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Exact solution on unsteady Couette flow of generalized Maxwell fluid with fractional derivative

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Summary. In this paper the unsteady Couette flow of a generalized Maxwell fluid with fractional derivative (GMF) is studied. The exact solution is obtained with the help of integral transforms (Laplace transform and Weber transform) and generalized Mittag-Leffler function. It was shown that the distribution and establishment of the velocity is governed by two non-dimensional parameters η, b and fractional derivative α of the model. The result of classical (Newtonian fluid and standard Maxwell fluid) Couette flow can be obtained as a special case of the result given by this paper, and the decaying of the unsteady part of GMF displays power law behavior, which has scale invariance.

1 Introduction

The non-Newtonian fluids are increasingly being considered more important and appropriate in technological applications than the Newtonian fluids. Strictly speaking, the linear relation between stress and the rate of strain does not exist for a lot of real fluids, such as blood, oils, paints and polymeric solution. In general, the analysis of the behavior of the fluid motion of the non-Newtonian fluids tends to be much more complicated and subtle in comparison with that of the Newtonian fluids. There has been a fairly large number of flows of Newtonian fluids for which a closed form analytical solution is possible. However, for non-Newtonian fluids such exact solutions are rare. In order to describe the rheological properties of wide classes of materials, the rheological constitutive equations with fractional derivatives have been introduced for a long time, which are discussed in the papers given by Friedrich [1], Bagley [2], Glöckle and Nonnenmacher [3], Rossikhin and Shitikova [4], [5], Mainardi [6], Mainardi and Gorenflo [7], Makris and Coustantinous [8] and the references therein. And they obtained the ideal results which are in good agreement with the experimental data. The starting point of the fractional derivative model of non-Newtonian fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional differential operator.

The fluid considered in this paper is a generalized Maxwell fluid with fractional derivative (GMF). In one dimension its constitutive equation may be expressed in terms of scalar form [1], [9], [10]

$$\tau + \kappa_0^\alpha D_t^\alpha \tau = G_0 \kappa^\beta {}_0D_t^\beta \gamma, \tag{1}$$

where τ is the shear stress, γ is the shear strain, $\kappa = \mu/G_0$ is the relaxation time and G_0 is a shear modulus, μ is a viscosity constant ($\kappa > 0$ and $\mu > 0$), α, β are fractional parameters such that $0 \le \alpha \le \beta \le 1$. And ${}_0D_t^\alpha$ is the Riemann-Liouville (R-L) differential operator defined as:

$${}_0D_t^\alpha[f(t)] = \frac{d}{dt}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(z)}{(t-z)^\alpha}dz\right\} \quad (0 < \alpha < 1). \tag{2}$$

Furthermore, Friedrich [1] proved that this kind of rheological constitutive equation shows fluid-like behavior only in the case that the derivative of stress is fractional and one of the strains is first order in time. Therefore, the following constitutive equation of GMF is used in this paper:

$$\tau + \kappa^\alpha {}_0D_t^\alpha \tau = \mu\dot{\gamma}, \tag{3}$$

where $\dot{\gamma} = d\gamma/dt$ is the rate of shear strain.

The unsteady Couette flow problem has been considered in several works for a long time containing various effects as in the book and paper given by Joseph [11], Bernardin [12] and Demirel [13]. In this paper we use the constitutive equation (3) to study the unsteady Couette flow of GMF. By using the Laplace transform, Weber transform and generalized Mittag-Leffler function, we get the exact solution of the problem. It was shown that the result of classical Couette flow can be contained as a special case of the result given by this paper, and the decaying of the unsteady part of GMF displays power law behavior, which has its scale invariance.

2 The model and the basic equations

Consider a GMF with constant density between two very long concentric circular cylinders of radius R_0 and $R_1(> R_0)$, set in motion by the inner cylinder's rotation about the common axis with the constant angular velocity ω, and the outer one is stationary. Obviously, the motion of GMF is axial symmetric, so we choose the cylindrical coordinates (r, θ, z). And the components of the velocity can be written as $V_r = 0$, $V_\theta = u(r, t)$, $V_z = 0$. Under the above assumptions, the constitutive equation of GMF is

$$\tau_{r\theta} + \kappa^\alpha {}_0D_t^\alpha[\tau_{r\theta}] = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u}{r}\right)\right]. \tag{4}$$

And the momentum equation is

$$\rho\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}), \tag{5}$$

where ρ is the density, $\tau_{r\theta}$ is the component of the shear stress. The initial and boundary conditions are as follows:

$$u(r, t) = 0, \quad t \le 0, \tag{6}$$

$$u(r, t) = U_0, \quad r = R_0, \quad t > 0, \tag{7}$$

$$u(r, t) = 0, \quad r = R_1, \tag{8}$$

where $U_0 = \omega R_0$ denotes the inner cylinder's rotation velocity.

3 The exact solution of the model

Let us define the dimensionless variables $u^* = u/U_0$, $r^* = r/R_0$, $t^* = \mu t/\rho R_0^2$ and $\tau^*_{r\theta} = \tau_{r\theta}/\tau_0$, where $\tau_0 = \mu U_0/R_0$ is the characteristic shear stress. In view of the definition of R-L fractional operator, it is obvious that the operator ${}_0D_t^\alpha$ has the fractional dimension $[\mu/\rho R_0^2]^\alpha$. Then the fractional non-dimensional equations and the initial-boundary conditions read as follows: (for simplicity, the dimensionless mark "*" will be neglected hereinafter)

$$\tau_{r\theta} + \eta_0 D_t^\alpha[\tau_{r\theta}] = r\frac{\partial}{\partial r}\left(\frac{u}{r}\right), \tag{9}$$

$$\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}), \tag{10}$$

$$u(r,t) = 0, \quad t \le 0, \tag{11}$$

$$u(r,t) = 1, \quad r = 1, \quad t > 0, \tag{12}$$

$$u(r,t) = 0, \quad r = R_1/R_0 = b, \quad t > 0, \tag{13}$$

where $\eta = [\kappa\mu/\rho R_0^2]^\alpha$.

We define the Laplace transform and its inverse by

$$\bar{u}(r,s) = L\{u(r,t),s\} = \int_0^\infty e^{-st}u(r,t)dt, \tag{14}$$

$$u(r,t) = L^{-1}\{\bar{u}(r,s),t\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{u}(r,s)e^{st}\,ds. \tag{15}$$

From [9] we have the Laplace transform of the fractional derivative

$$L\{{}_0D_t^\beta u(t),s\} = s^\beta L\{u(t)\} - {}_0D_t^{\beta-1}u(0). \tag{16}$$

Applying the Laplace transform to Eqs. (9)–(13), we get

$$\bar{\tau}_{r\theta} + \eta s^\alpha \bar{\tau}_{r\theta} = \frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r}, \tag{17}$$

$$s\bar{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\bar{\tau}_{r\theta}), \tag{18}$$

$$\bar{u}(r,s) = 0, \quad s = 0, \tag{19}$$

$$\bar{u}(r,s) = 1/s, \quad r = 1, \quad s > 0, \tag{20}$$

$$\bar{u}(r,s) = 0, \quad r = R_1/R_0 = b, \quad s > 0. \tag{21}$$

From Eqs. (17) and (18), we arrive at

$$s\bar{u} = \frac{1}{1+\eta s^\alpha}\left(\frac{\partial^2\bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2}\right). \tag{22}$$

In order to obtain the exact solution of the model, we define the Weber integral transformation and its inverse as [14]

$$\tilde{\bar{u}}(\lambda_i, s) = \int_1^b r\bar{u}(r,s)\varphi(\lambda_i, r)dr, \tag{23}$$

$$\bar{u}(r,s) = \sum_{i=1}^{\infty} \tilde{\bar{u}}(\lambda_i, s)\frac{\varphi(\lambda_i, r)}{N(\lambda_i)}, \tag{24}$$

where $\varphi(\lambda_i, r) = J_1(\lambda_i)Y_1(\lambda_i r) - J_1(\lambda_i r)Y_1(\lambda_i)$, and λ_i are the positive roots of the equation $\varphi(\lambda_i, b) = 0$, and

$$\frac{1}{N(\lambda_i)} = \frac{\pi^2}{2}\frac{\lambda_i^2 J_1^2(\lambda_i b)}{J_1^2(\lambda_i) - J_1^2(\lambda_i b)}, \tag{25}$$

where $J_1(x)$ and $Y_1(x)$ are Bessel functions of the first kind and second kind of order one, respectively. Applying the above transform to Eq. (22), we get

$$s\tilde{\bar{u}} = \frac{1}{1+\eta s^\alpha}\int_1^b \left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2}\right) r\varphi(\lambda_i, r)dr. \tag{26}$$

Now, we calculate the integration in the RHS of the formula (26),

$$\begin{aligned} I &\equiv \int_1^b \left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2}\right) r\varphi(\lambda_i, r)dr \\ &= r\varphi\frac{\partial \bar{u}}{\partial r}\Big|_1^b - \int_1^b r\frac{d\varphi}{dr}d\bar{u} - \int_1^b \varphi\frac{\bar{u}}{r}dr \\ &= r\left[\varphi\frac{\partial \bar{u}}{\partial r} - \bar{u}\frac{d\varphi}{dr}\right]_1^b + \int_1^b \left(\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} - \frac{\varphi}{r^2}\right) r\bar{u}\, dr. \end{aligned}$$

From the boundary condition and the Wronskian relationship of the Bessel function [14], we get

$$J_1(\lambda_i r)Y_1'(\lambda_i r) - J_1'(\lambda_i r)Y_1(\lambda_i r) = \frac{2}{\pi\lambda_i r}. \tag{27}$$

Thus,

$$r\left[\varphi\frac{\partial \bar{u}}{\partial r} - \bar{u}\frac{d\varphi}{dr}\right]_1^b = -\bar{u}\frac{d\varphi}{dr}\Big|_1^b = \frac{2}{s\pi}.$$

From the eigenvalue problem

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + (\lambda_i^2 - \frac{1}{r^2})\varphi = 0,$$

we get

$$\int_1^b \left(\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} - \frac{\varphi}{r^2}\right) r\bar{u}dr = -\int_1^b \lambda_i^2 r\bar{u}\varphi dr = -\lambda_i^2\tilde{\bar{u}},$$

so we have

$$s\tilde{\bar{u}} = \frac{1}{1+\eta s^\alpha}(\frac{2}{s\pi} - \lambda_i^2\tilde{\bar{u}}),$$

and then have

$$\tilde{\bar{u}} = \frac{2}{s\pi}\frac{1}{s(1+\eta s^{\alpha})+\lambda_i^2} = \frac{2}{s\pi\lambda_i^2} - \frac{1+\eta s^{\alpha}}{s(1+\eta s^{\alpha})+\lambda_i^2}\frac{2}{\pi\lambda_i^2}. \tag{28}$$

Making use of the inverse Weber transform to the above relationship, we obtain the expression

$$\bar{u} = \bar{u}_0(r,s) - \sum_{i=1}^{\infty}\bar{A}(\lambda_i,s)\frac{\pi J_1^2(\lambda_i b)\varphi(\lambda_i,r)}{J_1^2(\lambda_i)-J_1^2(\lambda_i b)}, \tag{29}$$

where

$$\bar{u}_0(r,s) = \frac{b^2-r^2}{(b^2-1)r}\frac{1}{s}, \quad \bar{A}(\lambda_i,s) = \frac{1+\eta s^{\alpha}}{s(1+\eta s^{\alpha})+\lambda_i^2}. \tag{30}$$

Obviously,

$$u_0(r,t) = L^{-1}\{\bar{u}_0(r,s)\} = \frac{b^2-r^2}{(b^2-1)r}.$$

We can obtain the exact solution to the model if the inverse Laplace transform of $\bar{A}(\lambda_i,s)$ is known. $\bar{A}(\lambda_i,s)$ can be rewritten as Taylor series, and then using the inverse Laplace transform term-by-term [15], [16], we get

$$\begin{aligned}
A(\lambda_i,t) &= L^{-1}\left\{\frac{1+\eta s^{\alpha}}{s(1+\eta s^{\alpha})+\lambda_i^2}\right\} \\
&= L^{-1}\left\{\frac{1}{s}\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda_i^2 s^{-1})^n}{(1+\eta s^{\alpha})^n}\right\} \\
&= L^{-1}\left\{\frac{1}{s}+\frac{1}{s}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(\lambda_i^2 s^{-1})^{n+1}}{(1+\eta s^{\alpha})^{n+1}}\right\} \\
&= 1 + L^{-1}\left\{\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(\lambda_i^2\eta^{-1})^{n+1}s^{-(n+2)}}{(\eta^{-1}+s^{\alpha})^{n+1}}\right\} \\
&= 1 + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n!}\lambda_i^{2(n+1)}\eta^{-(n+1)}t^{(n+1)(1+\alpha)}E_{\alpha,n+2+\alpha}^{(n)}(-\eta^{-1}t^{\alpha}),
\end{aligned} \tag{31}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function [17] defined as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \alpha>0, \quad \beta\in C, \quad z\in C.$$

Here, we used an important formula of the Laplace transform of generalized Mittag-Leffler function [15]

$$L^{-1}\left\{\frac{n!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{n+1}}\right\} = t^{n\lambda+\mu-1}E_{\lambda,\mu}^{(n)}(\pm ct^{\lambda}), \quad \mathrm{Re}(s) > |c|^{1/\lambda}. \tag{32}$$

Thus, the exact solution of the model is

$$u(r,t) = \frac{b^2-r^2}{(b^2-1)r} - \sum_{i=1}^{\infty}A(\lambda_i,t)\frac{\pi J_1^2(\lambda_i b)\varphi(\lambda_i,r)}{J_1^2(\lambda_i)-J_1^2(\lambda_i b)}, \tag{33}$$

where $A(\lambda_i,t)$ is expressed in Eq. (31).

4 Discussion

The establishment of flow in an annular cylindrical space after impulsive start of the rotation of the inner cylinder may be visualized via a plot of u vs. r for various values of non-dimensional time t when $b = R_1/R_0 = 2$, $\alpha = 0.5$ and $\eta = [k\mu/\rho R_0^2]^{1/2} = 1$ as shown in Fig. 1. We can find that the fluid is essentially at rest for small t except very near the moving cylinder and that the motion is propagated by viscosity from the moving toward the stationary cylinder. Moreover, as can be seen from Fig. 2, the establishment of flow in an annular cylindrical space after impulsive start of the rotation of the inner cylinder for different values of α when $t = 0.10$, the propagation of the motion for larger α is faster than for smaller one. In particular, when $\alpha = 1$, the model represents the standard (integer order) Maxwell fluid, the solution given by Eq. (33) becomes:

$$u(r,t) = \frac{b^2 - r^2}{(b^2-1)r} - \sum_i B(\lambda_i, t) \times \frac{\pi J_1^2(\lambda_i b)\varphi(\lambda_i, r)}{J_1^2(\lambda_i) - J_1^2(\lambda_i b)}, \tag{34}$$

$$\text{where } B(\lambda_i,t) = \begin{cases} \dfrac{1}{\eta_0}\left[\dfrac{1+\eta_0 a_{i1}}{a_{i1}-a_{i2}}\exp(a_{i1}t) - \dfrac{1+\eta_0 a_{i2}}{a_{i1}-a_{i2}}\exp(a_{i2}t)\right] & \text{for } 4\eta_0\lambda_i^2 < 1 \\ \exp\left(\dfrac{-t}{2\eta_0}\right)\left[\cos\dfrac{\sqrt{4\eta_0\lambda_i^2-1}}{2\eta_0}t + \dfrac{1}{\sqrt{4\eta_0\lambda_i^2-1}}\sin\dfrac{\sqrt{4\eta_0\lambda_i^2-1}}{2\eta_0}t\right] & \text{for } 4\eta_0\lambda_i^2 \geq 1 \end{cases},$$

in which $\eta_0 = \kappa\mu/\rho R_0^2$, a_{i1} and a_{i2} are roots of the equation $\eta_0 s^2 + s + \lambda_i^2 = 0$ when $4\eta_0\lambda_i^2 < 1$. And both a_{i1} and a_{i2} are less than zero.

Some positive roots λ_i of the equation $\varphi(\lambda_i, 2) = J_1(\lambda_i)Y_1(2\lambda_i) - J_1(2\lambda_i)Y_1(\lambda_i) = 0$ are listed in [19]. The first six roots are as follows:

i	1	2	3	4	5	6
λ_i	3.1966	6.3124	9.4445	12.5812	15.7199	18.8599

From Eq. (34) it is obvious that to draw a diagram of velocity distribution of Maxwell fluid two different values of η_0 should be distinguished, namely $\eta_0 < 1/4\lambda_i^2$ and $\eta_0 \geq 1/4\lambda_i^2$ which correspond to the first and second expression of $B(\lambda_i, t)$ in Eq. (34), respectively. Without loss of generality, due to the monotonicity of λ_i we can choose $\eta_0 = 1$ ($\eta_0 > 1/4\lambda_i^2$) and $\eta_0 = 0.02$ ($1/159.24 \approx 1/4\lambda_2^2 < \eta_0 < 1/4\lambda_1^2 \approx 1/40.96$) as model parameters to draw the velocity distribution. According to Eq. (34) and the second expression of $B(\lambda_i, t)$, the flow development

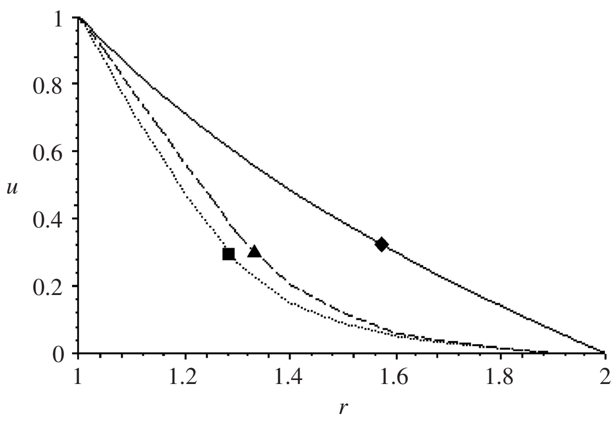

Fig. 1. Flow development of GMF in an annular cylindrical space for different times t when $\alpha = 0.5$ and $\eta = 1.0$. ■ $t = 0.03$, ▲ $t = 0.10$, ◆ $t \to \infty$

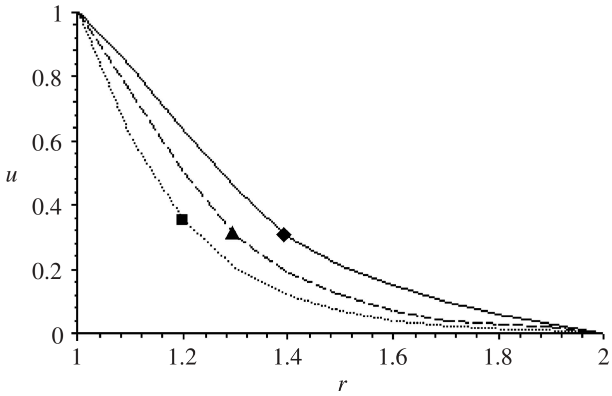

Fig. 2. Flow development of GMF in an annular cylindrical space for different values of α when $t = 0.10$ and $\eta = 1.0$. ■ $\alpha = 0.3$, ▲ $\alpha = 0.5$, ◆ $\alpha = 0.8$

for Maxwell fluid as a function of r for different times t when $\eta_0 = 1$ is shown in Fig. 3. From Fig. 3 we can find that at small times a velocity step jump exists in the flow field and it travels nearly at a constant speed, and its amplitude decays rapidly in time. At the same time it can be seen that at small times there are reflected waves in the flow field. According to Eq. (34) and the first expression of $B(\lambda_i, t)$, the flow development as a function of r for different times t when $\eta_0 = 0.02$ is shown in Fig. 4. The same characteristics of the flow field as we saw in Fig. 3 can be found in Fig. 4. However, it can be seen that there is a discrepancy between Fig. 3 and Fig. 4 due to the different values of the parameter η_0. Actually, because an exponential decaying term from the first expression of $B(\lambda_i, t)$ is added to $B(\lambda_i, t)$, the establishment of processes is so fast as to approach to steady state when $t = 0.4$ as shown in Fig. 4. In fact, because the condition $\eta_0 \geq 1/4\lambda_i^2$ can always be satisfied due to the property of monotone increasing λ_i, from the second expression of $B(\lambda_i, t)$ in Eq. (34) it can be seen that for a fixed time the oscillation of velocity distribution along the radial direction exists due to the existence of sine and cosine terms as shown in Figs. 3 and 4. All of the above mentioned visualized characteristics of the flow field coincided with the results given by Joseph [11] and Preziosi et al. [20]. In fact, the unsteady Couette flow considered in our paper is related to the well-known Stokes' first problem and theoretical model for the wave-speed meter which have been studied in the literature.

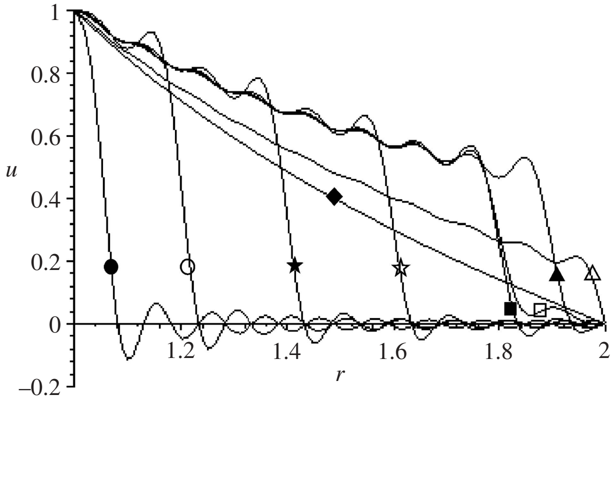

Fig. 3. Flow development of Maxwell fluid in an annular cylindrical space for different times t when $\eta_0 = 1.0$. ● $t = 0.05$, ○ $t = 0.2$, ★ $t = 0.4$, ☆ $t = 0.6$, ■ $t = 0.8$, ▲ $t = 0.9$, □ $t = 1.2$, △ $t = 3.0$, ◆ $t \to \infty$

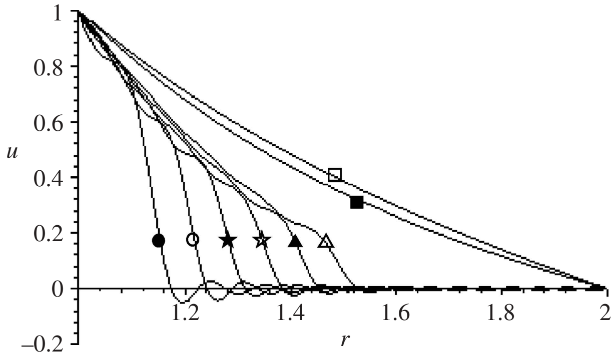

Fig. 4. Flow development of Maxwell fluid in an annular cylindrical space for different times t when $\eta_0 = 0.02$. ● $t = 0.02$, ○ $t = 0.03$, ★ $t = 0.04$, ☆ $t = 0.05$, ▲ $t = 0.06$, △ $t = 0.07$, ■ $t = 0.4$, □ $t \to \infty$

In particular, if $\alpha = 0$, the model represents the complete viscous Newtonian fluid,

$$A(\lambda_i, t) = L^{-1}\{\bar{A}(\lambda_i, s)\} = L^{-1}\left\{\frac{2}{2s + \lambda_i^2}\right\} = \exp\left(-\frac{\lambda_i^2}{2}t\right),$$

and the exact solution of the model can be written as

$$u(r, t) = \frac{b^2 - r^2}{(b^2 - 1)r} - \sum_{i=1}^{\infty} \exp\left(-\frac{\lambda_i^2}{2}t\right)\frac{\pi J_1^2(\lambda_i b)\varphi(\lambda_i, r)}{J_1^2(\lambda_i) - J_1^2(\lambda_i b)}, \tag{35}$$

which is another form of the analytic solution for the equation of Couette flow of a classical Newtonian fluid.

From Eq. (33) we find that the solution of the model is resolved into two parts. $u_0(r, t)$ is the steady part and the rest is the unsteady one. Comparing Eq. (35) with Eq. (33) we see that the steady part of the result of GMF has the same expression as that of the classical Newtonian fluid. Now, let us consider the establishment of steady state of GMF. In order to get the asymptotic expression of $A(\lambda_i, t)$ in Eq. (33) for $t \to \infty$, on the basis of the operational calculus theory we calculate the asymptotic expression of $\bar{A}(\lambda_i, s)$ in Laplace space for $s \to 0$. By means of Eq. (30) it is obvious that the following expression can be obtained:

$$\bar{A}(\lambda_i, s) \propto \frac{1}{\lambda_i^2}\{1 - \eta s^\alpha + \eta^2 s^{2\alpha} + \ldots\} \quad (0 < \alpha < 1, s \to 0). \tag{36}$$

Hence in virtue of [18] we arrive at

$$A(\lambda_i, t) \propto t^{-(1+\alpha)} \qquad (t \to \infty). \tag{37}$$

It is of interest to note that comparing (37) with (35) we can find that the decaying of unsteady part of GMF displays power law behavior, which has a typical property of its scale invariance.

To sum up, from Figs. 1 to 4 and from the formula (33) we can draw such a conclusion that for the unsteady Couette flow introducing and utilizing the model of GMF do not effect the steady state distribution of the velocity which coincides with the Newtonian fluid. In contrast with this, the establishment of a steady process is effected by the model of GMF. In particular, when $\alpha = 1$ for the Maxwell fluid this effect may be divided into two different types governed by the parameter η_0 of the model. When $\eta_0 = k\mu/\rho R_0^2 \geq 1/4\lambda_1^2 \approx 1/40.96$, the establishment of flow is shown in Fig. 3, when $1/4\lambda_{i+1}^2 < \eta_0 < 1/4\lambda_i^2$ $(i = 1, 2, 3, \ldots)$ the establishment of flow is shown in Fig. 4.

5 Conclusion

With the help of fractional calculus and Laplace and Weber transforms, we get the exact solution on unsteady Couette flow of GMF in this paper. Then we compare the solutions for different values of α, and analyze the effect of the fractional parameter α on the establishment of flow. The solution in the present paper contains some results of classical Couette flow.

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