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The refined theory of rectangular curved beams

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Summary. Through generalizing the method developed by the refined theory of straight beams, a refined theory of rectangular curved beams is derived by using Papkovich-Neuber (shortened form P-N) solution in polar coordinate system and Lur'e method without ad hoc assumptions. It is shown that the displacements and stresses of the beam can be represented by four displacement functions. For the beam under surface loads, the approximate governing differential equations are derived directly from the refined beam theory and are almost the same as those of other well-known theoretical models. To illustrate the application of the beam theory developed, a pure bending curved beam is examined, which indicates that the stress expressions derived are an exact solution and are consistent with the results gained by exact beam theory of elasticity.

1 Introduction

The curved beam theory has been studied for many years, and more and more works on curved beams with various cross-section shapes are investigated by the following researchers, i.e., Timoshenko [1], Love [2], Southwell [3], Freiberger and Smith [4]. The bending theory of Euler-Bernoulli curved beams has been well established [5]. The deflections and the stress resultants are commonly determined using Castigliano's theorem. These results, however, are not valid when the curved beams are thick, because the Euler-Bernoulli theory neglects the effect of transverse shear deformation. The more refined Timoshenko theory must be used instead. Chianese and Erdlach [6] and Kardomateas [7] studied a plane stress problem of curved beam with a rectangular cross-section. Using the mathematical similarity, Lim et al. [8] presented the exact relationships between the deflections and stress resultants of Timoshenko curved beams and that of the corresponding Euler-Bernoulli curved beams, which enable a straight conversion of the familiar Euler-Bernoulli's solutions into Timoshenko's.

Cheng [9] gave a refined plate theory from Boussinesq-Galerkin elasticity solution and Lur'e method [10] without ad hoc assumptions. A parallel development of Cheng's theory by Barrett and Ellis [11] has been obtained for the isotropic plates under transverse surface loadings (only homogeneous cases are considered in the previous works). Wang and Shi [12], Zhao and Wang [13] derived a new thick plate theory by using P-N solution and Lur'e method without ad hoc assumptions, and obtained the shear theory of plates from the refined plate theory. Recently, several extensions have been found in the rectangular beam problems of among elastic beams [14], [15], magnetoelastic beams [16], piezoelectric beams [17] and thermoelastic beams [18], and the refined theory of beams in the coupling fields has been obtained. Moreover, the exact equations for the beam without transverse surface loadings and the approximate equations for the beam under transverse loadings are derived from the refined beam theory, respectively.

This paper presents the theory for a curved beam of narrow rectangular cross-section by using the straight beam method developed by Gao and Wang [14]–[18]. In the next Section, based on elasticity theory, the refined theory of curved beams is derived by using P-N solution in polar coordinate system and Lur'e method [10] without employing ad hoc assumptions, and the displacements and stresses of the beam can be represented by four displacement functions. In Sect. 3, based on the refined theory of curved beams, the approximate governing differential equations are derived for the curved beam under surface loads. Finally, by comparing its form with that of the known exact beam theory of elasticity, an example is examined to illustrate the application of the refined theory.

2 Lur'e method

We consider an isotropic curved beam with a constant narrow rectangular cross-section and a circular axis as a plane stress problem in a fixed polar coordinate system (r, θ), denoting by a and b the inner and the outer radii of the boundary, and taking the width of the rectangular cross section as unity. In the absence of body force, the equilibrium equations of the elasticity plane stress problem expressed by displacements u_r and u_θ are

$$\left(\nabla^2 - \frac{1}{r^2}\right)u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1+v}{1-v}\frac{\partial \Theta}{\partial r} = 0, \quad \left(\nabla^2 - \frac{1}{r^2}\right)u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1+v}{1-v}\frac{1}{r}\frac{\partial \Theta}{\partial \theta} = 0, \tag{2.1}$$

where $\nabla^2 = \partial^2/\partial r^2 + 1/r \cdot \partial/\partial r + 1/r^2 \cdot \partial^2/\partial \theta^2$ is a two-dimensional Laplacian operator in polar coordinate system, $\Theta = \partial u_r/\partial r + u_r/r + 1/r \cdot \partial u_\theta/\partial \theta$, and v is Poisson's ratio.

P-N solution of the governing equations (2.1) can be obtained as

$$u_r = P_r - \frac{1+v}{4}\frac{\partial}{\partial r}(P_0 + rP_r), \quad u_\theta = P_\theta - \frac{1+v}{4}\frac{1}{r}\frac{\partial}{\partial \theta}(P_0 + rP_r), \tag{2.2}$$

where the functions $P_r(r, \theta)$, $P_\theta(r, \theta)$ and $P_0(r, \theta)$ satisfy the following equations:

$$\begin{aligned} &\frac{\partial^2 P_r}{\partial r^2} + \frac{1}{r}\frac{\partial P_r}{\partial r} + \frac{1}{r^2}\left(\partial_\theta^2 - 1\right)P_r - \frac{2}{r^2}\partial_\theta P_\theta = 0, \\ &\frac{\partial^2 P_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial P_\theta}{\partial r} + \frac{1}{r^2}\left(\partial_\theta^2 - 1\right)P_\theta + \frac{2}{r^2}\partial_\theta P_r = 0, \\ &\nabla^2 P_0 = 0, \end{aligned} \tag{2.3}$$

where $\partial_\theta = \partial/\partial\theta$. Based on Lur'e method [10], we treat Eqs. (2.3) as an ordinary differential equation in r with constant coefficients, and let

$$P_r = r^\lambda f(\theta), \quad P_\theta = r^\lambda g(\theta). \tag{2.4}$$

Substitution of expressions (2.4) into Eqs. (2.3) leads to two differential equations for $f(\theta)$ and $g(\theta)$

$$\left(\lambda^2 + \partial_\theta^2 - 1\right)f(\theta) - 2\partial_\theta g(\theta) = 0, \quad \left(\lambda^2 + \partial_\theta^2 - 1\right)g(\theta) + 2\partial_\theta f(\theta) = 0. \tag{2.5}$$

To obtain a solution of Eqs. (2.5), the expression

$$\lambda = \pm 1 \pm i\partial_\theta$$

must be fulfilled. In terms of expressions (2.4), one obtains the following symbolic solution of Eqs. (2.3):

$$
\begin{aligned}
P_r &= r\sin\Phi g_1(\theta) + r\cos\Phi g_2(\theta) + \frac{r_0^2}{r}\sin\Phi g_3(\theta) + \frac{r_0^2}{r}\cos\Phi g_4(\theta),\\
P_\theta &= r\cos\Phi g_1(\theta) - r\sin\Phi g_2(\theta) - \frac{r_0^2}{r}\cos\Phi g_3(\theta) + \frac{r_0^2}{r}\sin\Phi g_4(\theta),\\
P_0 &= r_0^2\sin\Phi f_3(\theta) + r_0^2\cos\Phi f_4(\theta),
\end{aligned}
\tag{2.6}
$$

where for simplicity we have put

$$\Phi = (\ln r/r_0)\partial_\theta, \quad r_0 = \sqrt{ab}. \tag{2.7}$$

$\sin\Phi$ and $\cos\Phi$ have the following symbolic expressions

$$\sin\Phi = \Phi - \frac{1}{3!}\Phi^3 + \frac{1}{5!}\Phi^5 - \cdots, \quad \cos\Phi = 1 - \frac{1}{2!}\Phi^2 + \frac{1}{4!}\Phi^4 - \cdots. \tag{2.8}$$

In the Appendix, we can show that the harmonic function P_0 always can satisfy the following expression without loss of generality:

$$P_0 + rP_r = r^2\sin\Phi g_1(\theta) + r^2\cos\Phi g_2(\theta). \tag{2.9}$$

Substituting Eqs. (2.6) and (2.9) into Eqs. (2.2), one obtains

$$
\begin{aligned}
u_r &= \frac{1-v}{2}r\sin\Phi g_1 - \frac{1+v}{4}r\partial_\theta\cos\Phi g_1 + \frac{1-v}{2}r\cos\Phi g_2\\
&\quad + \frac{1+v}{4}r\partial_\theta\sin\Phi g_2 + \frac{r_0^2}{r}\sin\Phi g_3 + \frac{r_0^2}{r}\cos\Phi g_4,\\
u_\theta &= r\cos\Phi g_1 - \frac{1+v}{4}r\partial_\theta\sin\Phi g_1 - r\sin\Phi g_2\\
&\quad - \frac{1+v}{4}r\partial_\theta\cos\Phi g_2 - \frac{r_0^2}{r}\cos\Phi g_3 + \frac{r_0^2}{r}\sin\Phi g_4.
\end{aligned}
\tag{2.10}
$$

The angle of rotation and the deflection of the neutral surface can be found to be

$$
\begin{aligned}
\psi &= -\frac{\partial u_\theta}{\partial r}\bigg|_{r=r_0} = -\left(g_1 - \frac{1+v}{4}g_1'' - \frac{5+v}{4}g_2' + g_3 + g_4'\right),\\
\frac{w}{r_0} &= \frac{u_r}{r_0}\bigg|_{r=r_0} = -\frac{1+v}{4}g_1' + \frac{1-v}{2}g_2 + g_4,
\end{aligned}
\tag{2.11}
$$

where the differential symbol " $'$ " denotes differentiation with respect to θ. Equations (2.11) can be expressed by the following expressions:

$$g_3 = -\psi - \frac{w'}{r_0} - g_1 + \frac{7-v}{4}g_2', \quad g_4 = \frac{w}{r_0} + \frac{1+v}{4}g_1' - \frac{1-v}{2}g_2. \tag{2.12}$$

From Eqs. (2.12) and (2.10), the final expressions for the displacements are

$$
\begin{aligned}
u_r = & -\frac{r_0^2}{r}\sin\Phi\psi + \frac{r_0^2}{r}\cos\Phi\frac{w}{r_0} - \frac{r_0^2}{r}\partial_\theta\sin\Phi\frac{w}{r_0} \\
& + \left(\frac{1-v}{2}r - \frac{r_0^2}{r}\right)\sin\Phi g_1 - \frac{1+v}{4}\frac{1}{r}\left(r^2 - r_0^2\right)\partial_\theta\cos\Phi g_1 \\
& + \frac{1-v}{2}\frac{1}{r}\left(r^2 - r_0^2\right)\cos\Phi g_2 + \left(\frac{1+v}{4}r + \frac{7-v}{4}\frac{r_0^2}{r}\right)\partial_\theta\sin\Phi g_2, \\
u_\theta = & \frac{r_0^2}{r}\cos\Phi\psi + \frac{r_0^2}{r}\sin\Phi\frac{w}{r_0} + \frac{r_0^2}{r}\partial_\theta\cos\Phi\frac{w}{r_0} \\
& + \frac{1}{r}\left(r^2 + r_0^2\right)\cos\Phi g_1 - \frac{1+v}{4}\frac{1}{r}\left(r^2 - r_0^2\right)\partial_\theta\sin\Phi g_1 \\
& - \left(r + \frac{1-v}{2}\frac{r_0^2}{r}\right)\sin\Phi g_2 - \left(\frac{1+v}{4}r + \frac{7-v}{4}\frac{r_0^2}{r}\right)\partial_\theta\cos\Phi g_2.
\end{aligned}
\tag{2.13}
$$

Using Hooke's law, from Eqs. (2.13) the stress components σ_r, σ_θ and $\tau_{r\theta}$ can be indicated as

$$
\begin{aligned}
\sigma_r = & \frac{E}{1+v}\left[\frac{r_0^2}{r^2}\sin\Phi\psi - \frac{r_0^2}{r^2}\partial_\theta\cos\Phi\psi - \frac{r_0^2}{r^2}\cos\Phi\frac{w}{r_0} - \frac{r_0^2}{r^2}\partial_\theta^2\cos\Phi\frac{w}{r_0}\right. \\
& + \left(\frac{1+v}{2} + \frac{r_0^2}{r^2}\right)\sin\Phi g_1 + \left(\frac{1+v}{4} - \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\cos\Phi g_1 \\
& + \frac{1+v}{4}\left(1 - \frac{r_0^2}{r^2}\right)\partial_\theta^2\sin\Phi g_1 + \left(\frac{1+v}{2} + \frac{1-v}{2}\frac{r_0^2}{r^2}\right)\cos\Phi g_2 \\
& \left. -\left(\frac{1+v}{4} + \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\sin\Phi g_2 + \left(\frac{1+v}{4} + \frac{7-v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta^2\cos\Phi g_2\right],
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\sigma_\theta = & \frac{E}{1+v}\left[-\frac{r_0^2}{r^2}\sin\Phi\psi + \frac{r_0^2}{r^2}\partial_\theta\cos\Phi\psi + \frac{r_0^2}{r^2}\cos\Phi\frac{w}{r_0} + \frac{r_0^2}{r^2}\partial_\theta^2\cos\Phi\frac{w}{r_0}\right. \\
& + \left(\frac{1+v}{2} - \frac{r_0^2}{r^2}\right)\sin\Phi g_1 + \left(\frac{3+3v}{4} + \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\cos\Phi g_1 \\
& - \frac{1+v}{4}\left(1 - \frac{r_0^2}{r^2}\right)\partial_\theta^2\sin\Phi g_1 + \left(\frac{1+v}{2} - \frac{1-v}{2}\frac{r_0^2}{r^2}\right)\cos\Phi g_2 \\
& \left. -\left(\frac{3+3v}{4} - \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\sin\Phi g_2 - \left(\frac{1+v}{4} + \frac{7-v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta^2\cos\Phi g_2\right],
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
\tau_{r\theta} = & \frac{E}{1+v}\left[-\frac{r_0^2}{r^2}\cos\Phi\psi - \frac{r_0^2}{r^2}\partial_\theta\sin\Phi\psi - \frac{r_0^2}{r^2}\sin\Phi\frac{w}{r_0} - \frac{r_0^2}{r^2}\partial_\theta^2\sin\Phi\frac{w}{r_0}\right. \\
& - \frac{r_0^2}{r^2}\cos\Phi g_1 - \left(\frac{1+v}{4} + \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\sin\Phi g_1 \\
& - \frac{1+v}{4}\left(1 - \frac{r_0^2}{r^2}\right)\partial_\theta^2\cos\Phi g_1 + \frac{1-v}{2}\frac{r_0^2}{r^2}\sin\Phi g_2 \\
& \left. -\left(\frac{1+v}{4} - \frac{5+v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta\cos\Phi g_2 + \left(\frac{1+v}{4} + \frac{7-v}{4}\frac{r_0^2}{r^2}\right)\partial_\theta^2\sin\Phi g_2\right],
\end{aligned}
\tag{2.16}
$$

where E is Young's modulus. Equations (2.13)–(2.16) are the displacement and stress expressions by four displacement functions ψ, w, g_1 and g_2.

3 Transverse surface loadings

Now let us consider the case that the curved beam is subjected only to the transverse surface loadings, i.e.

$$\tau_{r\theta}|_{r=a,b}=0, \quad \sigma_r|_{r=a}=0, \quad \sigma_r|_{r=b}=q. \tag{3.1}$$

It is well-known that Love's curved beam theory [2] based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation. It is also known as classic curved beam theory. The governing differential equation of Love's curved beam theory is as follows:

$$\frac{D}{R^3}\left(1+\partial_\theta^2\right)^2 w'' = -Rq, \tag{3.2}$$

where the flexural rigidity D and the radii of approximate neutral axis R of curved beams are

$$D=\frac{E(b-a)^3}{12}, \quad R=\frac{b-a}{\ln(b/a)}. \tag{3.3}$$

Now the governing equations of the refined beam theory will be derived. Substituting the stress expressions in Eqs. (2.14) and (2.16) into the boundary conditions (3.1) of the beam, we get the following equations:

$$\begin{aligned}
&e^{2\delta}(\partial_\theta SN-CS)\psi+e^{2\delta}(1+\partial_\theta^2)SN\frac{w}{r_0}\\
&\quad+\left[-e^{2\delta}CS+\left(\frac{1+v}{4}+\frac{5+v}{4}e^{2\delta}\right)\partial_\theta SN-\frac{1+v}{4}(1-e^{2\delta})\partial_\theta^2 CS\right]g_1\\
&\quad+\left[-\frac{1-v}{2}e^{2\delta}SN+\left(-\frac{1+v}{4}+\frac{5+v}{4}e^{2\delta}\right)\partial_\theta CS-\left(\frac{1+v}{4}+\frac{7-v}{4}e^{2\delta}\right)\partial_\theta^2 SN\right]g_2=0,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&-e^{-2\delta}(CS+\partial_\theta SN)\psi-e^{-2\delta}(1+\partial_\theta^2)SN\frac{w}{r_0}\\
&\quad-\left[e^{-2\delta}CS+\left(\frac{1+v}{4}+\frac{5+v}{4}e^{-2\delta}\right)\partial_\theta SN+\frac{1+v}{4}(1-e^{-2\delta})\partial_\theta^2 CS\right]g_1\\
&\quad+\left[\frac{1-v}{2}e^{-2\delta}SN+\left(-\frac{1+v}{4}+\frac{5+v}{4}e^{-2\delta}\right)\partial_\theta CS+\left(\frac{1+v}{4}+\frac{7-v}{4}e^{-2\delta}\right)\partial_\theta^2 SN\right]g_2=0,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&-e^{2\delta}(SN+\partial_\theta CS)\psi-e^{2\delta}(1+\partial_\theta^2)CS\frac{w}{r_0}\\
&\quad+\left[-\left(\frac{1+v}{2}+e^{2\delta}\right)SN+\left(\frac{1+v}{4}-\frac{5+v}{4}e^{2\delta}\right)\partial_\theta CS-\frac{1+v}{4}(1-e^{2\delta})\partial_\theta^2 SN\right]g_1\\
&\quad+\left[\left(\frac{1+v}{2}+\frac{1-v}{2}e^{2\delta}\right)CS+\left(\frac{1+v}{4}+\frac{5+v}{4}e^{2\delta}\right)\partial_\theta SN+\left(\frac{1+v}{4}+\frac{7-v}{4}e^{2\delta}\right)\partial_\theta^2 CS\right]g_2=0,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&e^{-2\delta}(SN-\partial_\theta CS)\psi-e^{-2\delta}(1+\partial_\theta^2)CS\frac{w}{r_0}\\
&\quad+\left[\left(\frac{1+v}{2}+e^{-2\delta}\right)SN+\left(\frac{1+v}{4}-\frac{5+v}{4}e^{-2\delta}\right)\partial_\theta CS+\frac{1+v}{4}(1-e^{-2\delta})\partial_\theta^2 SN\right]g_1\\
&\quad+\left[\left(\frac{1+v}{2}+\frac{1-v}{2}e^{-2\delta}\right)CS-\left(\frac{1+v}{4}+\frac{5+v}{4}e^{-2\delta}\right)\partial_\theta SN+\left(\frac{1+v}{4}+\frac{7-v}{4}e^{-2\delta}\right)\partial_\theta^2 CS\right]g_2\\
&\quad=\frac{(1+v)q}{E},
\end{aligned} \tag{3.7}$$

where the differential operators SN and CS are defined by

$$SN = \sin(\delta\partial_\theta), \quad CS = \cos(\delta\partial_\theta), \quad \delta = \ln\sqrt{b/a}.$$

Equations (3.4)–(3.7) can be expressed in matrix form as

$$\mathbf{AU} = \mathbf{Q}, \tag{3.8}$$

where the vector $\mathbf{U} = [\psi, w/r_0, g_1, g_2]^{\mathrm{T}}$, $\mathbf{Q} = [0, 0, 0, (1+v)q/E]^{\mathrm{T}}$ (the superscript "T" denotes the transpose), and $\mathbf{A}$ is a 4×4 differential operator matrix. The "determinant" of $\mathbf{A}$ is represented by the operator A_0, there is

$$A_0 = -\left(\frac{1+v}{2}\right)^2 \left(1+\partial_\theta^2\right)^2 \partial_\theta^2 \left[\sinh^2 2\delta - \frac{\sin^2(2\delta\partial_\theta)}{\partial_\theta^2}\right]. \tag{3.9}$$

Based on Lur'e method [10], the solutions of the preceding matrix equation are

$$A_0 \cdot w/r_0 = A_{42}^* \cdot (1+v)q/E, \quad A_0 \cdot \psi = A_{41}^* \cdot (1+v)q/E, \tag{3.10}$$

where A_{41}^* and A_{42}^* are algebraic complements of the matrix $\mathbf{A}$. The expressions of A_{41}^* and A_{42}^* are quite complicated, such that

$$\begin{aligned}
\left(\frac{4}{1+v}\right)^2 A_{42}^* &= 4e^{-4\delta}\partial_\theta SN \cdot CS\left[-2\partial_\theta(SN - \partial_\theta CS) + (CS + \partial_\theta SN)\left(1 - \partial_\theta^2\right)\right] \\
&+ 2e^{-2\delta}\left(e^{-2\delta} - e^{2\delta}\right)\partial_\theta^2(SN - \partial_\theta CS)\left[2SN \cdot CS + \partial_\theta\left(CS^2 - SN^2\right)\right] \\
&+ e^{-2\delta}\left(e^{-2\delta} - e^{2\delta}\right)\partial_\theta\left[-4SN \cdot CS^2\left(1 + \partial_\theta^2\right) - \partial_\theta(CS - \partial_\theta SN)\left(1 - \partial_\theta^2\right)\right] \\
&+ 4e^{-2\delta}\partial_\theta SN \cdot CS\left[\partial_\theta(CS + \partial_\theta SN)\left(1 + \partial_\theta^2\right) - 2\frac{1-v}{1+v}(SN - \partial_\theta CS)\left(1 + \partial_\theta^2\right)\right] \\
&\left(e^{-2\delta} - e^{2\delta}\right)\partial_\theta\left(1 + \partial_\theta^2\right)\left[-2\frac{1-v}{1+v}CS(SN + \partial_\theta CS) - \partial_\theta(CS - \partial_\theta SN)\right].
\end{aligned}$$

Equations (3.10) are the exact governing equations for the curved beam subjected to the transverse surface loadings. Since these equations are of infinite order, however, it is not applicable in most cases. Accordingly, the infinite-order governing equation should be truncated to a finite order. Of course the higher order terms are included in Eqs. (3.10), the higher precision would be obtained.

For the refined theories of rectangular plates and beams [12], [15], the same exact governing equations have been established. In order to compare their forms with those of other well-known straight plate and beam theories, certain approximate manipulations need to be made. If these governing equations are accurate up to the second-order terms with respect to plate or beam thickness, then they are almost the same as those of Reissner plate theory or Timoshenko beam theory. Furthermore, these equations are consistent with those of Kirchhoff plate theory or Euler-Bernoulli beam theory if all the terms containing plate or beam thickness are omitted.

The approximate manipulation given in rectangular plates and beams [12], [15] still holds good in curved beams, while the infinite-order terms in Eqs. (3.10) are associated with δ, but not beam thickness. Using the Taylor series of the trigonometric functions in Eqs. (2.8), and then dropping all the terms associated with δ^4 or higher orders, we arrive at the following equations:

$$\begin{aligned}
&\frac{2E\delta^3}{3}\left(1+\partial_\theta^2\right)^2 w'' = -r_0 q\left[1 - \delta + \frac{1}{2}\left(\frac{11}{3} - v\right)\delta^2 - \frac{1}{2}\left(\frac{3}{5} - v\right)\delta^3\right], \\
&\frac{2E\delta^3}{3}\left(1+\partial_\theta^2\right)^2 \psi' = -q\left(1 - \delta + \frac{4}{3}\delta^2 - \frac{1}{3}\delta^3\right).
\end{aligned} \tag{3.11}$$

Love's beam theory is applicable to only thin or slender beams and should not be applied to thick or deep beams, since it is based on the normality assumption of plane sections which are to remain plane and normal to the deformed centreline of the curved beam, implying that the effect of transverse shear deformation is neglected. In order to come to Love's result, we consider a slender curved beam with $\delta \ll 1$. Therefore, the terms containing δ in the right side of Eqs. (3.11) are omitted in this case, then Eqs. (3.11) reduce to

$$\frac{2E\delta^3}{3}\left(1+\partial_\theta^2\right)^2 w'' = -r_0 q, \quad \frac{2E\delta^3}{3}\left(1+\partial_\theta^2\right)^2 \psi' = -q. \tag{3.12}$$

Equations (3.12) form the governing equations for an approximate theory for curved beams under transverse loadings.

From expressions (3.3), Eqs. (3.12) are similar to the governing differential equation (3.2) of Love's curved beam theory. However, the two approaches are appreciably different, the coefficient of the right part of Eqs. (3.12) is r_0, but not R . For this problem, both of the beam theories equally overestimate the radii of neutral axis of curved beams less than 2% if $a < b < 2a$. So r_0 can be the radii of the neutral axis of narrow curved beams, the approximate governing differential equations of new curved beam theory are almost the same as those of classical curved beam theory [2].

4 Pure bending of curved beams

Considering a circular axis bent in the plane of curvature by couples M applied at the ends of a curved beam. In this case, the bending moment is constant along the length of the beam, and the stress distribution is the same in all radial cross sections. The boundary conditions are

$$\tau_{r\theta}\big|_{r=a,b} = 0, \quad \sigma_r\big|_{r=a,b} = 0, \quad \int_a^b \sigma_\theta \mathrm{d}r = 0, \quad \int_a^b \sigma_\theta r \mathrm{d}r = -M. \tag{4.1}$$

Conditions (4.1) indicate that the convex and concave boundaries of the beam are free from normal and tangential forces, and the normal stresses at the ends give rise to the couple M only, so the stress functions depend on r only. We assume that the functions ψ, w, g_1 and g_2 are

$$\psi = a_1\theta + a_0, \quad w = b_0 r_0, \quad g_1 = c_1\theta + c_0, \quad g_2 = d_0, \tag{4.2}$$

where the coefficients a_0, a_1, b_0, c_0, c_1 and d_0 are unknown constants to be determined later. Substitution of Eqs. (4.2) into Eqs. (2.14)~(2.16) yields

$$\begin{aligned} \sigma_r &= \frac{E}{1+v}\left[\frac{r_0^2}{r^2}\ln\frac{r}{r_0}a_1 - \frac{r_0^2}{r^2}a_1 - \frac{r_0^2}{r^2}b_0 + \left(\frac{1+v}{2} + \frac{r_0^2}{r^2}\right)\ln\frac{r}{r_0}c_1 \right.\\ &\quad \left. + \left(\frac{1+v}{4} - \frac{5+v}{4}\frac{r_0^2}{r^2}\right)c_1 + \left(\frac{1+v}{2} + \frac{1-v}{2}\frac{r_0^2}{r^2}\right)d_0\right], \\ \sigma_\theta &= \frac{E}{1+v}\left[-\frac{r_0^2}{r^2}\ln\frac{r}{r_0}a_1 + \frac{r_0^2}{r^2}a_1 + \frac{r_0^2}{r^2}b_0 + \left(\frac{1+v}{2} - \frac{r_0^2}{r^2}\right)\ln\frac{r}{r_0}c_1 \right.\\ &\quad \left. + \left(\frac{3+3v}{4} + \frac{5+v}{4}\frac{r_0^2}{r^2}\right)c_1 + \left(\frac{1+v}{2} - \frac{1-v}{2}\frac{r_0^2}{r^2}\right)d_0\right], \\ \tau_{r\theta} &= \frac{E}{1+v}\left[-\frac{r_0^2}{r^2}(a_1\theta + a_0) - \frac{r_0^2}{r^2}(c_1\theta + c_0)\right]. \end{aligned} \tag{4.3}$$

On substituting Eq. (4.3.3) into boundary condition (4.1.1), we obtain

$$a_1 = -c_1, \quad a_0 = -c_0. \tag{4.4}$$

From Eqs. (4.4) and (4.3), the final expressions for the stress components can be indicated as

$$\begin{aligned}
\sigma_r &= \frac{E}{1+v}\left[\frac{r_0^2}{r^2}\left(-b_0 - \frac{1+v}{4}c_1 + \frac{1-v}{2}d_0\right) + (1+2\ln r)\frac{1+v}{4}c_1 + \frac{1+v}{2}(d_0 - \ln r_0 c_1)\right],\\
\sigma_\theta &= \frac{E}{1+v}\left[-\frac{r_0^2}{r^2}\left(-b_0 - \frac{1+v}{4}c_1 + \frac{1-v}{2}d_0\right) + (3+2\ln r)\frac{1+v}{4}c_1 + \frac{1+v}{2}(d_0 - \ln r_0 c_1)\right],\\
\tau_{r\theta} &= 0.
\end{aligned} \tag{4.5}$$

These expressions constitute an exact solution of the problem only when the normal forces at the ends of curved beams are distributed in the manner given by Eqs. (4.5.2). For any other distribution of forces the stress distribution near the ends will be different from that given by solution (4.5), but at larger distances this solution may still be valid. Therefore, in the cases where Saint-Venant's principle holds, the refined theory of curved beams should be a very accurate one. More importantly, Eqs. (4.5) are identical with the corresponding results deduced by Timoshenko and Goodier [19], if the coefficients of Eqs. (4.5) are replaced by the coefficients A, B and C of the exact theory of curved beams [19], such that

$$\frac{E}{1+v}\left(-r_0^2 b_0 - \frac{1+v}{4}r_0^2 c_1 + \frac{1-v}{2}r_0^2 d_0\right) = A, \quad \frac{E}{4}c_1 = B, \quad \frac{E}{4}(d_0 - \ln r_0 c_1) = C. \tag{4.6}$$

So Eqs. (4.5) are changed into the corresponding results [19], which take the form

$$\begin{aligned}
\sigma_r &= -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r}\right),\\
\sigma_\theta &= -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a} + b^2\ln\frac{r}{b} + a^2\ln\frac{a}{r} + b^2 - a^2\right),\\
\tau_{r\theta} &= 0,
\end{aligned} \tag{4.7}$$

where

$$N = \left(b^2 - a^2\right)^2 - 4a^2b^2\left(\ln\frac{b}{a}\right)^2.$$

Similarly, for the cases of bending of curved beams by a horizontal load and a vertical load at the end, we can choose an appropriate form of functions ψ, w, g_1 and g_2, whereby the stress solutions of new beam theory are again transformed into those of exact beam theory of elasticity.

5 Conclusion

In the above Sections, a refined theory of curved beams has been deduced systematically and directly from elasticity theory by using P-N solution in polar coordinate system and Lur'e method without ad hoc assumptions, and the displacements and stresses of the beam can be represented by four displacement functions. For the beam under surface loads, the exact governing differential equations are derived by using the refined theory of curved beams. By omitting the higher order terms, the approximate governing differential equations of the refined beam theory are almost the same as those of Love's curved beam theory. For the pure bending curved beam, noticeably the stress expressions derived are consistent with the results gained by elasticity. By comparing their forms with that of other well-known beam theories, such as Love's curved beam theory and exact curved beam theory, the new curved beam theory should be a very accurate one, and can be degenerated or transformed into these curved beam theories.

Appendix

The method used in this Appendix is obtained by extending our previous work [16]. Next we will prove that when P_0 is defined according to Eq. (2.9), the general solution (2.2) is complete without loss of generality.

First, from the nonuniqueness of P-N solution, P_r and P_0 in Eqs. (2.2) can be changed to $\tilde{P}_r$ and $\tilde{P}_0$, respectively, and

$$\tilde{P}_r = P_r + \frac{\partial A}{\partial r}, \quad \tilde{P}_0 = P_0 + \frac{4}{1+v}A - r\frac{\partial A}{\partial r}, \tag{A.1}$$

in which P_r and P_0 have the form of expressions (2.6), and $A(r, \theta)$ is a harmonic function. Therefore, we can set

$$A = r_0^2 \sin \Phi a_1(\theta) + r_0^2 \cos \Phi a_2(\theta). \tag{A.2}$$

Now we come to prove that it is always possible to choose two functions a_1 and a_2 in Eq. (A.2) so that Eq. (2.9), i.e.

$$\tilde{P}_0 + r\tilde{P}_r = r^2 \sin \Phi \tilde{g}_1(\theta) + r^2 \cos \Phi \tilde{g}_2(\theta) \tag{A.3}$$

may hold, in which

$$\tilde{g}_1 = g_1, \quad \tilde{g}_2 = g_2. \tag{A.4}$$

Substituting Eqs. (A.1) and (A.4) into Eq. (A.3), we get the following expression:

$$P_0 + \frac{4}{1+v}A + rP_r = r^2 \sin \Phi g_1 + r^2 \cos \Phi g_2. \tag{A.5}$$

Then inserting Eqs. (2.6) into Eq. (A.5), it will be seen that

$$A = -\frac{1+v}{4} r_0^2 \sin \Phi (g_3 + f_3) - \frac{1+v}{4} r_0^2 \cos \Phi (g_4 + f_4). \tag{A.6}$$

From expression (A.6) we know Eq. (A.3) holds when the expressions

$$a_1 = -\frac{1+v}{4}(g_3 + f_3), \quad a_2 = -\frac{1+v}{4}(g_4 + f_4) \tag{A.7}$$

in Eq. (A.2) are fulfilled.

For convenience $\tilde{P}_r$ and $\tilde{P}_0$ will still be written as P_r and P_0, respectively. Thus Eq. (2.9) holds. Consequently, if P_0 is taken according to Eq. (2.9), the general solutions (2.2) hold without loss of generality.

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