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Tri-diagonal and penta-diagonal block matrices for efficient eigensolutions of problems in structural mechanics

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Summary. Eigenvalues and eigenvectors have many applications in structural mechanics and combinatorial optimization. In this paper, a set of matrices of special forms is studied for which the calculation of eigenvalues can be performed much easier than with the existing general methods. First tri-diagonal matrices are presented and then the relationships for calculating their eigenvalues are extended to the evaluation of the eigenvalues of block tri-diagonal matrices. Block penta-diagonal matrices are also studied in this paper. The eigensolution of different problems of structural mechanics is performed to show the simplicity of using the present formulations.

1 Introduction

Large eigenvalue problems arise in many scientific and engineering problems. While the basic mathematical ideas are independent of the size of the matrices, the numerical determination of eigenvalues and eigenvectors requires additional considerations as the dimensions and the sparsity of the matrices increase. Special methods are needed for an efficient solution of such problems. There are classes of matrices whose eigenvalues can be calculated more easily by factorization techniques. Some of these matrices were previously studied [1]–[4], and here some other matrices with canonical forms are introduced for which the eigensolutions can be obtained much simpler compared to the classic approaches.

Symmetry has been widely studied in science and engineering [5]–[9]. Eigenvalue problems arise in many scientific and engineering problems [10]–[13]. While the basic mathematical ideas are independent of the size of matrices, the numerical determination of eigenvalues and eigenvectors requires additional consideration as the dimensions and the sparsity of the matrices increase. Special methods are needed for an efficient solution of such problems.

Methods are developed for decomposing the graph models of structures in order to calculate the eigenvalues of matrices with special patterns [1]–[4]. The eigenvectors corresponding to such patterns are studied in [2]. The application of these methods is extended to the vibration of mass-spring systems [14], and free vibration of frames [15].

Eigenvalues and eigenvectors have many applications in structural mechanics and combinatorial optimization. In this paper, a set of matrices of special forms is studied for which the calculation of eigenvalues can be performed much easier than with the existing general methods. First tri-diagonal matrices are presented, and then with the relationships for calculating their eigenvalues are extended to the evaluation of the eigenvalues of block tri-diagonal matrices. Block penta-diagonal matrices are also studied in this paper. The eigensolution of different problems of structural mechanics is performed to show the simplicity of using the present formulations.

2 Eigensolution of the matrices with unequal diagonal entries

Consider a tri-diagonal matrix in the following form:

$$\mathbf{M}_n = \begin{bmatrix} a & b & & & & & \\ b & c & . & & & & \\ & b & . & . & & & \\ & & . & . & . & & \\ & & & . & . & b & \\ & & & & . & c & b \\ & & & & & b & a \end{bmatrix}. \tag{1}$$

This matrix is denoted in an abbreviated form by $\mathbf{M}_n = \mathbf{F}_n(a, b, c)$. Now consider a special form as $\mathbf{M}_n = \mathbf{F}_n(a, b, c)$, i.e., when $a = c$. For the matrices of this form, the eigenvalues can be obtained [16] by:

$$\lambda = a + 2b\cos\frac{k\pi}{n+1}, \quad k = 1 : n. \tag{2.1}$$

For the special case, when $a = 2$ and $b = 1$, the eigenvalues are

$$\lambda = \left[2\cos\frac{k\pi}{2(n+1)}\right]^2. \tag{2.2}$$

Now consider a matrix of the following form, in which not all the diagonal entries are identical, and the first and last diagonal entries can have different values than the remaining entries:

$$\mathbf{E} = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & 2 & 1 \\ & & & & 1 & 1 \end{bmatrix}_n. \tag{3}$$

Let $\mathbf{P}$ be a matrix of dimension $2n$ as $\mathbf{P} = \mathbf{F}_{2n}(2, 1, 2)$. As it is shown before, a matrix of the form $\mathbf{F}(a, b, c)$ can be decomposed into two submatrices [17]. The matrix $\mathbf{P}$ can similarly be expressed as:

$$
\mathbf{P}=\left[\begin{array}{cccc|cccc}
2 & 1 & & & & & & \\
1 & 2 & & & & & & \\
 & & & & & & & \\
 & & & 2 & 1 & & & \\
\hline
 & & & 1 & 2 & 1 & & \\
 & & & & & & & \\
 & & & & & & 2 & 1 \\
 & & & & & & 1 & 2
\end{array}\right]
\approx
\left[\begin{array}{cccc|cccc}
2 & 1 & & & & & & \\
1 & 2 & & & & & & \\
 & & & & & & & \\
 & & & 2 & & & & 1 \\
\hline
 & & & & 2 & 1 & & \\
 & & & & 1 & & & \\
 & & & & & & 2 & 1 \\
 & & & 1 & & & 1 & 2
\end{array}\right]
$$

$$
=\left[\begin{array}{c|c}
\mathbf{M} & \mathbf{N} \\
\hline
\mathbf{N} & \mathbf{M}
\end{array}\right]
\approx
\left[\begin{array}{c|c}
\mathbf{M}+\mathbf{N} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{M}-\mathbf{N}
\end{array}\right]
=
\left[\begin{array}{cccc|cccc}
2 & 1 & & & & & & \\
1 & & & & & & & \\
 & & & & & \mathbf{0} & & \\
 & & & 1 & & & & \\
 & & 1 & 3 & & & & \\
\hline
 & & & & 2 & 1 & & \\
 & & & & 1 & & & \\
 & \mathbf{0} & & & & & & \\
 & & & & & & & 1 \\
 & & & & & & 1 & 1
\end{array}\right]
=
\left[\begin{array}{c|c}
\mathbf{E} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{D}
\end{array}\right].
\tag{4}
$$

Considering λ for $\mathbf{M}$ from Eq. (1) and comparing with $\mathbf{P}$, one can substitute $2n$ in place of n, and for calculating the smallest eigenvalue, k should be replaced by $2n$, resulting in

$$
\lambda_2(\mathbf{E})=\left[2\cos\frac{n\pi}{(2n+1)}\right]^2. \tag{5}
$$

One of the applications of Eq. (5) is the calculation of the eigen-frequency and eigen-mode of a shear building which has equal masses and stiffnesses for all the stories, and the floors are considered as rigid (see Fig. 1). In this case, the stiffness and mass matrices have the following patterns:

$$
\mathbf{K}=\begin{bmatrix}
K_1+K_2 & -K_2 & & & & \\
-K_2 & K_2+K_3 & \cdot & & & \\
 & \cdot & \cdot & \cdot & & \\
 & & \cdot & \cdot & \cdot & \\
 & & & \cdot & K_{n-1}+K_n & -K_n \\
 & & & & -K_n & K_n
\end{bmatrix} \tag{6}
$$

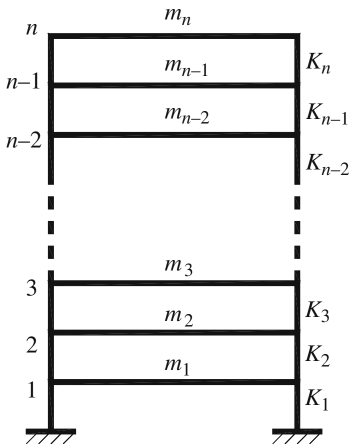

Fig. 1. A shear building with equal floor masses and stiffnesses

and

$$\mathbf{M} = \begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & m_{n-1} & \\ & & & & & m_n \end{bmatrix}. \tag{7}$$

Now consider $\det(\mathbf{K} - \mathbf{M}\omega^2) = 0$. Here the matrix $\mathbf{M}$ is a multiple of the unit matrix $\mathbf{I}$, and therefore it can be dealt with much simpler. It is sufficient to calculate the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$, and since $\mathbf{M}$ is diagonal, its inverse has the inverse of the diagonal entries of $\mathbf{M}$. Thus, one should find the eigenvalues of $\mathbf{K}$ and divide the corresponding diagonal entries by the diagonal entries of $\mathbf{M}$. Since all K_i s and m_i s are equal, therefore the matrix $\mathbf{K}$ is a multiple of $\mathbf{E}$, and we have

$$(K_i = K, m_i = m) \tag{8}$$

and

$$\lambda_k = 4K\cos^2\frac{i\pi}{2n+1}, \quad i = 1:n \tag{9}$$

leading to:

$$\omega = \sqrt{\frac{\lambda_k}{m}} = 2\cos\frac{i\pi}{2n+1}\sqrt{\frac{K}{m}} \Rightarrow \omega_{\min} = 2\cos\left(\frac{n\pi}{2n+1}\right)\sqrt{\frac{K}{m}}. \tag{10}$$

This result corresponds to the case when K and m for all the stories are the same. If this is not the case, then the differences are often small. As an example, the stiffness of the first story may be more or the mass of the top story may be less. In such cases, one can calculate ω from the above relationship and then improve the result using the Rayleigh quotient relationship as $\mathbf{u}^{\mathrm{t}}(\mathbf{M}^{-1}\mathbf{K})\mathbf{u}/\mathbf{u}^{\mathrm{t}}\mathbf{u}$. The method for finding $\mathbf{u}$ which is the corresponding eigenvector will be presented in the following.

As an example, consider a six story frame with $\mathbf{M} = [16, 13, 13, 13, 13, 11]$ and $\mathbf{K} = [8,6,6,6,6,5]$. Assuming $K_i = 6$ and $m_i = 13$, we obtain

$$\omega_{\min} = 2\cos\left(\frac{6\pi}{13}\right)\sqrt{\frac{6}{13}} = 0.1638$$

leading to

$$\mathbf{u} = \{-0.0368, -0.0715, -0.1020, -0.1266, -0.1438, -0.1527\}^t,$$

$$\frac{\mathbf{u}^t(\mathbf{M}^{-1}\mathbf{K})\mathbf{u}}{\mathbf{u}^t\mathbf{u}} = 0.2254,$$

where the exact value is 0.2256. It should be noted that the formation of $\mathbf{M}^{-1}$ produces no problem, since $\mathbf{M}$ is diagonal. It can be observed that the application of these relationships for a symmetric structure with different boundary (non-symmetric) conditions results in matrices with non-equal first and last diagonal entries.

In [18] it is proven that for a tri-diagonal matrix of the form

$$\mathbf{M}_n = \begin{bmatrix} -\alpha+b & c & & & & \\ a & b & \cdot & & & \\ & a & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c \\ & & & & \cdot & b & c \\ & & & & & a & -\beta+b \end{bmatrix} \tag{11}$$

the eigenvalues can be obtained as

$$\lambda = b + 2\sqrt{ac}\cos\theta, \quad \theta \neq m\pi, \tag{12}$$

where θ is obtained from the following relationship:

$$ac\sin(n+1)\theta + (\alpha+\beta)\sqrt{ac}\sin n\theta + \alpha\beta\sin(n-1)\theta = 0. \tag{13}$$

The corresponding eigenvectors can be calculated as

$$u_j = \frac{u_1}{\sin\theta}\rho^{j-1}\left[\sin j\theta + \frac{\alpha}{\sqrt{ac}}\sin(j-1)\theta\right], \quad j = 1:n, \tag{14}$$

where $\rho = \sqrt{a/c}$.

When $\theta = m\pi$ or $\alpha = \beta = \pm\sqrt{ac}$, then the eigenvalues and the corresponding eigenvectors should be calculated from different relationships as discussed in [18].

These relationships are the generalization of a special case previously given by Eqs. (3)–(5). For the form $\mathbf{F}$ we have:

$$\mathbf{M}_n = \mathbf{F}_n(a,b,c) \Rightarrow b^2\sin(n+1)\theta + 2b(c-a)\sin n\theta + (c-a)^2\sin(n-1)\theta = 0, \tag{15}$$

$$\lambda = c + 2b\cos\theta; \quad \theta \neq k\pi. \tag{16}$$

Due to symmetry

$$\rho = 1 \Rightarrow u_j = u_1\left[\sin j\theta + \frac{c-a}{b}\sin(j-1)\theta\right] \Big/ \sin\theta. \tag{17}$$

Depending on the magnitudes of a, b or c, these relationships can be expressed in simpler forms. This is another concern of the present paper.

The application of these equations starts with using Eq. (12) for calculating θ. From Eq. (13) the magnitude of λ is obtained, and Eq. (14) is employed to find the corresponding eigenvector. Obviously, if a, b or c are given, then λ can be calculated. However, for some mechanical problems, the values of a, b or c can be in parametric form, and then the aim will consist of calculating the parameter such that the determinant of the considered matrix becomes zero. This happens when calculating the buckling load and natural frequencies of the structures is required. Since the determinant should become zero, therefore at least one of the eigenvalues must have zero value. Thus substituting $\lambda = 0$ in Eq. (13), and calculating θ by Eq. (12), the parameter involved can be calculated (see Example 2).

The problem of shear buildings can be solved by these relationships. For this case we have:

$$\alpha = 0, \beta \neq 0 \Rightarrow \theta = \frac{2i\pi}{2n+1} \Rightarrow \lambda = c + 2b\cos\theta, \quad \mathrm{i} = 1:n, \tag{18}$$

leading to

$$\lambda = 2K + 2K\cos\left(\frac{2i\pi}{2n+1}\right) = 4K\cos^2\left(\frac{\mathrm{i}\pi}{2n+1}\right); \quad \omega = \sqrt{\frac{\lambda}{m}} = 2\cos\left(\frac{\mathrm{i}\pi}{2n+1}\right)\sqrt{\frac{K}{m}}, \tag{19}$$

$$u_j^{(k)} = \sin\left(\frac{(2k-1)\mathrm{j}\pi}{2n+1}\right); \quad (u^{(\mathrm{i})} = \{u_1^{(\mathrm{i})}, u_2^{(\mathrm{i})}, \ldots, u_{\mathrm{n}}^{(\mathrm{i})}\}^t). \tag{20}$$

The above relationships can also be generalized to block matrices. As an example, for adjacency of the three graph products introduced in [10] we have

$$\mathbf{B}^2\sin(n+1)\theta + 2(\mathbf{C}-\mathbf{A})\mathbf{B}\sin \mathrm{n}\theta + (\mathbf{C}-\mathbf{A})^2\sin(n-1)\theta = 0; \lambda = \mathbf{C} + 2\mathbf{B}\cos\theta. \tag{21}$$

For the adjacency matrices of these products

$$\mathbf{C} = \mathbf{A} \Rightarrow \sin(n+1) = 0 \Rightarrow \theta = \frac{k\pi}{n+1} \tag{22}$$

leading to

$$\lambda = \mathbf{A} + 2\mathbf{B}\cos\left(\frac{i\pi}{n+1}\right)(i = 1:n), \quad u_j^{(\mathrm{k})} = \sin\left(\frac{\mathrm{k}j\pi}{n+1}\right)(j = 1:n). \tag{23}$$

As another example, for calculating the eigenvalues of the Laplacian matrices of the strong Cartesian product of P ⊠ P one can add edges only to two opposite edges instead of all four edges of the model. If these added members are for the edges with more members, then we will have $\mathbf{A} = \mathbf{B}+\mathbf{C}$, and we will have

$$\mathbf{B}^2\sin(n+1)\theta - 2\mathbf{B}^2\sin n\theta + \mathbf{B}^2\sin(n-1)\theta = 0, \tag{24}$$

$$\sin(n+1)\theta - 2\sin n\theta + \sin(n-1)\theta = 0 \Rightarrow 2\sin n\theta\cos\theta - 2\sin n\theta = 0, \tag{25}$$

$$2\sin n\theta(\cos\theta - 1) = 0 \overset{\theta \neq \mathrm{m}\pi}{\Rightarrow} \sin n\theta = 0 \Rightarrow \theta = k\pi/n, \tag{26}$$

$$\lambda = \mathbf{C} + 2\mathbf{B}\cos\left(\frac{k\pi}{n}\right), \quad u_j^{(k)} = \sin\left(\frac{kj\pi}{n}\right) - \sin\left(\frac{k(j-1)\pi}{n}\right). \tag{27}$$

In this way, the eigenvectors are calculated and since we have added extra members, therefore the use of the Rayleigh quotient relationship can improve the results. In the following section examples are considered as applications in structural mechanics.

Now we consider $\mathbf{M}$ as a penta-diagonal matrix expressed as the sum of three Kronecker products $\mathbf{M} = \sum_{\mathrm{i}=1}^{3}\mathbf{A}_{\mathrm{i}} \otimes \mathbf{B}_{\mathrm{i}}$, where $\mathbf{M}$ is a penta-diagonal matrix (see [19] for definitions and proofs). A typical penta-diagonal matrix can be expressed in an F form as follows:

$$\mathbf{F}(a,b,c,d)=\begin{bmatrix} a & b & d & & & & & \\ b & c & b & d & & & & \\ d & b & c & . & . & & & \\ & d & . & . & . & . & & \\ & & . & . & . & . & d & \\ & & & . & . & . & b & d \\ & & & & d & b & c & b \\ & & & & & d & b & a \end{bmatrix}. \tag{28}$$

If $\mathbf{A}_i$ and $\mathbf{B}_i$ commute two by two with respect to multiplication, then $\mathbf{M}$ can be made *tri*-diagonal. As an example, consider the following matrix:

$$\mathbf{M}=\sum_{i=1}^{3}\mathbf{A}_i\otimes\mathbf{B}_i=\mathbf{I}\otimes\mathbf{B}_1+\mathbf{F}(0,1,0)\otimes\mathbf{B}_2+\mathbf{F}(0,0,1,1)\otimes\mathbf{B}_3. \tag{29}$$

Since $\mathbf{A}_i$s commute with respect to multiplication, therefore the first and third terms can be combined, resulting in a similar matrix:

$$\mathbf{M}\approx\mathbf{N}=\mathbf{I}\otimes(\mathbf{B}_1+\lambda(\mathbf{F}(0,0,1,1))\mathbf{B}_3)+\mathbf{F}(0,1,0)\otimes\mathbf{B}_2, \tag{30}$$

and this is a tri-diagonal matrix. If all the $\mathbf{B}_i$s become numbers in place of matrices, then one can use the relationships presented for tri-diagonal matrices.

4 Numerical examples

Example 1: The longitudinal frequency and the corresponding mode shape of a simply supported bar as shown in Fig. 2a is required. The same problem is repeated for a bar with one end as clamped and the other end free.

First the continuous problem is transformed into a discrete problem, and then the number of elements is taken to infinity. For the first problem, the bar is modeled as n masses and n springs as illustrated in Fig. 2b, where $m_i=\mathrm{mL}/n$ and $K_i=nEA/L$. The formation of the stiffness and mass matrices for this system reveals that $\mathbf{K}$ has an $\mathbf{F}$ form and $\mathbf{M}$ is diagonal, i.e.

$$\mathbf{K}=\frac{nEA}{L}\begin{bmatrix} 3 & -1 & & & & \\ -1 & 2 & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & 2 & -1 \\ & & & & -1 & 3 \end{bmatrix}=\mathbf{F}_n(3,-1,2),\quad \mathbf{M}=\frac{mL}{n}\mathbf{I}, \tag{31}$$

where $\mathbf{I}$ is a unit matrix. Since $\mathbf{M}$ is a multiple of $\mathbf{I}$, therefore it is sufficient to find the eigenvalues of $\mathbf{K}$. Using Eqs. (12)–(14) we have

$$\alpha=\beta=b=-1\Rightarrow 2\sin \mathrm{n}\theta(\cos\theta-1)=0, \tag{32}$$

$$\text{for } \theta\neq \mathrm{m}\pi\Rightarrow\sin\theta=0\Rightarrow\theta=\frac{k\pi}{n}\Rightarrow\lambda=2-2\cos\frac{k\pi}{n}=4\sin^2\frac{k\pi}{2n} \tag{33}$$

$$\Rightarrow\omega=\frac{n}{L}\sqrt{\frac{\lambda E}{\rho}}=\frac{2n}{L}\sin\left(\frac{k\pi}{2n}\right)\sqrt{\frac{E}{\rho}}\quad\text{when}\quad n\to\infty\Rightarrow\omega=\frac{2n}{L}\frac{k\pi}{2n}\sqrt{\frac{E}{\rho}}=\frac{k\pi}{L}\sqrt{\frac{E}{\rho}}. \tag{34}$$

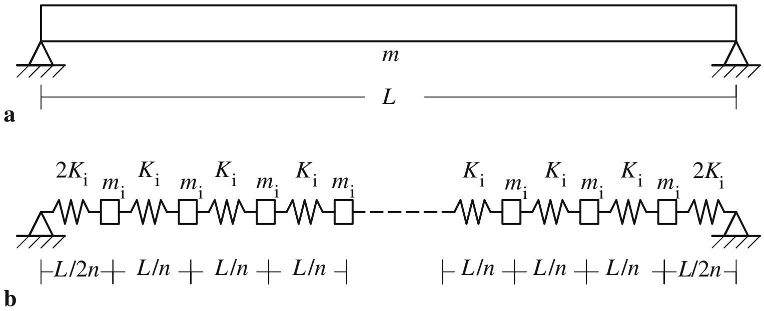

Fig. 2. A simply supported bar

Here, ρ is the mass for unit length. From Eq. (14) the mode shape is obtained as

$$u_j^{(k)} = \cos\frac{kj\pi}{L}, \quad k = 1 : n. \tag{35}$$

Now we assume the support at left hand side be clamped and the one in the right hand side be free, See Fig. 3. Then we will not have the end spring of length $L/2n$ with stiffness $2k_i$. Thus the last diagonal entry of the stiffness matrix will have unit value. This form can not be expressed in an **F** form and the previous methods [10] will not be applicable.

Here **K** is different and **M** does not change,

$$\mathbf{K} = \frac{nEA}{L}\begin{bmatrix} 3 & -1 & & & & \\ -1 & 2 & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}, \tag{36}$$

$$\alpha = -\beta = \mathrm{b}. \tag{37}$$

Using Eqs. (12) and (13), we have

$$\theta = \frac{(2k-1)\pi}{2n} \Rightarrow \lambda = 2 - 2\cos\frac{(2k-1)\pi}{2n}, \tag{38}$$

$$\text{If } n \to \infty \Rightarrow \lambda = \frac{(2k-1)\pi^2}{4n^2}; \quad \omega = \frac{n}{L}\sqrt{\frac{\lambda E}{\rho}} = \frac{(2k-1)\pi}{2L}\sqrt{\frac{E}{\rho}}. \tag{39}$$

Using Eq. (14) leads to the mode shapes

$$u_j^{(k)} = \sin\frac{(2k-1)j\pi}{2L}. \tag{40}$$

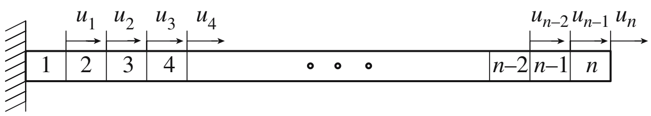

Fig. 3. Finite element discretization of a fixed-free bar

Example 2: Consider the second case of the previous example. For the fixed-free bar the longitudinal vibration mode is required using the finite element method. Unfortunately, the finite element modeling leads to a mass matrix which is not any more a multiple of the unit matrix. Considering n elements, the mass and stiffness matrices are obtained for each element using the kinetic and potential energies as

$$\mathbf{K} = \frac{EA}{L}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}, \quad \mathbf{M} = \frac{\rho AL}{6}\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & . & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & . & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix}. \tag{41}$$

It can be observed that **K** has the **F** form, but **M** is not in this form, since the last diagonal entry is 2 and not 4. Therefore, the previous methods are not applicable for this example. Here we want to obtain ω such that $\det(\mathbf{K}-\mathbf{M}\omega^2)=0$. Since **M** has not the same diagonal entries, therefore non-diagonal entries will also involve ω^2. Since the determinant is equated to zero, thus there should be a zero eigenvalue. Therefore, in Eqs. (12) and (13) we impose the condition $\lambda = 0$. Considering the matrix $(\mathbf{K}-\mathbf{M}\omega^2)$ from Eq. (12), we have:

$$(2-4\omega^2)\sin(\mathrm{n}+1)\theta - 2(1-2\omega^2)\cos\theta\sin\theta = 0 \tag{42}$$

leading to

$$\sin(n+1)\theta - \cos\theta\sin n\theta = 0 \Rightarrow \sin\theta\cos n\theta = 0, \tag{43}$$

$$\theta \neq m\pi \Rightarrow \cos n\theta = 0 \Rightarrow \theta = \frac{(2k-1)\pi}{2n}, \tag{44}$$

$$\lambda = 0 \Rightarrow c + 2b\cos\theta = 0 \Rightarrow \cos\theta = \frac{-c}{2b} = \frac{1-2\omega^2}{1+\omega^2} \Rightarrow \omega^2 = \frac{1-\cos\theta}{2+\cos\theta}, \tag{45}$$

$$(n\to\infty) \Rightarrow \begin{cases} 2+\cos\theta \to 3 \\ 1-\cos\theta \to \dfrac{\theta^2}{2} \end{cases} \Rightarrow \omega^2 = \frac{\theta^2}{6}, \tag{46}$$

$$\omega_k = n\omega\sqrt{\frac{6E}{\rho}} = \frac{(2k-1)\pi}{2L}\sqrt{\frac{E}{\rho}}. \tag{47}$$

Example 3: Consider the bar studied in the previous example with both ends fixed. We want to study the bending vibration of this model. Since the vibration is of bending type, therefore the finite difference equations correspond to a penta-diagonal matrix, while for longitudinal vibration the corresponding matrix becomes tri-diagonal which can easily be solved.

For bending vibration we have

$$\mathbf{M}_n = \mathbf{F}_n(5,-4,6,1). \tag{48}$$

This matrix is similar to $\mathbf{N}_{\mathrm{n}}$ expressed in the following form:

$$\mathbf{N}_n = \mathbf{F}_n\left(5+\left(1+2\cos\frac{2k\pi}{n+1}\right), -4, 5+\left(1+2\cos\frac{2k\pi}{n+1}\right)\right), \quad k=1:n. \tag{49}$$

For the tri-diagonal matrix obtained, it can be observed that $\alpha=\beta=0$, and substituting these values in the tri-diagonal matrices we obtain:

$$\sin(n+1)\theta = 0 \Rightarrow \theta = \frac{k\pi}{n}+1, \tag{50}$$

$$\lambda = c+2b\cos\theta = 6+2\cos\frac{2k\pi}{n+1}+2(-4)\cos\frac{k\pi}{n+1} = 16\sin^4\frac{k\pi}{2(n+1)}, \tag{51}$$

$$u_{\mathrm{j}}^{(\mathrm{i})} = \sin\frac{ij\pi}{n+1}\,(j=1:n), \tag{52}$$

leading to

$$\omega_k = (n+1)^2\sqrt{\frac{EI\lambda}{mL^4}}; \quad n\to\infty \Rightarrow \lambda = \frac{k^4\pi^4}{(n+1)^4}, \tag{53}$$

$$\omega_k = (k\pi)^2\sqrt{\frac{EI}{mL^4}}; \quad (k=1:n). \tag{54}$$

Example 4: In this example, the main aim is to derive a general relationship for calculating the eigenvalues of the Laplacian matrix for the strong Cartesian product $\mathrm{C_m} \boxtimes \mathrm{P_n}$. As shown in [10] and [19], the Laplacian matrix for such a product can be expressed as

$$\mathbf{L}_{mn} = \mathbf{I}_m \otimes 3\mathbf{F}_n(2,0,3) + \mathbf{B}_m \otimes \mathbf{F}_n(1,1,1). \tag{55}$$

Since $\mathbf{I}_m$ and $\mathbf{B}_m$ are commutative with respect to multiplication, therefore

$$\operatorname{eig}(\mathbf{L}_{mn}) = \bigcup_{k=1}^{m}\left\{\operatorname{eig}[3\mathbf{F}_n(2,0,3) - \left(1+2\cos\frac{2k\pi}{m}\right)\mathbf{F}_n(1,1,1)]\right\} = 0. \tag{56}$$

In the tri-diagonal matrix obtained, the equation corresponding to θ is as follows:

$$\left(-1-2\cos\frac{2k\pi}{m}\right)^2\sin(n+1)\theta - 6\left(1+2\cos\frac{2\mathrm{k}\pi}{m}\right)\sin n\theta + 9\sin(n-1)\theta = 0. \tag{57}$$

For calculating the second eigenvalue of this matrix,

$$\theta = \frac{k\pi}{n} \Rightarrow \lambda = 3\left(2+2\cos\frac{k\pi}{n}\right) = 12\cos^2\frac{(n-1)\pi}{2n}. \tag{58}$$

As an example, for $m=20$ and $n=15$, $\lambda=0.1311$ is obtained.

5 Concluding remarks

In many practical structures, for a symmetric model, the selected supports can be non-symmetric, resulting in matrices which are slightly different from the known canonical forms. In this paper, methods are developed to modify the solutions in such a way that full benefit can be made of the symmetry of the main structures.

First *tri*-diagonal matrices are studied, and similar relationships are derived for block *tri*-diagonal matrices. Block penta-diagonal matrices are then studied. The methods are illustrated using four practical structures.

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