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Inherent frame dependence of thermodynamic fields in a gas

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Summary. Grad's 13-moment theory – appropriate for rarefied gases – implies that a gas cannot perform a rigid rotation, if it conducts heat. Indeed, stationary heat conduction in a gas between co-axial cylinders at rest in a non-inertial frame exhibits azimuthal components of velocity and heat flux. The fields of velocity and heat flux are calculated. The effects are due to Coriolis terms in all transfer equations that result from the Boltzmann equation.

1 Introduction

In ordinary thermodynamics which is appropriate to gases under normal conditions, – $p = 1$ atm (say) – heat conduction is governed by Fourier's law and viscous friction by the Navier-Stokes laws. These laws assume the heat flux q_i and the deviatoric pressure tensor $\rho_{\langle ij \rangle}$ to be linearly dependent on the gradients of temperature and velocity so that we have[1]

$$q_i = -\frac{\kappa}{\frac{k}{m}} \frac{\partial \theta}{\partial x_i} \quad \text{and} \quad \rho_{\langle ij \rangle} = -2\mu \frac{\partial v_{<i}}{\partial x_{j>}}. \tag{1.1, 2}$$

θ stands for $\frac{k}{m}T$, where T is the absolute temperature, and κ and μ are the thermal conductivity and viscosity, respectively. For He^4 we have, assuming that its atoms are Maxwellian ones,

$$\kappa = \frac{1}{\alpha}\frac{15}{4}\frac{k}{m}\theta, \quad \mu = \frac{1}{\alpha}\theta \quad \text{and} \quad \alpha = 4.8\,10^9\ \frac{\mathrm{m}^3}{\mathrm{kg}}\frac{1}{\mathrm{s}}. \tag{2}$$

According to the Navier-Stokes equation (1.2) a gas in rigid rotation enclosed between two rotating coaxial cylinders does not have shear pressures. And it is well-known that this rigid rotation is a solution of the field equations irrespective of the heat flux between the cylinders. Another way of expressing this is by saying that in a rotating, non-inertial frame the gas and the cylinders can be at rest, even when the inner cylinder is heated and the outer one is kept at a fixed temperature. There is no shear pressure in this case and the heat flux is radial.

We prove here that this is no longer the case in a rarefied gas. There is still no shear stress, because none is applied, but the gas is not at rest with respect to the cylinders in a non-inertial

[1] Index notation is used throughout the paper. Round brackets for indices denote symmetrization and angular brackets define a symmetric, trace-less tensor. k is the Boltzmann constant and m is the atomic mass of the gas.

frame, at least not in the presence of heat flux. This is so because the rarefied gas does not obey the constitutive relations (1) of Navier-Stokes and Fourier. These equations need to be replaced by the equations of extended thermodynamics. The more rarefied a gas is, the more complex become the necessary equations, and the higher they grow in number, see Sect. 2 below. Here we shall exploit the simplest case of extension, in which the Navier-Stokes, Fourier equations are replaced by balance equations for the pressure deviator and the heat flux; these form part of Grad's 13-moment system of field equations.

Before we enter the formal part of the paper we wish to provide the reader with a suggestive argument so that he may intuitively understand the phenomena to be expected in a rarefied gas. For that purpose we consider two co-axial cylinders between which a radial temperature gradient is created, cf. Fig. 1a. We concentrate the attention on a small volume element of the linear dimensions of the mean free path of the atoms. First we consider the gas at rest in an inertial frame. In that case the free paths of the atoms are straight lines and because of the temperature gradient the atoms flying from top to bottom carry a bigger energy downwards across the plane *S*–*S* than is carried upwards by the atoms moving up; see Fig. 1b. Therefore a net flux of energy accompanies the passage of a pair of atoms through *S*–*S*, and that flux is proportional to the temperature gradient and opposite to it, just as predicted by Fourier's law. Next we consider the same situation for a gas in a non-inertial frame. Now the paths of free flight are curved by the Coriolis force, and there is a flux of energy through the plane *H*–*H* as well as through the plane *S*–*S*; see Fig. 1c. Thus the flux now has an additional component perpendicular to the gradient of temperature.

That argument was invented by Müller [1] in order to show that the principle of material objectivity is violated in the kinetic theory. Others, e.g., Biscari and Cercignani [2], have confirmed that observation. Here, however, we are not concerned with material objectivity; instead we are interested in a solution of the Grad 13-moment equations. Figure 1 should merely serve us to understand that the heat flux – and the velocity, for that matter – may have azimuthal components in a non-inertial frame.

The present paper is not the first one which we have written on this subject. There is the previous paper [3] in which we recognized that a rigid rotation of the rarefied gas is impossible. However, since we were then confused about assignable boundary values and about the role of the shear component of the pressure tensor, we could not find the proper solutions for the fields.

This paper is part of a continuing research effort which aims to find solutions of the moment-equations appropriate to boundary conditions. We expect important qualitative

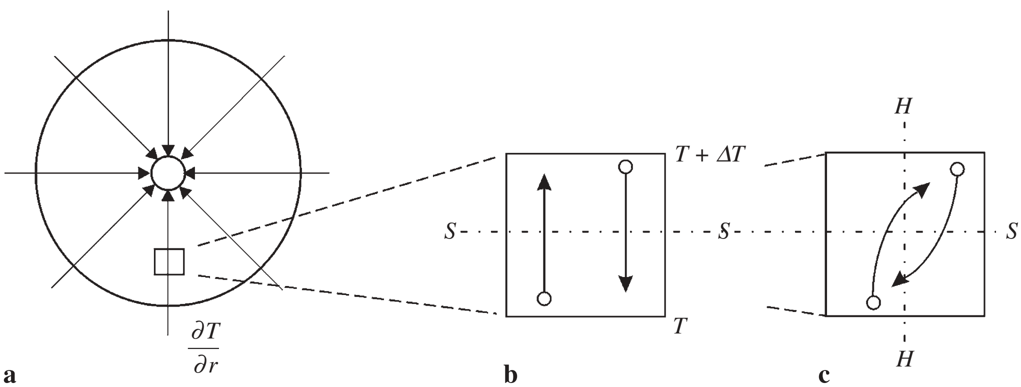

Fig. 1. On the frame dependence of the heat flux

results for rarefied gases in that field. Previous papers [4], [5] have already established interesting phenomena, viz. a distinction between kinetic and thermodynamic temperatures, and thermal boundary layers in a gas between parallel plates.

2 Extended thermodynamics in a non-inertial frame. In particular extended thermodynamics of 13 moments

The fields of extended thermodynamics of monatomic gases are moments of the distribution function $f(\underline{x}, \underline{c}, t)$ of the gas. $f(\underline{x}, \underline{c}, t)d\underline{c}$ represents the number density at $\underline{x}$ and t of the atoms that have velocities between $\underline{c}$ and $\underline{c} + d\underline{c}$. The moments of rank N are defined as

$$F_{i_1 i_2 \ldots i_N} = m \int c_{i_1} c_{i_2} \ldots c_{i_N} f \mathrm{d}\underline{c} \quad (N = 0, 1, \ldots) \tag{3}$$

such that F, $F_i = Fv_i$, F_{ij}, $\frac{1}{2}F_{ill}$ are the densities of mass, momentum, momentum flux, – i.e., pressure – and energy flux, respectively; v_i is the velocity of the gas.

The field equations of extended thermodynamics are the transfer equations of the moments which are dictated by the Boltzmann equation and which read[2], e.g., see [6] or [7],

$$\frac{\partial F_{i_1 i_2 \ldots i_N}}{\partial t} + \frac{\partial F_{i_1 i_2 \ldots i_N l}}{\partial x_l} - NF_{(i_1 i_2 \ldots i_{N-1}} i^0_{i_N)} - NF_{l(i_1 i_2 \ldots i_{N-1}} 2W_{i_N)l} = m\varphi(c_{i_1} c_{i_2} \ldots c_{i_N}). \tag{4}$$

The right-hand side is the moment of the collision operator that occurs in the Boltzmann equation. For Maxwellian molecules, which we shall consider, these collision terms are explicitly related to the moments of the distribution function itself.

The set of equations (4) is appropriate for a non-inertial frame in which the atoms are subject to an inertial acceleration of the form

$$i^c_i = 2W_{ik}\big(c_k - \dot{b}_k\big) - W^2_{ik}(x_k - b_k) + \dot{W}_{ik}(x_k - b_k) + \ddot{b}_i. \tag{5}$$

The acceleration consists – in that order – of the Coriolis-, centrifugal-, Euler-acceleration and the acceleration of relative translation of the frame to an inertial one. W_{ik} is the matrix of the angular velocity of the non-inertial frame and b_i is the distance vector between the origins of the frames; the dots on b_i and W_{ik} denote time derivatives. i^0_i in (4) represents the velocity-independent part of the inertial acceleration.

It is often appropriate to introduce internal moments

$$\rho_{i_1 i_2 \ldots i_N} = m \int C_{i_1} C_{i_2} \ldots C_{i_N} f \mathrm{d}\underline{c}, \tag{6}$$

where $C_i = c_i - v_i$ is the velocity of atoms relative to the velocity of the gas. Thus, ρ, ρ_{ij}, and $\frac{1}{2}\rho_{ill}$ are the mass density, the pressure tensor and the heat flux, respectively, while ρ_i is identically zero and ρ_{ll} is the density of internal energy. There is a one-to-one correspondence between the $\underline{F}$'s and the $\underline{\rho}$'s, viz.

$$F_{i_1 i_2 \ldots i_N} = \sum_{k=0}^{N} \binom{N}{k} \rho_{(i_1 i_2 \ldots i_{N-k}} v_{i_{N-k+1}} \ldots v_{i_N)} \tag{7}$$

so that the field equations (4) in terms of $\underline{\rho}$ read

[2] $F_{i_1 \, i_2 \ldots i_N}$ equals 0 for $N < 0$.

$$\frac{\mathrm{d}\rho_{i_1 i_2 \ldots i_N}}{\mathrm{d}t} + \rho_{i_1 i_2 \ldots i_N}\frac{\partial v_l}{\partial x_l} + N\rho_{(i_1 i_2 \ldots i_{N-1}}\left[\frac{\mathrm{d}v_{i_N)}}{\mathrm{d}t} - i^0_{i_N)} - 2W_{i_N)l}v_l\right] - N\rho_{k(i_1 i_2 \ldots i_{N-1}}2W_{i_N)k}$$
$$+\frac{\partial \rho_{i_1 i_2 \ldots i_N l}}{\partial x_l} + N\rho_{l(i_1 i_2 \ldots i_{N-1}}\frac{\partial v_{i_N)}}{\partial x_l} = m\varphi(C_{i_1}C_{i_2}\ldots C_{i_N}). \tag{8}$$

$\frac{\mathrm{d}}{\mathrm{d}t}$ is the material time derivative. We have $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}$.

Inspection shows that the equations of extended thermodynamics for $N > 0$ are all affected by inertial terms and we are interested in their effects. In order to be specific we choose a theory of 13 moments, where Eqs. (8) read explicitly

$$\begin{aligned}
&\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\partial v_l}{\partial x_l} = 0,\\
&+\frac{\partial \rho_{il}}{\partial x_l} + \rho\left[\frac{\mathrm{d}v_i}{\mathrm{d}t} - i^0_i - 2W_{ik}v_k\right] = 0,\\
&\frac{\mathrm{d}\rho_{ij}}{\mathrm{d}t} - 4\rho_{k(i}W_{j)k} + \rho_{ij}\frac{\partial v_l}{\partial x_l} + \frac{\partial \rho_{ijl}}{\partial x_l} + 2\rho_{k(i}\frac{\partial v_{j)}}{\partial x_k} = -\alpha\rho\rho_{\langle ij\rangle},\\
&\frac{\mathrm{d}\rho_{ill}}{\mathrm{d}t} - 6\rho_{k(il}W_{l)k} + \rho_{ijj}\frac{\partial v_l}{\partial x_l} + \frac{\partial \rho_{ijjl}}{\partial x_l} + 3\rho_{(il}\left[\frac{\mathrm{d}v_{l)}}{\mathrm{d}t} - i^0_{l)} - 2W_{l)k}v_k\right] + 3\rho_{k(ij}\frac{\partial v_{j)}}{\partial x_k} = -\frac{2}{3}\alpha\rho\rho_{ill}.
\end{aligned} \tag{9}$$

In this case of 13 moments there are only two collision terms and they have been calculated for Maxwellian molecules, e.g., cf. [6]. α is the constant given in (2), and $\alpha\rho$ is a typical value for the mean collision frequency of an atom.

The set of Eqs. (9) is not closed because of the occurrence of the third and fourth rank moments $\rho_{\langle ij\rangle l}$ and ρ_{ijjl}. We close the system by calculating those moments from the Grad 13-moment distribution, which represents an expansion of the distribution function in terms of Hermite polynomials

$$f^{\mathrm{G}} = f^{\mathrm{E}}\left(1 + \frac{1}{2p\theta}\rho_{\langle ij\rangle}C_iC_j - \frac{1}{2p\theta^2}\rho_{ijj}C_i\left(1 - \frac{1}{5\theta}C^2\right)\right). \tag{10}$$

Thus the additional moments read

$$\begin{aligned}
\rho_{\langle ij\rangle l} &= \frac{1}{5}\left(\rho_{ikk}\delta_{jl} + \rho_{jkk}\delta_{il} - \frac{2}{3}\rho_{lkk}\delta_{ij}\right),\\
\rho_{ijjl} &= 5p\theta\delta_{il} + 7\theta\rho_{\langle il\rangle}.
\end{aligned} \tag{11}$$

$p = \frac{1}{3}\rho_{ii} = \rho\theta$ is the pressure. f^{E} is the Maxwell distribution appropriate for equilibrium.

3 Heat conduction between co-axial cylinders

A simple case, in which a solution of the system (9), (11) can be found, is the case of stationary heat conduction in a gas between co-axial cylinders whose axes coincide with the axis of rotation of the frame; there is no translation of the frame, nor any angular acceleration.

If the rotational axis is in the x_3-direction we thus have

$$b_i \equiv 0 \quad \text{and} \quad W_{ij} \equiv \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

where ω is the constant angular velocity of the frame.

As boundary conditions we choose that the gas at the inner cylinder is heated at a prescribed rate and at the outer cylinder it is kept at a fixed temperature. Also at the cylinders the gas exhibits no slip nor, of course, can it penetrate the cylinders.

The symmetry of the problem suggests that we use cylindrical coordinates (r, ϑ, z). The metric tensor and the Christoffel symbols are then given by

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma^1_{22} = -r, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^m_{kn} = 0 \quad \text{else.} \tag{13}$$

As it is usual for such stationary processes of high symmetry we make an assumption on the character of the solution: a semi-inverse ansatz. In the present case a natural assumption of this type would be that

– all fields depend on r only,
– there is no velocity in the z-direction,
– there is no heat flux in the z-direction,
– there are no shear stresses in the z-direction.

We use physical cylindrical components of all vectors and tensors and thus the semi-inverse ansatz reads

$$\begin{gathered} p = p(r), \quad \theta = \theta(r), \quad v[i] = \begin{bmatrix} v[r] \\ v[\vartheta] \\ 0 \end{bmatrix}, \\ \rho[ijj] = 2\begin{bmatrix} q[r] \\ q[\vartheta] \\ 0 \end{bmatrix}, \quad \rho[ij] - pg[ij] = \begin{bmatrix} \sigma[rr] & \sigma[r\vartheta] & 0 \\ \sigma[r\vartheta] & \sigma[\vartheta\vartheta] & 0 \\ 0 & 0 & \sigma[zz] \end{bmatrix}. \end{gathered} \tag{14}$$

All fields may be functions of r. Square brackets indicate physical cylindrical components. Thus $q[i]$ are the components of the heat flux and $\sigma[ij]$ are the components of the deviatoric pressure tensor.

The mass balance requires $v[r] = 0$ and is thus satisfied. The remaining equations follow from (9), (11) and we obtain

$$\frac{\mathrm{d}(p + \sigma[rr])}{\mathrm{d}r} - \frac{1}{r}(\sigma[\vartheta\vartheta] - \sigma[rr]) = \frac{p}{\theta}\frac{1}{r}(r\omega + v[\vartheta])^2, \tag{15.1}$$

$$\sigma[r\vartheta] = \frac{D}{r^2}, \tag{15.2}$$

$$q[r] + \sigma[r\vartheta]v[\vartheta] = \frac{C}{r}, \tag{15.3}$$

$$-\frac{5}{2}\frac{\alpha p}{\theta}\sigma[rr] = -10\omega\sigma[r\vartheta] + \frac{4}{3}\frac{\mathrm{d}q[r]}{\mathrm{d}r} - \frac{5}{3}\sigma[r\vartheta]\frac{\mathrm{d}v[\vartheta]}{\mathrm{d}r} - \frac{2}{3}\frac{q[r]}{r} - \frac{25}{3}\frac{\sigma[r\vartheta]v[\vartheta]}{r}, \tag{15.4}$$

$$-\frac{5}{2}\frac{\alpha p}{\theta}\sigma[r\vartheta] = \frac{5}{r}(\sigma[rr] - \sigma[\vartheta\vartheta])(r\omega + v[\vartheta]) + \frac{\mathrm{d}q[\vartheta]}{\mathrm{d}r} - \frac{q[\vartheta]}{r} + \frac{5}{2}(p + \sigma[rr])\left(\frac{\mathrm{d}v[\vartheta]}{\mathrm{d}r} - \frac{v[\vartheta]}{r}\right), \tag{15.5}$$

$$-\frac{5}{2}\frac{\alpha p}{\theta}\sigma[\vartheta\vartheta] = 10\omega\sigma[r\vartheta] + \frac{2}{r}q[r] + 4\sigma[r\vartheta]\frac{\mathrm{d}v[\vartheta]}{\mathrm{d}r} + \frac{6}{r}\sigma[r\vartheta]v[\vartheta], \tag{15.6}$$

$$\begin{aligned}-\frac{4}{3}\frac{\alpha p}{\theta}q[r] &= -4\omega q[\vartheta] - \frac{2}{r}\sigma[rr](r\omega + v[\vartheta])^2 + (5p + 7\sigma[rr])\frac{\mathrm{d}\theta}{\mathrm{d}r} \\ &\quad + 2\theta\left[\frac{\mathrm{d}\sigma[rr]}{\mathrm{d}r} + \frac{1}{r}(\sigma[rr] - \sigma[\vartheta\vartheta])\right] + \frac{4}{5}q[\vartheta]\frac{\mathrm{d}v[\vartheta]}{\mathrm{d}r} - \frac{24}{5}\frac{q[\vartheta]v[\vartheta]}{r},\end{aligned} \tag{15.7}$$

$$-\frac{4}{3}\frac{\alpha p}{\theta}q[\vartheta] = \frac{4}{r}q[r](r\omega + v[\vartheta]) - \frac{2}{r}\sigma[r\vartheta](r\omega + v[\vartheta])^2 + 7\sigma[r\vartheta]\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{14}{5}q[r]\left(\frac{\mathrm{d}v[\vartheta]}{\mathrm{d}r} - \frac{v[\vartheta]}{r}\right). \tag{15.8}$$

D and C are constants which result from the integration of the ϑ-component of the momentum balance and of the energy balance, respectively. $\sigma[r\vartheta]$ is the shear stress on a cylinder in the circumferential direction such that

$$M' = 2\pi r^2\sigma[r\vartheta] = 2\pi D$$

is the torque per unit length in the z-direction. By (15.2) M' is constant, so that this torque may be applied at the outer or inner cylinder or it may be applied to a shaft to which the outer and inner cylinder are rigidly connected. Since we are interested in the free rotation of the cylinders, *we shall consider the case that there is no such torque*, so that the outer – and the inner – cylinder are free of the azimuthal shear. This means that we have to set $\sigma[r\vartheta] = 0$, or $D = 0$. This is the natural case to be considered, because the inner and the outer cylinder is then at rest and so is the gas, except if there is heat conduction.

In this case the system (15) may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{v[\vartheta]}{r}\right] = -\frac{5}{7}\left[\frac{2}{3}\frac{\alpha p}{\theta}\frac{1}{C}q[\vartheta] + \frac{2}{r}\left(\omega + \frac{v[\vartheta]}{r}\right)\right], \tag{16.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{q[\vartheta]}{r}\right] = \frac{25}{21}\frac{\alpha p}{\theta}\frac{1}{C}\left(p + \frac{4}{5}\frac{\theta}{\alpha p}\frac{C}{r^2}\right)q[\vartheta] + \left[\frac{25}{7}\frac{1}{r}\left(p + \frac{4}{5}\frac{\theta}{\alpha p}\frac{C}{r^2}\right) - 8\frac{\theta}{\alpha p}\frac{C}{r^3}\right]\left(\omega + \frac{v[\vartheta]}{r}\right), \tag{16.2}$$

$$\begin{aligned}&5p\left(1 + \frac{36}{25}\frac{\theta}{\alpha p^2}\frac{C}{r^2}\right)\frac{\mathrm{d}\theta}{\mathrm{d}r} - \frac{8}{5}\frac{\theta^2}{\alpha p^2}\frac{C}{r^2}\frac{\mathrm{d}p}{\mathrm{d}r} \\ &\quad = -\frac{4}{3}\frac{\alpha p}{\theta}\frac{C}{r} + \frac{36}{7}q[\vartheta]\left(\omega + \frac{v[\vartheta]}{r}\right) + \frac{8}{5}\frac{\theta}{\alpha p}\frac{C}{r}\left(\omega + \frac{v[\vartheta]}{r}\right)^2 + \frac{8}{21}\frac{\alpha p}{\theta}\frac{r}{C}q[\vartheta]^2,\end{aligned} \tag{16.3}$$

$$\frac{4}{5}\frac{1}{\alpha p}\frac{C}{r^2}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \left(1 - \frac{4}{5}\frac{\theta}{\alpha p^2}\frac{C}{r^2}\right)\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{p}{\theta}r\left(\omega + \frac{v[\vartheta]}{r}\right)^2, \tag{16.4}$$

$$\sigma[rr] = -\sigma[\vartheta\vartheta] = \frac{4}{5}\frac{\theta}{\alpha p}\frac{C}{r^2}. \tag{16.5}$$

One might be tempted to strengthen the semi-inverse ansatz by setting $v[\vartheta] = 0$, so that the gas rotates rigidly. This, however, is impossible. Indeed, if we set $v[\vartheta] = 0$ in (16), we obtain from (16.1)

$$q[\vartheta] = -3\frac{\theta}{\alpha p}\omega\frac{C}{r} \tag{17}$$

and therefore, by (15.3), (15.5) with $\sigma[r\vartheta] = 0$ and (16.5)

$$\omega C\left[\frac{14}{r}\frac{\theta}{p} - 3\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\theta}{p}\right)\right] = 0, \tag{18}$$

which, in view of (16.3, 4) cannot be satisfied unless

– $C = 0$, i.e., no heat conduction, or
– $\omega = 0$, i.e., the frame is inertial.

Therefore a rigid rotation of a gas is impossible in the presence of heat conduction. That is the main conclusion of this paper. It is the same conclusion that was already reached in [3]. But here we continue and derive the shape of all fields as function of r for realistic boundary conditions.

4 Solutions

We choose the parameters of the problem as follows:

$$r_i = 0.5\,10^{-2}\ \mathrm{m}, \quad r_e = 2\,10^{-2}\ \mathrm{m}, \quad \omega = 5\,\frac{1}{\mathrm{s}}, \tag{19}$$

and we solve Eqs. (16) for the boundary data

$$q[r](r_i) = 10^3\ \frac{\mathrm{W}}{\mathrm{m}^2}, \quad p(r_e) = x\ \frac{\mathrm{N}}{\mathrm{m}^2}, \quad \theta(r_e) = 300\ \mathrm{K}\,\frac{k}{\mathrm{m}}, \quad v[\vartheta](r_e) = 0. \tag{20.1–4}$$

The boundary value $q[\vartheta](r_e)$ is used as shooting parameter which we adjust so that $v[\vartheta](r_i) = 0$ holds. Thus there is no slip of the gas at both cylinders and that condition determines the boundary value $q[\vartheta](r_e)$ which could realistically not be prescribed independently. We choose He^4 as the gas, so that m in (20) is the atomic mass of Helium. The prescription (20.1) of the heat flux at r_i means that the parameter C in the equations equals $5\,\frac{\mathrm{W}}{\mathrm{m}}$. We choose x in the pressure relation (20.2) as 10, 20, 30, 50 and 100; all pressure values are thus appropriate to a rarefied gas.

The set of equations is easily solved by integration from r_e inwards to r_i, and we obtain the solutions shown in Fig. 2. The resulting values of the shooting parameter $q[\vartheta](r_e)$ are listed in the figure caption.

Inspection of the Figure shows that the azimuthal velocity lags behind the velocities of the cylinders in the bulk of the gas; more so, the lower the pressure $p(r_e)$ is. The velocity field has a narrow boundary layer near the outer cylinder. In order to appreciate the amount of lag we note that, with the data given, the outer cylinder has a speed of $0.1\,\frac{\mathrm{m}}{\mathrm{s}}$ with respect to an inertial frame.

The temperature increases toward the inner cylinder, as expected, since there is an outward heat flux imposed by the boundary conditions. At the inner cylinder the temperature has the value 306.9 K, up 6.9 K from 300 K at the outer cylinder. For the lowest pressure $p(r_e) = 10\,\frac{\mathrm{N}}{\mathrm{m}^2}$ the temperature increase is a little smaller.

The azimuthal heat flux $q[\vartheta]$ has a boundary layer to match the boundary layer of the velocity field. Note that, by (15.5), both boundary layers essentially compensate each other so as to ensure the vanishing of the shear stress. For an appreciation of the values of $q[\vartheta](r)$, shown in Fig. 2, we compare them with $q[r](r_i)$ which, by (20.1), amounts to $10^3\,\frac{\mathrm{W}}{\mathrm{m}^2}$. Thus the azimuthal heat flux is mostly smaller than 1‰ of the radial one in the circumstances considered.

The pressure grows with increasing radius because of the centrifugal forces. The non-homogeneous temperature field makes the pressure profile concave rather than convex, which it would be without heat conduction.

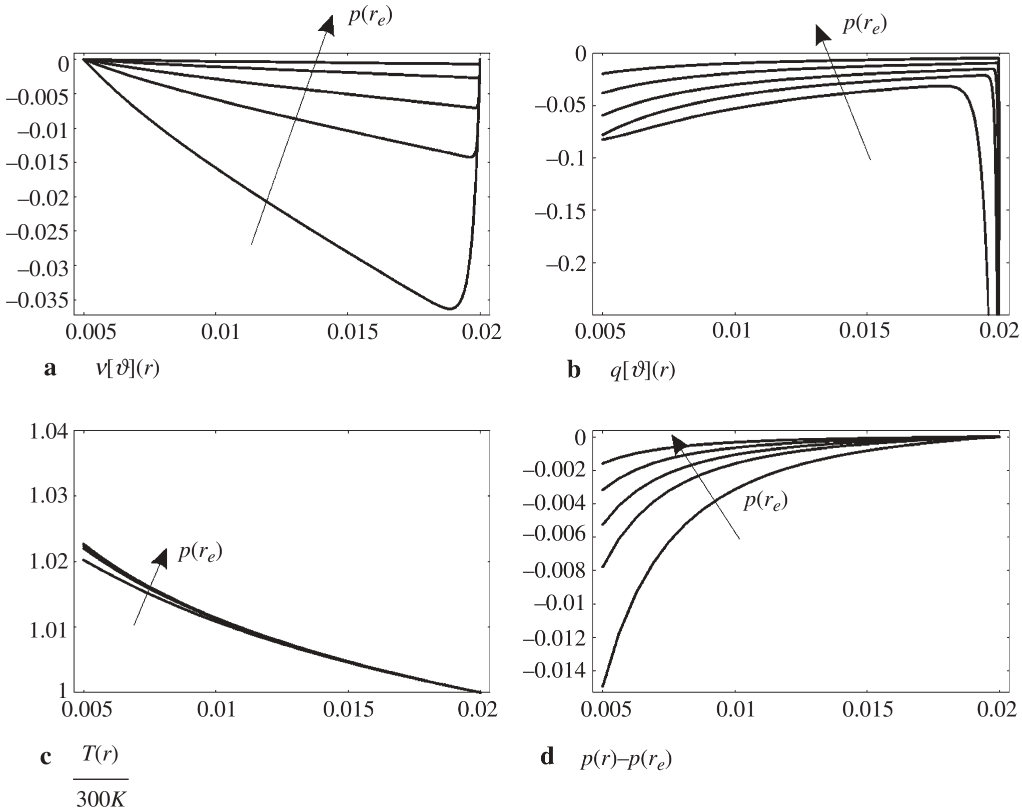

Fig. 2. Solutions in the domain $r_i \leq r \leq r_e$ for values $p(r_e) = (10, 20, 30, 50, 100)\frac{\mathrm{N}}{\mathrm{m}^2}$ increasing as indicated. **a** $v[\vartheta](r)$ in $\frac{\mathrm{m}}{\mathrm{s}}$; **b** $q[\vartheta](r)$ in $\frac{\mathrm{W}}{\mathrm{m}^2}$ with $q[\vartheta](r_e) = (-1.02, -0.72, -0.54, -0.34, -0.18)\frac{\mathrm{W}}{\mathrm{m}^2}$ for increasing pressure; **c** $T(r)$ in units of 300K; **d** $p(r) - p(r_e)$

Of course $p(r)$ is not the only contribution to the pressure tensor. There is also $\sigma[rr]$ so that the radial normal component of the pressure tensor is equal to

$$\rho[rr] = p + \sigma[rr], \quad \text{or by (16.5):} \quad \rho[rr] = p + \frac{4}{5}\frac{\theta}{\alpha p}\frac{C}{r^2}. \tag{21}$$

The second term in this expression dominates by far so that the normal pressure decreases with increasing r. Figure 3 shows graphs for $\rho[rr](r)$. The effect is due to heat conduction, because it depends on C.

Coming back to the temperature fields we note that these are – for all pressures – only minimally affected by the rotation of the frame and the consequent azimuthal fields $v[\vartheta](r)$ and $q[\vartheta](r)$. Figure 4 demonstrates that fact in showing two graphs – one on top of the other – for $\theta(r)$ and $p(r)$ in an inertial frame and in a non-inertial frame.

On the other hand, if we were to use the Fourier law, which according to (15.7) reads

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{1}{5p}\left\{-\frac{4}{3}\frac{C}{r}\frac{\alpha p}{\theta}\right\}, \tag{22}$$

and if we compare its solution with the solution of (16.4), we obtain a considerable difference as illustrated by Fig. 5, at least for the low pressure $p(r_e) = 10\frac{\mathrm{N}}{\mathrm{m}^2}$. For higher pressures that difference vanishes.

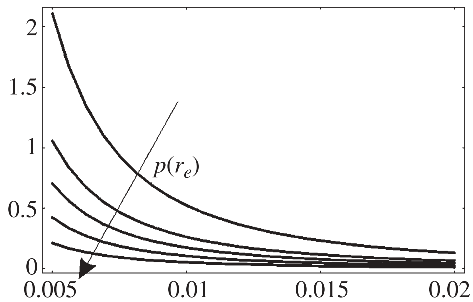

Fig. 3. $\rho[rr] - p(r_e)$ in $\frac{\mathrm{N}}{\mathrm{m}^2}$ for $p(r_e) = (10, 20, 30, 50, 100)\,\frac{\mathrm{N}}{\mathrm{m}^2}$

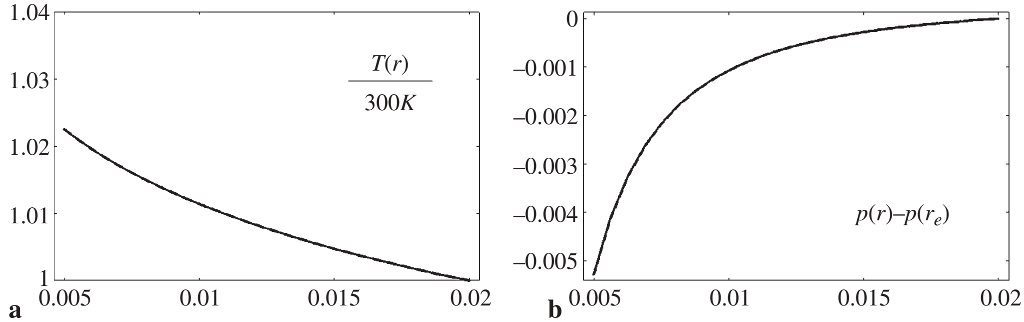

Fig. 4. **a** $\theta(r)$; **b** $p(r)$. Both figures show two curves, one on top of the other. They represent the temperatures and pressures for $p(r_e) = 30\,\frac{\mathrm{N}}{\mathrm{m}^2}$ in an inertial frame and in a non-inertial one, respectively

The difference between the temperature fields according to Fourier and Grad was recently illustrated by Müller and Ruggeri [4]. It is still noticeable for higher pressures, if the inner cylinder has a smaller radius than assumed in the present paper.

5 Discussion

It is customary to introduce the Knudsen number as a measure for the rarefaction of a gas. The Knudsen number is the ratio of the mean free path λ and the macroscopic dimension of the gas, which in the present case, by (19), is $1.5\ 10^{-2}$ m. The mean free path is taken to be $\lambda = c\tau$, where $c = \sqrt{1.66}\sqrt{\theta}$ is the speed of sound of a monatomic gas and $\tau = \frac{1}{\alpha}\frac{\theta}{p}$ is the mean time of free flight. Thus we have for $T = 300$ K

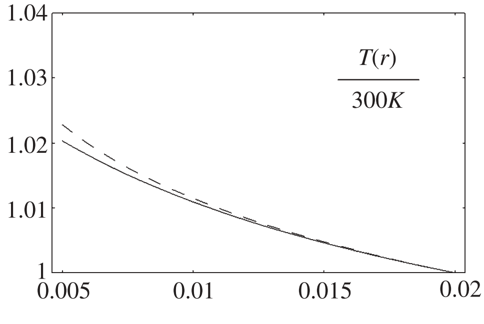

Fig. 5. Comparison of the Fourier solution (dashed) and the Grad solution (solid) for $p(r_e) = 10\,\frac{\mathrm{N}}{\mathrm{m}^2}$

$$\lambda = \sqrt{1.66}\frac{\theta^{\frac{3}{2}}}{\alpha}\frac{1}{p} = 0.13\frac{1}{p/\frac{\mathrm{N}}{\mathrm{m}^2}}\mathrm{m}. \tag{23}$$

Therefore the pressures which we have considered correspond to Knudsen numbers as shown in Table 1.

It is well-known that the Navier-Stokes-Fourier theory is the small-Knudsen-limit of the thermodynamic equations. The 13-moment theory is considered to be better for rarefied gases, i.e. larger Knudsen numbers. But it is uncertain how reliable the theory is for large Knudsen numbers. In that respect we have to consider the results for quite large Knudsen numbers – small pressures – with caution. It may well be that we need to treat such cases with extended thermodynamics of higher order. The equations are well-known, cf. [8] and [9], but boundary values present a problem.

Boundary layer flows are prone to be unstable. Therefore a linear stability analysis for our solution is indicated. That investigation, however, is postponed to a future time.

6 A simplified case

Inspection of Figs. 2c and d shows that the variation of temperatures is roughly 2% between the inner and the other cylinder, while the variation of pressures is less than one tenth of one percent. Therefore in the Eqs. (16.1, 2) we may approximate p and θ as constants and thus obtain a system of linear coupled differential equations for $v[\vartheta]$ and $q[\vartheta]$ with r-dependent coefficients.

We set

$$\tau = \frac{1}{\alpha}\frac{\theta(r_e)}{p(r_e)}, \quad Q = \frac{q[\vartheta]}{r}, \quad W = \omega + \frac{v[\vartheta]}{r} \tag{24}$$

and obtain the system in the form

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}r} &= -\frac{10}{21}\frac{1}{\tau}\frac{r}{C}Q - \frac{10}{7}\frac{1}{r}W,\\ \frac{\mathrm{d}Q}{\mathrm{d}r} &= \frac{25}{21}\left(p + \frac{4}{5}\tau\frac{C}{r^2}\right)\frac{1}{\tau}\frac{r}{C}Q + \left[\frac{25}{7}\left(p + \frac{4}{5}\tau\frac{C}{r^2}\right)\frac{1}{r} - 8\tau\frac{C}{r^3}\right]W.\end{aligned} \tag{25}$$

Table 1. Knudsen numbers for the relevant pressures

$p/\frac{\mathrm{N}}{\mathrm{m}^2}$	10	20	30	50	100
Kn	0.87	0.43	0.29	0.17	0.087

Table 2. Mean times of free flight

$p/\frac{\mathrm{N}}{\mathrm{m}^2}$	10	20	30	50	100
$\tau/10^{-6}s$	13.1	6.55	4.33	2.52	1.31

τ is a time of the order of magnitude of the mean time of free flight, and for the data (2) and (20) we have the values listed in Table 2.

We solve these equations for boundary values on $v[\vartheta]$ and $q[\vartheta]$ as before and obtain graphs, which are nearly identical to those of Figs. 2a and b. Therefore the "linearized" theory gives very nearly the same results as the exact theory, at least for the parameters under consideration.

7 Conclusions

The main conclusion is that a gas cannot be at rest in a rotating frame when there is heat conduction. Indeed, there is always an azimuthal velocity component although this becomes negligible small for a dense gas. An azimuthal heat flux "compensates" for the azimuthal velocity gradient.

The fields of temperature and pressure are only minimally affected by the rotation – at least for the very moderate angular velocity of 5 Hz of the frame which we have considered here. This makes it possible to simplify the problem by linearization.

There is a normal pressure $\sigma[rr]$ in the radial direction which exceeds the isotropic pressure p by far for the data considered.

Remark

A reviewer of a previous version of this paper has called our attention to a paper by Sharipov, Gramani Cumin and Kremer [10], where the authors deal with evaporation/condensation, and heat- and momentum transfer in a gas between two rotating cylinders. They employ a combination of Thermodynamics of Irreversible Processes and a direct solution of an (approximated) Boltzmann equation. The boundary values are not conventional ones – like no-slip and continuity of temperature and heat flux – rather they are imposed on the distribution function. A vanishing shear force – the natural boundary condition considered by us – is not an easily exploited option in that case.

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