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Two-to-one internal resonances in a shallow curved beam resting on an elastic foundation

H. R. Öz, Gaziantep, and **M. Pakdemirli**, Manisa, Turkey

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Summary. Vibrations of shallow curved beams are investigated. The rise function of the beam is assumed to be small. Sinusoidal and parabolic curvature functions are examined. The immovable end conditions result in mid-plane stretching of the beam which leads to nonlinearities. The beam is resting on an elastic foundation. The method of multiple scales, a perturbation technique, is used in search of approximate solutions of the problem. Two-to-one internal resonances between any two modes of vibration are studied. Amplitude and phase modulation equations are obtained. Steady state solutions and stability are discussed, and a bifurcation analysis of the amplitude and phase modulation equations are given. Conditions for internal resonance to occur are discussed, and it is found that internal resonance is possible for the case of parabolic curvature but not for that of sinusoidal curvature.

Nomenclature

a_n	real amplitudes
A	area of beam cross section
A_n	complex amplitudes
b_n	coefficients related to curvature function
d	volumetric density
D_0	derivative with respect to fast time scale
D_1	derivative with respect to slow time scale
E	Youngs' modulus of elasticity
f	coefficient related to excitation amplitude
f_1, f_2	bifurcation points in the force response diagrams
$\bar{F}$	dimensionless amplitude of excitation
F	ordered amplitude of excitation
I	moment of inertia of cross sectional area of beam
k	dimensional linear coefficient of elastic foundation
L	length of the beam
p_n, q_n, v_n	functions for alternative representations of complex amplitudes
$\boldsymbol{Q}$	a general operator representing quadratic nonlinearities
r	radius of gyration of the beam cross section
t^*	dimensional time
t	dimensionless time
T_0	fast time scale
T_1	slow time scale
u^*	dimensional longitudinal displacement

u	dimensionless longitudinal displacement
w^*	dimensional transverse displacement
w	dimensionless transverse displacement
w_1	O(ε) solution for the response
w_2	O(ε_2) solution for the response
W	parts of solution w_3 related to non-secular terms
x^*	dimensional spatial variable
x	dimensionless spatial variable
Y_n	linear mode shapes
Z_0^*	dimensional curvature function
Z_0	dimensionless curvature function
α	dimensionless linear coefficient of elastic foundation
α_1, α_2	coefficients in the amplitude and phase modulation equations
β_n	parameters used for calculating natural frequencies
δ	variational derivative
ε	perturbation parameter
ϕ_1, ϕ_2	parts of solution w_2 related to secular terms
γ, λ	phases
μ	ordered damping coefficient
$\bar{\mu}$	dimensionless damping coefficient
θ_n	phase functions in the polar form of complex amplitudes
ρ	detuning parameter for internal resonance
σ	detuning parameter for excitation frequency
ω_n	natural frequencies
Ω	dimensionless frequency of excitation

1 Introduction

The problem of beams on elastic foundations occupies an important place in modern structural and foundation engineering. They are used extensively in modern day structures like airplanes, rockets, missiles, boosters, use of soft filaments in aerospace structures, building activities in cold regions, foundations of heavy duty machines, underwater and embedded structures, in which the effect of the supporting medium has to be considered for adequate analysis. Bars with small curvatures are used in automobiles to reduce the effect of side impacts for safety reasons. These can be modeled as curved beams resting on an elastic foundation including the effect of stretching of the neutral axis. The stretching of the neutral axis introduces an integral type quadratic nonlinearity to the equations of motion. Rehfield [1] studied free vibrations of a shallow arc with an arbitrary rise function. Singh and Ali [2] investigated a moderately thick clamped beam with a sinusoidal rise function. Yamaki and Mori [3] analyzed a clamped buckled beam by using a combination of Galerkin and Harmonic Balance method.

Internal resonances occur frequently, and energy is easily transferred from the excited mode to the specified mode with 2:1 (two-to-one) or 3:1 (three-to-one) etc. resonances. Tien et al. [4], [5] examined the weakly nonlinear resonance response of a two-degree-of-freedom shallow arch subjected to simple harmonic excitation for the case of 1:1 and 2:1 internal resonance by using the method of averaging. Öz et al. [6] studied the effects of nonlinear elastic foundation, axial stretching and curvature by using the method of multiple scales. It was found that the nonlinearities due to curvature were of softening type whereas those of elastic foundation were of hardening type. Bi and Dai [7] investigated the dynamical behavior of a shallow arch subjected to periodic excitation with internal resonance by using a time-integration scheme in the numerical solutions and also applied a numerical simulation to obtain double-period cascading bifurcations leading to chaos and the steady state period-3 solution in the chaos region. Pakdemirli [8], [9] developed a general operator notation to investigate quadratic and cubic nonlinearities in a general manner. Different types of resonances and similar general operator notations were considered in [10]–[18]. These works, revealed some common properties of nonlinear systems. Two-to-one internal resonances in systems such as pendulums, ships having pitch and roll motions, arches, liquids in a cylindrical container, beams are discussed and results in literature are reviewed in [19]. Approximate analytical solutions, experimental verifications and control implementations can be found in that reference. For a system with arbitrary quadratic nonlinearities, 2:1 internal resonances were investigated in detail and general features of such systems were revealed by Pakdemirli and Özkaya [20]. The general conditions for such resonances to occur are derived in that analysis. Lacarbonara et al. [21] studied 1:1 internal resonances in a shallow beam with one end simply supported and a spring at the other end.

In this study, vibrations of simply supported curved shallow beams are investigated. Two types of curvature are considered, sinusoidal and parabolic curvatures. The beam is resting on a linear elastic foundation. The method of multiple scales is used in the analysis. The curvature function is assumed to be of order 1 and the amplitude of vibrations to be of order ε. Two-to-one internal resonance case is studied. Amplitude and phase modulation equations are obtained. Steady state solutions and stability are discussed. There are no 2:1 internal resonances for sinusoidal curvature. Internal resonance is obtained only for the beams with parabolic curvature.

2 Equation of motion

The geometry of the problem is given in Fig. 1. The kinetic and potential energies of the system are

$$T = \frac{1}{2}\rho A \int_0^L \dot{w}^{*^2} dx^*, \tag{1}$$

$$U = \frac{1}{2}EA \int_0^L \left(u^{*'} + \frac{1}{2} w^{*'^2} + Z_0^{*'} w^{*'} \right)^2 dx^* + \frac{1}{2}EI \int_0^L w^{*''^2} dx^* + \int_0^L \frac{1}{2} k w^{*^2} dx^*, \tag{2}$$

where w^* is the transverse displacement, u^* is the longitudinal displacement, Z_0^* is the arbitrary initial rise function (curvature), E is the modulus of elasticity, A is the cross-sectional area, I is the area moment of inertia with respect to the neutral axis, d is the volumetric density, and k is the linear spring constant for the elastic foundation. x^* and t^* are the spatial and time variables, respectively, and prime and dot denotes differentiation with respect to these variables. In Eq. (2), the first term is the energy due to the stretching of the beam, the second term is the energy due to the bending of the beam and the last term is the energy due to the elastic foundation.

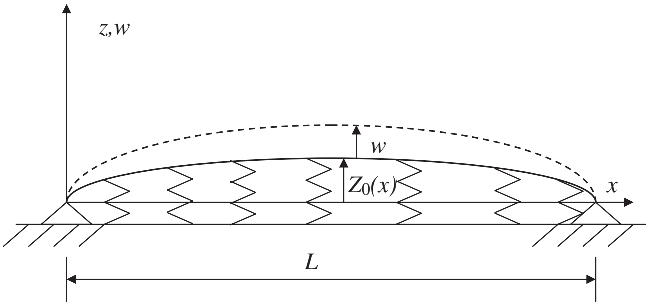

Fig. 1. Geometry of the problem

Defining the Lagrangian as $\mathcal{L} = T - U$ and invoking Hamilton's principle $\delta \int_{t_1^*}^{t_2^*} \mathcal{L}\, dt^* = 0$ leads to the following coupled equations of motion:

$$dA\ddot{w}^* + EIw^{*^{iv}} + kw^* = EA\left(\left(u^{*'} + \frac{1}{2}w^{*'^2} + Z_0^{*'}w^{*'}\right)\left(Z_0^{*'} + w^{*'}\right)\right)', \tag{3}$$

$$EA\left(u^{*'} + \frac{1}{2}w^{*'^2} + Z_0^{*'}w^{*'}\right)' = 0. \tag{4}$$

Integrating the last equation with the assumption of immovable ends, the longitudinal displacement can be eliminated. The final equation of motion reads

$$dA\ddot{w}^* + EIw^{*^{iv}} + kw^* = \frac{EA}{L}\int_0^L \left(Z_0^{*'}w^{*'} + \frac{1}{2}w^{*'^2}\right)dx^* \left(w^{*''} + Z_0^{*''}\right). \tag{5}$$

The associated boundary conditions are

$$w^*(0,t^*) = w^{*''}(0,t^*) = w^*(L,t^*) = \mathrm{w}^{*''}(L,t^*) = 0. \tag{6}$$

In deriving the equations, the following assumptions are made: (i) shallow beam; (ii) linear bending curvature; (iii) stretching of the beam due to the immovable end conditions. Note that the stretching type of nonlinearities dominates over other types of nonlinearities such as nonlinear curvature and inertia type nonlinearities [21].

The nondimensional quantities are defined as follows:

$$w = \frac{w^*}{r}, \quad x = \frac{x^*}{L}, \quad Z_0 = \frac{Z_0^*}{r}, \quad t = \frac{r}{L^2}\sqrt{\frac{E}{\rho}}t^*, \quad \alpha = \frac{kL^4}{EI}, \tag{7}$$

where r is the radius of gyration. Inserting non-dimensional quantities into Eqs. (5) and (6) and adding a damping and external excitation term finally leads to the mathematical model

$$\ddot{w} + w^{iv} + 2\overline{\mu}\dot{w} + \alpha w = \overline{F}\cos\Omega t + \int_0^1 \left(Z_0'w' + \frac{1}{2}w'^2\right)dx(Z_0'' + w''), \tag{8}$$

$$w(0,t) = w''(0,t) = w(1,t) = w''(1,t) = 0, \tag{9}$$

where $\overline{\mu}$ is the dimensionless viscoelastic damping coefficient, α is the dimensionless elastic foundation coefficient and $\overline{F}$ and Ω are the dimensionless external excitation amplitude and frequency, respectively. This same model was employed previously in examining primary resonances of the external excitation [6]. Here, the model will be employed in analyzing 2:1 internal resonances between the modes and external excitation of the higher frequency mode. Two different curvature functions will be used in the analysis, one with the sinusoidal variation and the other with the parabolic variation,

$$Z_0(x) = \sin \pi x, \tag{10}$$

$$Z_0(x) = 4x(1-x). \tag{11}$$

3 Perturbation analysis

The method of multiple scales will be used to solve the problem approximately [22], [23]. One assumes an expansion of the form

$$w(x,t;\varepsilon) = \varepsilon w_1(x,T_0,T_1) + \varepsilon^2 w_2(x,T_0,T_1) + O(\varepsilon^3), \tag{12}$$

where ε is a small parameter indicating that the amplitudes of vibrations are small (weakly nonlinear system) and $T_0 = t$ and $T_1 = \varepsilon t$ are the usual fast and slow time scales in the multiple scales method. Assume that $Z_0(x)$ is of order one, and the external excitation amplitude and damping is selected to balance the effect of quadratic nonlinearities, that is,

$$\overline{F} = \varepsilon^2 F, \quad \overline{\mu} = \varepsilon\mu, \quad Z_0 \sim O(1). \tag{13}$$

Derivatives with respect to time are written in terms of fast and slow time scales,

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \dots, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \dots, \tag{14}$$

where $D_n = \partial/\partial T_n$. Substituting (12)–(14) into (8)–(9) and separating terms at each order of ε, one has

Order ε:

$$\begin{aligned} &D_0^2 w_1 + w_1^{iv} + \alpha w_1 - Z_0'' \int_0^1 Z_0' w_1' dx = 0, \\ &w_1(0,t) = w_1''(0,t) = w_1(1,t) = w''{}_1(1,t) = 0; \end{aligned} \tag{15}$$

Order ε^2:

$$\begin{aligned} &D_0^2 w_2 + w_2^{iv} + \alpha w_2 - Z_0'' \int_0^1 Z_0' w_2' dx \\ &\qquad = -2D_0(D_1 w_1 + \mu w_1) + F\cos\Omega T_0 + \frac{1}{2} Z_0'' \int_0^1 w_1'^2 dx + w_1'' \int_0^1 Z_0' w_1' dx, \\ &w_2(0,t) = w_2''(0,t) = w_2(1,t) = w_2''(1,t) = 0. \end{aligned} \tag{16}$$

At order ε, the solution may be represented by

$$w_1(x,T_0,T_1) = \sum_{n=1}^{\infty} \{A_n(T_1) e^{i\omega_n T_0} + cc\} Y_n(x), \tag{17}$$

where cc stands for the complex conjugates of the preceding terms. ω_n are the natural frequencies and Y_n are the mode shapes. Inserting Eq. (17) into (15) yields

$$Y_n^{iv} - \beta_n^4 Y_n - b_n Z_0'' = 0, \tag{18}$$

$$Y_n(0) = Y_n''(0) = Y_n(1) = Y_n''(1) = 0, \tag{19}$$

where

$$b_n = \int_0^1 Z_0' Y_n' dx, \quad \beta_n^4 = \omega_n^2 - \alpha. \tag{20}$$

One assumes $\omega_m = 2\omega_n + \varepsilon\rho$ for 2:1 internal resonance and $\Omega = \omega_m + \varepsilon\sigma$ for the excitation. With this assumption, the higher mode is excited and some of the energy of this higher mode is transferred to the lower mode via internal resonances. Inserting Eq. (17) and the frequency relations above into order ε^2 equations, one gets

$$\begin{aligned}
&D_0^2 w_2 + w_2^{iv} + \alpha w_2 - Z_0'' \int_0^1 Z_0' w_2' dx \\
&\quad = \Bigg\{ -2i\omega_n (D_1 A_n + \mu A_n) Y_n + \bar{A}_n A_m e^{i\rho T_1} \\
&\quad\quad \times \left(Y_m'' \int_0^1 Z_0' Y_n' dx + Y_n'' \int_0^1 Z_0' Y_m' dx + Z_0'' \int_0^1 Y_n' Y_m' dx \right) \Bigg\} e^{i\omega_n T_0} \\
&\quad\quad + \Bigg\{ -2i\omega_m (D_1 A_m + \mu A_m) Y_m + A_n^2 e^{-i\rho T_1} \\
&\quad\quad \times \left(Y_n'' \int_0^1 Z_0' Y_n' dx + \frac{1}{2} Z_0'' \int_0^1 {Y_n'}^2 dx \right) + \frac{1}{2} F e^{i\sigma T_1} \Bigg\} e^{i\omega_m T_0} + NST,
\end{aligned} \tag{21}$$

where NST stands for non secular terms. The solution at this order can be cast into the below form:

$$w_2(x, T_0, T_1) = \phi_1(x, T_1) e^{i\omega_n T_0} + \phi_2(x, T_1) e^{i\omega_m T_0} + cc + W(x, T_0, T_1), \tag{22}$$

where cc stands for complex conjugate of the preceding terms and $W(x, T_0, T_1)$ stands for the solution of non-secular terms. Substituting Eq. (22) into Eq. (21) yields

$$\begin{aligned}
&{\phi_1}^{iv} - (\omega_n^2 - \alpha)\phi_1 - Z_0'' \int_0^1 Z_0' \phi_1' dx \\
&\quad = -2i\omega_n (D_1 A_n + \mu A_n) Y_n \\
&\quad\quad + \overline{A_n} A_m e^{i\rho T_1} \left(Y_m'' \int_0^1 Z_0' Y_n' dx + Y_n'' \int_0^1 Z_0' Y_m' dx + Z_0'' \int_0^1 Y_n' Y_m' dx \right),
\end{aligned} \tag{23}$$

$$\phi_1(0) = 0, \quad \phi_1(1) = 0, \quad \phi_1''(0) = 0, \quad \phi_1(1) = 0, \tag{24}$$

$$\begin{aligned}
&{\phi_2}^{iv} - (\omega_m^2 - \alpha)\phi_2 - Z_0'' \int_0^1 Z_0' \phi_2' dx = -2i\omega_m (D_1 A_m + \mu A_m) Y_m \\
&\quad + A_n^2 e^{-i\rho T_1} \left(Y_n'' \int_0^1 Z_0' Y_n' dx + \frac{1}{2} Z_0'' \int_0^1 {Y_n'}^2 dx \right) + \frac{1}{2} F e^{i\sigma T_1},
\end{aligned} \tag{25}$$

$$\phi_2(0) = 0, \quad \phi_2(1) = 0, \quad \phi_2''(0) = 0, \quad \phi_2''(1) = 0 \tag{26}$$

for secular terms. Since the homogeneous parts of Eqs. (23) and (25) possess nontrivial solutions, the non-homogeneous equations have a solution only if the following solvability conditions [22] are satisfied:

$$2i\omega_n(D_1A_n + \mu A_n) + \alpha_1\overline{A_n}A_m e^{i\rho T_1} = 0, \tag{27}$$

$$2i\omega_m(D_1A_m + \mu A_m) + \alpha_2 A_n^2 e^{-i\rho T_1} - \frac{f}{2}e^{i\sigma T_1} = 0, \tag{28}$$

where

$$\alpha_1 = -\int_0^1 Y_m'' Y_n dx \int_0^1 Z_0' Y_n' dx - \int_0^1 Y_n'' Y_n dx \int_0^1 Z_0' Y_m' dx - \int_0^1 Z_0'' Y_n dx \int_0^1 Y_n' Y_m' dx, \tag{29}$$

$$\alpha_2 = -\int_0^1 Y_n'' Y_m dx \int_0^1 Z_0' Y_n' dx - \frac{1}{2}\int_0^1 Z_0'' Y_m dx \int_0^1 Y_n'^2 dx, \tag{30}$$

$$f = \int_0^1 F Y_m dx. \tag{31}$$

For the mode shapes, $\int_0^1 Y_n^2 dx = 1$ normalization condition is applied. Two different curvature functions will be considered in the solutions. In reference [6], the shape function for sinusoidal curvature was treated. If $b_n = 0$ in Eq (20), the mode shapes at the first order are

$$Y_n = \sqrt{2}\sin n\pi x, \quad \beta_n = n\pi, \quad n = 2, 3, 4, \ldots\ . \tag{32}$$

If $b_n \neq 0$ then the solution for the first mode is

$$Y = \sqrt{2}\sin \pi x, \quad \beta = \sqrt[4]{\frac{3}{2}}\pi. \tag{33}$$

Inserting Eqs. (10), (32) and (33) into Eqs. (29) and (30) and evaluating the integrals, one obtains the following solvability conditions for the sinusoidal curvature case:

$$2i\omega_n(D_1A_n + \mu A_n) = 0, \tag{34}$$

$$2i\omega_m(D_1A_m + \mu A_m) - \frac{f}{2}e^{i\sigma T_1} = 0, \tag{35}$$

which means there is no 2:1 internal resonance. From Eq. (34) A_n decays in time, and energy transfer from the higher mode is impossible.

Secondly, the parabolic curvature case will be analyzed. In this case, Eq. (18) becomes

$$Y_n^{iv} - \beta_n^4 Y_n + 64\int_0^1 Y_n dx = 0. \tag{36}$$

One may assume the following function for the solution of the equation above:

$$Y_n = c_1\cos\beta_n x + c_2\sin\beta_n x + c_3\cosh\beta_n x + c_4\sinh\beta_n x + c_5. \tag{37}$$

Applying the boundary conditions (19) to the above solution yields

$$Y_n = c_5\left\{1 - \frac{1}{2}\cos\beta_n x - \frac{1}{2}\cosh\beta_n x - \frac{1-\cos\beta_n}{2\sin\beta_n}\sin\beta_n x - \frac{1-\cosh\beta_n}{2\sinh\beta_n}\sinh\beta_n x\right\}, \tag{38}$$

where c_5 can be calculated from similar normalization conditions. Substituting Eq. (38) into Eq. (36), the transcendental equation determining frequencies is obtained as follows:

$$-\beta_n^5 \sin\beta_n \sinh\beta_n + 32\Big\{(2\beta_n - \sin\beta_n - \sinh\beta_n)\sin\beta_n \sinh\beta_n - (1-\cos\beta_n)^2 \sinh\beta_n + (1-\cosh\beta_n)^2 \sin\beta_n\Big\} = 0. \qquad (39)$$

Numerical values for the first five natural frequencies will be given for different elastic foundation values in the numerical analysis section. For the parabolic curvature, the solvability conditions (27) and (28) are valid.

For a model with general quadratic nonlinearity, the sufficiency condition for 2:1 internal resonance to occur has been derived previously [20] as

$$\int_0^1 Y_n[\mathbf{Q}(Y_n, Y_m) + \mathbf{Q}(Y_m, Y_n)]dx \neq 0, \qquad (40)$$

where $\mathbf{Q}$ is a general quadratic nonlinearity operator. This term corresponds to α_1 in Eq. (27) which vanishes for a sinusoidal shape function eliminating the internal resonance but does not vanish for a parabolic shape function which means internal resonance is possible for this case. Therefore, the initial geometry of the beam is extremely important in the development of resonances.

4 Steady state solutions

Equations (27) and (28) represent the modulations in the complex amplitudes. Writing them in the polar form

$$A_n = \frac{1}{2}a_n(T_1)e^{i\theta_n(T_1)}, \quad A_m = \frac{1}{2}a_m(T_1)e^{i\theta_m(T_1)} \qquad (41)$$

and substituting into Eqs. (27) and (28), separating real and imaginary parts, one finally obtains

$$-\omega_n a_n \theta_n' + \frac{1}{4}\alpha_1 a_n a_m \cos\gamma = 0,$$

$$\omega_n a_n' + \mu\omega_n a_n + \frac{1}{4}\alpha_1 a_n a_m \sin\gamma = 0, \qquad (42)$$

$$-\omega_m a_m \theta_m' + \frac{1}{4}\alpha_2 a_n^2 \cos\gamma - \frac{f}{2}\cos\lambda = 0,$$

$$\omega_m a_m' + \mu\omega_m a_m - \frac{1}{4}\alpha_2 a_n^2 \sin\gamma - \frac{f}{2}\sin\lambda = 0,$$

where

$$\gamma = \rho T_1 + \theta_m - 2\theta_n, \quad \lambda = \sigma T_1 - \theta_m. \qquad (43)$$

For steady state solutions, $a_n' = a_m' = \gamma' = \lambda' = 0$, $\theta_n' = (\sigma + \rho)/2$, and $\theta_m' = \sigma$. Equations (42) become

$$-\omega_n a_n \frac{\sigma + \rho}{2} + \frac{1}{4}\alpha_1 a_n a_m \cos\gamma = 0,$$

$$\mu\omega_n a_n + \frac{1}{4}\alpha_1 a_n a_m \sin\gamma = 0, \tag{44}$$

$$-\omega_m a_m \sigma + \frac{1}{4}\alpha_2 a_n^2 \cos\gamma - \frac{f}{2}\cos\lambda = 0,$$

$$\mu\omega_m a_m - \frac{1}{4}\alpha_2 a_n^2 \sin\gamma - \frac{f}{2}\sin\lambda = 0.$$

The amplitude and phase modulation equations (42) can be reorganized in terms of new variables γ and λ, and hence the system becomes autonomous:

$$a_n' = -\mu a_n - \frac{1}{4\omega_n}\alpha_1 a_n a_m \sin\gamma = G_1(a_n, a_m, \lambda, \gamma),$$

$$a_m' = -\mu a_m + \frac{1}{4\omega_m}\alpha_2 a_n^2 \sin\gamma + \frac{f}{2\omega_m}\sin\lambda = G_2(a_n, a_m, \lambda, \gamma), \tag{45}$$

$$\lambda' = \sigma - \frac{1}{4\omega_m}\alpha_2 \frac{a_n^2}{a_m}\cos\gamma + \frac{f}{2a_m\omega_m}\cos\lambda = G_3(a_n, a_m, \lambda, \gamma),$$

$$\gamma' = \rho + \frac{1}{4\omega_m}\alpha_2 \frac{a_n^2}{a_m}\cos\gamma - \frac{f}{2a_m\omega_m}\cos\lambda - \frac{1}{2\omega_n}\alpha_1 a_m \cos\gamma = G_4(a_n, a_m, \lambda, \gamma).$$

The approximate solution for the case where internal resonances exist is

$$w(x,t) = \varepsilon\left[a_n \cos\left(\frac{\Omega}{2}t - \frac{\lambda+\gamma}{2}\right)Y_n(x) + a_m\cos(\Omega t - \lambda)Y_m(x)\right] + O(\varepsilon^2). \tag{46}$$

The amplitudes and the phases are now governed by Eqs. (45).

The analytical expressions for the steady state response amplitudes can be calculated. When a_n is trivial, from Eqs. (44)

$$a_n = 0, \quad a_m = \frac{f}{2\omega_m\sqrt{\sigma^2+\mu^2}}. \tag{47}$$

When a_n becomes nontrivial

$$a_n = \sqrt{\frac{8\omega_n\omega_m}{\alpha_1\alpha_2}(\sigma(\sigma+\rho) - 2\mu^2) \pm \frac{2}{\alpha_2}\sqrt{f^2 - \frac{16\omega_n^2\omega_m^2}{\alpha_1^2}\mu^2(\rho+3\sigma)^2}}, \tag{48}$$

$$a_m = \frac{2\omega_n}{|\alpha_1|}\sqrt{4\mu^2 + (\sigma+\rho)^2}, \tag{49}$$

which means a_m remains constant with respect to the excitation amplitude. This phenomenon is called saturation [24].

5 Stability analysis

In this section, the stability of the steady state equations will be investigated. One has to consider the amplitude and phase modulation equations given in Eq. (45) and construct the associated Jacobian matrix evaluated at the fixed points, i.e.,

$$\begin{bmatrix} \dfrac{\partial G_1}{\partial a_n} & \dfrac{\partial G_1}{\partial a_m} & \dfrac{\partial G_1}{\partial \lambda} & \dfrac{\partial G_1}{\partial \gamma} \\ \dfrac{\partial G_2}{\partial a_n} & \dfrac{\partial G_2}{\partial a_m} & \dfrac{\partial G_2}{\partial \lambda} & \dfrac{\partial G_2}{\partial \gamma} \\ \dfrac{\partial G_3}{\partial a_n} & \dfrac{\partial G_3}{\partial a_m} & \dfrac{\partial G_3}{\partial \lambda} & \dfrac{\partial G_3}{\partial \gamma} \\ \dfrac{\partial G_4}{\partial a_n} & \dfrac{\partial G_4}{\partial a_m} & \dfrac{\partial G_4}{\partial \lambda} & \dfrac{\partial G_4}{\partial \gamma} \end{bmatrix}_{\substack{a_n=a_{n0}\\ a_m=a_{m0}\\ \lambda=\lambda_0\\ \gamma=\gamma_0}}. \tag{50}$$

The eigenvalues of the Jacobian matrix should not have positive real parts to maintain stability. This approach is useful for the nontrivial solution but is not suitable for the trivial solution. To determine the stability of the trivial solution, an alternative form for the complex amplitude equations will be used as given below:

$$A_n = \frac{1}{2}(p_n - iq_n)\, e^{iv_n T_1}, \quad A_m = \frac{1}{2}(p_m - iq_m)\, e^{iv_m T_1}. \tag{51}$$

Substituting the new definitions into the solvability conditions (i.e., Eqs. (27) and (28)), one finally has

$$\begin{aligned} p_n' &= -\mu p_n - v_n q_n - \frac{\alpha_1}{4\omega_n}(q_n p_m - p_n q_m) = H_1(p_n, q_n, p_m, q_m), \\ q_n' &= -\mu q_n + v_n p_n - \frac{\alpha_1}{4\omega_n}(p_n p_m + q_n q_m) = H_2(p_n, q_n, p_m, q_m), \\ p_m' &= -\mu p_m - v_m q_m + \frac{\alpha_2}{2\omega_m} p_n q_n = H_3(p_n, q_n, p_m, q_m), \\ q_m' &= -\mu q_m + v_m p_m - \frac{\alpha_2}{4\omega_m}(p_n^2 - q_n^2) + \frac{f}{2\omega_m} = H_4(p_n, q_n, p_m, q_m), \end{aligned} \tag{52}$$

where

$$v_n = \frac{\rho + \sigma}{2}, \quad v_m = \sigma. \tag{53}$$

Fixed points of the equations are

$$p_n = 0, \quad q_n = 0, \quad p_m = p_{m0} = -\frac{f\sigma}{2\omega_m(\sigma^2 + \mu^2)}, \quad q_m = q_{m0} = \frac{f\mu}{2\omega_m(\sigma^2 + \mu^2)}. \tag{54}$$

The Jacobian matrix is evaluated at these fixed points,

$$\begin{bmatrix} \dfrac{\partial H_1}{\partial p_n} & \dfrac{\partial H_1}{\partial q_n} & \dfrac{\partial H_1}{\partial p_m} & \dfrac{\partial H_1}{\partial q_m} \\ \dfrac{\partial H_2}{\partial p_n} & \dfrac{\partial H_2}{\partial q_n} & \dfrac{\partial H_2}{\partial p_m} & \dfrac{\partial H_2}{\partial q_m} \\ \dfrac{\partial H_3}{\partial p_n} & \dfrac{\partial H_3}{\partial q_n} & \dfrac{\partial H_3}{\partial p_m} & \dfrac{\partial H_3}{\partial q_m} \\ \dfrac{\partial H_4}{\partial p_n} & \dfrac{\partial H_4}{\partial q_n} & \dfrac{\partial H_4}{\partial p_m} & \dfrac{\partial H_4}{\partial q_m} \end{bmatrix}_{\substack{p_n=0\\ q_n=0\\ p_m=p_{m0}\\ q_m=q_{m0}}}. \tag{55}$$

Eigenvalues of this matrix should not have positive real parts for maintaining stability. Eigenvalues can be calculated analytically for this degenerate case, and the solution is stable if

$$f < \frac{4\omega_n\omega_m}{|\alpha_1|}\sqrt{\mu^2+\sigma^2}\sqrt{4\mu^2+(\sigma+\rho)^2} \tag{56}$$

and unstable otherwise.

As will be discussed later, the lower mode is trivial and as the excitation amplitude is increased, it acquires a nontrivial response and the higher mode saturates. The transition point is a supercritical pitchfork bifurcation point. The general analytical expression for this bifurcation point can be obtained by equating Eq. (48) to zero as follows:

$$f_1 = \frac{4\omega_n\omega_m}{|\alpha_1|}\sqrt{\mu^2(\rho+3\sigma)^2+[\sigma(\rho+\sigma)-2\mu^2]^2}. \tag{57}$$

For some specific set of parameters, saddle node bifurcation points arise in force-response diagrams. The analytical expression for the saddle node bifurcation point can be obtained by equating the inside square root of expression (48) to zero as given below,

$$f_2 = \frac{4\omega_n\omega_m}{|\alpha_1|}\ \mu|\rho+3\sigma|. \tag{58}$$

6 Numerical results

In this section, numerical results will be given. For the sinusoidal curvature case, there were no 2:1 internal resonances. For the parabolic curvature case, the related equations were obtained. The frequency equation and values for the sinusoidal curvature case were given in [6]. For the parabolic curvature case, the natural frequency is obtained from Eq. (20) as below:

$$\omega_n = \sqrt{\beta_n^4+\alpha}. \tag{59}$$

The first five frequencies are presented in Table 1 for different elastic foundation values ($\alpha = 0, 10, 50, 100$ and 500). The values of the elastic foundation coefficient resulting in 2:1 internal resonances can be calculated by employing the following equation:

$$\alpha \cong \frac{\beta_m^4 - 4\beta_n^4}{3}. \tag{60}$$

Assuming a value for $\varepsilon = 0.1$, one can determine the modes at which there exist 2:1 internal resonances. There are 2:1 internal resonances between the second and the fourth modes when $\alpha = 6234$, and between the first and fourth modes when $\alpha = 8113$. The corresponding frequencies, α_1 and α_2 values (coupling coefficients in the solvability conditions) are presented in Table 2. Recalling the definition of α given in Eq. (7) and using $\alpha = 6234$, one can calculate the

Table 1. The first five frequencies versus elastic foundation coefficient

α	0	10	50	100	500
ω_1	12.2166	12.6192	14.1154	15.7875	25.4803
ω_2	39.4784	39.6049	40.1067	40.7252	45.3712
ω_3	88.8591	88.9153	89.140	89.420	91.6294
ω_4	157.914	157.945	158.072	158.230	159.489
ω_5	246.744	246.765	246.846	246.947	247.755

Table 2. Special elastic foundation values and corresponding mode numbers, frequencies and amplitude-phase modulation coefficients for 2:1 internal resonances to occur

α	n	m	ω_n	ω_m	α_1	α_2
8113	1	4	90.8969	181.7958	197.400	98.7000
6234	2	4	88.2754	176.552	158.863	79.4315

stiffness of the elastic foundation. For a steel member, $E = 2.07 \times 10^{11}$, having rectangular cross section with width of 2 cm, height of 0.5 cm and length of 1 m, the stiffness is $k = 268841$ $\mathrm{N/m^2}$. For $\alpha = 8113$, $k = 349873$ $\mathrm{N/m^2}$. Thus the elastic foundation becomes stiff in the case of 2:1 internal resonances.

In Figs. 2–9, solid lines denote stable and dashed lines denote unstable solutions. In Figs. 2–5, internal resonances between the first and fourth modes are considered for $\alpha = 8113$. In Fig. 2, when the excitation amplitude is gradually increased, the first mode remains unexcited but the amplitude of the fourth mode is increasing linearly until $f = f_1(0.14122)$ is reached. This point is a supercritical pitchfork bifurcation point, and the first mode acquires a nontrivial response after this point but the amplitude of the fourth mode remains constant (saturates).

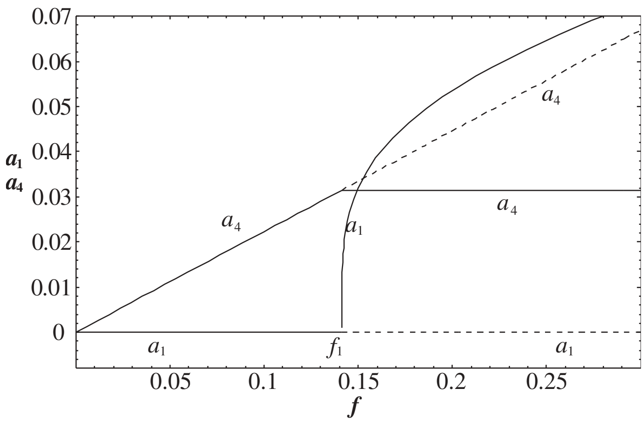

Fig. 2. Force-response curves for the externally excited (a_4) and internally excited (a_1) modes for parameter values $\alpha = 8113$, $\omega_4 = 181.7958$, $\omega_1 = 90.8969$, $\mu = 0.01$, $\sigma = 0.00219628$, $\varepsilon = 0.1$, $\rho = 0.0205695, f_1 = f_2 = 0.14122$ (solid: stable, dashed: unstable solutions)

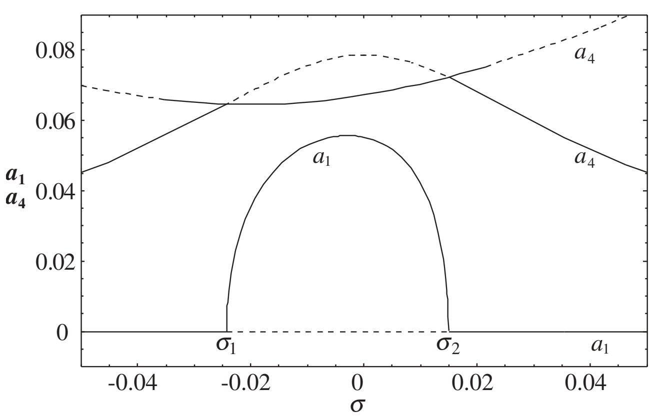

Fig. 3. Frequency-response curves for the externally excited (a_4) and internally excited (a_1) modes for parameter values $\alpha = 8113$, $\omega_4 = 181.7958$, $\omega_1 = 90.8969$, $\mu = 0.035$, $\sigma_1 = -0.0242908$, $\sigma_2 = 0.014928$, $\varepsilon = 0.1$, $\rho = 0.0205695, f = 1$ (solid: stable, dashed: unstable solutions)

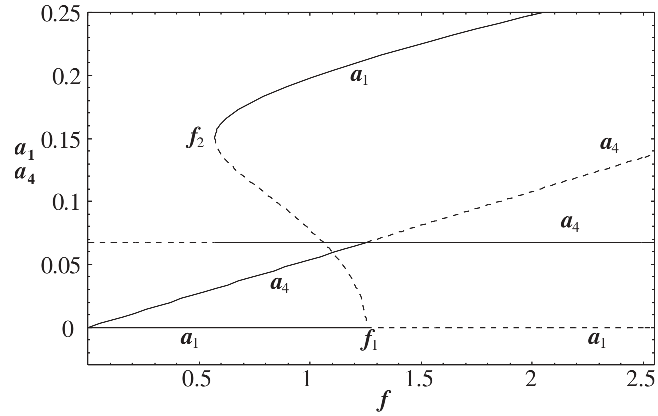

Fig. 4. Force-response curves for the externally excited (a_4) and internally excited (a_1) modes for parameter values $\alpha = 8113$, $\omega_4 = 181.7958$, $\omega_1 = 90.8969$, $\mu = 0.01$, $\sigma = 0.05$, $\varepsilon = 0.1$, $\rho = 0.0205695, f_1 = 1.25235$, $f_2 = 0.571146$ (solid: stable, dashed: unstable solutions)

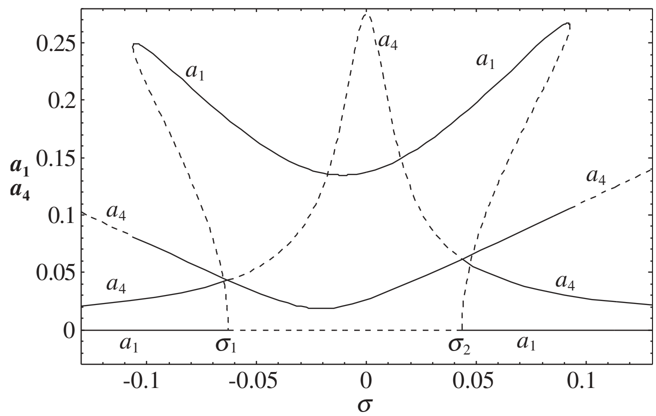

Fig. 5. Frequency-response curves for the externally excited (a_4) and internally excited (a_1) modes for parameter values $\alpha = 8113$, $\omega_4 = 181.7958$, $\omega_1 = 90.8969$, $\mu = 0.01$, $\sigma_1 = -0.0629447$, $\sigma_2 = 0.0434132$, $\varepsilon = 0.1$, $\rho = 0.0205695, f = 1$ (solid: stable, dashed: unstable solutions)

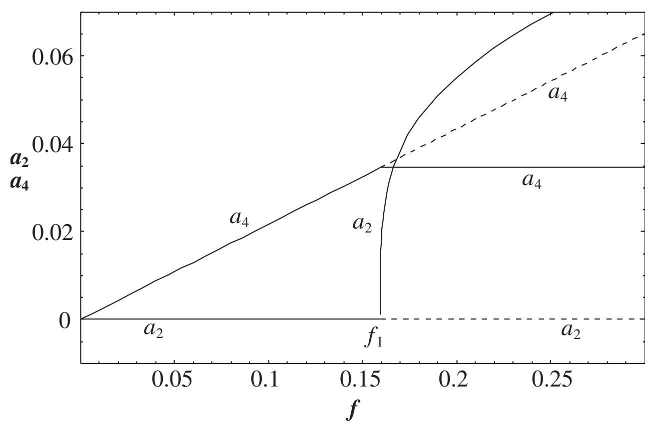

Fig. 6. Force-response curves for the externally excited (a_4) and internally excited (a_2) modes for parameter values $\alpha = 6234$, $\omega_4 = 176.552$, $\omega_2 = 88.2754$, $\mu = 0.01$, $\sigma = 0.00838994$, $\varepsilon = 0.1$, $\rho = 0.0154481, f_1 = f_2 = 0.159393$ (solid: stable, dashed: unstable solutions)

In Fig. 3, frequency response curves are presented. When $\sigma_1(-0.0242908) < \sigma < \sigma_2(0.014928)$, the trivial a_1 response loses its stability and a nontrivial stable solution exists. When a_1 is nontrivial, some of the energy is transferred to this mode from a_4. The locations of bifurcation points are shown in the figure.

For different sets of parameters, force response curves exhibit jump behavior as shown in Fig. 4. When $f = f_1(1.25235)$ is approached from left, the first mode jumps to the upper stable

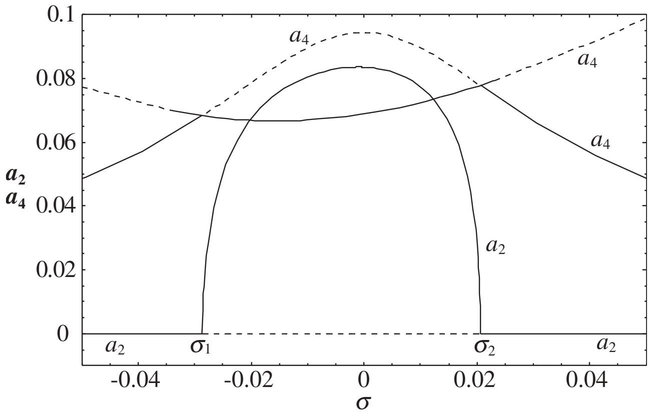

Fig. 7. Frequency-response curves for the externally excited (a_4) and internally excited (a_2) modes for parameter values $\alpha = 6234$, $\omega_4 = 176.552$, $\omega_2 = 88.2754$, $\mu = 0.03$, $\sigma = -0.0286459$, $\sigma_2 = 0.0206173$, $\varepsilon = 0.1$, $\rho = 0.0154481$, $f = 1$ (solid: stable, dashed: unstable solutions)

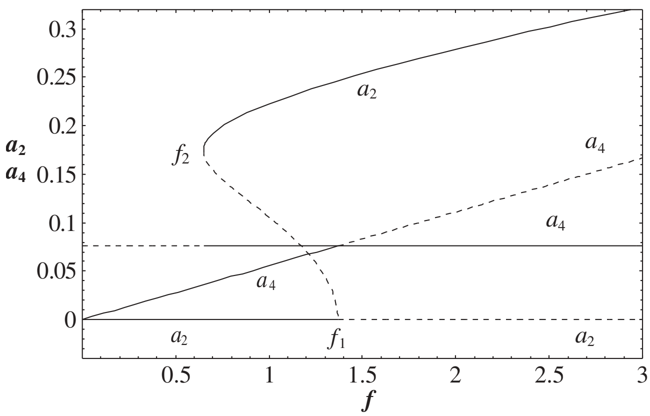

Fig. 8. Force-response curves for the externally excited (a_4) and internally excited (a_2) modes for parameter values $\alpha = 6234$, $\omega_4 = 176.552$, $\omega_2 = 88.2754$, $\mu = 0.01$, $\sigma = 0.05$, $\varepsilon = 0.1$, $\rho = 0.0154481$, $f_1 = 1.36937$, $f_2 = 0.649251$ (solid: stable, dashed: unstable solutions)

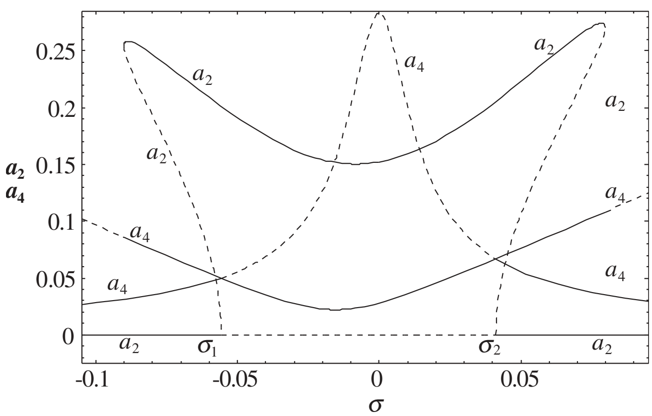

Fig. 9. Frequency-response curves for the externally excited (a_4) and internally excited (a_2) modes for parameter values $\alpha = 6234$, $\omega_4 = 176.552$, $\omega_2 = 88.2754$, $\mu = 0.01$, $\sigma_1 = -0.0557498$, $\sigma_2 = 0.0412136$, $\varepsilon = 0.1$, $\rho = 0.0154481$, $f = 1$ (solid: stable, dashed: unstable solutions)

branch and a_4 saturates. When the excitation amplitude is gradually decreased, the saddle node bifurcation point $f = f_2(0.571146)$ is reached and the nontrivial a_1 solution jumps to the trivial one. In the meantime, a_4 jumps from its constant value to the other stable solution represented by the inclined line.

In Fig. 5, different values are selected to obtain frequency-response curves. Jump occurs and stable solutions for both modes co-exist for $\sigma < \sigma_1(-0.0629447)$ and $\sigma > \sigma_2(0.0434132)$.

In Figs. 6–9, similar curves are obtained for $\alpha = 6234$. Two-to-one internal resonances occur between the second and the fourth modes in that case.

7 Concluding remarks

Two-to-one internal resonances are investigated for a shallow curved beam on an elastic foundation. The nonlinearities are mainly due to stretching of the beam during vibrations. Two different curvature functions are investigated, a sinusoidal and a parabolic one. Approximate solutions are obtained. There are no 2:1 internal resonances for sinusoidal curvature. The amplitude and phase modulations of the solution are derived for the parabolic curvature. Steady state solutions, their stability and bifurcation analysis are presented for the problem. Two-to-one internal resonances occur for some special values of the elastic foundation coefficient. Interactions between the second and fourth mode and between the first and fourth modes are treated numerically.

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Authors' addresses: H. R. Öz, Department of Mechanical Engineering, University of Gaziantep, 27310, Turkey; M. Pakdemirli, Department of Mechanical Engineering, Celal Bayar University, Muradiye Manisa 45140, Turkey (E-mail: mpak@bayar.edu.tr)