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Relations between material, intermediate and spatial generalized strain measures for anisotropic multiplicative inelasticity

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Summary. In this contribution we discuss the application of generalized strain measures to finite inelasticity based on the multiplicative decomposition of the total deformation gradient. The underlying symmetry properties of the material are modelled via the incorporation of structural tensors while the evolution of any inelastic spin is neglected. Appropriate pushforward and pullback transformations of particular generalized strain measures to different configurations enable the setup of anisotropic hyperelastic formats with respect to all configurations of interest. This rather general formalism turns out be convenient in view of for instance efficient numerical algorithms and computational applications.

1 Introduction

The application of different stretch representations and strain measures within nonlinear continuum mechanics is since several decades under discussion; see for instance the fundamental contributions by Murnaghan [1], Kauderer [2] or Richter [3] and Truesdell and Toupin [4] or Eringen [5] for an overview. When referring to generalized strain measures, we commonly think of sufficiently smooth monotone tensor functions with respect to an appropriate deformation tensors. The spectral decomposition theorem thereby allows convenient interpretation in terms of principal stretches. These ideas date back to the pioneering contributions by Seth [6] and Hill [7]. Particular emphasis on logarithmic strains has for instance been placed by Hoger [8], Sansour [9] and Xiao and Chen [10] among others. For a general overview we refer the reader to the monographs by Biot [11], Ogden [12] and Šilhavý [13] or Havner [14] and Lubarda [15] where inelastic material behavior is addressed.

The formulation of nonlinear constitutive response as based on the introduction of different deformation and strain measures as well as higher order terms, say, constitutes a traditional but still very active field of research; see for instance the classical modelling approaches elaborated in the monographs by Murnaghan [16] and Kauderer [17]. Apparently, for particular applications where the total strains might remain rather small it is – from the modelling point of view – attractive to introduce St. Venant-Kirchhoff type constitutive equations in terms of generalized strain measures. Conceptually speaking, the sought nonlinear response is incorporated via specific strain measures with respect to which the quadratic format of the strain energy function is retained. In this regard, the well-established framework of linear elasticity is adopted and combined with generalized strain measures which then constitutes an essential part of the constitutive modelling itself. It is well-known, however, that the region of ellipticity is rather restricted for these approaches; see, e.g., Bruhns et al. [18] for a detailed discussion based on an isotropic setting in terms of logarithmic strains. Nevertheless, one of the main advantages of a St. Venant-Kirchhoff type ansatz consists in the fact that the backbone of anisotropic linear elasticity can directly be combined with nonlinearities related to appropriate generalized strain measures; see for instance the monograph by Green and Adkins [19] and references cited therein. For detailed surveys the reader is also referred to the contributions by Mehrabadi and Cowin [20], [21] and Rychlewski [22]. A delightful representation of Hooke's law, say, in anisotropic linear elasticity is provided by the introduction of Kelvin modes, or in other words, the application of the spectral decomposition theorem to the elasticity tensor; see for instance [23]–[28].

Recently, Papadopoulos and Lu [29], [30] introduced generalized strain measures to a strain–space formulation of finite isotropic and anisotropic elasto–plasticity, see also the contributions by Miehe et al. [31] or Schröder et al. [32]. In this direction, we focus on a stress-space framework in the sequel and model the underlying symmetry of the material via structural tensors which – at least formally – enables us to overcome the structure of a St. Venant-Kirchhoff type ansatz. The essential point of departure thereby consists in applying the fundamental covariance principle to the Helmholtz free energy density. Following these ideas allows to set up a convenient formulation (especially for numerical applications) in terms of spatial arguments. Detailed background information of the fundamental covariance principle is provided in, e.g., the monograph by Marsden and Hughes [33], while its application to anisotropic response is elaborated by Lu and Papadopoulos [34] and Menzel and Steinmann [35], [36], see also [37].

In the following, particular emphasis is placed on the multiplicative decomposition of the deformation gradient into an elastic and inelastic contribution, see for instance [38] and references cited therein. Concerning notation, we distinguish between co– and contra–variant base vectors. The reader is referred to, e.g., [39], [40] or [41] for further background information in this regard. The Helmholtz free energy density is assumed to incorporate the total deformation gradient as well as the elastic distortion. Accordingly, different types of material behavior – typically elasticity, plasticity or viscoelasticty – are included within the proposed framework. Based on these deformation quantities, various strain tensors can be introduced – naming solely some of theses measures, classical strain tensors are typically identified with e.g. Biot, Green–Lagrange and Almansi strains, tensors of Green or Karni Reiner type, etc. In this study, however, we mainly focus on strain tensors of the Seth-Hill family. These generalized strain measures are introduced with respect to different configurations in the sequel. Moreover, we carefully distinguish between strain tensors of co- and contra-variant type. In this regard, the main goal of this work consists in the elaboration of these strain measures embedded into the framework of general covariance of the Helmholtz free energy density. This finally enables us to clearly develop pushforward and pullback transformations between generalized strains and correlated hyperelastic stress tensors in different configurations. As such, this study might on one hand seem slightly technical or rather formal but on the other hand reviews established formulations and gives new insight into the geometric interpretation of the relations between strain and stress quantities in one and the same as well as different configurations. With these elaborations in hand, evolution equations for the inelastic distortion or related strain measures, respectively, are discussed. By analogy with the transformations between different strain tensors, one observes similar relations for the driving forces entering these evolution equations. As a main result, essential equations for anisotropic inelasticity together with anisotropic elasticity in terms of entirely spatial arguments are derived which is of particular interest for efficient numerical simulations based on, for instance, finite element techniques.

The paper is organised as follows: In Sect. 2, we formally review essential kinematic relations based on a multiplicative decomposition of the total deformation gradient. Various stretch tensors are thereby introduced and particular emphasis is placed on the application of the spectral decomposition theorem. Based on these elaborations, the definition of Seth-Hill type strain measures is discussed in Sect. 3. Besides distinguishing between co- and contra-variant generalized strain measures, we additionally derive transformation relations between different strain measures in one and the same configuration as well as between strain measures in different configurations. Next, the Helmholtz free energy density is introduced in terms of appropriate strain tensors and an additional structural tensor, say; see Sect. 4. The Helmholtz free energy density is thereby defined in terms of two contributions – one with respect to the total deformation gradient while the other refers to the elastic distortion – so that classical modelling approaches as e.g. elasticity, plasticity or viscoelasticity are embodied. Appropriate stress tensors as well as typical evolution equations are elaborated in Sect. 5. An alternative approach of Green-Naghdi type is discussed in Sect. 6, and the paper is closed with a short summary in Sect. 7.

2 Deformation measures

Let $\boldsymbol{\varphi}(\boldsymbol{X},t) : \mathscr{B}_0 \times \mathbb{R} \to \mathscr{B}_t | \boldsymbol{X} \times t \mapsto \boldsymbol{x}$ represent the nonlinear motion of the body B of interest with corresponding linear tangent map $\boldsymbol{F} = \partial_{\boldsymbol{X}} \boldsymbol{\varphi}$. In the following we adopt the common ansatz that the elastic part of the deformation gradient is defined as $\boldsymbol{F}_\mathrm{e} \doteq \boldsymbol{F} \cdot \boldsymbol{F}_\mathrm{i}^{-1}$, namely

$$\begin{aligned} \boldsymbol{F} = \partial_{\boldsymbol{X}} \boldsymbol{\varphi} \doteq \boldsymbol{F}_\mathrm{e} \cdot \boldsymbol{F}_\mathrm{i} : T\mathscr{B}_0 \to T\mathscr{B}_t, \quad \boldsymbol{F}_\mathrm{i} : T\mathscr{B}_0 \to T\mathscr{B}_\mathrm{i}, \quad \boldsymbol{F}_\mathrm{e} : T\mathscr{B}_\mathrm{i} \to T\mathscr{B}_t \\ \text{with } \det(\boldsymbol{F}),\ \det(\boldsymbol{F}_\mathrm{e}),\ \det(\boldsymbol{F}_\mathrm{i}) > 0. \end{aligned} \tag{1}$$

The adopted multiplicative decomposition allows interpretation as a local rearrangement or rather material isomorphism and also gives rise to a generally incompatible, and possibly stress-free, intermediate configuration $\mathscr{B}_\mathrm{i}$. Furthermore, let the introduced configurations, which are embedded into the three-dimensional Euclidian space, be equipped with the co-variant and contra-variant metric tensors

$$\begin{aligned} &\boldsymbol{G} : T\mathscr{B}_0 \to T^*\mathscr{B}_0, \quad \boldsymbol{G}^{-1} = \det{}^{-1}(\boldsymbol{G}) \operatorname{cof}(\boldsymbol{G}) : T^*\mathscr{B}_0 \to T\mathscr{B}_0, \\ &\widehat{\boldsymbol{G}} : T\mathscr{B}_\mathrm{i} \to T^*\mathscr{B}_\mathrm{i}, \quad \widehat{\boldsymbol{G}}^{-1} = \det{}^{-1}(\widehat{\boldsymbol{G}}) \operatorname{cof}(\widehat{\boldsymbol{G}}) : T^*\mathscr{B}_\mathrm{i} \to T\mathscr{B}_\mathrm{i}, \\ &\boldsymbol{g} : T\mathscr{B}_t \to T^*\mathscr{B}_t, \quad \boldsymbol{g}^{-1} = \det{}^{-1}(\boldsymbol{g}) \operatorname{cof}(\boldsymbol{g}) : T^*\mathscr{B}_t \to T\mathscr{B}_t. \end{aligned} \tag{2}$$

For completeness, we additionally define contra-co-variant second order identities via

$$\boldsymbol{I} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F} = \boldsymbol{F}_\mathrm{i}^{-1} \cdot \boldsymbol{F}_\mathrm{i} : T\mathscr{B}_0 \to T\mathscr{B}_0, \tag{3.1}$$

$$\widehat{\boldsymbol{I}} = \boldsymbol{F}_\mathrm{e}^{-1} \cdot \boldsymbol{F}_\mathrm{e} = \boldsymbol{F}_\mathrm{i} \cdot \boldsymbol{F}_\mathrm{i}^{-1} : T\mathscr{B}_\mathrm{i} \to T\mathscr{B}_\mathrm{i}, \tag{3.2}$$

$$\boldsymbol{i} = \boldsymbol{F} \cdot \boldsymbol{F}^{-1} = \boldsymbol{F}_\mathrm{e} \cdot \boldsymbol{F}_\mathrm{e}^{-1} : T\mathscr{B}_t \to T\mathscr{B}_t. \tag{3.3}$$

Next, the polar decomposition theorem is (formally) applied to the linear tangent maps in Eq. (1) so that appropriate stretch tensors are conveniently defined via

$$\boldsymbol{F}=\boldsymbol{R}\cdot\boldsymbol{U}=\boldsymbol{v}\cdot\boldsymbol{R},\quad \boldsymbol{F}_\mathrm{i}=\boldsymbol{R}_\mathrm{i}\cdot\boldsymbol{U}_\mathrm{i}=\boldsymbol{v}_\mathrm{i}\cdot\boldsymbol{R}_\mathrm{i}\quad\text{and}\quad \boldsymbol{F}_\mathrm{e}=\boldsymbol{R}_\mathrm{e}\cdot\boldsymbol{U}_\mathrm{e}=\boldsymbol{v}_\mathrm{e}\cdot\boldsymbol{R}_\mathrm{e}. \tag{4}$$

The proper orthogonal part of the inelastic, i.e. irreversible, part is (usually) not uniquely determined, $\boldsymbol{F}=\boldsymbol{v}_\mathrm{e}\cdot\boldsymbol{R}_\mathrm{e}\cdot\boldsymbol{R}_\mathrm{i}\cdot\boldsymbol{U}_\mathrm{i}=\boldsymbol{v}_\mathrm{e}\cdot\boldsymbol{R}_\mathrm{e}\cdot\boldsymbol{Q}\cdot\boldsymbol{Q}^\mathrm{t}\cdot\boldsymbol{R}_\mathrm{i}\cdot\boldsymbol{U}_\mathrm{i}=\boldsymbol{v}_\mathrm{e}\cdot\boldsymbol{R}'_\mathrm{e}\cdot\boldsymbol{R}'_\mathrm{i}\cdot\boldsymbol{U}_\mathrm{i}$ with $\boldsymbol{Q}\cdot\boldsymbol{Q}^\mathrm{t}=\widehat{\boldsymbol{I}}$ and the stretches in Eq. (4) can alternatively be introduced as the square root of appropriate symmetric deformation tensors; see Eqs. (11)–(16) and (18) below. Moreover, application of the spectral decomposition theorem renders

$$\boldsymbol{F}=\lambda_k\boldsymbol{n}_k\otimes\boldsymbol{N}^k,\quad \boldsymbol{R}=\boldsymbol{n}_k\otimes\boldsymbol{N}^k,\quad \boldsymbol{U}=\lambda_k\boldsymbol{N}_k\otimes\boldsymbol{N}^k,\quad \boldsymbol{v}=\lambda_k\boldsymbol{n}_k\otimes\boldsymbol{n}^k \tag{5}$$

and

$$\boldsymbol{F}_\mathrm{i}=\lambda_{\mathrm{i}k}\widehat{\boldsymbol{n}}_{\mathrm{i}k}\otimes\boldsymbol{N}_\mathrm{i}^k,\quad \boldsymbol{R}_\mathrm{i}=\widehat{\boldsymbol{n}}_{\mathrm{i}k}\otimes\boldsymbol{N}_\mathrm{i}^k,\quad \boldsymbol{U}_\mathrm{i}=\lambda_{\mathrm{i}k}\boldsymbol{N}_{\mathrm{i}k}\otimes\boldsymbol{N}_\mathrm{i}^k,\quad \widehat{\boldsymbol{v}}_\mathrm{i}=\lambda_{\mathrm{i}k}\widehat{\boldsymbol{n}}_{\mathrm{i}k}\otimes\widehat{\boldsymbol{n}}_\mathrm{i}^k \tag{6}$$

as well as

$$\boldsymbol{F}_\mathrm{e}=\lambda_{\mathrm{e}k}\boldsymbol{n}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k,\quad \boldsymbol{R}_\mathrm{e}=\boldsymbol{n}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k,\quad \widehat{\boldsymbol{U}}_\mathrm{e}=\lambda_{\mathrm{e}k}\widehat{\boldsymbol{N}}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k,\quad \boldsymbol{v}_\mathrm{e}=\lambda_{\mathrm{e}k}\boldsymbol{n}_{\mathrm{e}k}\otimes\boldsymbol{n}_\mathrm{e}^k, \tag{7}$$

see also [42] for further elaborations. Here, and in the progression of this work, the summation convention over, e.g., $k=1,2,3$ is implied. The incorporated eigen-vectors are unit-vectors in the sense that

$$\boldsymbol{N}^{(k)}\cdot\boldsymbol{G}^{-1}\cdot\boldsymbol{N}^{(k)}=\boldsymbol{N}_{(k)}\cdot\boldsymbol{G}\cdot\boldsymbol{N}_{(k)}=\boldsymbol{N}_\mathrm{i}^{(k)}\cdot\boldsymbol{G}^{-1}\cdot\boldsymbol{N}_\mathrm{i}^{(k)}=\boldsymbol{N}_{\mathrm{i}(k)}\cdot\boldsymbol{G}\cdot\boldsymbol{N}_{\mathrm{i}(k)}=1 \tag{8}$$

and

$$\widehat{\boldsymbol{N}}_\mathrm{e}^{(k)}\cdot\widehat{\boldsymbol{G}}^{-1}\cdot\widehat{\boldsymbol{N}}_\mathrm{e}^{(k)}=\widehat{\boldsymbol{N}}_{\mathrm{e}(k)}\cdot\widehat{\boldsymbol{G}}\cdot\widehat{\boldsymbol{N}}_{\mathrm{e}(k)}=\widehat{\boldsymbol{n}}_\mathrm{i}^{(k)}\cdot\widehat{\boldsymbol{G}}^{-1}\cdot\widehat{\boldsymbol{n}}_\mathrm{i}^{(k)}=\widehat{\boldsymbol{n}}_{\mathrm{i}(k)}\cdot\widehat{\boldsymbol{G}}\cdot\widehat{\boldsymbol{n}}_{\mathrm{i}(k)}=1 \tag{9}$$

as well as

$$\boldsymbol{n}_\mathrm{e}^{(k)}\cdot\boldsymbol{g}^{-1}\cdot\boldsymbol{n}_\mathrm{e}^{(k)}=\boldsymbol{n}_{\mathrm{e}(k)}\cdot\boldsymbol{g}\cdot\boldsymbol{n}_{\mathrm{e}(k)}=\boldsymbol{n}^{(k)}\cdot\boldsymbol{g}^{-1}\cdot\boldsymbol{n}^{(k)}=\boldsymbol{n}_{(k)}\cdot\boldsymbol{g}\cdot\boldsymbol{n}_{(k)}=1, \tag{10}$$

with the notation (k) indicating that the summation over k is (here) excluded.

Based on these relations, it is now straightforward to introduce different deformation tensors, for instance

$$\begin{aligned}\boldsymbol{C}&=\boldsymbol{F}^{-1}\star\,\boldsymbol{g}=\lambda_k^2\boldsymbol{N}^k\otimes\boldsymbol{N}^k,\quad &\boldsymbol{B}&=\boldsymbol{F}^{-1}\star\,\boldsymbol{g}^{-1}=\lambda_k^{-2}\boldsymbol{N}_k\otimes\boldsymbol{N}_k,\\ \boldsymbol{c}&=\boldsymbol{F}\quad\star\,\boldsymbol{G}=\lambda_k^{-2}\boldsymbol{n}^k\otimes\boldsymbol{n}^k,\quad &\boldsymbol{b}&=\boldsymbol{F}\quad\star\,\boldsymbol{G}^{-1}=\lambda_k^2\boldsymbol{n}_k\otimes\boldsymbol{n}_k\end{aligned} \tag{11}$$

and

$$\begin{aligned}\boldsymbol{C}_\mathrm{i}&=\boldsymbol{F}_\mathrm{i}^{-1}\star\,\widehat{\boldsymbol{G}}=\lambda_{\mathrm{i}k}^2\boldsymbol{N}_\mathrm{i}^k\otimes\boldsymbol{N}_\mathrm{i}^k,\quad &\boldsymbol{B}_\mathrm{i}&=\boldsymbol{F}_\mathrm{i}^{-1}\star\,\widehat{\boldsymbol{G}}^{-1}=\lambda_{\mathrm{i}k}^{-2}\boldsymbol{N}_{\mathrm{i}k}\otimes\boldsymbol{N}_{\mathrm{i}k},\\ \widehat{\boldsymbol{c}}_\mathrm{i}&=\boldsymbol{F}_\mathrm{i}\quad\star\,\boldsymbol{G}=\lambda_{\mathrm{i}k}^{-2}\widehat{\boldsymbol{n}}_\mathrm{i}^k\otimes\widehat{\boldsymbol{n}}_\mathrm{i}^k,\quad &\widehat{\boldsymbol{b}}_\mathrm{i}&=\boldsymbol{F}_\mathrm{i}\quad\star\,\boldsymbol{G}^{-1}=\lambda_{\mathrm{i}k}^2\widehat{\boldsymbol{n}}_{\mathrm{i}k}\otimes\widehat{\boldsymbol{n}}_{\mathrm{i}k}\end{aligned} \tag{12}$$

as well as

$$\begin{aligned}\widehat{\boldsymbol{C}}_\mathrm{e}&=\boldsymbol{F}_\mathrm{e}^{-1}\star\,\boldsymbol{g}=\lambda_{\mathrm{e}k}^2\widehat{\boldsymbol{N}}_\mathrm{e}^k\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k,\quad &\widehat{\boldsymbol{B}}_\mathrm{e}&=\boldsymbol{F}_\mathrm{e}^{-1}\star\,\boldsymbol{g}^{-1}=\lambda_{\mathrm{e}k}^{-2}\widehat{\boldsymbol{N}}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_{\mathrm{e}k},\\ \boldsymbol{c}_\mathrm{e}&=\boldsymbol{F}_\mathrm{e}\quad\star\,\widehat{\boldsymbol{G}}=\lambda_{\mathrm{e}k}^{-2}\boldsymbol{n}_\mathrm{e}^k\otimes\boldsymbol{n}_\mathrm{e}^k,\quad &\boldsymbol{b}_\mathrm{e}&=\boldsymbol{F}_\mathrm{e}\star\,\widehat{\boldsymbol{G}}^{-1}=\lambda_{\mathrm{e}k}^2\boldsymbol{n}_{\mathrm{e}k}\otimes\boldsymbol{n}_{\mathrm{e}k},\end{aligned} \tag{13}$$

wherein the notation $\star$ abbreviates the linear action in terms of the preceding tensor, see also remark 1 and 2. These tensors are essentially characterized by metric coefficients with respect to particular configurations, which provides a nice geometrical interpretation of the above transformations, and can alternatively be expressed in terms of the right and left stretch tensors as reviewed in Eqs. (5)–(7). In this context, the action of the proper orthogonal tensors $\boldsymbol{R},\boldsymbol{R}_\mathrm{i}$ and $\boldsymbol{R}_\mathrm{e}$ on the considered metric tensors are a priori included and the remaining relations read

$$\boldsymbol{C} = \boldsymbol{U}^{-1} \star \boldsymbol{G}, \quad \boldsymbol{B} = \boldsymbol{U}^{-1} \star \boldsymbol{G}^{-1}, \quad \boldsymbol{c} = \boldsymbol{v} \star \boldsymbol{g}, \quad \boldsymbol{b} = \boldsymbol{v} \star \boldsymbol{g}^{-1} \tag{14}$$

and

$$\boldsymbol{C}_{\mathrm{i}} = \boldsymbol{U}_{\mathrm{i}}^{-1} \star \boldsymbol{G}, \quad \boldsymbol{B}_{\mathrm{i}} = \boldsymbol{U}_{\mathrm{i}}^{-1} \star \boldsymbol{G}^{-1}, \quad \widehat{\boldsymbol{c}}_{\mathrm{i}} = \widehat{\boldsymbol{v}}_{\mathrm{i}} \star \widehat{\boldsymbol{G}}, \quad \widehat{\boldsymbol{b}}_{\mathrm{i}} = \widehat{\boldsymbol{v}}_{\mathrm{i}} \star \widehat{\boldsymbol{G}}^{-1} \tag{15}$$

as well as

$$\widehat{\boldsymbol{C}}_{\mathrm{e}} = \widehat{\boldsymbol{U}}_{\mathrm{e}}^{-1} \star \widehat{\boldsymbol{G}}, \quad \widehat{\boldsymbol{B}}_{\mathrm{e}} = \widehat{\boldsymbol{U}}_{\mathrm{e}}^{-1} \star \widehat{\boldsymbol{G}}^{-1}, \quad \boldsymbol{c}_{\mathrm{e}} = \boldsymbol{v}_{\mathrm{e}} \star \boldsymbol{g}, \quad \boldsymbol{b}_{\mathrm{e}} = \boldsymbol{v}_{\mathrm{e}} \star \boldsymbol{g}^{-1}, \tag{16}$$

compare remarks 1 and 2.

Remark 1: In order to represent the transformations in Eqs. (11)–(16) in more detail, we introduce the dual quantities of the linear tangent maps and stretch tensors, respectively. Placing emphasis on the overall motion of the considered body B, one consequently obtains

$$\begin{aligned} &\boldsymbol{F}^{\mathrm{d}} = \lambda_k \boldsymbol{N}^k \otimes \boldsymbol{n}_k, \quad \boldsymbol{F}^{-\mathrm{d}} = \lambda_k^{-1} \boldsymbol{n}^k \otimes \boldsymbol{N}_k, \quad \boldsymbol{R}^{\mathrm{d}} = \boldsymbol{N}^k \otimes \boldsymbol{n}_k, \quad \boldsymbol{R}^{-\mathrm{d}} = \boldsymbol{n}^k \otimes \boldsymbol{N}_k, \\ &\boldsymbol{U}^{\mathrm{d}} = \lambda_k \boldsymbol{N}^k \otimes \boldsymbol{N}_k, \quad \boldsymbol{U}^{-\mathrm{d}} = \lambda_k^{-1} \boldsymbol{N}^k \otimes \boldsymbol{N}_k, \quad \boldsymbol{v}^{\mathrm{d}} = \lambda_k \boldsymbol{n}^k \otimes \boldsymbol{n}_k, \quad \boldsymbol{v}^{-\mathrm{d}} = \lambda_k^{-1} \boldsymbol{n}^k \otimes \boldsymbol{n}_k, \end{aligned} \tag{17}$$

so that

$$\begin{aligned} &\boldsymbol{C} = \boldsymbol{F}^{\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{F} = \boldsymbol{U}^{\mathrm{d}} \cdot \boldsymbol{G} \cdot \boldsymbol{U}, \qquad \boldsymbol{B} = \boldsymbol{F}^{-1} \cdot \boldsymbol{g}^{-1} \cdot \boldsymbol{F}^{-\mathrm{d}} = \boldsymbol{U}^{-1} \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{U}^{-\mathrm{d}}, \\ &\boldsymbol{c} = \boldsymbol{F}^{-\mathrm{d}} \cdot \boldsymbol{G} \cdot \boldsymbol{F}^{-1} = \boldsymbol{v}^{-\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{v}^{-1}, \quad \boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{F}^{\mathrm{d}} = \boldsymbol{v} \cdot \boldsymbol{g}^{-1} \cdot \boldsymbol{v}^{\mathrm{d}}. \end{aligned} \tag{18}$$

By analogy with Eq. (17) we obtain the definitions of $\boldsymbol{F}^{-1}$, $\boldsymbol{R}^{-1}$, $\boldsymbol{U}^{-1}$ and $\boldsymbol{v}^{-1}$ which are obvious and not additionally summarized. With these relations in hand, a corresponding outline in terms of $\boldsymbol{F}_{\mathrm{i}}$ and $\boldsymbol{F}_{\mathrm{e}}$ is straightforward and therefore omitted.

Remark 2: Deformation tensors as, e.g., highlighted in Eqs. (11)–(13) are usually introduced as either co- or contra-variant tensors. Alternatively, one could also define mixed-variant deformation tensors, see, e.g., [33]. In this context, we first introduce the transposition of mixed-variant tensors and second set up appropriate deformation tensors. By analogy with remark 1, special emphasis is placed on the overall motion of the considered body B. In this regard, the relation between the transposed and the dual of the deformation gradient – to be specific $\boldsymbol{F}^{\mathrm{t}} = \boldsymbol{G}^{-1} \cdot \boldsymbol{F}^{\mathrm{d}} \cdot \boldsymbol{g}$ – together with the previously introduced spectral decomposition theorem results in

$$\begin{aligned} &\boldsymbol{F}^{\mathrm{t}} = \lambda_k \boldsymbol{N}_k \otimes \boldsymbol{n}^k, \qquad \boldsymbol{F}^{-\mathrm{t}} = \lambda_k^{-1} \boldsymbol{n}_k \otimes \boldsymbol{N}^k, \\ &\boldsymbol{R}^{\mathrm{t}} = \boldsymbol{N}_k \otimes \boldsymbol{n}^k = \boldsymbol{R}^{-1}, \quad \boldsymbol{R}^{-\mathrm{t}} = \boldsymbol{n}_k \otimes \boldsymbol{N}^k = \boldsymbol{R}, \\ &\boldsymbol{U}^{\mathrm{t}} = \boldsymbol{U}, \qquad \boldsymbol{v}^{\mathrm{t}} = \boldsymbol{v}. \end{aligned} \tag{19}$$

Moreover, we additionally note the relations

$$\boldsymbol{G}^{-1} \cdot \boldsymbol{N}^k = \boldsymbol{N}_k, \quad \boldsymbol{G} \cdot \boldsymbol{N}_k = \boldsymbol{N}^k, \quad \boldsymbol{g}^{-1} \cdot \boldsymbol{n}^k = \boldsymbol{n}_k, \quad \boldsymbol{g} \cdot \boldsymbol{n}_k = \boldsymbol{n}^k. \tag{20}$$

Now, with the transposition operation being defined, the well-established transformations between right and left stretch tensors

$$\boldsymbol{U} = \boldsymbol{R}^{\mathrm{t}} \cdot \boldsymbol{v} \cdot \boldsymbol{R}, \quad \boldsymbol{U}_{\mathrm{i}} = \boldsymbol{R}_{\mathrm{i}}^{\mathrm{t}} \cdot \widehat{\boldsymbol{v}}_{\mathrm{i}} \cdot \boldsymbol{R}_{\mathrm{i}}, \quad \widehat{\boldsymbol{U}}_{\mathrm{i}} = \boldsymbol{R}_{\mathrm{e}}^{\mathrm{t}} \cdot \boldsymbol{v}_{\mathrm{e}} \cdot \boldsymbol{R}_{\mathrm{e}} \tag{21}$$

become evident – the inverse relation, as e.g. $\boldsymbol{v} = \boldsymbol{R} \cdot \boldsymbol{U} \cdot \boldsymbol{R}^{\mathrm{t}}$, being obvious. Finally, these elaborations allow on the one hand to reiterate Eqs. (3.1) and (3.3) as

$$
\begin{aligned}
\boldsymbol{F}^{-1}\cdot\boldsymbol{F} = \boldsymbol{F}^{\mathrm{t}}\cdot\boldsymbol{F}^{-\mathrm{t}} = \boldsymbol{R}^{\mathrm{t}}\cdot\boldsymbol{R} &= \boldsymbol{N}_k\otimes\boldsymbol{N}^k = \boldsymbol{I},\\
\boldsymbol{F}\cdot\boldsymbol{F}^{-1} = \boldsymbol{F}^{-\mathrm{t}}\cdot\boldsymbol{F}^{\mathrm{t}} = \boldsymbol{R}\cdot\boldsymbol{R}^{\mathrm{t}} &= \boldsymbol{n}_k\otimes\boldsymbol{n}^k = \boldsymbol{i},\\
\boldsymbol{F}^{\mathrm{d}}\cdot\boldsymbol{F}^{-\mathrm{d}} = \boldsymbol{R}^{\mathrm{d}}\cdot\boldsymbol{R}^{-\mathrm{d}} &= \boldsymbol{N}^k\otimes\boldsymbol{N}_k = \boldsymbol{I}^{\mathrm{d}},\\
\boldsymbol{F}^{-\mathrm{d}}\cdot\boldsymbol{F}^{\mathrm{d}} = \boldsymbol{R}^{-\mathrm{d}}\cdot\boldsymbol{R}^{\mathrm{d}} &= \boldsymbol{n}^k\otimes\boldsymbol{n}_k = \boldsymbol{i}^{\mathrm{d}},
\end{aligned}
\tag{22}
$$

and on the other hand enable us to introduce the sought mixed–variant deformation tensors via

$$
\begin{aligned}
\boldsymbol{C}^{\natural} &= \boldsymbol{F}^{\mathrm{t}}\cdot\boldsymbol{F} = \boldsymbol{U}\cdot\boldsymbol{U} = \lambda_k^2\boldsymbol{N}_k\otimes\boldsymbol{N}^k = \boldsymbol{G}^{-1}\cdot\boldsymbol{C},\\
\boldsymbol{B}^{\natural} &= \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-\mathrm{t}} = \boldsymbol{U}^{-\mathrm{d}}\cdot\boldsymbol{U}^{-\mathrm{d}} = \lambda_k^{-2}\boldsymbol{N}_k\otimes\boldsymbol{N}^k = \boldsymbol{B}\cdot\boldsymbol{G},\\
\boldsymbol{c}^{\natural} &= \boldsymbol{F}^{-\mathrm{t}}\cdot\boldsymbol{F}^{-1} = \boldsymbol{v}^{-\mathrm{d}}\cdot\boldsymbol{v}^{-\mathrm{d}} = \lambda_k^{-2}\boldsymbol{n}_k\otimes\boldsymbol{n}^k = \boldsymbol{g}^{-1}\cdot\boldsymbol{c},\\
\boldsymbol{b}^{\natural} &= \boldsymbol{F}\cdot\boldsymbol{F}^{\mathrm{t}} = \boldsymbol{v}\cdot\boldsymbol{v} = \lambda_k^2\boldsymbol{n}_k\otimes\boldsymbol{n}^k = \boldsymbol{b}\cdot\boldsymbol{g},
\end{aligned}
\tag{23}
$$

so that $\boldsymbol{B}\cdot\boldsymbol{C} = \boldsymbol{B}^{\natural}\cdot\boldsymbol{C}^{\natural} = \boldsymbol{C}^{\natural}\cdot\boldsymbol{B}^{\natural} = \boldsymbol{I}$ and $\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{b}^{\natural}\cdot\boldsymbol{c}^{\natural} = \boldsymbol{c}^{\natural}\cdot\boldsymbol{b}^{\natural} = \boldsymbol{i}$. With these relations in hand, a corresponding outline in terms of $\boldsymbol{F}_{\mathrm{i}}$ and $\boldsymbol{F}_{\mathrm{e}}$ is straightforward and therefore omitted.

3 Generalized strain measures

The geometric interpretation of strain measures (in one configuration) consists in the point of view that

"strain means change of metric with time".

In order to compare the metric coefficients, according to a particular particle of the body B, at two different states of deformation, we must analyse these quantities (or powers thereof) in one and the same configuration. In this regard, the transformations highlighted in Eqs. (11)–(13) are of cardinal importance.

One classical strain tensor family is provided by the so-called Seth-Hill strain measures, see e.g., Seth [6] and Hill [7]. Adopting this framework, the corresponding class of co-variant strains in $\mathscr{B}_0$ reads

$$
\boldsymbol{E}_{(m)}(\boldsymbol{C};\boldsymbol{G}) = \begin{cases}
\dfrac{1}{m}[\lambda_k^m - 1]\boldsymbol{N}^k\otimes\boldsymbol{N}^k = \dfrac{1}{m}[\boldsymbol{G}\cdot\boldsymbol{U}^m - \boldsymbol{G}] \\[2ex]
\qquad\qquad = \dfrac{1}{m}\Big[\boldsymbol{G}\cdot[\boldsymbol{G}^{-1}\cdot\boldsymbol{C}]^{\frac{m}{2}} - \boldsymbol{G}\Big] \\[2ex]
\dfrac{1}{2}\ln(\lambda_k^2)\boldsymbol{N}^k\otimes\boldsymbol{N}^k = \dfrac{1}{2}\ln(\boldsymbol{G}\cdot\boldsymbol{U}^2) \\[2ex]
\qquad\qquad = \dfrac{1}{2}\ln(\boldsymbol{C}) \quad \text{if } m = 0
\end{cases}
\tag{24}
$$

with $m \geq 0$. Herein powers of the pullback of the spatial co-variant metric $\boldsymbol{g}$ are compared with the material co-variant metric $\boldsymbol{G}$. It is clearly seen that the spectral decomposition theorem (which is the standard representation for strain measure of the Seth-Hill family) renders strain eigen-values that allow interpretation as being monotone functions in terms of the principal stretches; to be specific, $E_{(m)k}(\lambda_k)$ with $E_{(m)k}|_{\lambda_k=1}=0$ and $\partial_{\lambda_k}E_{(m)k}|_{\lambda_k=1}=1$ which is assumed to hold throughout for the subsequent strain measures. Commonly applied representations are e.g., the logarithmic Hencky strains ($m=0$), the Biot strain measure ($m=1$) and the right Cauchy-Green strain tensor ($m=2$). The scalar quantity m in Eq. (24) is restricted to remain non-negative. Strain measures in $\mathscr{B}_0$ that incorporate the inverse stretch are consequently introduced as contra-variant tensors, namely

$$\boldsymbol{K}_{(m)}(\boldsymbol{B};\boldsymbol{G}^{-1})=\begin{cases}\dfrac{1}{m}[1-\lambda_k^{-m}]\boldsymbol{N}_k\otimes\boldsymbol{N}_k=\dfrac{1}{m}[\boldsymbol{G}^{-1}-\boldsymbol{U}^{-m}\cdot\boldsymbol{G}^{-1}]\\ \qquad\qquad\qquad\qquad\qquad\quad=\dfrac{1}{m}\Big[\boldsymbol{G}^{-1}-\boldsymbol{G}^{-1}\cdot[\boldsymbol{G}\cdot\boldsymbol{B}]^{\frac{m}{2}}\Big]\\ -\dfrac{1}{2}\ln(\lambda_k^{-2})\boldsymbol{N}_k\otimes\boldsymbol{N}_k=-\dfrac{1}{2}\ln(\boldsymbol{U}^{-2}\cdot\boldsymbol{G}^{-1})\\ \qquad\qquad\qquad\qquad\qquad\quad=-\dfrac{1}{2}\ln(\boldsymbol{B})\quad\text{if } m=0\end{cases}\tag{25}$$

with $m\geq 0$. Herein powers of the pullback of the spatial contra-variant metric $\boldsymbol{g}^{-1}$ are compared with the material contra-variant metric $\boldsymbol{G}^{-1}$.

A similar setup in $\mathscr{B}_\mathrm{i}$ is obtained if the pullback of the spatial metric tensors is not performed via the total deformation gradient $\boldsymbol{F}$ but in terms of solely its reversible part $\boldsymbol{F}_\mathrm{e}$. In this context, the following strain measures take the interpretation of an elastic setting with respect to $\mathscr{B}_\mathrm{i}$. By analogy with Eq. (24) we obtain

$$\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}(\widehat{\boldsymbol{C}}_\mathrm{e};\widehat{\boldsymbol{G}})=\begin{cases}\dfrac{1}{m}[\lambda_{\mathrm{e}k}^{m}-1]\widehat{\boldsymbol{N}}_\mathrm{e}^k\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k=\dfrac{1}{m}[\widehat{\boldsymbol{G}}\cdot\widehat{\boldsymbol{U}}_\mathrm{e}^{m}-\widehat{\boldsymbol{G}}]\\ \qquad\qquad\qquad\qquad\qquad\quad=\dfrac{1}{m}\Big[\widehat{\boldsymbol{G}}\cdot[\widehat{\boldsymbol{G}}^{-1}\cdot\widehat{\boldsymbol{C}}_\mathrm{e}]^{\frac{m}{2}}-\widehat{\boldsymbol{G}}\Big]\\ \dfrac{1}{2}\ln(\lambda_{\mathrm{e}k}^{2})\widehat{\boldsymbol{N}}_\mathrm{e}^k\otimes\widehat{\boldsymbol{N}}_\mathrm{e}^k=\dfrac{1}{2}\ln(\widehat{\boldsymbol{G}}\cdot\widehat{\boldsymbol{U}}_\mathrm{e}^{2})\\ \qquad\qquad\qquad\qquad\qquad\quad=\dfrac{1}{2}\ln(\widehat{\boldsymbol{C}}_\mathrm{e})\quad\text{if } m=0\end{cases}\tag{26}$$

and Eq. (25) corresponds to

$$\widehat{\boldsymbol{K}}_{\mathrm{e}(m)}(\widehat{\boldsymbol{B}}_\mathrm{e};\widehat{\boldsymbol{G}}^{-1})=\begin{cases}\dfrac{1}{m}[1-\lambda_{\mathrm{e}k}^{-m}]\widehat{\boldsymbol{N}}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_{\mathrm{e}k}=\dfrac{1}{m}[\widehat{\boldsymbol{G}}^{-1}-\widehat{\boldsymbol{U}}_\mathrm{e}^{-m}\cdot\widehat{\boldsymbol{G}}^{-1}]\\ \qquad\qquad\qquad\qquad\qquad\quad=\dfrac{1}{m}\Big[\widehat{\boldsymbol{G}}^{-1}-\widehat{\boldsymbol{G}}^{-1}\cdot[\widehat{\boldsymbol{G}}\cdot\widehat{\boldsymbol{B}}_\mathrm{e}]^{\frac{m}{2}}\Big]\\ -\dfrac{1}{2}\ln(\lambda_{\mathrm{e}k}^{-2})\widehat{\boldsymbol{N}}_{\mathrm{e}k}\otimes\widehat{\boldsymbol{N}}_{\mathrm{e}k}=-\dfrac{1}{2}\ln(\widehat{\boldsymbol{U}}_\mathrm{e}^{-2}\cdot\widehat{\boldsymbol{G}}^{-1})\\ \qquad\qquad\qquad\qquad\qquad\quad=-\dfrac{1}{2}\ln(\widehat{\boldsymbol{B}}_\mathrm{e})\quad\text{if } m=0\end{cases}\tag{27}$$

whereby the restriction $m\geq 0$ is assumed to hold for both types of strain measures, i.e. for the co-variant tensor $\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}$ as well as for the contra-variant tensor $\widehat{\boldsymbol{K}}_{\mathrm{e}(m)}$.

While the previously highlighted strain measures are defined in terms of pullback operations of metric tensors in $\mathscr{B}_t$ to $\mathscr{B}_0$ or to $\mathscr{B}_\mathrm{i}$, respectively, we can alternatively introduce pushforward transformations of metric tensors in either $\mathscr{B}_0$ or $\mathscr{B}_\mathrm{i}$ to $\mathscr{B}_t$. By comparing the obtained deformation tensors with metric tensors in $\mathscr{B}_t$ renders alternative strain measures which, conceptually speaking, are related to the inverse motion problem; for a general discussion on configurational balance relation for the problem at hand the reader is referred to the outline in Menzel and Steinmann [43] and references cited therein. By analogy with Eqs. (24) and (25), we obtain the co–variant spatial strain family

$$\boldsymbol{e}_{(m)}(\boldsymbol{c};\boldsymbol{g}) = \begin{cases} \dfrac{1}{m}[1-\lambda_k^{-m}]\boldsymbol{n}^k \otimes \boldsymbol{n}^k = \dfrac{1}{m}[\boldsymbol{g} - \boldsymbol{g}\cdot\boldsymbol{v}^{-m}] \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{m}\left[\boldsymbol{g} - \boldsymbol{g}\cdot[\boldsymbol{g}^{-1}\cdot\boldsymbol{c}]^{\frac{m}{2}}\right] \\ -\dfrac{1}{2}\ln(\lambda_k^{-2})\boldsymbol{n}^k \otimes \boldsymbol{n}^k = -\dfrac{1}{2}\ln(\boldsymbol{g}\cdot\boldsymbol{v}^{-2}) \\ \qquad\qquad\qquad\qquad\quad = -\dfrac{1}{2}\ln(\boldsymbol{c}) \quad \text{if } m = 0 \end{cases} \tag{28}$$

and the contra-variant counterpart reads

$$\boldsymbol{k}_{(m)}(\boldsymbol{b};\boldsymbol{g}^{-1}) = \begin{cases} \dfrac{1}{m}[\lambda_k^{m}-1]\boldsymbol{n}_k \otimes \boldsymbol{n}_k = \dfrac{1}{m}\left[\boldsymbol{v}^m\cdot\boldsymbol{g}^{-1} - \boldsymbol{g}^{-1}\right] \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{m}\left[\boldsymbol{g}^{-1}\cdot[\boldsymbol{g}\cdot\boldsymbol{b}]^{\frac{m}{2}} - \boldsymbol{g}^{-1}\right] \\ \dfrac{1}{2}\ln(\lambda_k^{2})\boldsymbol{n}_k \otimes \boldsymbol{n}_k = \dfrac{1}{2}\ln(\boldsymbol{v}^2\cdot\boldsymbol{g}^{-1}) \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{2}\ln(\boldsymbol{b}) \quad \text{if } m = 0 \end{cases} \tag{29}$$

for $m \geq 0$.

Finally, we place emphasis on the reversible part of the deformation gradient $\boldsymbol{F}_\mathrm{e}$ instead of the total tangent map $\boldsymbol{F}$ and obtain similarly to Eqs. (26) and (27) the spatial co-variant strain measures

$$\boldsymbol{e}_{\mathrm{e}(m)}(\boldsymbol{c}_\mathrm{e};\boldsymbol{g}) = \begin{cases} \dfrac{1}{m}[1-\lambda_{\mathrm{e}k}^{-m}]\boldsymbol{n}_\mathrm{e}^k \otimes \boldsymbol{n}_\mathrm{e}^k = \dfrac{1}{m}\left[\boldsymbol{g} - \boldsymbol{g}\cdot\boldsymbol{v}_\mathrm{e}^{m}\right] \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{m}\left[\boldsymbol{g} - \boldsymbol{g}\cdot[\boldsymbol{g}^{-1}\cdot\boldsymbol{c}_\mathrm{e}]^{\frac{m}{2}}\right] \\ -\dfrac{1}{2}\ln(\lambda_{\mathrm{e}k}^{-2})\boldsymbol{n}_\mathrm{e}^k \otimes \boldsymbol{n}_\mathrm{e}^k = -\dfrac{1}{2}\ln(\boldsymbol{g}\cdot\boldsymbol{v}_\mathrm{e}^{-2}) \\ \qquad\qquad\qquad\qquad\quad = -\dfrac{1}{2}\ln(\boldsymbol{c}_\mathrm{e}) \quad \text{if } m = 0 \end{cases} \tag{30}$$

with the corresponding contra-variant representation consequently taking the format

$$\boldsymbol{k}_{\mathrm{e}(m)}(\boldsymbol{b}_\mathrm{e};\boldsymbol{g}^{-1}) = \begin{cases} \dfrac{1}{m}[\lambda_{\mathrm{e}k}^{m}-1]\boldsymbol{n}_{\mathrm{e}k} \otimes \boldsymbol{n}_{\mathrm{e}k} = \dfrac{1}{m}\left[\boldsymbol{v}_\mathrm{e}^m\cdot\boldsymbol{g}^{-1} - \boldsymbol{g}^{-1}\right] \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{m}\left[\boldsymbol{g}^{-1}\cdot[\boldsymbol{g}\cdot\boldsymbol{b}_\mathrm{e}]^{\frac{m}{2}} - \boldsymbol{g}^{-1}\right] \\ \dfrac{1}{2}\ln(\lambda_{\mathrm{e}k}^{2})\boldsymbol{n}_{\mathrm{e}k} \otimes \boldsymbol{n}_{\mathrm{e}k} = \dfrac{1}{2}\ln(\boldsymbol{v}_\mathrm{e}^2\cdot\boldsymbol{g}^{-1}) \\ \qquad\qquad\qquad\qquad\quad = \dfrac{1}{2}\ln(\boldsymbol{b}_\mathrm{e}) \quad \text{if } m = 0 \end{cases} \tag{31}$$

for $m \geq 0$.

Remark 3: The previously highlighted list of Seth-Hill-type strain measures is by far not complete – for instance the deformation tensors $\boldsymbol{C}_\mathrm{i}$ and $\boldsymbol{B}_\mathrm{i}$ have, up to now, not been incorporated. In this regard, a typical example is provided by, e.g.,

$$\boldsymbol{E}_{\mathrm{i}(2)}(\boldsymbol{C};\boldsymbol{C}_\mathrm{i}) \doteq \boldsymbol{F}_\mathrm{i}^{-1} \star \widehat{\boldsymbol{E}}_{\mathrm{e}(2)}(\widehat{\boldsymbol{C}}_\mathrm{e};\widehat{\boldsymbol{G}}) = \frac{1}{2}[\boldsymbol{C} - \boldsymbol{C}_\mathrm{i}] \tag{32}$$

with $\boldsymbol{E}_{\mathrm{i}(2)} \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{C} \neq \boldsymbol{C} \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{E}_{\mathrm{i}(2)}$ while $\widehat{\boldsymbol{E}}_{\mathrm{e}(m)} \cdot \widehat{\boldsymbol{G}}^{-1} \cdot \widehat{\boldsymbol{C}}_\mathrm{e} = \widehat{\boldsymbol{C}}_\mathrm{e} \cdot \widehat{\boldsymbol{G}}^{-1} \cdot \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}$ constantly holds. Since absolute tensor representations (besides spectral decompositions) are applied in this work and any additional assumptions on isotropy are avoided, we predominantly restrict ourselves to the strains highlighted in Eqs. (24)–(31). Moreover, appropriate evolution equations for, e.g., the irreversible part of the deformation gradient will be expressed in terms of appropriate strain measures in this work. In other words, the inelastic spin of $\boldsymbol{F}_\mathrm{i}$ will be neglected. Please note that the presented (kinematical) framework is not restricted to one single intermediate configuration but also allows extension to the combination of several intermediate configurations, see, e.g. [44] for an outline. Even though it seems to be natural to introduce mixed-variant strain measures, compare remark 2, we refer to the highlighted co- or contra-variant strain tensors in the following.

Remark 4: Seth-Hill-type strain measures represent solely one particular family of strain tensors. The general definition of strain measures is commonly introduced in terms of the strain eigen-values, s_k say, which are monotone functions with respect to appropriate principal stretches, λ_k say, with $s_k(\lambda_k)|_{\lambda_k=1} = 0$ and $\partial_{\lambda_k} s_k(\lambda_k)|_{\lambda_k=1} = 1$. This enables us to define admissible strain families in terms of for instance combinations of Seth-Hill-type strain measures like

$$\boldsymbol{\Xi}_{(m)} = \frac{1}{2}[\boldsymbol{E}_{(m)} + \boldsymbol{G} \cdot \boldsymbol{K}_{(m)} \cdot \boldsymbol{G}] = \begin{cases} \frac{1}{2m}[\boldsymbol{G} \cdot \boldsymbol{U}^m - \boldsymbol{U}^{-m} \cdot \boldsymbol{G}] \\ \frac{1}{4}[\ln(\boldsymbol{G} \cdot \boldsymbol{U}^2) - \ln(\boldsymbol{G} \cdot \boldsymbol{U}^{-2})] \quad \text{if } m = 0 \end{cases} \tag{33}$$

or

$$\boldsymbol{\Gamma}_{(m,n)} = \frac{1}{2}[\boldsymbol{E}_{(m)} + \boldsymbol{E}_{(n)}] = \begin{cases} \frac{1}{2m}\boldsymbol{G} \cdot \boldsymbol{U}^m + \frac{1}{2n}\boldsymbol{G} \cdot \boldsymbol{U}^n - [\frac{1}{2m} + \frac{1}{2n}]\boldsymbol{G} \\ \frac{1}{2m}\boldsymbol{G} \cdot \boldsymbol{U}^m + \frac{1}{4}\ln(\boldsymbol{G} \cdot \boldsymbol{U}^2) - \frac{1}{2m}\boldsymbol{G} \quad \text{if } m = 0 \end{cases} \tag{34}$$

for $m, n \geq 0$ – with the outline for $m > 0$ but $n = 0$ as well as $\boldsymbol{\Gamma}_{(0,0)} = \boldsymbol{E}_{(0)}$ being obvious; see e.g., [45] or [46] where several applications based on the introduction of $\boldsymbol{\Xi}_{(2)}$ are elaborated. The setup of similar strains of contra-variant nature, derivations with respect to other configurations and the combination of more than solely two strain tensors are straightforward and therefore omitted.

Furthermore, the spectral decomposition enables us to represent strain measures as isotropic tensor functions based on the eigenvalues of other strain tensors, for instance,

$$\boldsymbol{E}_{(m)} = \boldsymbol{E}_{(m)}(\boldsymbol{E}_{(n)}) = \begin{cases} \frac{1}{m}\Big[[nE_{(n)k} + 1]^{\frac{m}{n}} - 1\Big]\boldsymbol{N}^k \otimes \boldsymbol{N}^k \\ \frac{1}{m}\Big[[\exp(2E_{(n)k})]^{\frac{m}{2}} - 1\Big]\boldsymbol{N}^k \otimes \boldsymbol{N}^k \quad \text{if } n = 0 \\ \frac{1}{2}\ln\Big([nE_{(n)k} + 1]^{\frac{2}{n}}\Big)\boldsymbol{N}^k \otimes \boldsymbol{N}^k \quad \text{if } m = 0 \end{cases} \tag{35}$$

wherein $E_{(n>0)k} = \frac{1}{n}[\lambda^n - 1]$ and $E_{(0)k} = \frac{1}{2}\ln(\lambda_k^2)$ abbreviate the set of eigen–values with respect to $\boldsymbol{E}_{(n)}$, $m \geq 0$ as well as $n \geq 0$ and the case $m = n = 0$ being trivial.

Finally, note that general strain tensors are not restricted to represent quantities in solely one single configuration – even though the geometric interpretation of strain as change of metric with time is then formally lost. In this regard, Böck and Holzapfel [47] recently introduced the strain tensor $\boldsymbol{\Upsilon} = \frac{1}{2}[\boldsymbol{F} - \boldsymbol{F}^{-\mathrm{t}}]$ possessing the property $\|\boldsymbol{\Upsilon}^{\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{\Upsilon}\| \to \infty$ for $\det(\boldsymbol{F}) \to \infty$ or $\det(\boldsymbol{F}) \to 0$, the latter case (possibly) becoming more important for, e.g., numerical applications.

Remark 5: A volumetric/isochoric split of generalized strain measures follows straightforwardly by replacing the considered strain tensor with its unimodular part. To give an example, let $\{\bullet\}$ abbreviate a representative strain measure so that $\{\bullet\} \mapsto \{\bullet\}^{\mathrm{iso}} = \det^{-1/3}(\{\bullet\})\{\bullet\}$ represents the corresponding unimodular part; see [29],[30] and [48] for detailed outlines.

3.1 Relations between different strain measures in one configuration

The previously highlighted introduction of Seth–Hill–type strain measures suggests that there exist transformations which map co–variant strains onto contra–variant representations within one strain family and within one configuration. Taking the particular format of the linear tangent maps as reviewed in Eqs. (5)–(7) into account, one observes the fundamental relation

$$\boldsymbol{E}_{(m)} \cdot \boldsymbol{G}^{-1} = \boldsymbol{C}^{\frac{m}{2}} \cdot \boldsymbol{K}_{(m)} \quad \leftrightarrow \quad \boldsymbol{K}_{(m)} \cdot \boldsymbol{G} = \boldsymbol{B}^{\frac{m}{2}} \cdot \boldsymbol{E}_{(m)} \tag{36}$$

for those strain measures set up in Eqs. (24) and (25) with respect to $\mathscr{B}_0$. As an interesting side aspect, Eq. (36) generally holds for $m \geq 0$. Similarly, we obtain the corresponding strain tensors in the intermediate configuration $\mathscr{B}_{\mathrm{i}}$

$$\widehat{\boldsymbol{E}}_{\mathrm{e}(m)} \cdot \widehat{\boldsymbol{G}}^{-1} = \widehat{\boldsymbol{C}}_{\mathrm{e}}^{\frac{m}{2}} \cdot \widehat{\boldsymbol{K}}_{\mathrm{e}(m)} \quad \leftrightarrow \quad \widehat{\boldsymbol{K}}_{\mathrm{e}(m)} \cdot \widehat{\boldsymbol{G}} = \widehat{\boldsymbol{B}}_{\mathrm{e}}^{\frac{m}{2}} \cdot \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}, \tag{37}$$

recall Eqs. (26) and (27). When placing emphasis on the spatial configuration $\mathscr{B}_t$, Eqs. (28)–(31) consequently result in

$$\begin{aligned} \boldsymbol{e}_{(m)} \cdot \boldsymbol{g}^{-1} &= \boldsymbol{c}^{\frac{m}{2}} \cdot \boldsymbol{k}_{(m)} \quad \leftrightarrow \quad \boldsymbol{k}_{(m)} \cdot \boldsymbol{g} = \boldsymbol{b}^{\frac{m}{2}} \cdot \boldsymbol{e}_{(m)}, \\ \boldsymbol{e}_{\mathrm{e}(m)} \cdot \boldsymbol{g}^{-1} &= \boldsymbol{c}_{\mathrm{e}}^{\frac{m}{2}} \cdot \boldsymbol{k}_{\mathrm{e}(m)} \quad \leftrightarrow \quad \boldsymbol{k}_{\mathrm{e}(m)} \cdot \boldsymbol{g} = \boldsymbol{b}_{\mathrm{e}}^{\frac{m}{2}} \cdot \boldsymbol{e}_{\mathrm{e}(m)}. \end{aligned} \tag{38}$$

Remark 6: Please note that the above transformations allow alternative representation in terms of appropriate stretch tensors, to give an example

$$\begin{aligned} \boldsymbol{E}_{(m)} \cdot \boldsymbol{G}^{-1} &= \boldsymbol{C}^{\frac{m}{2}} \cdot \boldsymbol{K}_{(m)} &&= \boldsymbol{U}^{\mathrm{d}\frac{m}{2}} \cdot \boldsymbol{G} \cdot \boldsymbol{U}^{\frac{m}{2}} \cdot \boldsymbol{K}_{(m)} \\ &= \boldsymbol{U}^{\mathrm{d}\frac{m}{2}} \cdot \boldsymbol{G} \cdot \boldsymbol{K}_{(m)} \cdot \boldsymbol{U}^{\mathrm{d}\frac{m}{2}} &&= \boldsymbol{U}^{\mathrm{d}\frac{m}{2}} \cdot \boldsymbol{G} \cdot \boldsymbol{K}_{(m)} \cdot \boldsymbol{G} \cdot \boldsymbol{U}^{\frac{m}{2}} \cdot \boldsymbol{G}^{-1}, \end{aligned} \tag{39}$$

whereby use of Eq. (18) has been made.

3.2 Relations between different strain measures in different configurations

In addition to the transformation relations of strain measures in one configuration the by far more interesting task consists in the correlations between strain measures in different configurations. Without these connections, typical pushforward and pullback transformations between the introduced configurations of interest (namely application of the fundamental covariance principle with respect to, for instance, the Helmholtz free energy), as commonly applied in computational inelasticity, would not be accessible.

In this context, we first observe

$$\boldsymbol{R} \star \boldsymbol{E}_{(m)} \cdot \boldsymbol{g}^{-1} = \boldsymbol{g} \cdot \boldsymbol{k}_{(m)}, \qquad \boldsymbol{R} \star \boldsymbol{K}_{(m)} \cdot \boldsymbol{g} = \boldsymbol{g}^{-1} \cdot \boldsymbol{e}_{(m)} \tag{40}$$

as well as

$$\boldsymbol{R}_{\mathrm{e}} \star \widehat{\boldsymbol{E}}_{\mathrm{e}(m)} \cdot \boldsymbol{g}^{-1} = \boldsymbol{g} \cdot \boldsymbol{k}_{\mathrm{e}(m)}, \qquad \boldsymbol{R}_{\mathrm{e}} \star \widehat{\boldsymbol{K}}_{\mathrm{e}(m)} \cdot \boldsymbol{g} = \boldsymbol{g}^{-1} \cdot \boldsymbol{e}_{\mathrm{e}(m)}, \tag{41}$$

which becomes obvious from the underlying spectral decomposition theorem and, practically speaking, identifies identical eigenvalues

$$E_{(m)k} = k_{(m)k}, \qquad K_{(m)k} = e_{(m)k}, \qquad \widehat{E}_{\mathrm{e}(m)k} = k_{\mathrm{e}(m)k}, \qquad \widehat{K}_{\mathrm{e}(m)k} = e_{\mathrm{e}(m)k}, \tag{42}$$

wherein the abbreviation $E_{(m)k}$ collects the eigenvalues of $\boldsymbol{E}_{(m)}$, etc.; compare remark 6. Apparently, Eqs. (40)–(42) hold for $m \geq 0$. Based on these relations, taking the transformations between the underlying eigenvectors into account and recalling Eqs. (5)–(7), we second obtain the sought transformations, namely on the one hand by analogy with Eq. (40),

$$\begin{aligned} \boldsymbol{F} \star \boldsymbol{E}_{(m)} \cdot \boldsymbol{g}^{-1} &= \boldsymbol{c} \cdot \boldsymbol{k}_{(m)}, & \boldsymbol{F} \star \boldsymbol{K}_{(m)} \cdot \boldsymbol{g} &= \boldsymbol{b} \cdot \boldsymbol{e}_{(m)}, \\ \boldsymbol{F}^{-1} \star \boldsymbol{k}_{(m)} \cdot \boldsymbol{G} &= \boldsymbol{B} \cdot \boldsymbol{E}_{(m)}, & \boldsymbol{F}^{-1} \star \boldsymbol{e}_{(m)} \cdot \boldsymbol{G}^{-1} &= \boldsymbol{C} \cdot \boldsymbol{K}_{(m)}, \end{aligned} \tag{43}$$

while on the other hand the representation which stems from Eq. (41) reads

$$\begin{aligned} \boldsymbol{F}_{\mathrm{e}} \star \widehat{\boldsymbol{E}}_{\mathrm{e}(m)} \cdot \boldsymbol{g}^{-1} &= \boldsymbol{c}_{\mathrm{e}} \cdot \boldsymbol{k}_{\mathrm{e}(m)}, & \boldsymbol{F}_{\mathrm{e}} \star \widehat{\boldsymbol{K}}_{\mathrm{e}(m)} \cdot \boldsymbol{g} &= \boldsymbol{b}_{\mathrm{e}} \cdot \boldsymbol{e}_{\mathrm{e}(m)}, \\ \boldsymbol{F}_{\mathrm{e}}^{-1} \star \boldsymbol{k}_{\mathrm{e}(m)} \cdot \widehat{\boldsymbol{G}} &= \widehat{\boldsymbol{B}}_{\mathrm{e}} \cdot \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}, & \boldsymbol{F}_{\mathrm{e}}^{-1} \star \boldsymbol{e}_{\mathrm{e}(m)} \cdot \widehat{\boldsymbol{G}}^{-1} &= \widehat{\boldsymbol{C}}_{\mathrm{e}} \cdot \widehat{\boldsymbol{K}}_{\mathrm{e}(m)}. \end{aligned} \tag{44}$$

Remark 7: Please note that the above transformations allow alternative representation in terms of appropriate stretch tensors, to give an example

$$\begin{aligned} \boldsymbol{F} \star \boldsymbol{E}_{(m)} \cdot \boldsymbol{g}^{-1} &= \boldsymbol{v}^{-\mathrm{d}} \cdot \boldsymbol{R} \star \boldsymbol{E}_{(m)} \cdot \boldsymbol{v}^{-1} \cdot \boldsymbol{g}^{-1} = \boldsymbol{v}^{-\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g} \cdot \boldsymbol{v}^{-1} \cdot \boldsymbol{g}^{-1} \\ &= \boldsymbol{v}^{-\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{v}^{-\mathrm{d}} = \boldsymbol{v}^{-\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{v}^{-1} \cdot \boldsymbol{k}_{(m)} = \boldsymbol{c} \cdot \boldsymbol{k}_{(m)}, \end{aligned} \tag{45}$$

whereby use of Eqs. (18) and (40) has been made.

4 Helmholtz free energy density

In order to incorporate both types of generalized strains, namely those measures referring to the total deformation as well as those quantities which allow interpretation as an elastic setting with respect to $\mathscr{B}_{\mathrm{i}}$, we assume the following additive split of the Helmholtz free energy density:

$$\psi_0(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{i}}, \boldsymbol{A}; \boldsymbol{X}) = \psi_0^0(\boldsymbol{E}_{(m)}, \boldsymbol{G}^{-1}, \boldsymbol{A}; \boldsymbol{X}) + \psi_0^{\mathrm{e}}(\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}, \widehat{\boldsymbol{G}}^{-1}, \widehat{\boldsymbol{A}}; \boldsymbol{X}), \tag{46}$$

wherein $\boldsymbol{A}$ and $\widehat{\boldsymbol{A}} = \boldsymbol{F}_{\mathrm{i}} \star \boldsymbol{A}$ represent an additional contra-variant second order tensor which is assumed to be symmetric. Further arguments – which enable for instance the modelling of different hardening effects – are neglected for the sake of simplicity. With Eq. (46) in hand, the fundamental covariance relation apparently allows us to rewrite the Helmholtz free energy density as

$$\psi_0(\boldsymbol{F},\boldsymbol{F}_\mathrm{i},\boldsymbol{A};\boldsymbol{X}) = \psi_0^0(\boldsymbol{E}_{(m)},\boldsymbol{G}^{-1},\boldsymbol{A};\boldsymbol{X}) + \psi_0^\mathrm{e}(\boldsymbol{E}_{\mathrm{e}(m)},\boldsymbol{B}_\mathrm{i},\boldsymbol{A};\boldsymbol{X}) \tag{47.1}$$

$$= \psi_0^0(\widehat{\boldsymbol{E}}_{(m)},\widehat{\boldsymbol{b}}_\mathrm{i},\widehat{\boldsymbol{A}};\boldsymbol{X}) + \psi_0^\mathrm{e}(\widehat{\boldsymbol{E}}_{\mathrm{e}(m)},\widehat{\boldsymbol{G}}^{-1},\widehat{\boldsymbol{A}};\boldsymbol{X}) \tag{47.2}$$

$$= \psi_0^0(\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g},\boldsymbol{b},\boldsymbol{a};\boldsymbol{X}) + \psi_0^\mathrm{e}(\boldsymbol{c}_\mathrm{e}\cdot\boldsymbol{k}_{\mathrm{e}(m)}\cdot\boldsymbol{g},\boldsymbol{b}_\mathrm{e},\boldsymbol{a};\boldsymbol{X}) \tag{47.3}$$

with $\boldsymbol{E}_{\mathrm{e}(m)} = \boldsymbol{F}_\mathrm{i}^{-1}\star\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}$, $\widehat{\boldsymbol{E}}_{(m)} = \boldsymbol{F}_\mathrm{i}\star\boldsymbol{E}_{(m)}$ and $\boldsymbol{a} = \boldsymbol{F}\star\boldsymbol{A}$. The reader is referred to, e.g., [33] and references cited therein, for background information on the covariance principle. A particular application of this fundamental principle consists is the invariance of the Helmholtz free energy density under superposed material isometries (onto the arguments of ψ_0 with respect to the representation highlighted in Eq. (47.1). Conceptually speaking, the Helmholtz free energy is characterized by an isotropic tensor function determined via two sets of invariants. For clarity's sake, however, but without loss of generality we will assume $\boldsymbol{A}$ to remain constant during a deformation process in the progression of this work. Each set of invariants consequently includes solely five invariants, to be specific

$$\begin{aligned}
I_j &= \boldsymbol{I} : [\boldsymbol{E}_{(m)}\cdot\boldsymbol{G}^{-1}]^j &&= \widehat{\boldsymbol{I}} : [\widehat{\boldsymbol{E}}_{(m)}\cdot\widehat{\boldsymbol{b}}_\mathrm{i}]^j \\
&= \boldsymbol{i} : \big[[\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}]\cdot\boldsymbol{b}\big]^j &&= \boldsymbol{i} : [\boldsymbol{g}\cdot\boldsymbol{k}_{(m)}]^j, \\
I_{\alpha+3} &= \boldsymbol{I} : \Big[\boldsymbol{E}_{(m)}\cdot[\boldsymbol{G}^{-1}\cdot\boldsymbol{E}_{(m)}]^{\alpha-1}\cdot\boldsymbol{A}\Big] &&= \widehat{\boldsymbol{I}} : \Big[\widehat{\boldsymbol{E}}_{(m)}\cdot[\widehat{\boldsymbol{b}}_\mathrm{i}\cdot\widehat{\boldsymbol{E}}_{(m)}]^{\alpha-1}\cdot\widehat{\boldsymbol{A}}\Big] \\
&= \boldsymbol{i} : \Big[[\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}]\cdot[\boldsymbol{b}\cdot\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}]^{\alpha-1}\cdot\boldsymbol{a}\Big],
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
I_{\mathrm{e}j} &= \boldsymbol{I} : [\boldsymbol{E}_{\mathrm{e}(m)}\cdot\boldsymbol{B}_\mathrm{i}]^j &&= \widehat{\boldsymbol{I}} : [\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}\cdot\widehat{\boldsymbol{G}}^{-1}]^j \\
&= \boldsymbol{i} : \big[[\boldsymbol{c}_\mathrm{e}\cdot\boldsymbol{k}_{\mathrm{e}(m)}\cdot\boldsymbol{g}]\cdot\boldsymbol{b}_\mathrm{e}\big]^j &&= \boldsymbol{i} : [\boldsymbol{g}\cdot\boldsymbol{k}_{\mathrm{e}(m)}]^j, \\
I_{\mathrm{e}\alpha+3} &= \boldsymbol{I} : \Big[\boldsymbol{E}_{\mathrm{e}(m)}\cdot[\boldsymbol{B}_\mathrm{i}\cdot\boldsymbol{E}_{\mathrm{e}(m)}]^{\alpha-1}\cdot\boldsymbol{A}\Big] &&= \widehat{\boldsymbol{I}} : \Big[\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}\cdot[\widehat{\boldsymbol{G}}^{-1}\cdot\widehat{\boldsymbol{E}}_{\mathrm{e}(m)}]^{\alpha-1}\cdot\widehat{\boldsymbol{A}}\Big] \\
&= \boldsymbol{i} : \Big[[\boldsymbol{c}_\mathrm{e}\cdot\boldsymbol{k}_{\mathrm{e}(m)}\cdot\boldsymbol{g}]\cdot[\boldsymbol{b}_\mathrm{e}\cdot\boldsymbol{c}_\mathrm{e}\cdot\boldsymbol{k}_{\mathrm{e}(m)}\cdot\boldsymbol{g}]^{\alpha-1}\cdot\boldsymbol{a}\Big],
\end{aligned} \tag{49}$$

with $j = 1,2,3$ and $\alpha = 1,2$, compare Eq. (18). The underlying (elastic) symmetry group $\mathbb{G}$ of the considered body B, as based on the subsequent hyperelastic formats of representative stress tensors, based on for instance ψ_0^0, is defined via

$$\mathbb{G} = \{\boldsymbol{Q}\in\mathbb{O}^3 \,|\, \boldsymbol{Q}\star\boldsymbol{A} = \boldsymbol{A}\}, \tag{50}$$

see, e.g., [49] or the contributions in [50]. A typical example is provided by transversal isotropic symmetry so that $\boldsymbol{A} \doteq \boldsymbol{v}_0\otimes\boldsymbol{v}_0$ while orthotropy can be monitored via $\boldsymbol{A} \doteq \boldsymbol{v}_0\otimes\boldsymbol{v}_0 - \boldsymbol{w}_0\otimes\boldsymbol{w}_0$ whereby $\boldsymbol{v}_0\cdot\boldsymbol{G}\cdot\boldsymbol{w}_0 = 0$; see, e.g., [51] and [52] for the definition of ("general") structural tensors characterizing crystalline and non–crystalline symmetries.

Remark 8: It is obvious that the Helmholtz free energy density as represented in Eq. (46) can be generalized by replacing the single argument $\boldsymbol{A}$ in ψ_0^0 and ψ_0^e with series $\boldsymbol{A}_{1,\ldots,q}$ and $\widehat{\boldsymbol{A}}_{q+1,\ldots,r}$ of tensors, respectively. Moreover, both sets also allow consideration as internal variables such that these arguments would evolve during the deformation process. In this regard, the reader is referred to [35] and [53].

Remark 9: Please note that the highlighted format of the Helmholtz free energy in, e.g., Eq. (46) captures several standard approaches of typical (isothermal) 'constitutive models', namely hyperelasticity ($\psi_0 = \psi_0^0$), plasticity ($\psi_0 = \psi_0^{\mathrm{e}}$) or viscoelasticity ($\psi_0 = \psi_0^0 + \psi_0^{\mathrm{e}}$, the generalization to several viscosity terms, $\psi_0(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{i}_{1,\dots,s}}; \boldsymbol{X}) \doteq \psi_0^0(\boldsymbol{F}; \boldsymbol{X}) + \sum_{s=1}^{S} \psi_{0_s}^{\mathrm{e}}(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{i}_s}; \boldsymbol{X})$, being straightforward).

5 Coleman–Noll entropy principle

The pointwise format of the isothermal Clausius–Duhem inequality reads

$$\mathscr{D}_0 = \boldsymbol{\Pi}^{\mathrm{d}} : \mathrm{D}_t \boldsymbol{F} - \mathrm{D}_t \psi_0 = \boldsymbol{\Pi}^{\mathrm{d}} : \mathrm{D}_t \boldsymbol{F} - \left.\frac{\partial \psi_0}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}_{\mathrm{i}}} : \mathrm{D}_t \boldsymbol{F} - \left.\frac{\partial \psi_0}{\partial \boldsymbol{F}_{\mathrm{i}}}\right|_{\boldsymbol{F}} : \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}} \geq 0, \tag{51}$$

whereby the notation D_t characterises the material time derivative. From Eq. (46) and $\widehat{\boldsymbol{C}}_{\mathrm{e}} = \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}} \cdot \boldsymbol{C} \cdot \boldsymbol{F}_{\mathrm{i}}^{-1}$, recall Eq. (1) and Eqs. (11)–(13), we obtain

$$\left.\frac{\partial \psi_0}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}_{\mathrm{i}}} = \frac{\partial \psi_0^0}{\partial \boldsymbol{E}_{(m)}} : \frac{\partial \boldsymbol{E}_{(m)}}{\partial \boldsymbol{C}} : \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{F}} + \frac{\partial \psi_0^{\mathrm{e}}}{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}} : \frac{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}}{\partial \widehat{\boldsymbol{C}}_{\mathrm{e}}} : \left.\frac{\partial \widehat{\boldsymbol{C}}_{\mathrm{e}}}{\partial \boldsymbol{F}}\right|_{\boldsymbol{F}_{\mathrm{i}}} \tag{52}$$

as well as

$$\left.\frac{\partial \psi_0}{\partial \boldsymbol{F}_{\mathrm{i}}}\right|_{\boldsymbol{F}} = \frac{\partial \psi_0^{\mathrm{e}}}{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}} : \frac{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}}{\partial \widehat{\boldsymbol{C}}_{\mathrm{e}}} : \left.\frac{\partial \widehat{\boldsymbol{C}}_{\mathrm{e}}}{\partial \boldsymbol{F}_{\mathrm{i}}}\right|_{\boldsymbol{F}}, \tag{53}$$

compare, e.g., [54]. In order to abbreviate notation, we introduce symmetric stress tensors of second Piola–Kirchhoff–type which are essentially based on commonly applied projection operators, to be specific

$$\begin{aligned}
\tfrac{1}{2}\boldsymbol{S}^0 &= \boldsymbol{T}^0_{(m)} : \mathbf{P}_{(m)}, & \tfrac{1}{2}\widehat{\boldsymbol{S}}^{\mathrm{e}} &= \widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} : \widehat{\mathbf{P}}_{(m)} \quad \text{with} \\
\boldsymbol{T}^0_{(m)} &= \left.\frac{\partial \psi_0^0}{\partial \boldsymbol{E}_{(m)}}\right|_{\boldsymbol{F}_{\mathrm{i}}}, & \widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} &= \frac{\partial \psi_0^{\mathrm{e}}}{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}} \quad \text{and} \\
\mathbf{P}_{(m)} &= \frac{\partial \boldsymbol{E}_{(m)}}{\partial \boldsymbol{C}}, & \widehat{\mathbf{P}}^{\mathrm{e}}_{(m)} &= \frac{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}}{\partial \widehat{\boldsymbol{C}}_{\mathrm{e}}},
\end{aligned} \tag{54}$$

see also remark 10. Based on these notations, the computation of the derivatives $\partial_{\boldsymbol{F}} \boldsymbol{C}$, $\partial_{\boldsymbol{F}} \hat{\boldsymbol{C}}_{\mathrm{e}}\big|_{\boldsymbol{F}_{\mathrm{i}}}$ and $\partial_{\boldsymbol{F}_{\mathrm{i}}} \hat{\boldsymbol{C}}_{\mathrm{e}}\big|_{\boldsymbol{F}}$ finally renders

$$\mathscr{D}_0 = \left[\boldsymbol{\Pi}^{\mathrm{d}} - \boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{S}^0 - \boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right] : \mathrm{D}_t \boldsymbol{F} + \left[\widehat{\boldsymbol{C}}_{\mathrm{e}} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right] : \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}} \geq 0. \tag{55}$$

It is now straightforward to adopt the standard argumentation of rational thermodynamics and to introduce a hyperelastic constitutive equation for the Piola stress tensor

$$\boldsymbol{\Pi}^{\mathrm{d}} \doteq \boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{S}^0 + \boldsymbol{g} \cdot \boldsymbol{F} \cdot \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}} \quad (\text{dual: } \mathrm{D}_t \boldsymbol{F}) \tag{56}$$

in addition to the remaining part of the dissipation inequality

$$\mathscr{D}_0 = \left[\widehat{\boldsymbol{C}}_{\mathrm{e}} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right] : \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}} \geq 0. \tag{57}$$

Now, standard pullback operations to the material setting result in

$$\begin{aligned} \boldsymbol{M}^{\mathrm{d}} &= \boldsymbol{C} \cdot \boldsymbol{S}^0 + \boldsymbol{C} \cdot \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}} \quad (\text{dual: } \boldsymbol{L}), \\ \boldsymbol{S} &= \boldsymbol{S}^0 + \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}} \qquad\qquad (\text{dual: } \mathrm{D}_t \boldsymbol{C}), \\ \mathscr{D}_0 &= \left[\boldsymbol{C} \cdot \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right] : \boldsymbol{L}_{\mathrm{i}} \geq 0, \end{aligned} \tag{58}$$

whereby symmetry relations and the following notations have been applied:

$$\boldsymbol{L} = \boldsymbol{F}^{-1} \cdot \mathrm{D}_t \boldsymbol{F} \quad \rightarrow \quad \mathrm{D}_t \boldsymbol{C} = 2[\boldsymbol{C} \cdot \boldsymbol{L}]^{\mathrm{sym}}, \quad \boldsymbol{L}_{\mathrm{i}} = \boldsymbol{F}_{\mathrm{i}}^{-1} \cdot \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}}. \tag{59}$$

Likewise, standard pushforward transformations to the spatial setting consequently yield

$$\begin{aligned} \boldsymbol{m}^{\mathrm{d}} &= \boldsymbol{g} \cdot \tau^0 + \boldsymbol{g} \cdot \boldsymbol{\tau}^{\mathrm{e}} \quad (\text{dual} : \boldsymbol{l}), \\ \boldsymbol{\tau} &= \boldsymbol{\tau}^0 + \boldsymbol{\tau}^{\mathrm{e}} \qquad\qquad (\text{dual: } \mathrm{L}_t \boldsymbol{g}), \\ \mathscr{D}_0 &= \left[\boldsymbol{g} \cdot \boldsymbol{F}_{\mathrm{e}} \cdot \widehat{\boldsymbol{S}}^{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{e}}^{\mathrm{d}}\right] : \boldsymbol{l}_{\mathrm{i}} = \left[\boldsymbol{F}_{\mathrm{e}}^{\mathrm{d}} \cdot \boldsymbol{g} \cdot \boldsymbol{\tau}^{\mathrm{e}} \cdot \boldsymbol{F}^{-1}\right] : \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}} \geq 0, \end{aligned} \tag{60}$$

with

$$\boldsymbol{l} = \mathrm{D}_t \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \quad \rightarrow \quad \mathrm{L}_t \boldsymbol{g} = 2[\boldsymbol{g} \cdot \boldsymbol{l}]^{\mathrm{sym}}, \quad \boldsymbol{l}_{\mathrm{i}} = \boldsymbol{F}_{\mathrm{e}} \cdot \mathrm{D}_t \boldsymbol{F}_{\mathrm{i}} \cdot \boldsymbol{F}^{-1} \tag{61}$$

being obvious, and the Kirchhoff–type stresses allow similar representation as the second Piola–Kirchhoff–type tensors in Eq. (54), namely

$$\begin{array}{lllllllll} \frac{1}{2}\boldsymbol{F} \star \boldsymbol{S}^0 & = & \frac{1}{2}\boldsymbol{\tau}^0 & = & \boldsymbol{t}^0_{(m)} : \mathbf{p}_{(m)}, & \frac{1}{2}\boldsymbol{F}_{\mathrm{e}} \star \widehat{\boldsymbol{S}}^{\mathrm{e}} & = & \frac{1}{2}\boldsymbol{\tau}^{\mathrm{e}} & = & \boldsymbol{t}^{\mathrm{e}}_{(m)} : \mathbf{p}^{\mathrm{e}}_{(m)} \quad \text{with} \\ \boldsymbol{F} \star \boldsymbol{T}^0_{(m)} & = & \boldsymbol{t}^0_{(m)} & = & \dfrac{\partial \psi^0_0}{\partial [\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g}]}, & \boldsymbol{F}_{\mathrm{e}} \star \widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} & = & \boldsymbol{t}^{\mathrm{e}}_{(m)} & = & \dfrac{\partial \psi^{\mathrm{e}}_0}{\partial [\boldsymbol{c}_{\mathrm{e}} \cdot \boldsymbol{k}_{\mathrm{e}}(m) \cdot \boldsymbol{g}]} \quad \text{and} \\ \boldsymbol{F} \star \mathbf{P}_{(m)} & = & \mathbf{p}_{(m)} & = & \dfrac{\partial [\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g}]}{\boldsymbol{g}}, & \boldsymbol{F}_{\mathrm{e}} \star \widehat{\mathbf{P}}^{\mathrm{e}}_{(m)} & = & \mathbf{p}^{\mathrm{e}}_{(m)} & = & \dfrac{\partial [\boldsymbol{c}_{\mathrm{e}} \cdot \boldsymbol{k}_{\mathrm{e}(m)} \cdot \boldsymbol{g}]}{\boldsymbol{g}}, \end{array} \tag{62}$$

compare Eq. (48) and (49) and remark 10. The abbreviation $\mathrm{L}_t\{\bullet\}$, as applied in Eqs. (60) and (61), denotes the Lie derivative of the quantity $\{\bullet\}$, i.e. $\mathrm{L}_t \boldsymbol{g} = \boldsymbol{F}^{-\mathrm{d}} \cdot \mathrm{D}_t \boldsymbol{C} \cdot \boldsymbol{F}^{-1} = [\boldsymbol{g} \cdot \boldsymbol{l}^{\mathrm{d}}] + [\boldsymbol{g} \cdot \boldsymbol{l}^{\mathrm{d}}]^{\mathrm{t}}$ with $\mathrm{D}_t \boldsymbol{g} = \boldsymbol{0}$.

Remark 10: For convenience of the reader the general format of the projection tensors as introduced in Eqs. (54) and (62) is highlighted in the following. To give an example, the particular operator $\mathbf{P}_{(m)} = \partial_{\boldsymbol{C}} \boldsymbol{E}_{(m)}$ reads

$$\begin{aligned} \mathbf{P}_{(m)} = {} & \lambda_k^{m-2} \boldsymbol{N}^k \otimes \boldsymbol{N}^k \otimes \boldsymbol{N}_k \otimes \boldsymbol{N}_k \\ & + \frac{1}{m} \frac{\lambda_k^m - \lambda_l^m}{\lambda_k^2 - \lambda_l^2} [\boldsymbol{N}^k \otimes \boldsymbol{N}^l \otimes \boldsymbol{N}_k \otimes \boldsymbol{N}_l + \boldsymbol{N}^k \otimes \boldsymbol{N}^l \otimes \boldsymbol{N}_l \otimes \boldsymbol{N}_k] \end{aligned} \tag{63}$$

for $l \neq k$ and $m > 0$, wherein use of $E_{(m)k} - E_{(m)l} = \frac{1}{m}[\lambda_k^m - \lambda_l^m]$ has been made. For a general review the reader is referred to, e.g., [12] or [48] where the case of equal principal stretches is additionally addressed; in this regard, see also the contribution by [55] and the general elaborations in [56].

5.1 Relations between different stress tensors in one configuration

Based on the relations between different generalized strain measures as highlighted in Sect. 3.1, it is now straightforward to derive transformations between correlated stress tensors in one configuration. To be specific, Eq. (36) results in

$$\boldsymbol{T}_{(m)} = \left.\frac{\partial \psi_0^0}{\partial \boldsymbol{K}_{(m)}}\right|_{\boldsymbol{F}_\mathrm{i}} : \frac{\partial \boldsymbol{K}_{(m)}}{\partial \boldsymbol{E}_{(m)}} = \left[\boldsymbol{B}^{\frac{m}{2}} \cdot \boldsymbol{Y}_{(m)} \cdot \boldsymbol{G}^{-1}\right]^{\mathrm{sym}} \quad \text{with} \quad \boldsymbol{Y}_{(m)} = \left.\frac{\partial \psi_0^0}{\partial \boldsymbol{K}_{(m)}}\right|_{\boldsymbol{F}\mathrm{i}} \tag{64}$$

so that $\boldsymbol{Y}_{(m)} = [\boldsymbol{C}^{\frac{m}{2}} \cdot \boldsymbol{T}_{(m)} \cdot \boldsymbol{G}]^{\mathrm{sym}}$, while similar elaborations based on Eq. (37) render

$$\widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} = \frac{\partial \psi_0^{\mathrm{e}}}{\partial \widehat{\boldsymbol{K}}_{\mathrm{e}(m)}} : \frac{\partial \widehat{\boldsymbol{K}}_{\mathrm{e}(m)}}{\partial \widehat{\boldsymbol{E}}_{\mathrm{e}(m)}} = \left[\widehat{\boldsymbol{B}}_{\mathrm{e}}^{\frac{m}{2}} \cdot \widehat{\boldsymbol{Y}}^{\mathrm{e}}_{(m)} \cdot \widehat{\boldsymbol{G}}^{-1}\right]^{\mathrm{sym}} \quad \text{with} \quad \widehat{\boldsymbol{Y}}^{\mathrm{e}}_{(m)} = \frac{\partial \psi_0^{\mathrm{e}}}{\partial \widehat{\boldsymbol{K}}_{\mathrm{e}(m)}} \tag{65}$$

or $\widehat{\boldsymbol{Y}}_{(m)} = [\widehat{\boldsymbol{C}}_{\mathrm{e}}^{\frac{m}{2}} \cdot \widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} \cdot \widehat{\boldsymbol{G}}]^{\mathrm{sym}}$, respectively. By analogy with theses derivations, we observe for the corresponding spatial stress tensors

$$\boldsymbol{t}_{(m)} = \left.\frac{\partial \psi_0^0}{\partial \boldsymbol{e}_{(m)}}\right|_{\boldsymbol{F}_\mathrm{i}} : \frac{\partial \boldsymbol{e}_{(m)}}{\partial[\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g}]} = \left[\boldsymbol{b} \cdot \boldsymbol{c}^{\frac{m}{2}} \cdot \boldsymbol{z}_{(m)}\right]^{\mathrm{sym}} \quad \text{with} \quad \boldsymbol{z}_{(m)} = \left.\frac{\partial \psi_0^0}{\partial \boldsymbol{e}_{(m)}}\right|_{\boldsymbol{F}\mathrm{i}} \tag{66}$$

so that $\boldsymbol{z}_{(m)} = [\boldsymbol{b}^{\frac{m}{2}} \cdot \boldsymbol{c} \cdot \boldsymbol{t}_{(m)}]^{\mathrm{sym}}$, as well as

$$\boldsymbol{t}^{\mathrm{e}}_{(m)} = \frac{\partial \psi_0^{\mathrm{e}}}{\partial \boldsymbol{e}_{\mathrm{e}(m)}} : \frac{\partial \boldsymbol{e}_{\mathrm{e}(m)}}{\partial[\boldsymbol{c}_{\mathrm{e}} \cdot \boldsymbol{k}_{\mathrm{e}(m)} \cdot \boldsymbol{g}]} = \left[\boldsymbol{b}_{\mathrm{e}} \cdot \boldsymbol{c}_{\mathrm{e}}^{\frac{m}{2}} \cdot \boldsymbol{z}^{\mathrm{e}}_{(m)}\right]^{\mathrm{sym}} \quad \text{with} \quad \boldsymbol{z}^{\mathrm{e}}_{(m)} = \frac{\partial \psi_0^{\mathrm{e}}}{\partial \boldsymbol{e}_{\mathrm{e}(m)}} \tag{67}$$

or $\boldsymbol{z}^{\mathrm{e}}_{(m)} = [\boldsymbol{b}_{\mathrm{e}}^{\frac{m}{2}} \cdot \boldsymbol{c}_{\mathrm{e}} \cdot \boldsymbol{t}^{\mathrm{e}}_{(m)}]^{\mathrm{sym}}$, respectively, whereby Eq. (38) has been taken into account.

Remark 11: By analogy with Eq. (35), one can also relate different stress tensors in terms of different generalized strain measures with respect to one configuration. The lines of derivation in this regard are commonly based on the comparison of energetically conjugated quantities as, e.g., $\boldsymbol{T}^0_{(m)} : \mathrm{D}_t\boldsymbol{E}_{(m)} = \boldsymbol{T}^0_{(n)} : \mathrm{D}_t\boldsymbol{E}_{(n)}$ and $\widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} : \mathrm{D}_t\widehat{\boldsymbol{E}}_{\mathrm{e}(m)} = \widehat{\boldsymbol{T}}^{\mathrm{e}}_{(n)} : \mathrm{D}_t\widehat{\boldsymbol{E}}_{\mathrm{e}(n)}$ for $m, n \geq 0$, respectively. We do not place emphasis on these transformations in this work but refer the reader to the elaborations in [57]–[62] for detailed revisions.

5.2 Relations between different stress tensors in different configurations

The transformations between the stress tensors $\{\boldsymbol{S}^0, \boldsymbol{\tau}^0\}$, $\{\widehat{\boldsymbol{S}}^{\mathrm{e}}, \boldsymbol{\tau}^{\mathrm{e}}\}$, $\{\boldsymbol{T}^0, \boldsymbol{t}^0\}$ and $\{\widehat{\boldsymbol{T}}^{\mathrm{e}}, \boldsymbol{t}^{\mathrm{e}}\}$ are determined by direct pushforward and pullback operations which have already been highlighted in Sect. 5. Based on the relations between different generalized strain measures as reviewed in Sect. 3.2, it is now straightforward to derive further transformations between correlated stresses in different configurations. To be specific, Eq. (43) results in

$$\boldsymbol{z}_{(m)} = \boldsymbol{Y}_{(m)} : \frac{\partial \boldsymbol{K}_{(m)}}{\partial \boldsymbol{e}_{(m)}} = \left[\boldsymbol{F} \cdot \boldsymbol{G}^{-1} \cdot \boldsymbol{Y}_{(m)} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{g}^{-1}\right]^{\mathrm{sym}} \tag{68}$$

so that $\boldsymbol{Y}_{(m)} = [\boldsymbol{G} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{z}_{(m)} \cdot \boldsymbol{g} \cdot \boldsymbol{F}]^{\mathrm{sym}}$, while similar elaborations based on Eq. (44) render

$$\boldsymbol{z}^{\mathrm{e}}_{(m)} = \widehat{\boldsymbol{Y}}^{\mathrm{e}}_{(m)} : \frac{\partial \widehat{\boldsymbol{K}}_{\mathrm{e}(m)}}{\partial \boldsymbol{e}_{\mathrm{e}(m)}} = \left[\boldsymbol{F}_{\mathrm{e}} \cdot \widehat{\boldsymbol{G}}^{-1} \cdot \widehat{\boldsymbol{Y}}^{\mathrm{e}}_{(m)} \cdot \boldsymbol{F}_{\mathrm{e}}^{-1} \cdot \boldsymbol{g}^{-1}\right]^{\mathrm{sym}} \tag{69}$$

or $\widehat{\boldsymbol{Y}}^{\mathrm{e}}_{(m)} = [\widehat{\boldsymbol{G}} \cdot \boldsymbol{F}_{\mathrm{e}}^{-1} \cdot \boldsymbol{z}^{\mathrm{e}}_{(m)} \cdot \boldsymbol{g} \cdot \boldsymbol{F}_{\mathrm{e}}]^{\mathrm{sym}}$, respectively.

With these relations and those elaborations highlighted in Sect. 5.1 in hand, we can also relate the remaining stress tensors. For completeness, we finally mention the rather lengthy expressions

$$
\begin{aligned}
\boldsymbol{z}_{(m)} &= \left[\boldsymbol{F}\cdot\boldsymbol{G}^{-1}\cdot\left[\boldsymbol{C}^{\frac{m}{2}}\cdot\boldsymbol{T}_{(m)}\cdot\boldsymbol{G}\right]^{\mathrm{sym}}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{g}^{-1}\right]^{\mathrm{sym}},\\
\boldsymbol{z}^{\mathrm{e}}_{(m)} &= \left[\boldsymbol{F}_{\mathrm{e}}\cdot\widehat{\boldsymbol{G}}^{-1}\cdot\left[\widehat{\boldsymbol{C}}_{\mathrm{e}}^{\frac{m}{2}}\cdot\widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)}\cdot\widehat{\boldsymbol{G}}\right]^{\mathrm{sym}}\cdot\boldsymbol{F}_{\mathrm{e}}^{-1}\cdot\boldsymbol{g}^{-1}\right]^{\mathrm{sym}}
\end{aligned}
\tag{70}
$$

and

$$
\begin{aligned}
\boldsymbol{T}_{(m)} &= \left[\boldsymbol{B}^{\frac{m}{2}}\cdot\left[\boldsymbol{G}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{z}_{(m)}\cdot\boldsymbol{g}\cdot\boldsymbol{F}\right]^{\mathrm{sym}}\cdot\boldsymbol{G}^{-1}\right]^{\mathrm{sym}},\\
\widehat{\boldsymbol{T}}^{\mathrm{e}}_{(m)} &= \left[\widehat{\boldsymbol{B}}_{\mathrm{e}}^{\frac{m}{2}}\cdot\left[\widehat{\boldsymbol{G}}\cdot\boldsymbol{F}_{\mathrm{e}}^{-1}\cdot\boldsymbol{z}^{\mathrm{e}}_{(m)}\cdot\boldsymbol{g}\cdot\boldsymbol{F}_{\mathrm{e}}\right]^{\mathrm{sym}}\cdot\widehat{\boldsymbol{G}}^{-1}\right]^{\mathrm{sym}}.
\end{aligned}
\tag{71}
$$

5.3 Associated inelasticity

According to the (reduced) dissipation inequality (57), namely

$$
\mathscr{D}_0 = \left[\widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\right] : \widehat{\boldsymbol{L}}_{\mathrm{i}} \ge 0 \quad \text{with} \quad \widehat{\boldsymbol{L}}_{\mathrm{i}} = \mathrm{D}_t\boldsymbol{F}_{\mathrm{i}}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1},
\tag{72}
$$

associated evolution equations are straightforwardly set up via

$$
\widehat{\boldsymbol{L}}_{\mathrm{i}} = \mathrm{D}_t\lambda\,\frac{\partial\phi\left(\widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}},\widehat{\boldsymbol{H}};\boldsymbol{X}\right)}{\partial\left[\widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\right]} = \mathrm{D}_t\lambda\widehat{\boldsymbol{B}}_{\mathrm{e}}\cdot\frac{\partial\phi\left(\widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}},\widehat{\boldsymbol{H}};\boldsymbol{X}\right)}{\partial\widehat{\boldsymbol{S}}^{\mathrm{e}}},
\tag{73}
$$

wherein $\mathrm{D}_t\lambda \ge 0$ denotes a multiplier which is either derived from further constitutive assumptions, as for instance for viscous response, or determined from restricting the incorporated potential or rather yield function $\phi \le 0$ so that $\phi\,\mathrm{D}_t\lambda = 0$. The tensorial variable $\widehat{\boldsymbol{H}}$ denotes an additional argument that conveniently enables the formulation of, e.g., anisotropic flow rules. On the one hand, the introduction of the scalar–valued potential ϕ, from which we derive associated flow rules, restricts the type of evolution equation as compared to the general anisotropic case or rather a comprehensive tensor function with respect to the arguments incorporated into ϕ. On the other hand, the associated format is considered to be sufficiently general for the problem at hand. Alternatively to the representation in Eq. (73), transformation of the derived relation to the material or spatial configuration, respectively, renders

$$
\boldsymbol{L}_{\mathrm{i}} = \mathrm{D}_t\lambda\,\frac{\partial\phi\left(\boldsymbol{C}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}},\boldsymbol{H};\boldsymbol{X}\right)}{\partial\left[\boldsymbol{C}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right]} = \mathrm{D}_t\lambda\boldsymbol{B}\cdot\frac{\partial\phi\left(\boldsymbol{C}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}},\boldsymbol{H};\boldsymbol{X}\right)}{\partial\left[\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{-\mathrm{d}}\right]}
\tag{74}
$$

and

$$
\boldsymbol{l}_{\mathrm{i}} = \mathrm{D}_t\lambda\,\frac{\partial\phi(\boldsymbol{g}\cdot\boldsymbol{\tau}^{\mathrm{e}},\boldsymbol{h};\boldsymbol{X})}{\partial[\boldsymbol{g}\cdot\boldsymbol{\tau}^{\mathrm{e}}]} = \mathrm{D}_t\lambda\,\boldsymbol{g}^{-1}\cdot\frac{\partial\phi(\boldsymbol{g}\cdot\boldsymbol{\tau}^{\mathrm{e}},\boldsymbol{h};\boldsymbol{X})}{\partial\boldsymbol{\tau}_{\mathrm{e}}},
\tag{75}
$$

wherein $\boldsymbol{H} = \boldsymbol{F}_{\mathrm{i}}^{-1}\star\widehat{\boldsymbol{H}}$ and $\boldsymbol{h} = \boldsymbol{F}_{\mathrm{e}}\star\widehat{\boldsymbol{H}}$. An alternative evolution equation in terms of spatial arguments is provided by $-\frac{1}{2}\mathrm{L}_t\boldsymbol{b}_{\mathrm{e}} = [\boldsymbol{l}_{\mathrm{i}}\cdot\boldsymbol{b}_{\mathrm{e}}]^{\mathrm{sym}} = -\frac{1}{2}\mathrm{D}_t\lambda\,\partial_{\boldsymbol{\xi}^{\mathrm{e}}}\phi$, wherein $\boldsymbol{\xi}^{\mathrm{e}} = -\partial_{\boldsymbol{b}_{\mathrm{e}}}\psi_0^{\mathrm{e}}$; see [35] for a detailed discussion. Assuming an isotropic setting, i.e. $\boldsymbol{a}$ and $\boldsymbol{h}$ vanish, we obtain the well–established representation $-\frac{1}{2}\mathrm{L}_t\boldsymbol{b}_{\mathrm{e}} = \mathrm{D}_t\lambda\,\boldsymbol{g}^{-1}\cdot\partial_{\boldsymbol{\tau}_{\mathrm{e}}}\phi\cdot\boldsymbol{b}_{\mathrm{e}}$ which allows similar representation with respect to the intermediate and material configuration via straightforward pullback transformations.

Besides the fact that the Helmholtz free energy density is defined in terms of material, intermediate and spatial generalized strain measures in addition to appropriate metric tensors and structural tensors, the evolution equations (74) and (75) represent well–established formats of associated flow rules for anisotropic finite inelasticity.

Remark 12: The stress tensors of Mandel type which are incorporated into the potential ϕ turn out to be non–symmetric for the general elastically anisotropic case. The eigenvalues of these tensors, however, remain real since the Mandel type stresses are determined by the product of two symmetric tensors with one of them being positive definite, namely $\boldsymbol{C}$, $\widehat{\boldsymbol{C}}_{\mathrm{e}}$ and $\boldsymbol{g}$ for the problem at hand; compare, e.g., [63]. Due to the mixed–variant nature of these stress tensors, trace operations are defined with respect to appropriate identity tensors, so that for instance the deviatoric part of these stresses allows representation as

$$\begin{aligned}
[\boldsymbol{C}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{\mathrm{d}}]^{\mathrm{dev}} &= \boldsymbol{C}\cdot\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{\mathrm{d}} - \tfrac{1}{3}[\boldsymbol{C}:[\boldsymbol{F}_{\mathrm{i}}^{-1}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}\cdot\boldsymbol{F}_{\mathrm{i}}^{\mathrm{d}}]]\boldsymbol{I}^{\mathrm{d}},\\
[\widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}}]^{\mathrm{dev}} &= \widehat{\boldsymbol{C}}_{\mathrm{e}}\cdot\widehat{\boldsymbol{S}}^{\mathrm{e}} - \tfrac{1}{3}[\widehat{\boldsymbol{C}}_{\mathrm{e}}:\widehat{\boldsymbol{S}}^{\mathrm{e}}]\widehat{\boldsymbol{I}}^{\mathrm{d}},\\
[\boldsymbol{g}\cdot\boldsymbol{\tau}^{\mathrm{e}}]^{\mathrm{dev}} &= \boldsymbol{g}\cdot\boldsymbol{\tau}^{\mathrm{e}} - \tfrac{1}{3}[\boldsymbol{g}:\boldsymbol{\tau}^{\mathrm{e}}]\boldsymbol{i}^{d}.
\end{aligned} \tag{76}$$

6 Green–Naghdi type inelasticity

In contrast to the previous Section, where finite inelasticity has been discussed in a rather general context, we next place emphasis on a particular approach which dates back to the pioneering work by Green and Naghdi [64], see also the review article by Naghdi [65]. The fundamental kinematic assumption thereby consists in accepting ψ_0^{e} to take the representation

$$\begin{aligned}
\psi_0^{\mathrm{e}}(\boldsymbol{F},\boldsymbol{E}_{\mathrm{i}},\boldsymbol{A};\boldsymbol{X}) &= \psi_0^{\mathrm{e}}(\boldsymbol{E}_{(m)} - \boldsymbol{E}_{\mathrm{i}},\boldsymbol{G}^{-1},\boldsymbol{A};\boldsymbol{X})\\
&= \psi_0^{\mathrm{e}}(\widehat{\boldsymbol{E}}_{(m)} - \widehat{\boldsymbol{E}}_{\mathrm{i}},\widehat{\boldsymbol{b}}_{\mathrm{i}},\widehat{\boldsymbol{A}};\boldsymbol{X})\\
&= \psi_0^{\mathrm{e}}(\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g} - \boldsymbol{e}_{\mathrm{i}},\boldsymbol{b},\boldsymbol{a};\boldsymbol{X})
\end{aligned} \tag{77}$$

so that

$$\begin{aligned}
I_j &= \boldsymbol{I} : \left[[\boldsymbol{E}_{(m)}-\boldsymbol{E}_{\mathrm{i}}]\cdot\boldsymbol{G}^{-1}\right]^j = \widehat{\boldsymbol{I}} : \left[\left[\widehat{\boldsymbol{E}}_{(m)}-\widehat{\boldsymbol{E}}_{\mathrm{i}}\right]\cdot\widehat{\boldsymbol{b}}_{\mathrm{i}}\right]^j\\
&= \boldsymbol{i} : \left[[\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}-\boldsymbol{e}_{\mathrm{i}}]\cdot\boldsymbol{b}\right]^j = \boldsymbol{i} : [\boldsymbol{g}\cdot\boldsymbol{k}_{(m)}-\boldsymbol{e}_{\mathrm{i}}\cdot\boldsymbol{b}]^j,\\
I_{\alpha+3} &= \boldsymbol{I} : \left[[\boldsymbol{E}_{(m)}-\boldsymbol{E}_{\mathrm{i}}]\cdot\left[\boldsymbol{G}^{-1}\cdot[\boldsymbol{E}_{(m)}-\boldsymbol{E}_{\mathrm{i}}]\right]^{\alpha-1}\cdot\boldsymbol{A}\right]\\
&= \widehat{\boldsymbol{I}} : \left[\left[\widehat{\boldsymbol{E}}_{(m)}-\widehat{\boldsymbol{E}}_{\mathrm{i}}\right]\cdot\left[\widehat{\boldsymbol{b}}_{\mathrm{i}}\cdot\left[\widehat{\boldsymbol{E}}_{(m)}-\widehat{\boldsymbol{E}}_{\mathrm{i}}\right]\right]^{\alpha-1}\cdot\widehat{\boldsymbol{A}}\right]\\
&= \boldsymbol{i} : \left[[\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}-\boldsymbol{e}_{\mathrm{i}}]\cdot[\boldsymbol{b}\cdot[\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}-\boldsymbol{e}_{\mathrm{i}}]]^{\alpha-1}\cdot\boldsymbol{a}\right],
\end{aligned} \tag{78}$$

with $j = 1,2,3$ and $\alpha = 1,2$, which is in contrast to Eqs. (47)–(49); see also remark 3. The incorporated symmetric strain measures $\boldsymbol{E}_{(m)}$, $\widehat{\boldsymbol{E}}_{(m)}$ and $\boldsymbol{c}\cdot\boldsymbol{k}_{(m)}\cdot\boldsymbol{g}$ are thereby referred to the total deformation while $\boldsymbol{E}_{\mathrm{i}}$, $\widehat{\boldsymbol{E}}_{\mathrm{i}} = \boldsymbol{F}_{\mathrm{p}}\star\boldsymbol{E}_{\mathrm{i}}$ and $\boldsymbol{e}_{\mathrm{i}} = \boldsymbol{F}\star\boldsymbol{E}_{\mathrm{i}}$ take the interpretation as symmetric internal variables similar to the generally nonsymmetric irreversible distortion $\boldsymbol{F}_{\mathrm{i}}$. Conceptually speaking, this approach recaptures the formal structure of a small strain inelastic setting except that generalized strain measures – based on the total deformation – are incorporated and appropriate metric tensors are taken into account. Apparently, the representations in Eq. (77) in terms of material and spatial arguments are preferable compared to the intermediate format since neither $\boldsymbol{F}_\mathrm{i}$ nor $\boldsymbol{F}_\mathrm{e}$ are directly accessible within this framework. This aspect is also reflected by the considered metric tensors which differ from those applied in Eq. (47). In this context, and also for the sake of brevity, we will mainly place emphasis on formulations based on either material or spatial arguments, respectively, in the following.

6.1 Associated inelasticity

Similarly to Eq. (72), the isothermal dissipation inequality, as based on the ansatz highlighted in Eq. (77), now results in

$$\mathscr{D}_0 = -\left.\frac{\partial \psi_0^\mathrm{e}}{\partial \boldsymbol{E}_\mathrm{i}}\right|_{\boldsymbol{F}} : \mathrm{D}_t \boldsymbol{E}_\mathrm{i} = -\left.\frac{\partial \psi_0^\mathrm{e}}{\partial \widehat{\boldsymbol{E}}_\mathrm{i}}\right|_{\boldsymbol{F}} : \mathrm{L}_t^\mathrm{i} \widehat{\boldsymbol{E}}_\mathrm{i} = -\left.\frac{\partial \psi_0^\mathrm{e}}{\partial \boldsymbol{e}_\mathrm{i}}\right|_{\boldsymbol{F}} : \mathrm{L}_t \boldsymbol{e}_\mathrm{i} \geq 0 \tag{79}$$

with

$$-\left.\frac{\partial \psi_0^\mathrm{e}}{\partial \boldsymbol{E}_\mathrm{i}}\right|_{\boldsymbol{F}} = -\frac{\partial \psi_0^\mathrm{e}}{\partial [\boldsymbol{E}_{(m)} - \boldsymbol{E}_\mathrm{i}]} : \left.\frac{\partial [\boldsymbol{E}_{(m)} - \boldsymbol{E}_\mathrm{i}]}{\partial \boldsymbol{E}_\mathrm{i}}\right|_{\boldsymbol{F}} = \frac{\partial \psi_0^\mathrm{e}}{\partial [\boldsymbol{E}_{(m)} - \boldsymbol{E}_\mathrm{i}]} = \boldsymbol{T}^\mathrm{e}_{(m)} \tag{80}$$

as well as

$$-\left.\frac{\partial \psi_0^\mathrm{e}}{\partial \boldsymbol{e}_\mathrm{i}}\right|_{\boldsymbol{F}} = -\frac{\partial \psi_0^\mathrm{e}}{\partial [\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g} - \boldsymbol{e}_\mathrm{i}]} : \left.\frac{\partial [\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g} - \boldsymbol{e}_\mathrm{i}]}{\partial \boldsymbol{e}_\mathrm{i}}\right|_{\boldsymbol{F}} = \frac{\partial \psi_0^\mathrm{e}}{\partial [\boldsymbol{c} \cdot \boldsymbol{k}_{(m)} \cdot \boldsymbol{g} - \boldsymbol{e}_\mathrm{i}]} = \boldsymbol{t}^\mathrm{e}_{(m)}, \tag{81}$$

respectively, wherein the notation introduced in Sect. 5 for stresses derived from generalized strain measures has been adopted. By analogy with Eq. (74), we consequently define associated flow rules via

$$\mathrm{D}_t \boldsymbol{E}_\mathrm{i} = \mathrm{D}_t \lambda \frac{\partial \phi\left(\boldsymbol{T}^\mathrm{e}_{(m)}, \boldsymbol{C}, \boldsymbol{H}; \boldsymbol{X}\right)}{\partial \boldsymbol{T}^\mathrm{e}_{(m)}} = \mathrm{D}_t \lambda \left[\boldsymbol{C}^{\frac{m}{2}} \cdot \frac{\partial \phi\left(\boldsymbol{T}^\mathrm{e}_{(m)}, \boldsymbol{C}, \boldsymbol{H}; \boldsymbol{X}\right)}{\partial \boldsymbol{Y}^\mathrm{e}_{(m)}} \cdot \boldsymbol{G}\right]^\mathrm{sym} \tag{82}$$

wherein $\boldsymbol{Y}^\mathrm{e}_{(m)} = \partial_{\boldsymbol{K}_{(m)}} \psi_0^\mathrm{e}\big|_{\boldsymbol{E}_\mathrm{i}}$ and use of the symmetry of $\partial_{\boldsymbol{Y}^\mathrm{e}_{(m)}} \phi$ has been made; compare Sect. 5.1. Similarly to Eq. (75), the spatial representation of the sought evolution equation results in

$$\mathrm{L}_t \boldsymbol{e}_\mathrm{i} = \mathrm{D}_t \lambda \frac{\partial \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)}, \boldsymbol{g}, \boldsymbol{h}; \boldsymbol{X}\right)}{\partial \boldsymbol{t}^\mathrm{e}_{(m)}} = \mathrm{D}_t \lambda \left[\boldsymbol{c} \cdot \boldsymbol{b}^{\frac{m}{2}} \cdot \frac{\partial \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)}, \boldsymbol{g}, \boldsymbol{h}; \boldsymbol{X}\right)}{\partial \boldsymbol{z}^\mathrm{e}_{(m)}}\right]^\mathrm{sym} \tag{83}$$

wherein $\boldsymbol{z}^\mathrm{e}_{(m)} = \partial_{\boldsymbol{e}_{(m)}} \psi_0^\mathrm{e}\big|_{\boldsymbol{e}_\mathrm{i}}$ and use of the symmetry of $\partial_{\boldsymbol{z}^\mathrm{e}_{(m)}} \phi$ has been made; compare Sect. 5.1. Based on the elaborations highlighted in Sect. 5.2, we finally identify the relations

$$\begin{aligned}
\mathrm{D}_t \boldsymbol{E}_\mathrm{i} &= \mathrm{D}_t \lambda \left[\boldsymbol{C}^{\frac{m}{2}} \cdot \left[\boldsymbol{F}^{-1} \cdot \boldsymbol{g}^{-1} \cdot \frac{\partial \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)}, \boldsymbol{g}, \boldsymbol{h}; \boldsymbol{X}\right)}{\partial \boldsymbol{z}^\mathrm{e}_{(m)}} \cdot \boldsymbol{F}\right]^\mathrm{sym}\right]^\mathrm{sym} \\
&= \mathrm{D}_t \lambda \left[\boldsymbol{C}^{\frac{m}{2}} \cdot \left[\boldsymbol{F}^{-1} \cdot \boldsymbol{g}^{-1} \cdot \left[\boldsymbol{c}^{\frac{m}{2}} \cdot \boldsymbol{b} \cdot \frac{\partial \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)}, \boldsymbol{g}, \boldsymbol{h}; \boldsymbol{X}\right)}{\partial \boldsymbol{t}^\mathrm{e}_{(m)}}\right]^\mathrm{sym} \cdot \boldsymbol{F}\right]^\mathrm{sym}\right]^\mathrm{sym} \\
&= \mathrm{D}_t \lambda \boldsymbol{F}^\mathrm{d} \cdot \frac{\partial \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)}, \boldsymbol{g}, \boldsymbol{h}; \boldsymbol{X}\right)}{\partial \boldsymbol{t}^\mathrm{e}_{(m)}} \cdot \boldsymbol{F} = \boldsymbol{F}^\mathrm{d} \cdot \mathrm{L}_t \boldsymbol{e}_i \cdot \boldsymbol{F}
\end{aligned} \tag{84}$$

and

$$\begin{aligned}
\mathrm{L}_t\boldsymbol{e}_\mathrm{i} &= \mathrm{D}_t\lambda\left[\boldsymbol{c}\cdot\boldsymbol{b}^{\frac{m}{2}}\cdot\left[\boldsymbol{g}\cdot\boldsymbol{F}\cdot\frac{\partial\phi\left(\boldsymbol{T}^\mathrm{e}_{(m)},\boldsymbol{C},\boldsymbol{H};\boldsymbol{X}\right)}{\partial\boldsymbol{Y}^\mathrm{e}_{(m)}}\cdot\boldsymbol{G}\cdot\boldsymbol{F}^{-1}\right]^\mathrm{sym}\right]^\mathrm{sym}\\
&= \mathrm{D}_t\lambda\left[\boldsymbol{c}\cdot\boldsymbol{b}^{\frac{m}{2}}\cdot\left[\boldsymbol{g}\cdot\boldsymbol{F}\cdot\left[\boldsymbol{B}^{\frac{m}{2}}\cdot\frac{\partial\phi\left(\boldsymbol{T}^\mathrm{e}_{(m)},\boldsymbol{C},\boldsymbol{H};\boldsymbol{X}\right)}{\partial\boldsymbol{T}^\mathrm{e}_{(m)}}\cdot\boldsymbol{G}^{-1}\right]^\mathrm{sym}\cdot\boldsymbol{G}\cdot\boldsymbol{F}^{-1}\right]^\mathrm{sym}\right]^\mathrm{sym} \qquad (85)\\
&= \mathrm{D}_t\lambda\boldsymbol{F}^{-\mathrm{d}}\cdot\frac{\partial\phi\left(\boldsymbol{T}^\mathrm{e}_{(m)},\boldsymbol{C},\boldsymbol{H};\boldsymbol{X}\right)}{\partial\boldsymbol{T}^\mathrm{e}_{(m)}}\cdot\boldsymbol{F}^{-1} = \boldsymbol{F}^{-\mathrm{d}}\cdot\mathrm{D}_t\boldsymbol{E}_i\cdot\boldsymbol{F}^{-1}.
\end{aligned}$$

As previously mentioned in this Section, the proposed Green–Naghdi type framework takes the interpretation as an inelastic setting with respect to the reference configuration with the formal structure of a small strain setting being recaptured. This idea is also reflected by the flow rules in Eq. (81) since the covariant metric $\boldsymbol{C}(\boldsymbol{g})$ is additionally incorporated besides the symmetric generalised stress tensor $\boldsymbol{T}^\mathrm{e}_{(m)}$ ($\boldsymbol{t}^\mathrm{e}_{(m)}$). Analogous expressions have apparently not been introduced into the previously highlighted evolution equations (73)–(75) since, e.g., trace operations of the Mandel type stress tensors incorporated into these relations are performed with identity tensors which are redundant, compare remarks 12 and 13.

Remark 13: Alternatively, the potential ϕ could also be introduced with respect to different metric tensors, e.g., $\phi = \phi\left(\boldsymbol{T}^\mathrm{e}_{(m)},\boldsymbol{G},\boldsymbol{H};\boldsymbol{X}\right) = \phi\left(\boldsymbol{t}^\mathrm{e}_{(m)},\boldsymbol{c},\boldsymbol{h};\boldsymbol{X}\right)$. This ansatz, however, would not reflect the idea that corresponding trace operations in terms of spatial arguments are related to the spatial co–variant metric as, e.g., in Eqs. (75) and (76). Contrary, the particular choice highlighted in Eq. (82) renders for instance the deviatoric stresses to take the representations

$$\left[\boldsymbol{T}^\mathrm{e}_{(m)}\right]^\mathrm{dev} = \boldsymbol{T}^\mathrm{e}_{(m)} - \tfrac{1}{3}\left[\boldsymbol{C}:\boldsymbol{T}^\mathrm{e}_{(m)}\right]\boldsymbol{B} \quad \text{and} \quad \left[\boldsymbol{t}^\mathrm{e}_{(m)}\right]^\mathrm{dev} = \boldsymbol{t}^\mathrm{e}_{(m)} - \tfrac{1}{3}\left[\boldsymbol{g}:\boldsymbol{t}^\mathrm{e}_{(m)}\right]\boldsymbol{g}^{-1}. \qquad (86)$$

7 Summary

The main goal of this work was the elaboration of pushforward and pullback transformations between different Seth–Hill type strain measures embedded into established frameworks for finite inelasticity. A fundamental backbone for the highlighted derivations was provided by application of the spectral decomposition theorem as well as absolute tensor representations. These considerations enabled us to develop equivalent sets of material, intermediate and spatial generalized strain tensors. Based on the fundamental covariance of the Helmholtz free energy, correlated stress tensors – settled in different configurations – have been introduced. With these elaborations in hand, two different associated inelasticity approaches have been discussed whereby neither the particular elasticity law nor the incorporated flow rule were restricted to isotropic response. Summarizing, the basic sets of constitutive equations of finite inelasticity based on material, intermediate and spatial generalized strain measures have been highlighted. From the modelling point of view, this variability or rather general framework broadens the spectrum of possible particular formulations for finite inelasticity which, for instance, is of special interest for numerical applications and serves as a convenient platform for the implementation of efficient algorithmic settings.

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