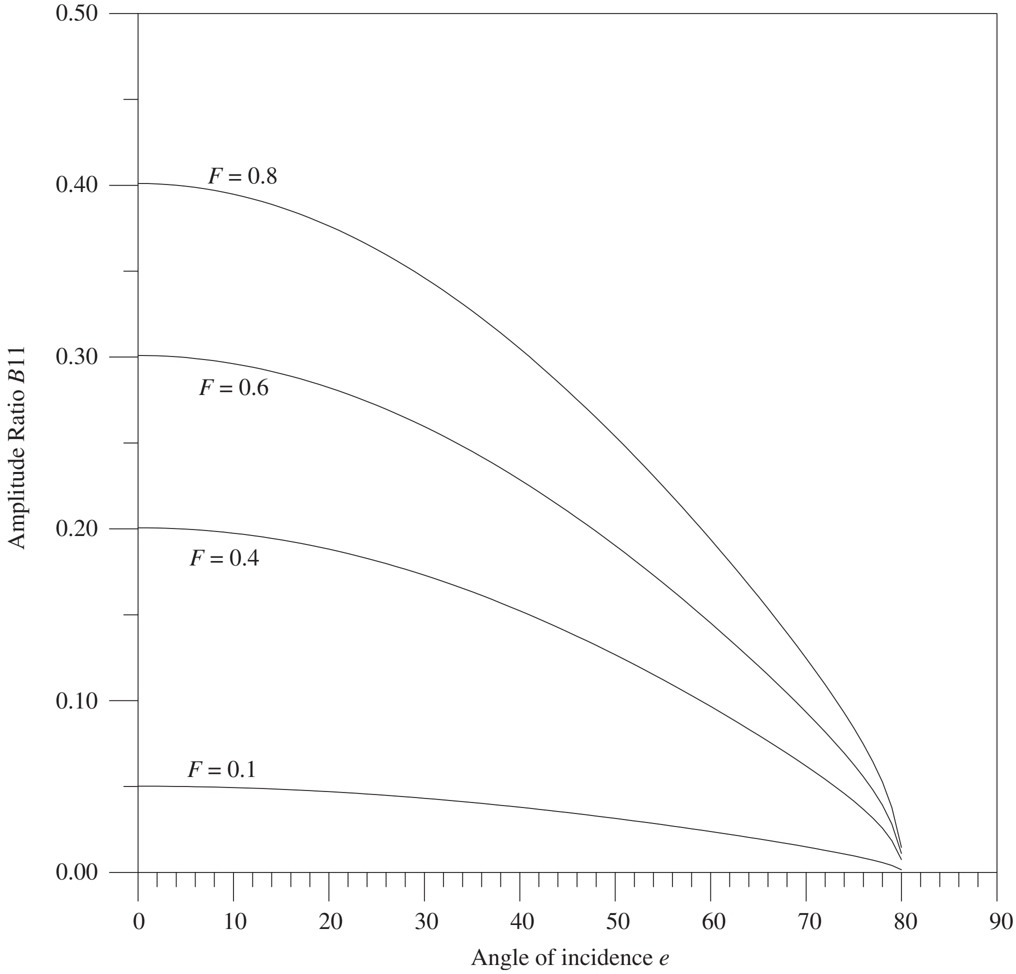

Fig. 13. Variation of the amplitude ratio $B1$ with the angle of incidence when the frequency $\omega\, c/\beta_1 = 0.1, 0.4, 0.6, 0.8$, where $F = \omega\, c/\beta_1$

$$B_1 = \frac{\Delta_{B_1}}{\Delta_1}, \quad D_1 = \frac{\Delta_{D_1}}{\Delta_1}, \quad B_1' = \frac{\Delta_{B_1'}}{\Delta_1'}, \quad D_1' = \frac{\Delta_{D_1'}}{\Delta_1'}, \tag{30}$$

where the values of $\Delta_{B_1}, \Delta_{D_1}, \Delta_{B_1'}, \Delta_{D_1'}$, Δ_1 and Δ_1' are

$$\Delta_{B_1} = \frac{c}{2}\left[-\mu_1(1+B)\left(s^2 + \frac{k^*\omega \sin e}{\beta_1}\right) + (1-B)L\eta s\, q_1 + L\eta D\left(q^2 - qq_1 + \frac{k^* N\omega \sin f}{L\beta_2}\right)\right],$$

$$\Delta_{D_1} = \frac{c}{2}\left[-\mu_1(1+B)\left(s^2 - qs_1 + \frac{k^*\omega \sin e}{\beta_1}\right) - \mu_1(1-B)s\, s_1 + LD\eta\left(q^2 + \frac{k^* N\omega \sin f}{L\beta_2}\right)\right],$$

$$\Delta_{B_1'} = \frac{c}{2}\left[-\mu_1(1+B)\left(s^2 - \frac{k^*\omega \sin\ e}{\beta_1}\right) + (1-B)L\eta s\, q_1' + LD\eta\left(q^2 - qq_1' - \frac{k^* N\omega \sin f}{L\beta_2}\right)\right],$$

$$\Delta_{D_1'} = \frac{c}{2}\left[-\mu_1(1+B)\left(s^2 - qs_1' - \frac{k^*\omega \sin e}{\beta_1}\right) - \mu_1(1-B)s\, s_1' + LD\eta\left(q^2 - \frac{k^* N\omega \sin f}{L\beta_2}\right)\right],$$

$$\Delta_1 = \mu_1 s_1 + L\eta q_1, \quad \Delta_1' = \mu_1 s_1' + L\eta q_1'.$$

From Eqs. (30) it follows that the coefficients for the first order approximation of the corrugation are proportional to the amplitude of the corrugated interface.

7 Particular cases

(a) When the sandy and anisotropy factors of the two media are neglected, then they become isotropic elastic solid half-spaces. The problem then reduces to that of reflection and refraction of *SH*-waves incident at a corrugated interface between two elastic half-spaces. Plugging $\eta = 1$ and $N = L = \mu_2$ in Eqs. (4), (9) and (10), the values of s, q, s_n, q_n, s'_n and q'_n reduce to

$$s = \imath \frac{\omega}{\beta_1} \cos e, \qquad q = \imath \frac{\omega}{\beta_1} \left(\frac{\beta_1^2}{\beta_2^2} - \sin^2 e \right)^{1/2},$$

$$s_n = \imath \frac{\omega}{\beta_1} \cos e_n, \quad q_n = \imath \frac{\omega}{\beta_1} \left(\frac{\beta_1^2}{\beta_2^2} - \sin^2 e_n \right)^{1/2},$$

$$s'_n = \imath \frac{\omega}{\beta_1} \cos e'_n, \quad q'_n = \imath \frac{\omega}{\beta_1} \left(\frac{\beta_1^2}{\beta_2^2} - \sin^2 e'_n \right)^{1/2}.$$

The coefficients B_1, D_1, B'_1 and D'_1, in this case, become

$$B_1 = \frac{c}{2\Delta_1} \left[-\mu_1 (1+B) \left(s^2 + \frac{k^* \omega \sin\ e}{\beta_1} \right) + \mu_2 s q_1 (1-B) + \mu_2 D \left(q^2 - q q_1 + \frac{k^* \omega \sin e}{\beta_1} \right) \right],$$

$$D_1 = \frac{c}{2\Delta_1} \left[-\mu_1 (1+B) \left(s^2 - q s_1 + \frac{k^* \omega \sin e}{\beta_1} \right) - \mu_1 s\, s_1 (1-B) + \mu_2 D \left(q^2 + \frac{k^* \omega \sin e}{\beta_1} \right) \right],$$

$$B'_1 = \frac{c}{2\Delta'_1} \left[-\mu_1 (1+B) \left(s^2 - \frac{k^* \omega \sin\ e}{\beta_1} \right) + \mu_2 s q'_1 (1-B) + \mu_2 D \left(q^2 - q q'_1 - \frac{k^* \omega \sin e}{\beta_1} \right) \right],$$

$$D'_1 = \frac{c}{2\Delta'_1} \left[-\mu_1 (1+B) \left(s^2 - q s'_1 - \frac{k^* \omega \sin e}{\beta_1} \right) - \mu_1 s s'_1 (1-B) + \mu_2 D \left(q^2 - \frac{k^* \omega \sin e}{\beta_1} \right) \right],$$

$$\Delta_1 = \mu_1 s_1 + \mu_2 q_1, \qquad \Delta'_1 = \mu_1 s'_1 + \mu_2 q'_1.$$

These formulae give the reflection and refraction coefficients for the first order approximation of the corrugated interface between two uniform elastic half spaces. It can be verified that by removing the sandy and anisotropy behaviors as explained above, the boundary conditions (12) and (13) match with those of [1].

Further, if we replace the corrugated interface by a plane interface, i.e., if we put $\zeta = 0$ in Eqs. (15), (16), (18) to (21), the problem reduces to that of reflection and refraction of *SH*-waves incident at a plane interface $z = 0$ between two homogeneous isotropic elastic half-spaces. In this case, B_1, D_1, B'_1 and D'_1 will vanish, since they are proportional to c. Setting $\frac{\mu_1}{\mu_2} = m_1$ and $\frac{\beta_1}{\beta_2} = m_2$, the reflection and transmission coefficients at the plane interface, as given by Eq. (17), become

$$B = \frac{m_1 \cos e - \sqrt{m_2^2 - \sin^2 e}}{m_1 \cos e + \sqrt{m_2^2 - \sin^2 e}}, \quad D = \frac{2 m_1 \cos e}{m_1 \cos e + \sqrt{m_2^2 - \sin^2 e}},$$

which are identical to those given in [30, p. 284] for the considered problem.

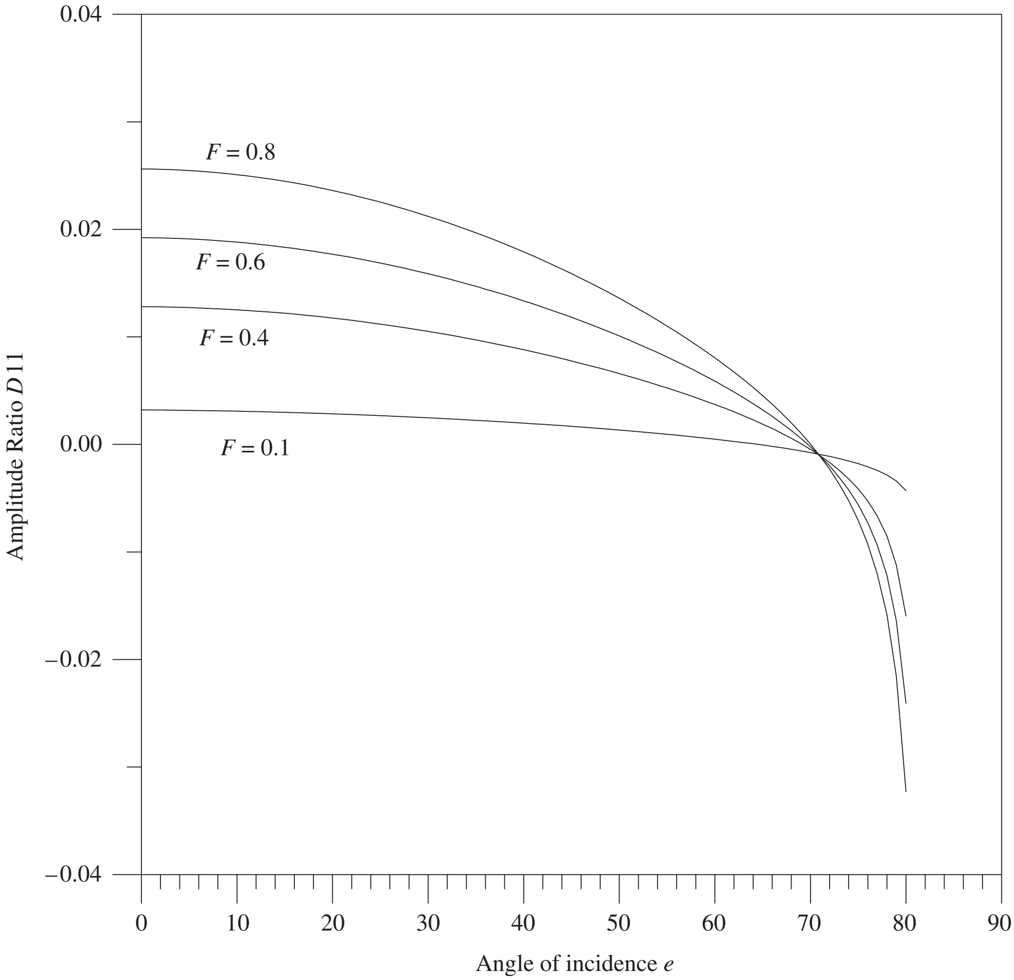

Fig. 14. Variation of the amplitude ratio $D11$ with the angle of incidence when the frequency $\omega\, c/\beta_1 = 0.1, 0.4, 0.6, 0.8$, where $F = \omega\, c/\beta_1$

(b) When we set $\eta = 1$, the half-space H_1 becomes homogeneously elastic. Thus, in this case, plugging the values of s and q and using the relation $N^* = N/\mu_1$ and $L^* = L/\mu_1$, the reflection and refraction coefficients at the plane interface B and D and at the corrugated interface B_1, D_1, B_1' and D_1' for the first order approximation of the corrugated interface between uniform elastic and anisotropic media can be obtained from Eq. (30).

5 Energy partition equation

The expression for the energy flux for each of the incident, reflected and refracted waves is obtained by multiplying the total energy per unit volume, which is twice the mean kinetic energy density, by the velocity of propagation and the area of the wave front involved (see [31, pp. 35, 166–167]). The area of the wave front is proportional to the cosine of the angle between the normal and the vertical. Thus, using Snell's law, Spectrum theorem and dividing the energy

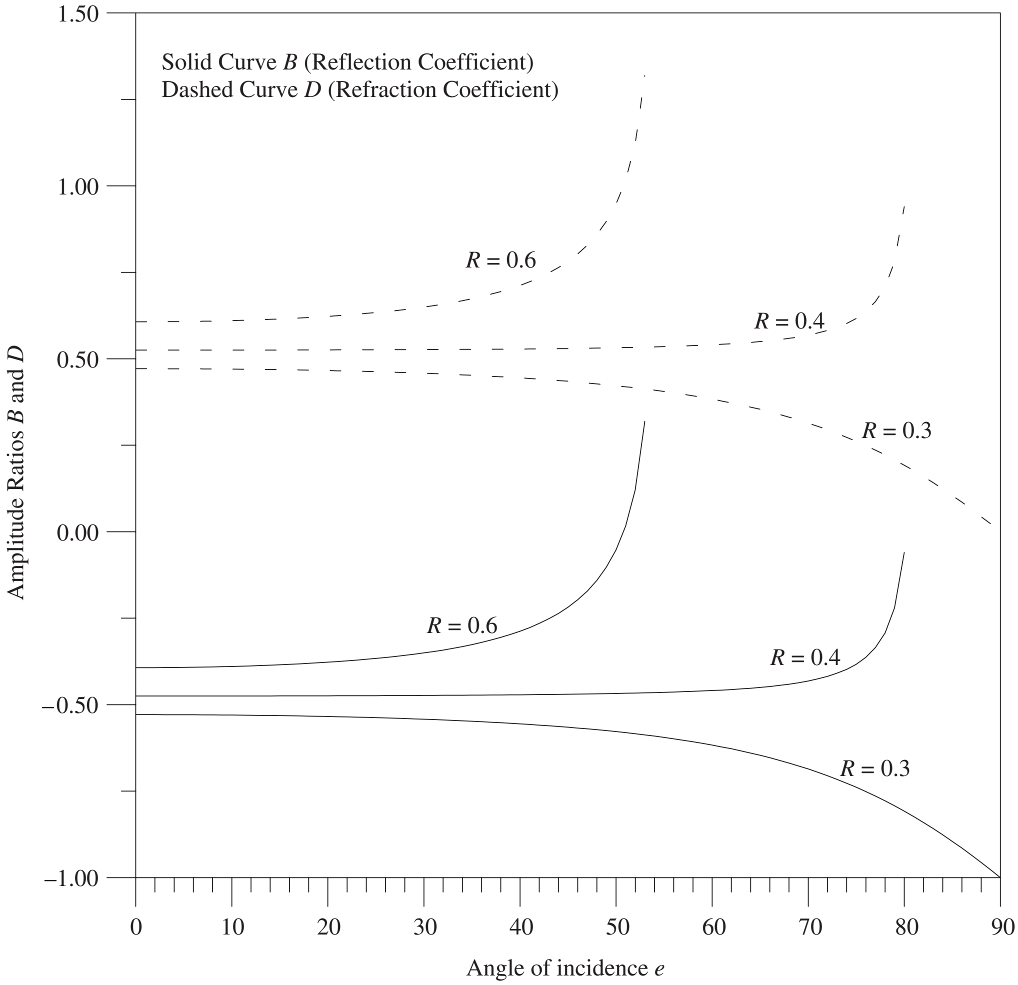

Fig. 15. Variation of the amplitude ratios B and D with the angle of incidence when $\rho_1/\rho_2 = 0.3, 0.4, 0.6$, where $R = \rho_1/\rho_2$

flux of the incident wave, the energy equation for the incident, regularly reflected, irregularly reflected, regularly refracted and irregularly refracted SH-waves for the n^{th} order approximation of the corrugation can be written as [5]

$$\begin{aligned} 1 = & |B^2| + \sum_{n=1}^{\infty} \frac{\cos e_n}{\cos e} |B_n^2| + \sum_{n=1}^{\infty} \frac{\cos e'_n}{\cos e} |B_n'^2| + \frac{\rho_2 \beta_2 \cos f}{\rho_1 \beta_1 \cos e} |D^2| \\ & + \sum_{n=1}^{\infty} \frac{\rho_2 \beta_2 \cos f_n}{\rho_1 \beta_1 \cos e} |D_n^2| + \sum_{n=1}^{\infty} \frac{\rho_2 \beta_2 \cos f'_n}{\rho_1 \beta_1 \cos e} |D_n'^2| \,. \end{aligned} \tag{31}$$

The partition of energy at a plane interface between the sandy and anisotropic half-spaces can be readily deduced from Eq. (31) by putting the values of the coefficients B_n, D_n, B'_n and D'_n equal to zero, as they are proportional to the amplitude of the corrugated interface,

$$1 = |B^2| + \frac{\rho_2\, \beta_2^2 \tan e}{\rho_1\, \beta_1^2 \tan f} |D^2| \,.$$

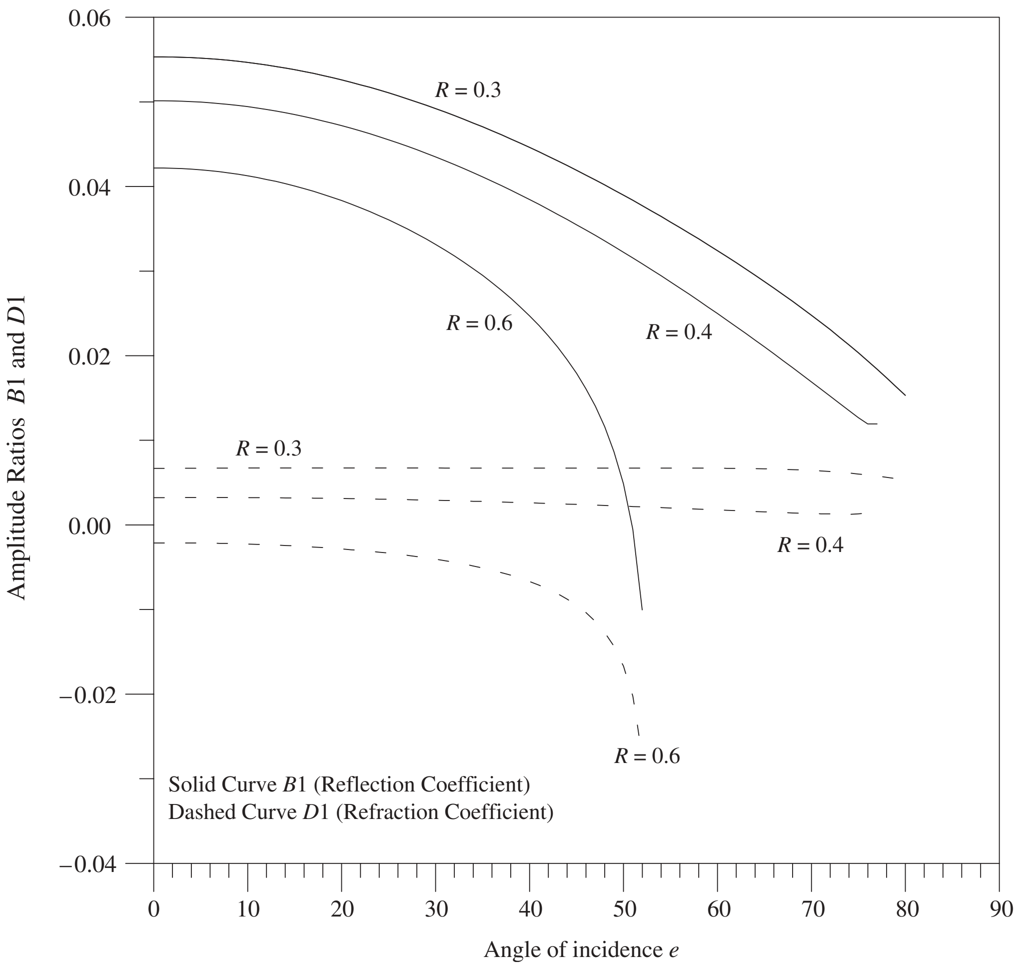

Fig. 16. Variation of the amplitude ratios $B1$ and $D1$ with the angle of incidence when $\rho_1/\rho_2 = 0.3, 0.4, 0.6$, where $R = \rho_1/\rho_2$

This relation is identical to that for the energy partition for the *SH*-wave incident at a plane interface between the sandy layer and anisotropic half-spaces, given in [25] for the considered problem.

In the present formulation, i.e., when $n = 1$, from Eq. (31) it follows that

$$\sum_{i=1}^{6} E_i \approx 1,$$

where E_1 and E_2 are the ratios of energy transmitted by the regularly reflected and refracted waves to the energy transmitted by the incident wave. Similarly, E_3, E_5 and E_4, E_6 are ratios of energy transmitted by irregularly reflected and irregularly refracted waves to the energy transmitted by the incident wave for the first order approximation of the corrugation. The energy ratios are then given by

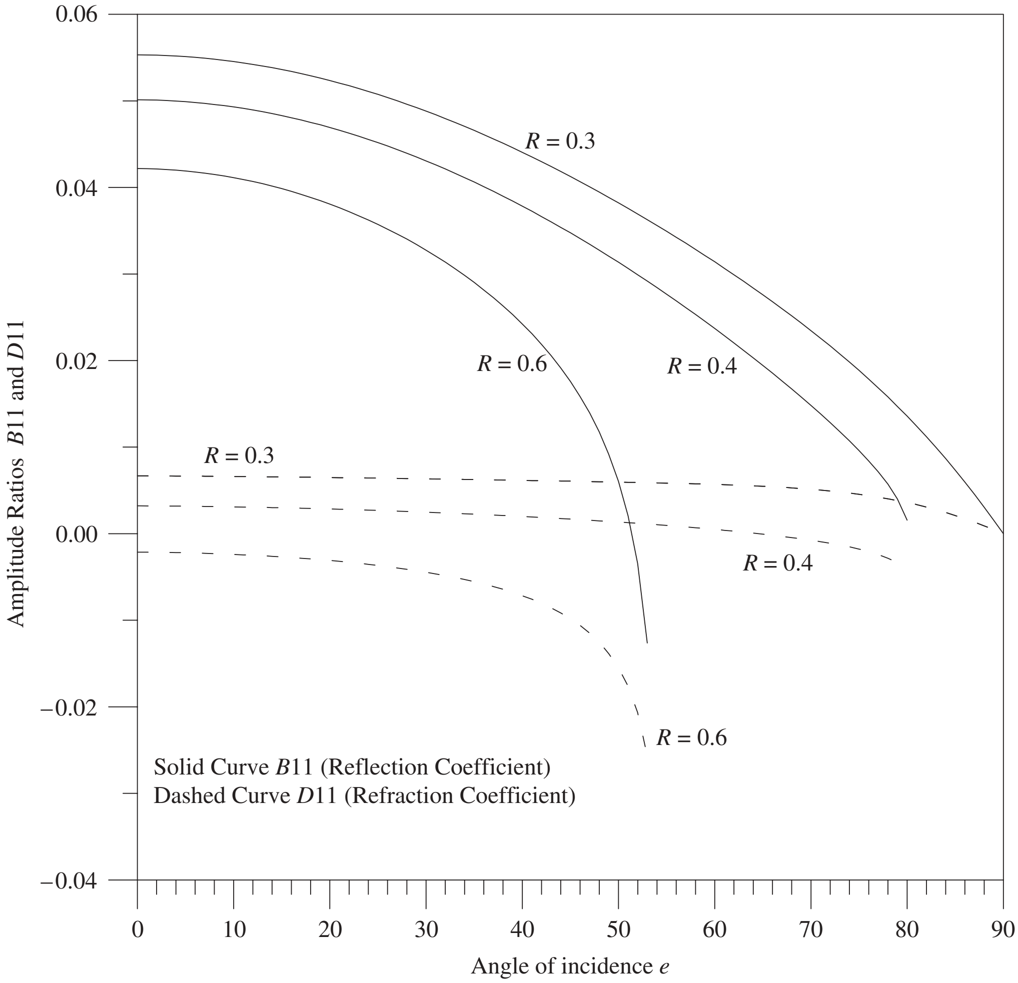

Fig. 17. Variation of the amplitude ratios $B11$ and $D11$ with the angle of incidence when $\rho_1/\rho_2 = 0.3, 0.4, 0.6$, where $R = \rho_1/\rho_2$

$$E_1 = |B^2|, \quad E_2 = \frac{\rho_2\beta_2 \cos f}{\rho_1\beta_1 \cos e} \mid D^2 \mid, \quad E_3 = \frac{\cos e_1}{\cos e} \mid B_1^2 \mid,$$

$$E_4 = \frac{\rho_2\beta_2 \cos f_1}{\rho_1\beta_1 \cos e} \mid D_1^2 \mid, \quad E_5 = \frac{\cos e_1'}{\cos e} \mid B_1'^2 \mid, \quad E_6 = \frac{\rho_2\beta_2 \cos f_1'}{\rho_1\beta_1 \cos e} \mid D_1'^2 \mid .$$

8 Numerical results and discussion

In order to study the effects of the sandy parameter, the anisotropy, the corrugation of the interface, the frequency and the angle of the incidence on the reflection and transmission coefficients, when a plane SH-wave is obliquely incident at the corrugated interface between the two half spaces H_1 and H_2, we computed these coefficients for the model considered in Sect. 6. For this purpose, we selected the following numerical values of the elastic parameters (see

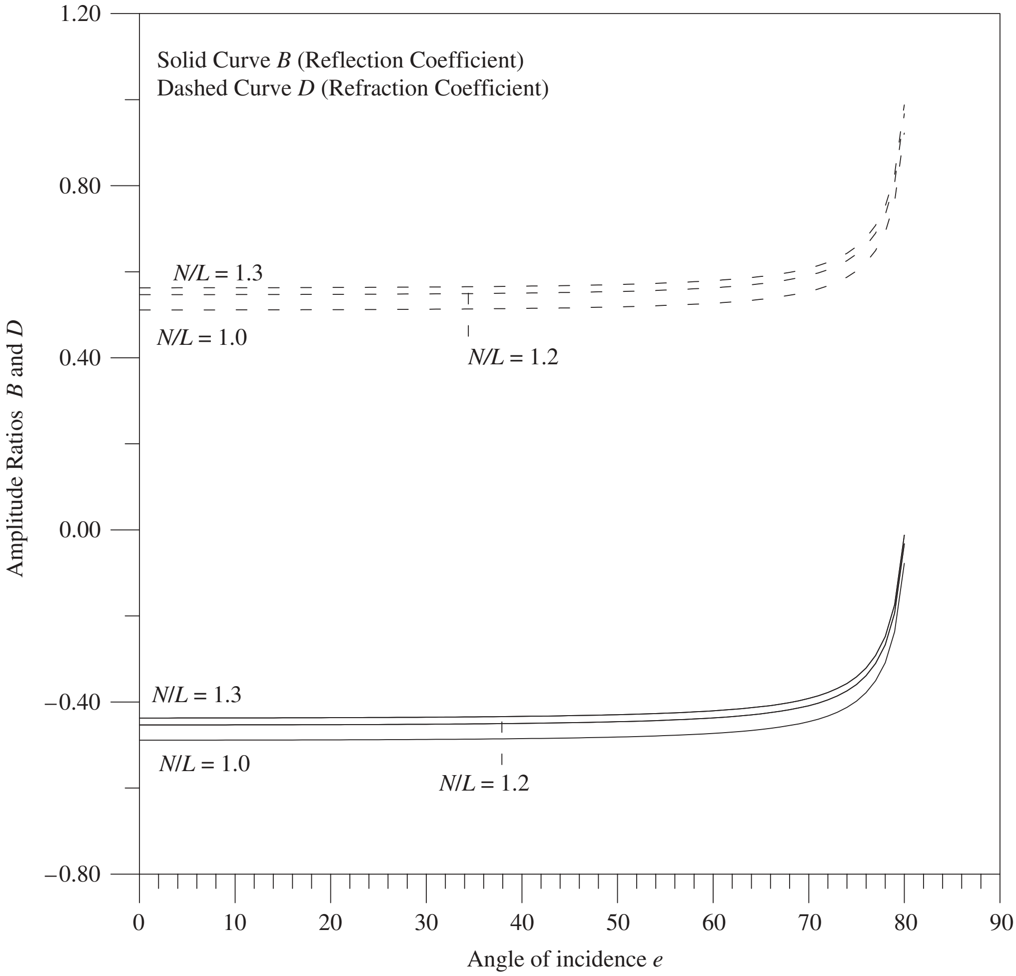

Fig. 18. Variation of the amplitude ratios B and D with the angle of incidence when the anisotropy factor $N/L = 1.0, 1.2, 1.3$

Chakraborty and Chandra [25]): $\frac{N}{\mu_1} = 2.95$, $\frac{L}{\mu_1} = 2.75$, $\frac{\rho_2}{\rho_1} = 2.5$, and $\omega c/\beta_{h_1} = 0.10$, where c is the amplitude of corrugation, $\eta = 1.15$, the angle of incidence $e = 45^0$, and $k^*c = 0.00125$.

(i) The effect of sandiness of the half-space H_1: To study this effect on the reflection and transmission coefficients at both the plane and corrugated interface, we selected, $\eta = 1.00, 1.10, 1.15, 1.20$ and 1.30. Figures 2 and 3 show the reflection and transmission coefficients as functions of the angle of incidence at a plane interface between H_1 and H_2. Note the significant effect η on these coefficients. Note that the values of the coefficients decrease slowly with the increase of sandiness. Also, with the increase of the angle of incidence (and that of the sandiness η from 1.00 to 1.15), the values of the coefficients increase. Furthermore, when the sandiness η varies from 1.20 to 1.30, the values of the reflection and transmission coefficients decrease considerably with the increase of the angle of incidence.

Figures 4 to 7 depict the reflection and transmission coefficients as functions of the angle of incidence e for the first-order approximation of the corrugated interface. The effect of

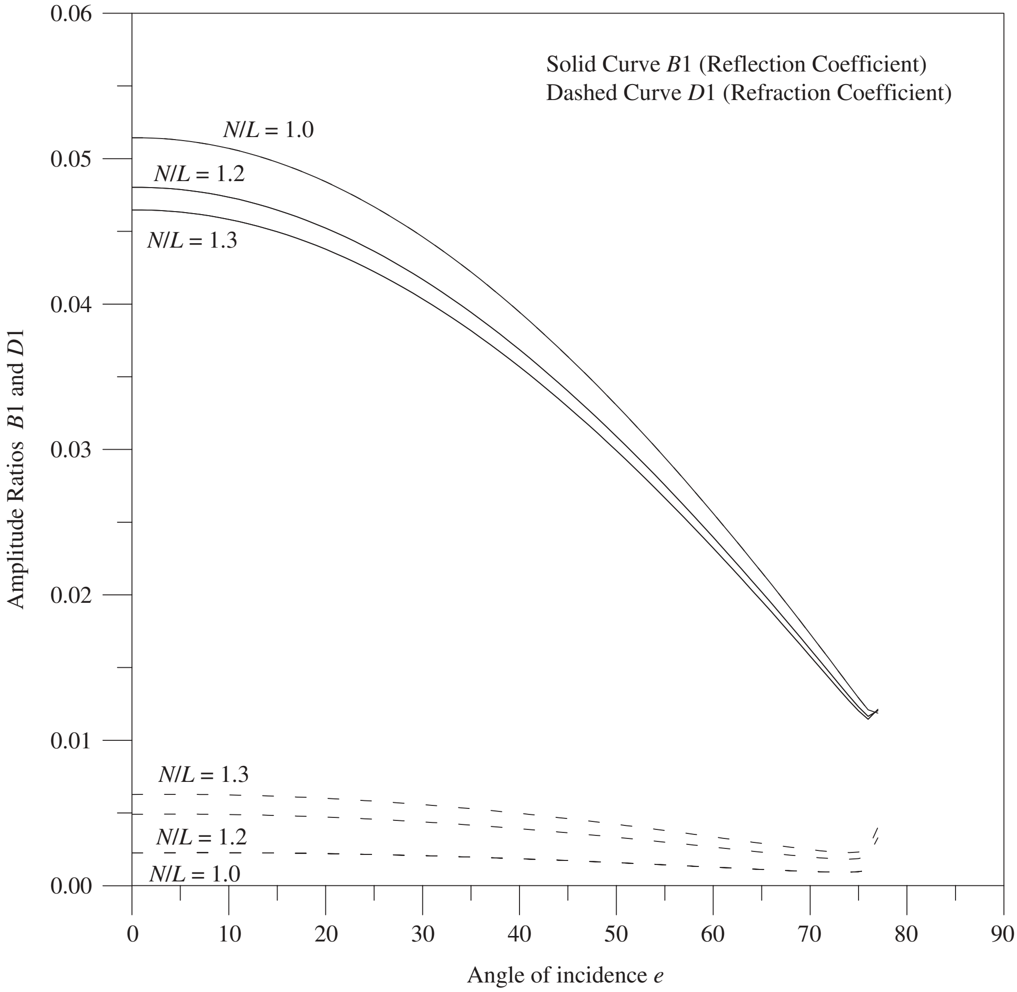

Fig. 19. Variation of the amplitude ratios B and D with the angle of incidence when the anisotropy factor $N/L = 1.0, 1.2, 1.3$

sandiness on these coefficients is clearly visible. The values of the reflection and transmission coefficients B_1, B_1' and D_1, D_1' decrease monotonically with the increase of the angle of incidence e and increase monotonically with the increase of sandiness η. The behavior of all these coefficients at the corrugated interface is similar. We also note from these figures that the values of D_1 and D_1' are less than those of B_1 and B_1'. However, the critical angles are different for different values of η.

(ii) The effect of corrugation: As B and D are the reflection and transmission coefficients, respectively, for the plane interface, it is obvious that they are independent of the corrugation parameter k^*c. This is also clear from the analytical results shown in Eq. (17). Here, we found that the coefficients B_1, B_1', D_1 and D_1' are strongly affected by the corrugation k^*c. Figures 8 and 9 show B_1, D_1, B_1' and D_1' as functions of the corrugation k^*c. Note that the values of B_1 and D_1 increase with increase of k^*c, while the values of B_1' and D_1' decrease with increase of k^*c. Also, the values of B_1 and D_1' are greater than those of D_1 and B_1'.

(iii) The effect of frequency: Figures 10 to 14 show the effect of the frequency on the reflection and transmission coefficients for both the plane and the corrugated interfaces. Note that

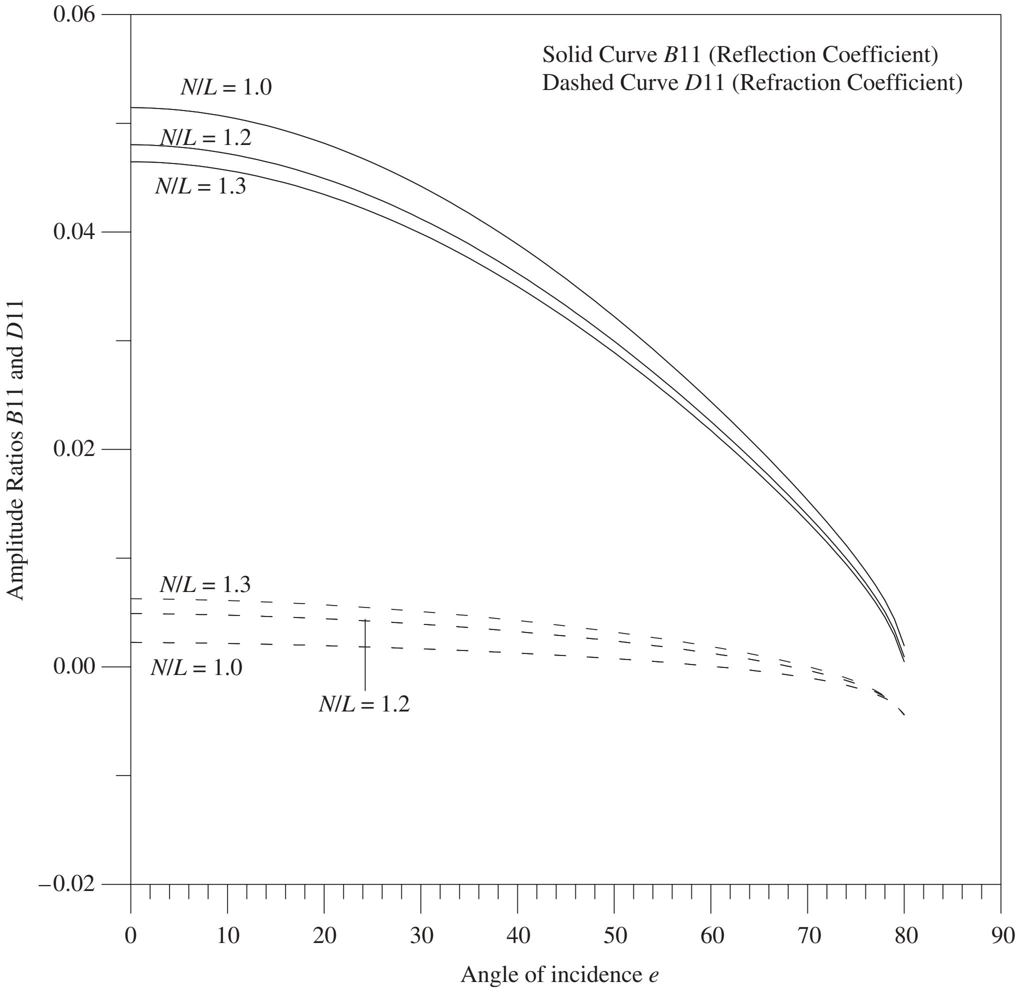

Fig. 20. Variation of the amplitude ratio B and D with the angle of incidence when the anisotropy factor $N/L = 1.0, 1.2, 1.3$

the amplitude of the transmitted wave at the plane interface is greater than that of the reflected wave at the plane interface, whereas the opposite behavior is found in the case of the reflection and transmission coefficients for the first order approximation of corrugation. Also, each coefficient B_1, D_1, B_1' and D_1' decreases with the increase of the angle of incidence e and also increases with the increase of frequency $\omega c/\beta_{h_1}$.

(iv) The effect of densities of the half-spaces H_1 and H_2: Figures 15 to 17 show the variation of B, D and B_1, D_1 and B_1', D_1' with the angle of incidence e, respectively, when the density of the upper half-space H_1 increases slightly. It is found that the values of these coefficients are affected by the variation of the density ρ_1 of the half-space H_1. The values of the coefficients B, D and B_1' increase while those of B_1, D_1 and D_1' decrease with the increase of density ρ_1 of the upper half-space H_1.

(v) The effect of anisotropy: Figures 18 to 20 show the variation of B, D, B_1, D_1, B_1' and D_1' with the angle of incidence e for different values of the anisotropy factor N/L in the lower half-space H_2. These values are taken as $N/L = 1.0, 1.2$ and 1.3. Note that there is a significant effect of anisotropy of the half-space H_2 on each coefficient. The values of the

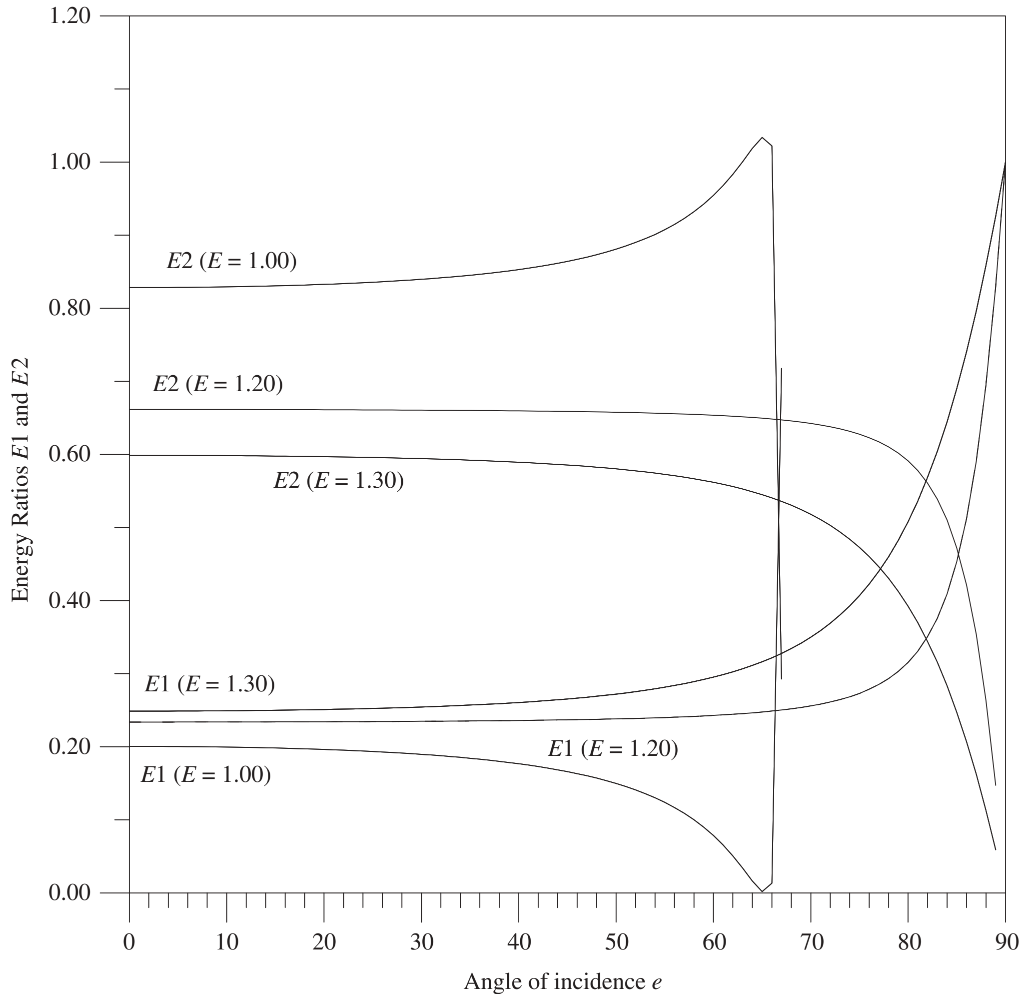

Fig. 21. Variation of the energy ratios E_1 and E_2 for different sandy factors $\eta = 1.0, 1.2, 1.3$, where $E = \eta$

reflection and transmission coefficients at the plane interface B and D increase with increase of anisotropy factor N/L. The values of the reflection coefficients B_1 and B_1' for the first order approximation of the corrugation decrease with increase of anisotropy factor N/L, while the transmission coefficients D_1 and D_1' for the first order approximation of the corrugation increase with increase of the anisotropy factor N/L. The values of the reflection coefficients at the corrugated interface decrease with increase of the anisotropy factor and the angle of incidence, whereas those of the transmission coefficients at the corrugated interface decrease with the increase of the angle of incidence and decrease of anisotropy factor.

(vi) Partition of energy: Figures 21 to 23 depict the variation of the reflected and transmitted energy ratios of SH-waves with the angle of incidence e at the plane and the corrugated interfaces for the first order approximation of corrugation. Note that the values of the energy ratios of the refracted wave at the plane interface E_2 are maximum in comparison to all other energy ratios. Also, with the increase of sandiness factor of the upper half-space H_1, the values of the energy ratios of the reflected wave increase, while those of the

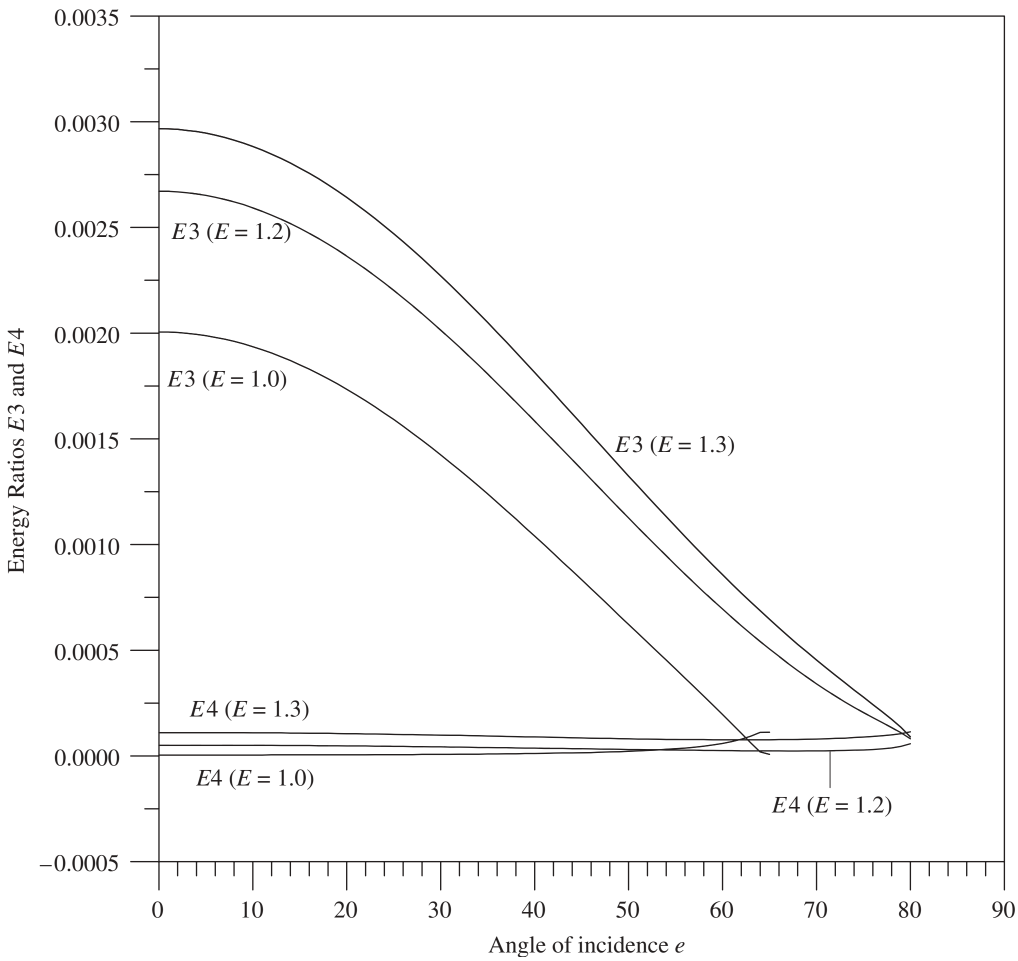

Fig. 22. Variation of the energy ratios E_3 and E_4 for different sandy factors $\eta = 1.0, 1.2, 1.3$, where $E = \eta$

refracted waves decrease. The sum of the energy ratios is found to be less than one. This is obvious, since we are considering only the coefficients of the first order approximation of corrugation.

9 Conclusions

Using Rayleigh's method, the formulae for the reflection and transmission coefficients due to an incident *SH*-wave at a corrugated interface separating a dry sandy layer and anisotropic elastic semi-infinite media are obtained in closed form.

The reflection and transmission coefficients for the first order approximation of the corrugation given by Eqs. (22) and (23) depend on ζ_{-n} and ζ_n, i.e, on ζ, the amplitude of the corrugation of the interface.

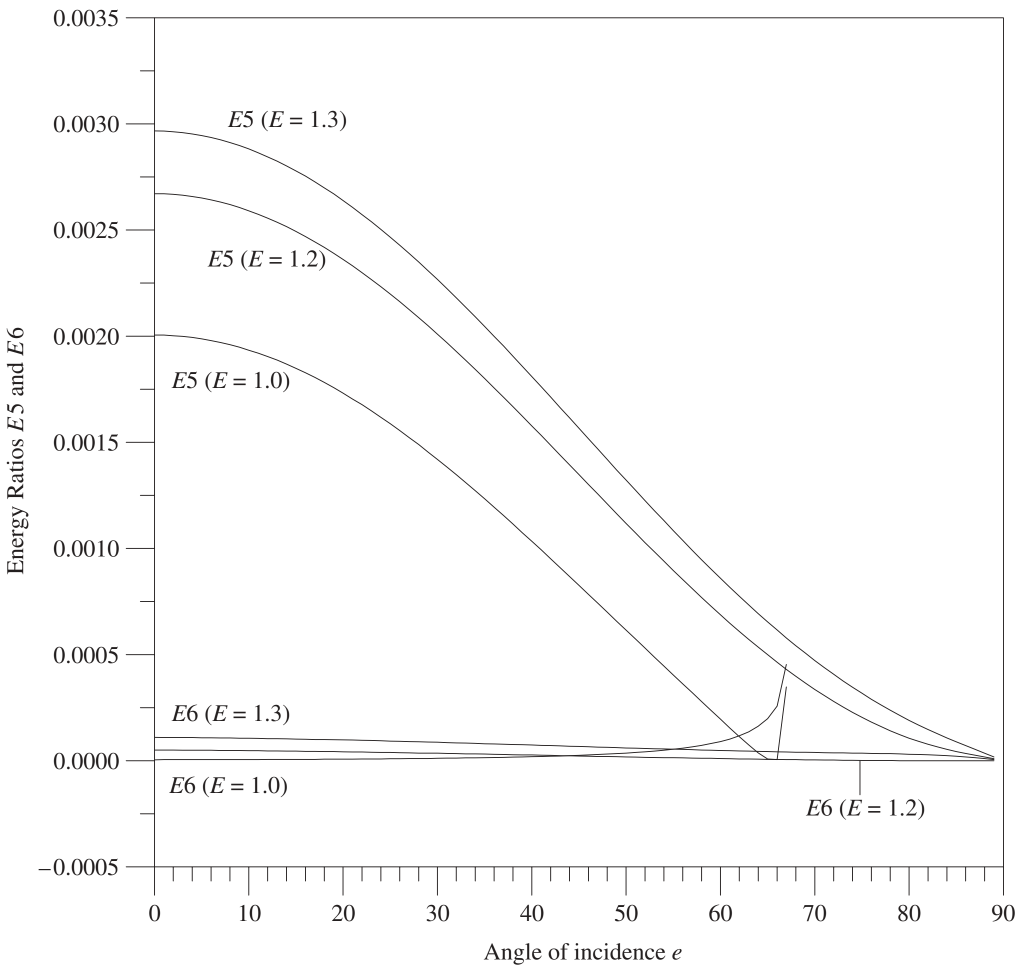

Fig. 23. Variation of the energy ratios E_5 and E_6 for different sandy factors $\eta = 1.0, 1.2, 1.3$, where $E = \eta$

From Eqs. (17) and (22) and numerical results we conclude that the reflection and transmission coefficients strongly depend upon the sandy material, the anisotropy, the frequency and the angle e of the incident SH-wave.

The reflection and transmission coefficients for the first order approximation of the corrugation are found to be affected by the sandy factor η. The effect is found least near normal incidence. However, different critical angles occur at different values of η.

As the corrugation parameter increases, the values of B_1 and D_1 increase while that of B_1' and D_1' decrease.

The reflection and transmission coefficients are found to be influenced by the frequency of the incident wave. The values of each coefficient B_1, D_1, B_1' and D_1' at the corrugated interface decrease with the increase of the angle of incidence and increase with the increase of the frequency.

The increase of the density ρ_1 of the upper half-space H_1 has a pronounced effect on the reflection and transmission coefficients. The values of the coefficients B, D and B_1' increase with the increase of density ρ_1 in the upper half-space H_1 while those of B_1, D_1 and D_1' decrease with the increase of density ρ_1 in the upper half-space H_1.

There is a significant effect of the anisotropy in the lower half-space H_2 on each coefficient, whether for the plane interface or for the corrugated interface. The coefficients B_1, B_1' decrease with the increase of the anisotropy and the angle of incidence whereas those of D_1, D_1' decrease with the angle of incidence e and increase with the anisotropy.

With the increase of the sandiness parameter η in the upper half-space H_1, the values of energy ratios of the reflected waves E_1, E_3 and E_5 increase while those of the refracted waves E_2, E_4 and E_6 decrease.

The problems treated by Asano [1], Savarensky [30] and Chakraborty and Chandra [25] were obtained as particular cases.

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