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A mathematical model of thermoviscoelastic FGM thin plates and Ritz approximate solutions

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Summary. According to the constitutive relation of linear thermoviscoelasticity, a mathematical model of viscoelastic FGM thin plates under thermal loads is set up with the help of Laplace transformation method and the introduction of "structural functions" and "thermal functions". The corresponding simplified Gurtin's type variational principle of FGM thin plates is presented by means of convolution bilinear forms. By combining the Ritz method in the spatial domain and the Legendre interpolation method in the temporal domain, the influence of temperature variation and effects of graded parameters on the quasi-static responses of the FGM plate are investigated.

1 Introduction

Functionally graded materials (FGMs) have attracted considerable attention as a special class of advanced inhomogeneous composite materials in many engineering applications since they were first reported in 1980's (see Yamanoushi, Koizumi and Hiraii [1], Koizumi [2]). FGMs were initially designed as thermal barrier materials for aerospace structural applications and fusion reactors. Due to superior thermo-mechanical performance, FGMs are now developed for general use as structural components in extremely high-temperature environments. Substantial research work has been done on elastic behavior of the FGMs, particularly by Williamson, Rabin and Drake [3], Obata and Noda [4], Praveen and Reddy [5], Loy, Lam and Reddy [6], Reddy [7], Woo and Meguid [8], Han and Liu [9], He, Liew and Ng [10], Yang and Shen [11].

Viscoelastic behavior of the materials should be considered when they serve in high-temperature environments. Up to now there are few works on viscoelastic behavior of FGMs. Paulino and Jin [12] have shown that the correspondence principle can still be used to obtain the viscoelastic solution for a class of FGMs exhibiting relaxation (or creep) functions with separable kernels in space and time. By using the revisited correspondence principle for FGMs, they have subsequently studied crack problems of FGM strips subjected to antiplane shear conditions (see Paulino and Jin [13], [14]). Yang [15] performed a stress analysis in FGM cylinders where steady-state creep conditions are considered only for homogeneous materials.

With regard to variational principles on homogeneous viscoelastic bodies, Reddy [16] directly constructed a simplified Gurtin's type functional for viscoelastic dynamic problems by using a convolution bilinear form. Luo [17] further generalized the simplified Gurtin's type variational principle. Dall'Asta and Menditto [18] studied the inverse variational problem on a perturbed viscoelastic body. However, variational principles of special structures are rare. Usually, the special feature of structures will make the ruling operator of a problem to become more complex or/and nonsymmetrical, and the operator is different from that of a 3D-viscoelastic body essentially. Hence, this greatly increases the difficulty constructing the corresponding functional. Dall'Asta and Menditto [19] pointed out this difficulty when they studied the variational problem of a perturbed viscoelastic body. Cheng and Zhang [20] and Zhang and Cheng [21] analyzed the inverse variational problem for the static and dynamic analysis of viscoelastic thin plates. Cheng and Lu [22] analyzed the inverse variational problem for the static and dynamic analysis of viscoelastic Timoshenko beams. By using the variational integral method, Sheng and Cheng [23] gave the convolution-type functional and presented the corresponding generalized variational principles and potential energy principle of viscoelastic solids with voids.

There are few works on variational principles of homogeneous thermoviscoelastic bodies. By the Laplace transformation method, Brilla [24] formulated a variational principle for linear uncoupled thermoviscoelastic plates under clamped or simply supported boundary conditions, and analyzed properties of eigenvalues and the convergence of the finite-element method. Altay and Dokmeci [25] presented a differential type of variational principles in terms of the Laplace transformed field variables for linear coupled thermoviscoelastic analysis of high-frequency motions of thin plates. For the above variational principles were constructed in the Laplace space by the classical Cartesian bilinear forms, the numerical error is inevitably enhanced. To the authors' knowledge, numerical results whether on the above classical variational principles or on modern convolution-type variational principles of thermoviscoelastic bodies are seldom reported (see Othman [26]).

Up to now variational principles on thermoviscoelastic functionally graded plates (FGPs) have not been reported in the open literature. In the present study, for thermoviscoelastic FGMs with material functions having separable kernels in space and time, according to the integral type constitutive relation of linear thermoviscoelasticity, a mathematical model of viscoelastic FGM thin plates under thermal loads is set up with the help of Laplace transformation method and the introduction of "structural functions" and "thermal functions". The corresponding simplified Gurtin's type variational principle of thermoviscoelastic FGM thin plates is presented by means of the modern convolution bilinear forms as well as the classical Cartesian bilinear forms. By combining the Ritz method in the spatial domain and the Legendre interpolation method in the temporal domain, the influence of temperature variation and effects of graded parameters on the quasi-static responses of the FGPs are investigated. By using the property of the Legendre series, two approaches in the temporal domain are presented to overcome the difficulty of numerical dates storage as a result of convolution type constitutive relations in direct methods.

2 Mathematical model

Consider a thermoviscoelastic FGM thin plate with the thickness h. Assume that the coordinate plane ox_1x_2 coincides with the undeformed mid-plane and the ox_3-axis is perpendicular to the mid-plane. Hence, the undeformed plate occupies the region to be $V = \{(x_1, x_2, x_3):\ (x_1, x_2) \in \Omega, |x_3| \leq h/2\}$, and its edge is $\partial\Omega = \partial\Omega_u + \partial\Omega_\sigma$, in which $\partial\Omega_u$ and $\partial\Omega_\sigma$ are the portions of the given edge displacements and given edge forces, respectively. Assume there is no body force applied on the plate, but there exist edge forces $(\overline{X_1}(t), \overline{X_2}(t), 0)$ that are parallel to the mid-plane at the edge of the plate and transverse load $q(x_\alpha, t)$ effecting on the plate.

Letting the displacements at any point in the mid-plane be $u_i(x_\alpha, t)$ and the stress $\sigma_{ij}(x_k, t)$ and strain $\varepsilon_{ij}(x_k, t)$. (Here and afterward, the Greek subscript has the ranges 1 and 2, and the Latin subscript has the ranges 1, 2 and 3), then we have the equations and conditions as follows.

2.1 Constitutive equations

For an anisotropic thermoviscoelastic material, the Boltzmann relaxation law is given as (see Christensen [27])

$$\sigma_{ij} = G^{xt}_{ijkl}(x_1, x_2, x_3, t) \otimes \varepsilon_{kl} - \varphi^{xt}_{ij}(x_1, x_2, x_3, t) \otimes \theta, \tag{1}$$

in which $G^{xt}_{ijkl}(x_1, x_2, x_3, t)$ and $\varphi^{xt}_{ij}(x_1, x_2, x_3, t)$ are relaxation functions and thermal strain functions of the material, $\theta(x_i, t)$ denotes the infinitesimal temperature deviation from the base temperature T_0, and the symbol $\otimes$ expresses the linear Boltzmann operator defined as (see Leitman and Fisher [28])

$$g(t) \otimes u(t) = g(0)u(t) + \dot{g}(t) * u(t) = g(0)u(t) + \int_0^t \dot{g}(t - \tau)u(\tau)\mathrm{d}\tau. \tag{2}$$

Assume that the material of the plate is homogeneous in-plane paralleling to the mid-plane, and its properties change only along the ox_3-axis. In other words usually relaxation functions and thermal strain functions depend on spatial variable x_3 and temporal variable t. For simplicity a class of viscoelastic materials with separable kernel functions in space and time was often used to study the behaviors of structures. Schovanec and Walton [29], [30], Herrmann and Schovanec [31], [32] and Alex and Schovanec [33] employed a separable form for the relaxation functions to investigate a series of crack problems in nonhomogeneous viscoelastic media, such as stationary cracks, quasi-static crack propagation, dynamic crack propagation and energy release rate of quasi-static mode I crack propagation. Paulino and Jin [12]–[14], [34] studied crack problems of FGMs exhibiting relaxation (or creep) functions with separable kernels in space and time. In the present study, the following separable material functions are used:

$$G^{xt}_{ijkl}(x_1, x_2, x_3, t) = f(x_3)G_{ijkl}(t), \quad \varphi^{xt}_{ij}(x_1, x_2, x_3, t) = k(x_3)\varphi_{ij}(t). \tag{3}$$

From (1), (2) and (3), we can obtain

$$\sigma_{ij} = f(x_3)G_{ijkl}(t) \otimes \varepsilon_{kl} - k(x_3)\varphi_{ij}(t) \otimes \theta. \tag{4}$$

According to Kirchhoff's theory of plates, $\sigma_{33} = 0$ and $\varepsilon_{\alpha 3} = 0$. By the Laplace transformation and its inverse transformation method, it is not difficult to obtain

$$\varepsilon_{33} = -g_{\alpha\beta} \otimes \varepsilon_{\alpha\beta} + [k(x_3)/f(x_3)]g'(t) \otimes \theta, \tag{5}$$

in which

$$g_{\alpha\beta}(t) \equiv L^{-1}[\bar{G}_{33\alpha\beta}/s\bar{G}_{3333}], \quad g'(t) = L^{-1}(\overline{\varphi_{33}}/s\overline{G_{3333}}). \tag{6}$$

Substituting (5) into (4) yields

$$\sigma_{\alpha\beta} = f(x_3)(G_{\alpha\beta\gamma\delta} + G'_{\alpha\beta\gamma\delta}) \otimes \varepsilon_{\gamma\delta} - k(x_3)(\varphi_{\alpha\beta} + \varphi'_{\alpha\beta}) \otimes \theta, \tag{7}$$

where

$$G'_{\alpha\beta\gamma\delta} \equiv -G_{\alpha\beta33} \otimes g_{\gamma\delta}, \quad \varphi'_{\alpha\beta} = -G'_{\alpha\beta33}(t) \otimes g'(t) \tag{8}$$

Obviously the functions $G'_{\alpha\beta\gamma\delta}(t)$ and $\varphi'_{\alpha\beta}(t)$ depend on the relaxation functions $G_{ijkl}(t)$ and $\varphi_{ij}(t)$. Although they are not independent material functions, the functions $G'_{\alpha\beta\gamma\delta}$ and $\varphi'_{\alpha\beta}$ play a key role in establishing the initial-boundary-value problem. As $G'_{\alpha\beta\gamma\delta}$ and $\varphi'_{\alpha\beta}$ express structural and thermal features of a thermoviscoelastic FGM thin plate, we call them "structural functions" and "thermal functions" of the plate, respectively.

For the isotropic thermoviscoelastic materials, we have (see Christensen [27])

$$G_{ijkl}(t) = (G_2 - G_1)\delta_{ij}\delta_{kl}/3 + G_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2, \quad \varphi_{ij} = \varphi\delta_{ij}, \tag{9}$$

in which G_1 and G_2 are the relaxation functions of isotropic materials, and φ is the thermal strain function of the isotropic material. Hence it is not difficult to get

$$g_{21} = g_{12} = 0, \quad g \equiv g_{11} = g_{22} = L^{-1}[(arG_2 - \bar{G}_1)/s(2\bar{G}_1 + \bar{G}_2)], \tag{10.1}$$

$$G'_{12\gamma\delta} = G'_{21\gamma\delta} = G'_{\alpha\beta12} = G'_{\alpha\beta21} = 0, \tag{10.2}$$

$$G'_{1111} = G'_{1122} = G'_{2211} = G'_{2222} = -(G_2 - G_1) \otimes g/3, \tag{10.3}$$

$$g' = L^{-1}[3\overline{\varphi}/s\overline{(2G_1 + G_2)}], \tag{10.4}$$

$$\varphi'_{12} = \varphi'_{21} = 0, \quad \varphi' = \varphi'_{11} = \varphi'_{22} = -(G_2 - G_1) \otimes g'/3. \tag{10.5}$$

So we can get the constitutive equation of isotropic thermoviscoelastic FGM plates

$$\sigma_{\alpha\beta} = f(x_3)(G_1 \otimes \varepsilon_{\alpha\beta} + \delta_{\alpha\beta}G_3 \otimes \varepsilon_{\gamma\gamma}) - k(x_3)\delta_{\alpha\beta}\Phi \otimes \theta \tag{11}$$

in which

$$G_3 = G_1 \otimes g, \quad \Phi = \varphi + \varphi'. \tag{12}$$

The functions G_3 and Φ are called "structural function" and "thermal function" of the isotropic plate, respectively.

2.2 Geometry and motion equations

In the linear theory of plates, we have

$$\varepsilon_{\alpha\beta} = -u_{3,\alpha\beta}x_3. \tag{13}$$

The motion equation of the plate is given by

$$(Q_\alpha + u_{3,\beta}N_{\alpha\beta})_{,\alpha} + q = \rho h u_{3,tt}, \quad Q_\alpha = M_{\alpha\beta,\beta}, \tag{14}$$

in which the internal forces are defined by

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}x_3 dx_3, \quad N_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dx_3, \quad Q_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha3} dx_3. \tag{15}$$

Based on Eqs. (13)–(15), at the same time, letting $\theta(x_i, t) = \psi'(x_3)\psi(x_\alpha, t)$, it is not difficult to obtain

$$\rho(x_3)hu_{3,tt} = A(G_1 + G_3) \otimes \nabla^4 u_3 + B\Phi \otimes \nabla^2\psi + u_{3,\beta\alpha}N_{\alpha\beta} + q, \tag{16}$$

in which

$$A = -\int_{-h/2}^{h/2} x_3^2 f(x_3)\,dx_3, \quad B = -\int_{-h/2}^{h/2} x_3 k(x_3)\psi'(x_3)\,dx_3. \tag{17}$$

2.3 Boundary and initial conditions

Assume that on $\partial\Omega_u$ the displacements are given and that on $\partial\Omega_\sigma$ the forces are given, then we have the following boundary conditions:

$$u_3 = \tilde{u}_3, \quad u_{3,n} = \tilde{\vartheta}, \quad (x_\alpha, t) \in \partial\Omega_u \times [0,T], \tag{18.1}$$

$$M_n = \tilde{M}_n, \quad V_n = \tilde{V}_n, \quad (x_\alpha, t) \in \partial\Omega_\sigma \times [0,T], \tag{18.2}$$

in which $\tilde{u}_3$ and $\tilde{\vartheta}$ are the known deflection and rotation angle on $\partial\Omega_u$, $\tilde{M}_n$ and $\tilde{V}_n$ the known bending moment and shear force on $\partial\Omega_\sigma$, and M_n and V_n are given as

$$\begin{aligned} M_n &= A(G_1 \otimes u_{3,nn} + G_3 \otimes \nabla^2 u_3) + B\Phi \otimes \psi, \\ V_n &= A[(G_1 + G_3) \otimes (\nabla^2 u_3)_{,n} + G_1 \otimes (u_{3,ns} - u_{3,s}/\rho_s)_{,s}] + B\Phi \otimes \psi_{,n} + N_n u_{3,n} + N_s u_{3,s}, \end{aligned} \tag{19}$$

where ρ_s is the radius of curvature of the edge, s is the arc length, and N_n and N_s are normal and tangential membrane force, respectively. In fact, from the definitions (15) of bending moments and shear forces and the constitutive equation (11), it is not difficult to obtain the expressions (19) by the method similar to deriving the corresponding relations of elastic thin plates (see Chien Weizhang [35]).

Assuming that the material and structure are in natural states when $t \in \left(-\infty, 0^-\right]$ and letting u_3^0 and $\dot{u}_3^0$ be the values of u_3 and $\dot{u}_3 = u_{3,t}$ at the initial time $t = 0^+$, then the initial conditions are

$$\begin{aligned} &u_3 = \dot{u}_3 = 0 && \text{in } \bar{\Omega} \times (-\infty, 0^-], \\ &u_3|_{t=0} = u_3^0 \quad \dot{u}_3|_{t=0} = \dot{u}_3^0, && \text{in } \bar{\Omega} \times t = 0^+, \end{aligned} \tag{20}$$

in which both the functions u_3^0 and $\dot{u}_3^0$ are known functions only in x_α.

Thus, the governing equation (16), the boundary conditions (18) and the initial conditions (20) form the initial-boundary-value problem for the static-dynamic analysis of thermoviscoelastic FGM thin plates. It should be pointed out that the effects of mid-plane strains induced by inhomogeneous properties of the FGP on the prediction of the deflection u_3 are assumed to be infinitesimal and may be omitted. Some works (see Shen [36]) have shown that the various traditional simplified theories of homogenous plates are accurate enough to predict the global responses of FGPs, such as displacements, buckling loads, and so on.

3 Variational principles

Now, the displacement u_3 and the coordinates x_1, x_2, x_3 are replaced by w, x, y, z, respectively. The variational principle holds as follows:

If $N_{\alpha\beta}(x_1, x_2, t) = N_{\alpha\beta}(x_1, x_2, T-t), \forall t \in [0, T]$, the solution of the problem (16), (18), (20) is equivalent to searching the stationary point of the functional Π among all w satisfying (18.1), and Π is given by

$$\Pi = \Pi_w + \Pi_\theta + \Pi_n + \Pi_b + \Pi_t, \tag{21}$$

$$\Pi_w = -\frac{A}{2}\iint\limits_{\Omega}[(G_1+G_3)\otimes(w_{,xx}+w_{,yy})*(w_{,xx}+w_{,yy})$$
$$+2G_1\otimes(w_{,xy}*w_{,xy}-w_{,xx}*w_{,yy})]\,dxdy,$$

$$\Pi_\theta = -B\iint\limits_{\Omega}\Phi\otimes\psi*(\nabla^2 w)\,dxdy, \quad \Pi_n = \frac{1}{2}\iint\limits_{\Omega}(N_{\alpha\beta}w_{,\alpha})*w_{,\beta}\,dxdy,$$

$$\Pi_b = -\iint\limits_{\Omega}q*w dxdy + \int\limits_{\partial\Omega_\sigma}(\tilde{M}_n*w_{,n}-\tilde{V}_n*w)ds,$$

$$\Pi_t = \iint\limits_{\Omega}\rho(z)h\left[\frac{1}{2}\dot{w}*\dot{w}+(w|_{t=0}-w_0)w|_{t=T}-\dot{w}_0 w|_{t=T}\right]dxdy.$$

Observing that the Boltzmann operator has the property

(i) $(A\otimes B)*C = A\otimes B*C = A\otimes(B*C) = A\otimes C*B,$ (22.1)

(ii) if $S(t) = S(T-t), \quad \forall t\in[0,T], \quad$ then $(SA)*B = (SB)*A$ (22.2)

it can be obtained

$$\delta\Pi_w = -\iint\limits_{\Omega}A(G_1+G_3)\otimes\nabla^4 w*\delta w dxdy - \int\limits_{\partial\Omega_\sigma}[AG_1\otimes w_{,nn}+AG_3\otimes(\nabla^2 w)]*\delta w_{,n}ds$$
$$+\int\limits_{\partial\Omega_\sigma}[A(G_1+G_3)\otimes(\nabla^2 w)_{,n}+AG_1\otimes(w_{,ns}-w_{,s}/\rho_s)_{,s}+\frac{1}{2}B\Phi\otimes\psi_{,n}]*\delta w ds, \tag{23.1}$$

$$\delta\Pi_\theta = -B\iint\limits_{\Omega}\Phi\otimes(\nabla^2\psi)*\delta w dxdy - B\int\limits_{\partial\Omega_\sigma}\Phi\otimes\psi*\delta w_{,n}ds + B\int\limits_{\partial\Omega_\sigma}\Phi\otimes\psi_{,n}*\delta w ds, \tag{23.2}$$

$$\delta\Pi_b = -\iint\limits_{\Omega}q*\delta w dxdy + \int\limits_{\partial\Omega_\sigma}(\tilde{M}_n*\delta w_{,n}-\tilde{V}_n*\delta w)ds, \tag{23.3}$$

$$\delta\Pi_t = \iint\limits_{\Omega}\rho(z)h[\ddot{w}*\delta w+(w|_{t=0}-w_0)\delta\dot{w}|_{t=T}+(\dot{w}|_{t=0}-\dot{w}_0)\delta w|_{t=T}]dxdy, \tag{23.4}$$

$$\delta\Pi_n = -\iint\limits_{\Omega}[(w_{,\beta\alpha}N_{\alpha\beta})*\delta w]dxdy + \int\limits_{\partial\Omega_\sigma}(N_n w_{,n}+N_s w_{,s})*\delta w ds. \tag{23.5}$$

Substituting (23) into $\delta\Pi = \delta\Pi_w + \delta\Pi_\theta + \delta\Pi_n + \delta\Pi_b + \delta\Pi_t = 0$, we obtain the variational equation

$$\delta\Pi = -\iint_{\Omega} [A(G_1+G_3)\otimes\nabla^4 w + B\Phi\otimes\nabla^2\psi + w_{,\beta\alpha}N_{\alpha\beta} + q - \rho(z)hw_{,tt}] * \delta w dxdy$$

$$- \int_{\partial\Omega_\sigma} (M_n - \tilde{M}_n) * \delta w_{,n} ds + \int_{\partial\Omega_\sigma} (V_n - \tilde{V}_n) * \delta w ds \tag{24}$$

$$+ \iint_{\Omega} \rho(z)h[(w|_{t=0} - w_0)\delta\dot{w}|_{t=T} + (\dot{w}|_{t=0} - \dot{w}_0)\delta w|_{t=T}]dxdy = 0.$$

Observing the arbitrariness of $\delta w, (\delta w)_{,n}, \delta\dot{w}|_{t=T}, \delta w|_{t=T}$ and using the Titchmarsh theorem (see Leitman [28]) and the fundamental preliminary theorem (see Chien Weizhang [35]) of the calculus of variations, this yields Eq. (16), the boundary conditions (18.2) and the initial conditions (20).

The variational principle is essentially a simplified Gurtin's type variational principle, in which both the classical Cartesian bilinear form and the modern convolution bilinear form are used simultaneously. But the key to find the functional Π is the introduction of the structural function G_3 and the thermal function Φ.

4 The quasi-static analysis

As applications, we consider quasi-static responses of thermoviscoelastic FGPs under transverse mechanical loads and thermal loads. The functional Π (21) can be degenerated into

$$\Pi = -\frac{A}{2}\iint_{\Omega} (G_1+G_2)\otimes[(w_{,xx}+w_{,yy})*(w_{,xx}+w_{,yy})$$

$$+ 2G_1\otimes(w_{,xy}*w_{,xy} - w_{,xx}*w_{,yy})]dxdy$$

$$- B\iint_{\Omega} \Phi\otimes\psi*(w_{,xx}+w_{,yy})dxdy - \iint_{\Omega} q*wdxdy + \int_{\partial\Omega_\sigma} (\tilde{M}_n*w_{,n} - \tilde{V}_n*w)ds. \tag{25}$$

Letting Poisson's ratio $\gamma(t) = const$, basing on the relations between $\bar{E}(s)$ and $\bar{\gamma}(s)$ (see, Christensen [27])

$$\bar{\gamma}(s) = (\bar{G}_2(s) - \bar{G}_1(s))/s(\bar{G}_1(s) + 2\bar{G}_2(s)), \tag{26.1}$$

$$\bar{E}(s) = 3\bar{G}_1(s)\bar{G}_2(s)/(\bar{G}_1(s) + 2\bar{G}_2(s)), \tag{26.2}$$

it is easy to obtain

$$G_3(t) = \frac{\gamma}{1-\gamma}G_1(t), \quad G_1(t)+G_3(t) = \frac{1}{1-\gamma}G_1(t) = \frac{E(t)}{1-\gamma^2}, \quad \Phi(t) = \frac{1-2\gamma}{1-\gamma}\varphi(t), \tag{27}$$

where $E(t)$ is a uniaxial relaxation function. At the same time, we introduce the dimensionless variables and parameters as follows:

$$\xi = x/R_c, \quad \eta = y/R_c, \quad \zeta = z/h, \quad W = w/h, \quad \hat{\varphi} = \varphi/phi(0),$$

$$A' = 12A/h^3 = -12 \int_{-1/2}^{1/2} \zeta^2 f(\zeta) d\zeta, \quad B' = B/h^3 = -\int_{-1/2}^{1/2} \zeta k(\zeta)\psi'(\zeta) d\zeta,$$

$$e(\tau) = \frac{D(\tau)}{D(0)} = \frac{E(\tau)}{E(0)}, \quad Q(t) = \frac{R_c^4 q(t)}{D(0)h}, \quad \hat{\psi} = \frac{(1-2\gamma)\varphi(0)h\psi R_c^2}{(1-\gamma)D(0)}, \tag{28}$$

$$D(\tau) = \frac{E(\tau)h^3}{12(1-\gamma^2)}, \quad \tilde{V}_n^* = \frac{R_c^3 V_n^*}{D(0)h}, \quad \tilde{M}_n^* = \frac{R_c^2 \tilde{M}_n}{D(0)h}, \quad \tau = t/t_c, \; S = T/t_c,$$

where R_c is the characteristic length of the plate, and t_c is the characteristic time of the material, so the functional Π becomes

$$\Pi = -\frac{A'}{2}\iint_\Omega e \otimes [(W_{,\xi\xi} + W_{,\eta\eta}) * (W_{,\xi\xi} + W_{,\eta\eta}) + 2(1-\gamma)(W_{,\xi\eta} * W_{,\xi\eta} - W_{,\xi\xi} * W_{,\eta\eta})] d\xi d\eta$$

$$- B'\int_\Omega \hat{\varphi} \otimes \hat{\psi} * (W_{,\xi\xi} + W_{,\eta\eta}) d\xi d\eta - \iint_\Omega Q * W d\xi d\eta + \int_{\partial\Omega_\sigma} (\tilde{M}_n^* * W_{,n} - \tilde{V}_n^* * W) ds. \tag{29.1}$$

For plates only with the simply-supported or/and clamped edge, Π may be simplified as

$$\Pi = -\frac{A'}{2}\iint_\Omega e \otimes [(W_{,\xi\xi} + W_{,\eta\eta}) * (W_{,\xi\xi} + W_{,\eta\eta})] d\xi d\eta.$$

$$- B'\iint_\Omega \hat{\varphi} \otimes \hat{\psi} * (W_{,\xi\xi} + W_{,\eta\eta}) d\xi d\eta - \iint_\Omega Q * W d\xi d\eta. \tag{29.2}$$

The general solution of deformation for a simply supported plate can be written as

$$W(\xi, \eta, \tau, S) = C_{ijk}(S) \sin(i\pi\xi) \sin(j\pi\eta) L_k(\tau, S), \quad i \le m, j \le n, k \le r, \tag{30}$$

where $L_k(\tau, S)$ is an orthonormalization system of Legendre polynomials in the interval [0, S] given by

$$L_k(\tau, S) = \sqrt{\frac{2k+1}{S}} \frac{S^k}{2^{2k} k!} \frac{d^k}{d\tau^k} \left\{ \left[\left(\frac{2\tau}{S} - 1 \right)^2 - 1 \right]^k \right\}, \quad k = 0, 1, \ldots, r. \tag{31}$$

It is not difficult to prove that the Legendre polynomials have the following property:

$$L_k(\tau, S) * L_s(\tau, S) = (-1)^k \delta_{ks}, \tag{32}$$

where δ_{ks} is the Kronecker symbol and the repeated index k does not denote a summation. Observing the simply supported boundary conditions $\tilde{M}_n^* = \tilde{V}_n^* = 0$, substituting (30) into (29.2), we can get

$$C_{ijk}(S) = \frac{4Q(S) * L_k(S, S) - 4(i^2 + j^2) B' \hat{\varphi} \otimes \hat{\psi} * L_k(S, S)}{-1^{k+1} ij (i^2 + j^2)^2 \pi^6 A' e(S)} (1 - \cos i\pi)(1 - \cos j\pi). \tag{33}$$

During the above calculations, the uniformly distributed transverse load $Q(\tau)$ and the temperature field $\hat{\psi}(\xi, \eta, \tau)$ are known functions. Substituting (33) into (30), the maximal deflection of the thermoviscoelastic FGM square plate is obtained,

$$W_{\max}(\tau,S) = W(\xi,\eta,\tau,S)|_{\xi=\eta=1/2}$$
$$= \frac{4Q(S) * L_k(S,S) - 4(i^2+j^2)B'\hat{\varphi} \otimes \hat{\psi} * L_k(S,S)}{(-1)^{\frac{i+j}{2}+k} ij(i^2+j^2)^2 \pi^6 A' e(S)} L_k(\tau,S)(1-\cos i\pi)(1-\cos j\pi). \tag{34}$$

In computation, two numerical approaches can be used to obtain the deflection. One is called the method $\tau-S$, in which if the integral upper limit S is given, then the time history of the deflection $W_{\max}(\tau,S)$ is calculated by the formula (34). The other is called as method $S–S$, in which the different upper limit S is given to get the deflection $W_{\max}(S,S)$, and the time history of the deflection is plotted with $W_{\max}(S,S)$ vs. S. In the following computation the plate sizes are taken as $\xi=\eta=1$.

4.1 Reponses of homogenous thermoviscoelastic plates under transverse mechanical loads

Let the temperature variation function $\hat{\psi}(\xi,\eta,\tau)=0$ and the graded functions $f(\zeta)=k(\zeta)=1$, the problem can be degenerated into the case of homogenous viscoelastic plates (see Cheng and Zhang [20], Zhang and Cheng [21]). Figure 1 shows the comparison between the numerical results obtained by the above two methods and the exact solution presented by Cheng and Zhang [20], in which $Q(\tau)=1+e^{-0.05\tau}$, $e(\tau)=0.5+0.5e^{-0.06\tau}$, $m=n=5$, $r=7$. It can be seen from Fig. 1 that in the time interval [0,100] a large difference exists between the exact solution and the numerical result obtained by the method τ–S even if the Legendre polynomial term r is increased bigger enough. On the contrary, the numerical curve obtained by the method $S–S$ is very close to the exact solution. Hence the method $S–S$ is better than the method τ–S for capturing the initial response of the plate. Figure 2 reveals that the numerical results obtained by the two methods are all approximate to the exact result in the interval [100, 200]. But the numerical curve predicted by the method τ–S is more close to the analytical solution. So as for predicting the steady state response of the plate the method τ–S is better than method S–S.

Based on the initial and steady state analysis, the method $S–S$ is preferred in the following computation. The result is similar to the conclusion that the higher-order accuracy of DQ method at the end of a time step can be obtained if the Legendre and Radau grids are used (see Fung [37]). Figure 3 reveals that the numerical curves obtained by the method $S–S$ get more and more close to the exact solution with the increase of the Legendre polynomial term r.

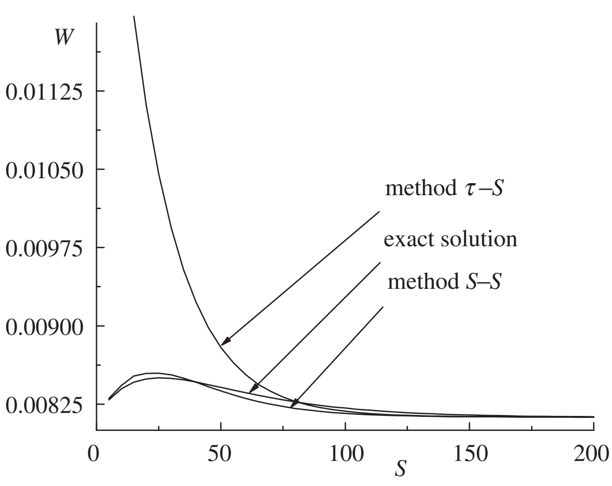

Fig. 1. The maximal deflection of the homogenous viscoelastic plate

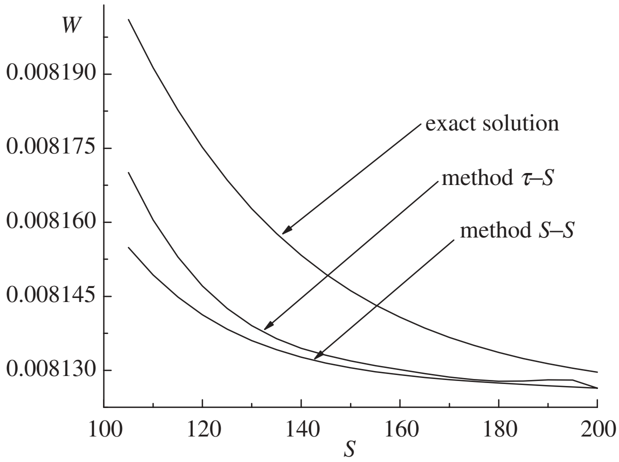

Fig. 2. The local magnification of Fig. 1

4.2 Reponses of thermoviscoelastic FGPs under thermal loads

Responses of thermoviscoelastic FGPs subjected to two types of temperature change across the thickness are studied. If the material is a standard linear solid, the relaxation function and the thermal function can be written as $e(\tau) = 0.5 + 0.5e^{-\alpha_1 \tau}$, $\hat{\varphi}(\tau) = 0.5 + 0.5e^{-\alpha_2 \tau}$, in which α_1 and α_2 are the reciprocal of the mechanical relaxation time and the reciprocal of the thermal relaxation time, respectively. The functionally gradient functions are $f(\zeta) = e^{-\beta_1 \zeta}$, $k(\zeta) = e^{\beta_2 \zeta}$, in which β_1 and β_2 are graded parameters of FGMs which show the heterogeneous effect along the thickness. In computation we take $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, the mechanical load $Q(\tau) = 0$, and the indices $m = n = 5$, $r = 7$.

First, a uniform temperature variation across the thickness is applied. Assume a uniform temperature variation across the thickness as

$$\hat{\theta} = \psi_0 \psi'(\zeta) \hat{\psi}(\xi, \eta, \tau), \quad \psi'(\zeta) = 1, \quad \hat{\psi}(\xi, \eta, \tau) = (1 - e^{-0.05\tau}). \tag{35}$$

Figure 4 shows the maximal deflection of the thermoviscoelastic FGP under the uniform temperature field when the graded parameter β changes from 0.3 to 0.5. Obviously the

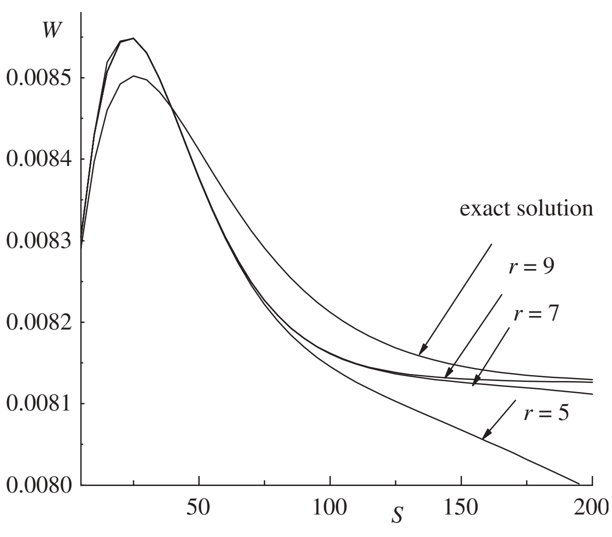

Fig. 3. The convergence of the method $S\text{–}S$

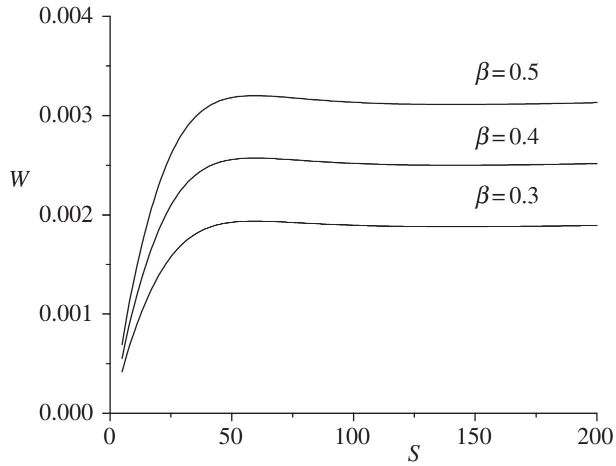

Fig. 4. The effect of the graded parameter β on the maximal deflection of FGP under the uniform temperature field

deflection increases with the increment of the graded parameter β. In other words, the greater the difference in material property across the thickness is, the bigger the deflection of the FGP under the uniform temperature field is. During the above calculation, let $\alpha = 0.06$, $\psi_0 = 10$.

Next a nonhomogeneous temperature variation across the plate thickness is applied. For simplicity, the temperature field is assumed as

$$\hat{\theta} = \psi'(\zeta)\hat{\psi}(\xi, \eta, \tau), \tag{36}$$

where the in-plane spatial part of the field is homogeneous and the temporal part of the field can be simulated by

$$\hat{\psi}(\xi, \eta, \tau) = (1 - e^{-0.05\tau}), \tag{37.1}$$

and the thickness part of the field $\psi'(\zeta)$ obeys the heat conduction equation and the boundary conditions (see, Eslami and Javaheri [38], Javaheri and Eslami [39])

$$\frac{d}{d\zeta}\left[\lambda(\zeta)\frac{d\psi'(\zeta)}{d\zeta}\right] = 0, \tag{37.2}$$

$$\zeta = -1/2, \;\; \psi' = \psi'_t, \quad \zeta = 1/2, \;\; \psi' = \psi'_b, \tag{37.3}$$

where ψ'_t and ψ'_b is the upside temperature and underside temperature of the plate, respectively. The coefficient of thermal conduction λ is a function of the thickness direction ζ. The solution for the temperature distribution across the FGP thickness becomes

$$\psi'(\zeta) = \psi'_b\left[\Delta\psi'\frac{K(\zeta) - K(1/2)}{K(-1/2) - K(1/2)} + 1\right], \tag{38.1}$$

in which

$$K(\zeta) = \int \lambda^{-1}(\zeta)d\zeta, \qquad \Delta\psi' = (\psi'_t - \psi'_b)/\psi'_b. \tag{38.2}$$

If $\lambda(\zeta) = 1$, then (38.1) is reduced to the linear temperature distribution of homogeneous plates,

$$\psi'(\zeta) = \psi'_b[\Delta\psi'(1/2 - \zeta) + 1]. \tag{39}$$

If $\lambda(\zeta) = e^{\beta_3\zeta}$, then (38.1) can be transformed into the nonlinear temperature distribution of FGPs,

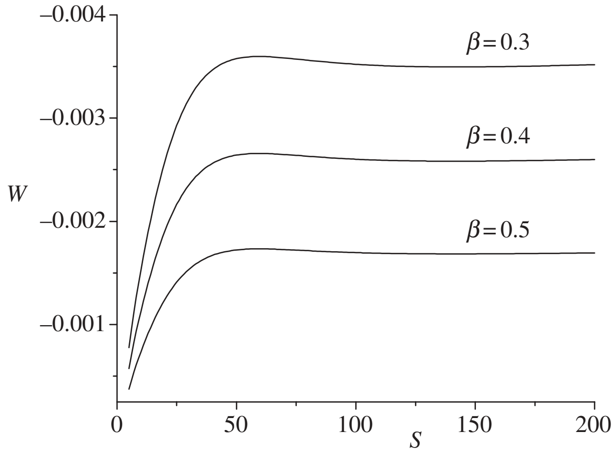

Fig. 5. The effect of the graded parameter β on the maximal deflection of FGP subject to nonlinear temperature change across the thickness

$$\psi' = \psi'_b \left[\Delta\psi' \frac{e^{-\beta_3 \zeta} - e^{-\beta_3/2}}{e^{\beta_3/2} - e^{-\beta_3/2}} + 1 \right], \tag{40}$$

in which β_3 is the graded parameter of the thermal conductivity coefficient.

Figure 5 shows the effect of the graded parameter β on the maximal deformation of the thermoviscoelastic FGP under the nonlinear temperature change (40) across the thickness when $\alpha = 0.06$, $\psi'_b = 10$, $\Delta\psi' = 1$, and $\beta_3 = \beta$. Obviously large differences exist between responses to the uniform temperature field (35) and responses to the nonlinear temperature field (40). Negative deflection shows that the deformed FGP locates on the top of the undeformed mid-plane. And the increase of the graded parameter β will help to the rigidity of FGPs under nonlinear temperature fields.

Figure 6 shows the effect of the temperature difference $\Delta\psi'$ on the maximal deflection of thermoviscoelastic FGPs subject to the nonlinear temperature change (40) across the thickness when $\alpha = 0.06$, $\psi'_b = 10$, and $\beta_3 = \beta = 0.5$. The deflection becomes larger and larger when the temperature difference $\Delta\psi'$ enhances from 1 to 3.

Finally, the effect of the relaxation time on the deflection of the thermoviscoelastic FGP is studied when the temperature field is uniform or nonlinearly changes across the thickness. The relaxation time t_{cr} and the parameter α are related by $\alpha = 1/t_{cr}$. Figures 7 and 8 show the effect

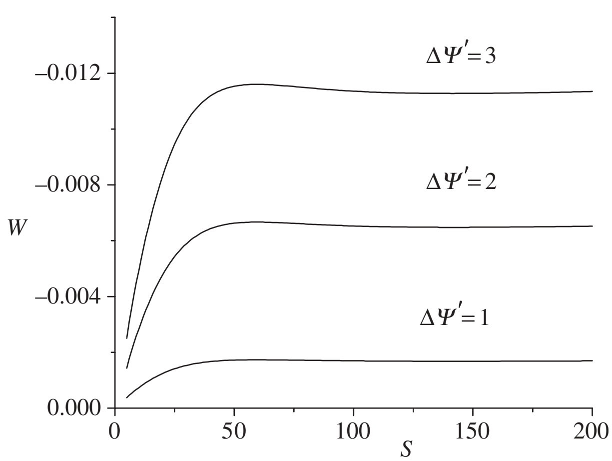

Fig. 6. The effect of $\Delta\psi'$ on the maximal deflection of FGP subject to nonlinear temperature change across the thickness

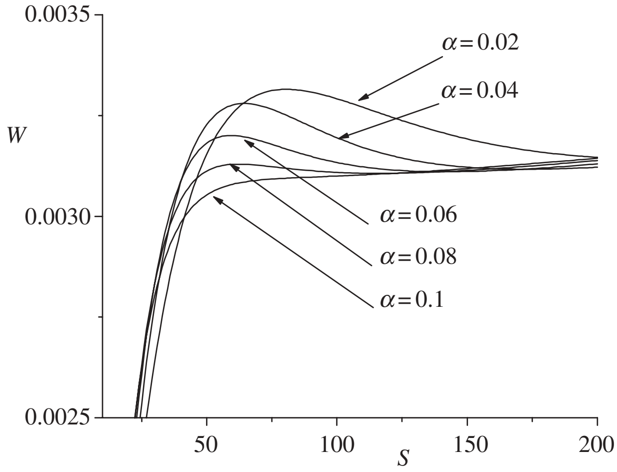

Fig. 7. The effect of the parameter α on the maximal deflection of the FGP subject to the uniform temperature field

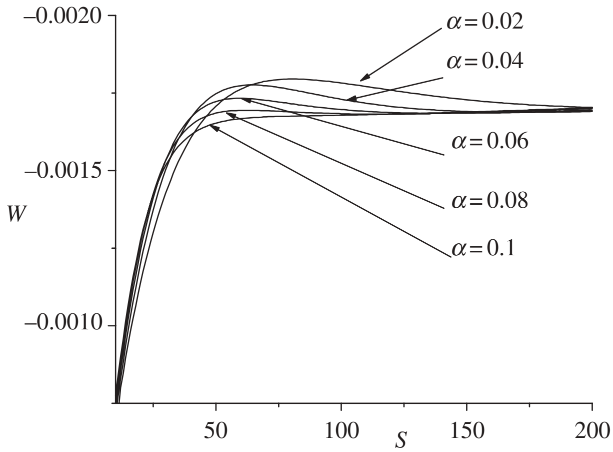

Fig. 8. The effect of the parameter α on the maximal deflection of the FGP subject to the nonlinear temperature field

of the parameter α on the deflection of the thermoviscoelastic FGP subject to the uniform temperature field (35) or nonlinear temperature field (40) across the thickness. The larger the parameter α is, the earlier the thermoviscoelastic FGP attains its steady state. That is to say, the thermoviscoelastic FGP reaches its final state more quickly with the decrease of the relaxation time. In Fig. 7 $\psi_0 = 10$, $\beta = 0.5$, and in Fig. 8 $\psi_b' = 10$, $\Delta\psi' = 1$, and $\beta_3 = \beta = 0.5$.

5 Conclusions

The mathematical model (16), (18), (20) of thermoviscoelastic FGPs is set up. The corresponding simplified Gurtin's type variational principle (21) is presented by means of modern convolution bilinear forms as well as classical Cartesian bilinear forms. The influence of the temperature variation and graded parameters on quasi-static responses of thermoviscoelastic FGPs are investigated by combining the Ritz method in the spatial domain and the Legendre interpolation method in the temporal domain. By using the property of the Legendre series, two numerical approaches in the temporal domain are introduced to calculate the deformation.

Numerical results show that the method $S-S$ is superior to the method $\tau-S$. For the thermoviscoelastic FGP subjected to the uniform temperature field, its steady state locates downside the undeformed mid-plane, and enhancing the graded parameter β will increase the deformation. But for the thermoviscoelastic FGP subject to the nonlinear temperature field across the thickness, its steady state locates upside the undeformed mid-plane, and enhancing the graded parameter β will reduce the deformation. When the temperature difference enhances, the deflection of the thermoviscoelastic FGP becomes larger and larger. So the temperature variation and graded parameter play a very important role in quasi-static responses of FGPs. The thermoviscoelastic FGP reaches its steady state earlier when the relaxation time becomes smaller.

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