Acta Mechanica 180, 117–132 (2005)
DOI 10.1007/s00707-005-0260-0

Acta Mechanica
Printed in Austria

Homotopy analysis of Couette and Poiseuille flows for fourth-grade fluids

A. M. Siddiqui, York, Pennsylvania, and **M. Ahmed, S. Islam** and **Q. K. Ghori,** Islamabad, Pakistan

Received September 16, 2004; revised April 15, 2005
Published online: August 25, 2005

Summary. The steady flow of a fluid, called a fourth-grade fluid, between two parallel plates is considered. Depending upon the relative motion of the plates we analyze four types of flows: Couette flow, plug flow, Poiseuille flow and generalized Couette flow. In each case, the nonlinear differential equation describing the velocity field is solved using perturbation technique and homotopy analysis method. The pressure distribution is also found. It is observed that the homotopy analysis method is more efficient and flexible than the perturbation technique.

1 Introduction

Couette flows are generated by the action of boundaries in relative motion. Typical examples of commonly used boundaries are two parallel plates or two coaxial cylinders or a flat plate and a convex cone with its apex touching the plate. In this paper, we consider steady plane Couette flows obtained in the region between parallel plates sliding with respect to each other.

The fluid between the plates is of fourth grade which is a non–Newtonian fluid and fails to obey Newton's viscosity law. Such a fluid cannot be described as simply as Newtonian fluids. Non–Newtonian fluids are not of rare occurrence. As remarked in [1], they are to be found close at hand everywhere. Many solid-liquid and liquid-liquid suspensions, solutions of macromolecules, molten plastics, mammalian whole blood and synovial fluid (fluid found in health joints) are treated as non–Newtonian fluids. The study of such fluids is therefore of wide interest and significance for researchers in biological and non-biological fields.

The classification of non–Newtonian fluids as second grade or higher grade fluids is based on the differential type of constitutive equations involving the Rivlin-Ericksen tensor. It has been shown by many authors that in several problems in which the flow is slow enough, in the visco-elastic sense, the results given by Oldroyd's constitutive equations are substantially those of the second or third order Rivlin-Ericksen constitutive equation. As remarked in [2], it seems reasonable to use second or third-order Rivlin-Ericksen equations in carrying out the calculations. This is particularly so in view of the fact that the calculations would generally be still simpler. For this reason, in the present paper we consider a fourth-grade fluid.

In obtaining plane Couette flows using a fourth grade fluid, four different problems depending on the relative motion of the sliding plates are considered: (i) one plate is moving and the other is at rest, giving simple Couette flow; (ii) both plates are moving with same speed in the same direction, giving plug flow; (iii) both plates are stationary, and the fluid is forced under constant pressure gradient, producing Poiseuille flow, and (iv) either of the two plates is moving with constant speed in the presence of an external pressure gradient, generating generalized plane Couette flow.

The steady plane flow problems of a fourth grade fluid are usually solved by perturbation techniques. In [2], Erdogan used such a technique to solve the nonlinear differential equation describing the steady pipe flow of a fourth-grade fluid. Recently, the homotopy analysis method has been widely used to tackle nonlinear partial differential equations. Regarding the homotopy analysis method and its applications we refer to [3] and [4]. Recently, Ayub, Rashid and Hayat [5] applied this method to obtain some exact flows of a third-grade fluid past a porous plate. In another paper [6], Hayat, Khan and Ayub used it to give some explicit analytic solutions of an Oldroyd 6-constant fluid.

In this paper, we use the perturbation method as well as the homotopy analysis method to obtain the solutions of steady plane Couette flows, and the corresponding results are compared. The organization of the paper is as follows: Section 2 contains the basic equations. Section 3 gives the solution of the four problems of Couette flows using perturbation technique. In Sect. 4.1, the basic idea of the homotopy analysis method is discussed and the method is then used in Sects. 4.2 to 4.5 to analyze the four problems of Couette flows. Section 5 is reserved for conclusions.

2 Basic equations

Let us consider two infinite parallel plates separated by a constant distance $2d$. We use an (x, y) coordinate system, where x is in the direction of motion of the fluid between the plates and the y-axis is perpendicular to the plates.

The basic equations governing the motion of an incompressible fluid, neglecting the thermal effects and body forces, are

$$\text{div } \mathbf{v} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \text{ div } \boldsymbol{\tau}, \tag{2}$$

where ρ is the constant density, $\mathbf{v}$ the velocity vector, p the pressure, $\boldsymbol{\tau}$ the stress tensor, and $\frac{D}{Dt}$ denotes the material derivative.

As discussed in [7]–[9], the stress tensor $\boldsymbol{\tau}$ defining a fourth-grade fluid is given by

$$\boldsymbol{\tau} = \sum_{i=1}^{4} \mathbf{S}_i, \tag{3}$$

where

$$\mathbf{S}_1 = \mu\mathbf{A}_1, \quad \mathbf{S}_2 = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

$$\mathbf{S}_3 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\text{tr } \mathbf{A}_2)\mathbf{A}_1,$$

$$\mathbf{S}_4 = \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3(A_2^2) + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2)$$
$$+ \gamma_5((\text{ tr}\mathbf{A}_2)\mathbf{A}_2) + \gamma_6(\text{ tr } \mathbf{A}_2)\mathbf{A}_1^2 + (\gamma_7 \text{ tr } \mathbf{A}_3 + \gamma_8 \text{ tr } (\mathbf{A}_2\mathbf{A}_1))A_1,$$

where μ is the coefficient of viscosity and α_1, α_2, β_1, β_2, β_3, γ_1, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7 and γ_8 are material constants. The Rivlin-Ericksen tensors $\mathbf{A}_n$ are defined by

$\mathbf{A}_o = I,$ the identity tensor,

$$\text{and } \mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + A_{n-1}(\nabla \mathbf{v}) + (\nabla \mathbf{v})^t \mathbf{A}_{n-1}, \quad n \geq 1. \tag{4}$$

Since the flow is one-dimensional, we assume that

$$\mathbf{v} = (u(y), 0, 0). \tag{5}$$

For steady one-dimensional flow of a fourth grade fluid, Eq. (2) in component form yields:
x-component:

$$-\frac{dp}{dx} + \mu \frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = 0. \tag{6}$$

y-component:

$$-\frac{dp}{dy} + (2\alpha_1 + \alpha_2)\frac{d}{dy}\left(\left(\frac{du}{dy}\right)^2\right) + 4\left(\gamma_3 + \gamma_4 + \gamma_5 + \frac{\gamma_6}{2}\right)\frac{d}{dy}\left(\frac{du}{dy}\right)^4 = 0. \tag{7}$$

Introducing the generalized pressure p^* by the relation

$$p^* = -p + (2\alpha_1 + \alpha_2)\left(\frac{du}{dy}\right)^2 + 4\left(\gamma_3 + \gamma_4 + \gamma_5 + \frac{\gamma_6}{2}\right)\left(\frac{du}{dy}\right)^4 \tag{8}$$

and substituting p^* in Eq. (7), we find that

$$\frac{dp^*}{dy} = 0, \tag{9}$$

showing that $p^* = p^*(x)$. Consequently, Eq. (6) reduces to the single equation

$$-\frac{dp^*}{dx} + \mu \frac{d^2u}{dy^2} + 6(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} = 0. \tag{10}$$

This is a second-order nonlinear ordinary differential equation. We note that in Eq. (10) no contribution comes from $\mathbf{S}_2$ and $\mathbf{S}_4$.

3 Perturbation method

The perturbation method is the traditional technique to solve nonlinear problems. The basic theme of the method is to expand the required solution in powers of the parameter ϵ (small or large), present in the differential equation, substitute the assumed solution in the equation and equate the coefficients of like powers of ϵ on the both sides of the equation to obtain a system of differential equations and solve them using the initial/boundary conditions.

In the following, we use the perturbation method to solve Eq. (10) with boundary conditions corresponding to different problems of Couette flow.

3.1 Plane Couette flow problem

Let us assume that of the two plates the upper plate moves with constant speed U, while the lower plate remains at rest and that there is no pressure gradient. Then Eq. (10) governing the flow of the fourth-grade fluid between the plates becomes

$$\mu\frac{d^2u}{dy^2} + 6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = 0. \tag{11}$$

The corresponding boundary conditions are

$$\begin{aligned} u(y) &= 0 \quad \text{at } y=0, \\ u(y) &= U \quad \text{at } y=2d. \end{aligned} \tag{12}$$

Taking $\epsilon = 6(\beta_2+\beta_3)/\mu$ as small parameter, we consider the perturbation expansion

$$u(y) \sim u_\circ(y) + \epsilon u_1(y) + \epsilon^2 u_2(y) + \dots\ . \tag{13}$$

Using Eq. (13) in Eqs. (11), (12) and then equating like powers of ϵ we obtain the following problems:

Zeroth-order problem

$$\frac{d^2u_\text{o}}{dy^2} = 0 \tag{14}$$

with boundary conditions

$$\begin{aligned} u_\text{o}(y) &= 0 \quad \text{at } y=0, \\ u_\text{o}(y) &= U \quad \text{at } y=2d, \end{aligned} \tag{15}$$

the corresponding solution being

$$u_\text{o}(y) = \frac{U}{2d}y. \tag{16}$$

First-order problem

$$\frac{d^2u_1}{dy^2} + \left(\frac{du_\text{o}}{dy}\right)^2\frac{d^2u_\text{o}}{dy^2} = 0 \tag{17}$$

subject to the boundary conditions

$$\begin{aligned} u_1(y) &= 0 \quad \text{at } y=0, \\ u_1(y) &= 0 \quad \text{at } y=2d. \end{aligned} \tag{18}$$

Substituting u_o in Eq. (17), the solution of (17) satisfying conditions (18) is given by

$$u_1(y) = 0. \tag{19}$$

Second-order problem

$$\frac{d^2u_2}{dy^2} + \left(\frac{du_\text{o}}{dy}\right)^2\frac{d^2u_1}{dy^2} + 2\frac{du_\text{o}}{dy}\frac{du_1}{dy}\frac{d^2u_\text{o}}{dy^2} = 0 \tag{20}$$

the boundary conditions being

$$\begin{aligned} u_2(y) &= 0 \quad \text{at } y=0, \\ u_2(y) &= 0 \quad \text{at } y=2d. \end{aligned} \tag{21}$$

Substituting for u_o and u_1 from Eqs. (16) and (19), respectively, and using Eq. (21) we obtain

$$u_2(y) = 0. \tag{22}$$

Thus, the perturbation solution up to second order is given by

$$u(y) = \frac{U}{2d}y. \tag{23}$$

We remark that the solution (23) for the plane Couette flow using a fourth-grade fluid comes out to be the same as for the Newtonian fluid and also for the Oldroyd 6-constant model obtained by Hayat et al in [6].

3.2 Plug flow problem

For plug flow we assume that both the plates move with constant speed U, and the pressure gradient is zero. Therefore the flow is due to the motion of both the plates. In this case the equation governing the motion will remain the same as Eq. (11), however the boundary conditions, with x-axis in the lower plate, will take the form

$$\begin{aligned} u(y) &= U \quad \text{at } y = 0, \\ u(y) &= U \quad \text{at } y = 2d. \end{aligned} \tag{24}$$

Here we take ϵ as in Sect. 3.1 and substitute Eq. (13) into Eq. (11) using the boundary conditions (24). Then we obtain:
Zeroth-order problem

$$\frac{d^2u_o}{dy^2} = 0 \tag{25}$$

and the boundary conditions are

$$\begin{aligned} u_o(y) &= U \quad \text{at } y = 0, \\ u_o(y) &= U \quad \text{at } y = 2d. \end{aligned} \tag{26}$$

The solution of the zeroth-order problem becomes

$$u_o(y) = U. \tag{27}$$

First-order problem

$$\frac{d^2u_1}{dy^2} + \left(\frac{du_o}{dy}\right)^2 \frac{d^2u_o}{dy^2} = 0 \tag{28}$$

with the boundary conditions

$$\begin{aligned} u_1(y) &= 0 \quad \text{at } y = 0, \\ u_1(y) &= 0 \quad \text{at } y = 2d. \end{aligned} \tag{29}$$

Making use of u_o in (28) and solving the resulting equation with conditions (29) for first-order solution, we get

$$u_1(y) = 0. \tag{30}$$

Second-order problem

$$\frac{d^2u_2}{dy^2} + \left(\frac{du_o}{dy}\right)^2 \frac{d^2u_1}{dy^2} + 2\frac{du_o}{dy}\frac{du_1}{dy}\frac{d^2u_o}{dy^2} = 0 \tag{31}$$

under the boundary conditions

$$\begin{aligned} u_2(y) &= 0 \quad \text{at } y = 0, \\ u_2(y) &= 0 \quad \text{at } y = 2d. \end{aligned} \tag{32}$$

Substituting the zeroth-and first-order solutions into Eq. (31) and solving the resulting equation which satisfies the boundary conditions (32), we obtain

$$u_2(y) = 0. \tag{33}$$

Thus, the perturbation solution of the plug flow problem up to second order can be written as

$$u(y) = U. \tag{34}$$

3.3 Fully developed plane Poiseuille flow problem

For fully developed plane Poiseuille flow, we assume that the upper plate and the lower plates are stationary, and the fluid is forced under constant pressure gradient. Then the equation of motion (10) in the presence of the constant pressure gradient A takes the form

$$\frac{d^2u}{dy^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = \frac{A}{\mu}, \qquad \text{where } A = \frac{dp^*}{dx}. \tag{35}$$

Taking the x-axis midway between the plates, the boundary conditions are

$$\begin{aligned} u(y) &= 0 \quad \text{at } y = -d, \\ u(y) &= 0 \quad \text{at } y = d. \end{aligned} \tag{36}$$

Proceeding as before, we obtain:
Zeroth-order problem

$$\frac{d^2u_o}{dy^2} = \frac{A}{\mu} \tag{37}$$

along with the boundary conditions

$$\begin{aligned} u_o(y) &= 0 \quad \text{at } y = -d, \\ u_o(y) &= 0 \quad \text{at } y = d. \end{aligned} \tag{38}$$

We obtain the solution of zeroth-order problem given by

$$u_o(y) = \frac{A}{2\mu}\left[y^2 - d^2\right]. \tag{39}$$

First-order problem

$$\frac{d^2u_1}{dy^2} + \left(\frac{du_o}{dy}\right)^2\frac{d^2u_o}{dy^2} = 0 \tag{40}$$

under the boundary conditions

$$\begin{aligned} u_1(y) &= 0 \quad \text{at } y = -d, \\ u_1(y) &= 0 \quad \text{at } y = d. \end{aligned} \tag{41}$$

When we substitute u_o in Eq. (40), the first-order solution is given by

$$u_1(y) = \frac{A^3}{12\mu^3}\left[d^4 - y^4\right]. \tag{42}$$

Second-order problem
The second-order problem along with the boundary conditions is given by

$$\frac{d^2u_2}{dy^2}+\left(\frac{du_o}{dy}\right)^2\frac{d^2u_1}{dy^2}+2\frac{du_o}{dy}\frac{du_1}{dy}\frac{d^2u_o}{dy^2}=0, \tag{43}$$

$$\begin{aligned} u_2(y)&=0 \quad \text{at } y=-d,\\ u_2(y)&=0 \quad \text{at } y=d. \end{aligned} \tag{44}$$

Using the expressions of u_o and u_1 in Eq. (43), we obtain the second-order solution in the form

$$u_2(y)=\frac{1}{18}\frac{A^5}{\mu^5}\left[y^6-d^6\right]. \tag{45}$$

Therefore, the solution of the fully developed plane Poiseuille flow problem up to second order takes the form

$$u(y)=\frac{A}{2\mu}\left[y^2-d^2\right]+\epsilon\left[\frac{A^3}{12\mu^3}\left[d^4-y^4\right]\right]+\epsilon^2\left[\frac{1}{18}\frac{A^5}{\mu^5}\left[y^6-d^6\right]\right]. \tag{46}$$

3.4 Generalized plane Couette flow problem

In generalized plane Couette flow either of the two plates is moving with constant speed U, and an external pressure gradient A is present. In contrast to the previous problem this flow is not symmetric with respect to the central line. So we take the origin on the lower plate. The corresponding equation governing the motion of the fluid with pressure gradient A is

$$\frac{d^2u}{dy^2}+\frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=\frac{A}{\mu}, \quad \text{where } A=\frac{dp^*}{dx}, \tag{47}$$

subject to the boundary conditions

$$\begin{aligned} u(y)&=0 \quad \text{at } y=0,\\ u(y)&=U \quad \text{at } y=2d. \end{aligned} \tag{48}$$

As in the earlier problems, the zeroth-order, first-order and second-order problems are:
Zeroth-order problem

$$\frac{d^2u_o}{dy^2}=\frac{A}{\mu} \tag{49}$$

along with the boundary conditions

$$\begin{aligned} u_o(y)&=0 \quad \text{at } y=0,\\ u_o(y)&=U \quad \text{at } y=2d. \end{aligned} \tag{50}$$

First-order problem

$$\frac{d^2u_1}{dy^2}+\left(\frac{du_o}{dy}\right)^2\frac{d^2u_o}{dy^2}=0 \tag{51}$$

under the boundary conditions

$$\begin{aligned} u_1(y)&=0 \quad \text{at } y=0,\\ u_1(y)&=0 \quad \text{at } y=2d. \end{aligned} \tag{52}$$

Second-order problem
The second-order problem along with the boundary conditions is given by

$$\frac{d^2u_2}{dy^2} + \left(\frac{du_o}{dy}\right)^2 \frac{d^2u_1}{dy^2} + 2\frac{du_o}{dy}\frac{du_1}{dy}\frac{d^2u_o}{dy^2} = 0, \tag{53}$$

$$\begin{aligned} u_2(y) &= 0 \quad \text{at } y = 0, \\ u_2(y) &= 0 \quad \text{at } y = 2d. \end{aligned} \tag{54}$$

The corresponding solutions are

$$u_o(y) = \frac{U}{2d}y - \frac{A}{2\mu}\left(2dy - y^2\right), \tag{55}$$

$$u_1(y) = -\frac{1}{12}\frac{A^3}{\mu^3}y^4 + Gy^3 - Hy^2 + My, \tag{56}$$

where

$$G = \frac{1}{6}\frac{A^3}{\mu^3} - \frac{1}{6}\frac{A^2}{\mu^2}\frac{U}{d},$$

$$H = \frac{1}{2}\frac{A^3}{\mu^3}d^2 + \frac{1}{2}\frac{A^2}{\mu^2}U - \frac{1}{4}\frac{A}{\mu}\frac{U^2}{d},$$

$$M = 8\frac{A^3}{\mu^3}d^3 - \frac{1}{3}\frac{A^2}{\mu^2}Ud + \frac{A}{2\mu}\frac{U^2}{d},$$

and

$$u_2(y) = -\frac{1}{18}\frac{A^5}{\mu^5}y^6 - ly^5 + my^4 - ny^3 + k_1y^2 - k_2y, \tag{57}$$

where

$$l = \frac{1}{3}\frac{A^5}{\mu^5}d - \frac{1}{6}\frac{A^4}{\mu^4}\frac{U}{d},$$

$$m = \frac{5}{6}\frac{A^5}{\mu^5}d^2 - \frac{5}{6}\frac{A^4}{\mu^4}U + \frac{5}{24}\frac{A^3}{\mu^3}\frac{U^2}{d^2},$$

$$n = \frac{11}{3}\frac{A^5}{\mu^5}d^3 - \frac{29}{18}\frac{A^4}{\mu^4}Ud + \frac{11}{12}\frac{A^3}{\mu^3}\frac{U^2}{d} - \frac{A^2}{8\mu^2}\frac{U^3}{d^3},$$

$$k_1 = \frac{17}{2}\frac{A^5}{\mu^5}d^4 - \frac{16}{3}\frac{A^4}{\mu^4}Ud^2 + \frac{17}{12}\frac{A^3}{\mu^3}U^2 - \frac{A^2}{2\mu^2}\frac{U^3}{d^2} + \frac{A}{32\mu}\frac{U^4}{d^4},$$

$$k_2 = \frac{13}{24}\frac{A^5}{\mu^5}d^5 - \frac{74}{9}\frac{A^4}{\mu^4}Ud^3 + \frac{5}{6}\frac{A^3}{\mu^3}U^2d - \frac{A^2}{2\mu^2}\frac{U^3}{d} + \frac{1}{16}\frac{A}{\mu}\frac{U^4}{d^3}.$$

Thus, the final solution obtained by the perturbation method up to second order is

$$\begin{aligned} u(y) = {} & \frac{U}{2d}y - \frac{A}{2\mu}\left(2dy - y^2\right) + \epsilon\left[-\frac{1}{12}\frac{A^3}{\mu^3}y^4 + Gy^3 - Hy^2 + My\right] \\ & + \epsilon^2\left[-\frac{1}{18}\frac{A^5}{\mu^5}y^6 - ly^5 + my^4 - ny^3 + k_1y^2 - k_2y\right]. \end{aligned} \tag{58}$$

4 Homotopy analysis method

4.1 Basic idea

To explain the basic idea of the homotopy analysis method, we consider the differential equation

$$\mathscr{A}[u(y)] = 0, \tag{59}$$

where $\mathscr{A}$ is a nonlinear operator and $u(y)$ is an unknown function of the independent variable y. Let $u_o(y)$ denote an initial approximation of $u(y)$ and let $\mathscr{L}$ denote the auxiliary linear operator with the property

$$\mathscr{L}f = 0 \quad \text{when}\, f = 0. \tag{60}$$

We then construct the so-called homotopy

$$\mathscr{H}[\phi(y;q);q] = (1-q)\mathscr{L}[\phi(y;q) - u_o(y)] + q\mathscr{A}[\phi(y;q)], \tag{61}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(y;q)$ a function of y and q. When $q = 0$ then Eq. (61) takes the form

$$\mathscr{H}[\phi(y;q);q]|_{q=0} = \mathscr{L}[\phi(y;0) - u_o(y)].$$

From Eq. (60) it follows that

$$\phi(y;0) = u_o(y)$$

is the solution of

$$\mathscr{H}[\phi(y;q);q]|_{q=0} = 0.$$

Again, when $q = 1$, Eq. (61) shows that

$$\mathscr{H}[\phi(y;q);q]|_{q=1} = \mathscr{A}[\phi(y;1)],$$

and therefore from Eq. (59) it follows that

$$\phi(y;1) = u(y)$$

is the solution of

$$\mathscr{H}[\phi(y;q);q]|_{q=1} = 0.$$

Thus, when the embedding parameter q varies from 0 to 1, the solution $\phi(y;q)$ of the equation

$$\mathscr{H}[\phi(y;q);q] = 0,$$

which depends upon the embedding parameter q, varies from the initial approximation $u_o(y)$ to the solution $u(y)$of Eq. (59). In topology, Eq. (61) describing a continuous variation is called deformation.

In what follows we revisit the Couette flow problems discussed in Sect. 3 and apply the homotopy analysis method to obtain their solutions.

4.2 Plane Couette flow problem

As discussed in Sect. 3.1, Eq. (11) governs the plane Couette flow subject to boundary conditions (12). In this case, Eq. (61) becomes

$$(1-q)\mathscr{L}[\widetilde{u}(y,q) - u_o(y)] = qh\left[\frac{\partial^2\widetilde{u}(y,q)}{\partial y^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{\partial\widetilde{u}(y,q)}{\partial y}\right)^2 \frac{\partial^2\widetilde{u}(y,q)}{\partial y^2}\right]. \tag{62}$$

The boundary conditions (12) yield

$$\widetilde{u}(0,q)=0,\quad \widetilde{u}(2d,q)=U. \tag{63}$$

The system (62) and (63) constructs the homotopy and for brevity we call it the zeroth-order deformation equations.

We choose

$$\mathscr{L}=\frac{\partial^2}{\partial y^2} \tag{64}$$

as an auxiliary linear operator, and

$$u_{\mathrm{o}}(y)=\frac{U}{2d}y \tag{65}$$

as the initial guess which satisfies the boundary conditions (12); h is an auxiliary parameter and q is an embedding parameter such that $q\in[0,1]$. In Eq. (62) we set $q=0$ to obtain

$$\widetilde{u}(y,0)=u_{\mathrm{o}}(y), \tag{66}$$

and $q=1$ to obtain

$$\widetilde{u}(y,1)=u(y). \tag{67}$$

By virtue of Eqs. (66) and (67), the variation of q from 0 to 1 is just the continuous variation of $\widetilde{u}(y,q)$ from the initial guess approximation $u_{\mathrm{o}}(y)$ to the unknown solution $u(y)$ of (11) subject to (12).

Assume that the deformation $\widetilde{u}(y,q)$ governed by Eqs. (62) and (63) is smooth enough, so that

$$u_{\mathrm{o}}^{(k)}(y)=\left.\frac{\partial^k\widetilde{u}(y,q)}{\partial q^k}\right|_{q=0},\quad k\geq 1, \tag{68}$$

namely, the k-th order deformation derivative exists. Then according to Eq. (66) and Taylor's formula, we have

$$\widetilde{u}(y,q)=u_{\mathrm{o}}(y)+\sum_{k=1}^{\infty}\left[\frac{u_{\mathrm{o}}^{(k)}(y)}{k!}\right]q^k. \tag{69}$$

Defining

$$u_k(y)=\frac{u_{\mathrm{o}}^{(k)}(y)}{k!} \tag{70}$$

and using Eqs. (67), (69) and (70) we get at $q=1$

$$u(y)=\sum_{k=0}^{\infty}u_k(y), \tag{71}$$

which gives the relationship between the initial guess approximation $u_{\mathrm{o}}(y)$ and the unknown solution of $u(y)$. Now differentiating the zeroth-order deformation (62) and (63) k times with respect to q and then setting $q=0$ we have

$$L[u_k(y)-\chi_k u_{k-1}(y)]=hR_k(y),\quad k\geq 1, \tag{72}$$

with the boundary conditions

$$u_k(0)=u_k(2d)=0. \tag{73}$$

Here

$$R_k(y) = u''_{k-1} + \frac{6(\beta_2+\beta_3)}{\mu}\sum_{j=0}^{k-1} u''_{k-1-j}\sum_{i=0}^{j} u'_i u'_{j-i} \tag{74}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k \geq 2, \end{cases} \tag{75}$$

primes denote the derivative with respect to y. We call Eqs. (72) and (73) k-th order deformation equations ($k \geq 1$). For the first-order solution of the problem with $k = 1$ in (72) and (73), we obtain the corresponding solution

$$u_1(y) = 0. \tag{76}$$

Similarly, setting $k = 2$, the second-order solution obtained by homotopy analysis method is

$$u_2(y) = 0. \tag{77}$$

Finally, the homotopy solution up to second-order of the Couette flow problem with fourth-grade fluid is

$$u(y) = \frac{U}{2d}y. \tag{78}$$

Here we can see that the homotopy analysis method solution does not involve the auxiliary parameter h, and the solutions from the two methods are in complete agreement.

4.3 Plug flow problem

Here we apply the homotopy analysis technique to solve the problem (11) with boundary conditions (24). The zeroth-order deformation for this case is again of the form of Eq. (62):

$$(1-q)\mathscr{L}[\widetilde{u}(y,q) - u_o(y)] = qh\left[\frac{\partial^2\widetilde{u}(y,q)}{\partial y^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{\partial\widetilde{u}(y,q)}{\partial y}\right)^2\frac{\partial^2\widetilde{u}(y,q)}{\partial y^2}\right] \tag{79}$$

with the boundary conditions

$$\widetilde{u}(0,q) = U, \quad \widetilde{u}(2d,q) = U. \tag{80}$$

As before, let

$$\mathscr{L} = \frac{\partial^2}{\partial y^2} \tag{81}$$

be an auxiliary linear operator and

$$u_o(y) = U \tag{82}$$

an initial guess approximation which satisfies the conditions (24), h an auxiliary parameter, q an embedding parameter such that $q \in [0,1]$. From Eqs. (79) and (80) we observe that the k-th order deformation equations for the plug flow problem will be the same as in the case of plane Couette flow. Proceeding as before, we obtain first-and second-order homotopy solutions in the form

$$u_1(y) = 0 \tag{83}$$

and

$$u_2(y) = 0, \tag{84}$$

giving the homotopy analysis solution up to second order in the form

$$u(y) = U. \tag{85}$$

The solution (85) for the plug flow problem is the same as obtained in Sect. 3.2 by the perturbation method.

4.4 Fully developed plane Poiseuille flow problem

To apply the homotopy analysis method to the problem (35) subject to conditions (36), we first select an auxiliary linear operator

$$\mathscr{L} = \frac{\partial^2}{\partial y^2}. \tag{86}$$

Then, we construct for Eq. (35) the family of differential equations giving the zeroth-order deformation, namely

$$(1-q)\mathscr{L}[\widetilde{u}(y,q) - u_o(y)] = qh\left[\frac{\partial^2\widetilde{u}(y,q)}{\partial y^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{\partial\widetilde{u}(y,q)}{\partial y}\right)^2\frac{\partial^2\widetilde{u}(y,q)}{\partial y^2} - \frac{A}{\mu}\right] \tag{87}$$

together with the boundary conditions

$$\widetilde{u}(-d,q) = 0, \quad \widetilde{u}(d,q) = 0. \tag{88}$$

In view of the boundary conditions (36), we choose

$$u_o(y) = \frac{A}{2\mu}\left[y^2 - d^2\right] \tag{89}$$

as initial guess approximation. When $q = 0$, Eqs. (87) and (88) give

$$\widetilde{u}(y,0) = u_o(y), \tag{90}$$

and when $q = 1$, they give

$$\widetilde{u}(y,1) = u(y). \tag{91}$$

By virtue of Eqs. (90) and (91), $\widetilde{u}(y,q)$ varies from the initial guess $u_o(y)$ to the exact solution $u(y)$ as the embedding parameter q increases from 0 to 1.

To obtain the k-th order deformation for the system (87) and (88), we differentiate k times with respect to q and set $q = 0$, then for $k \geq 1$ we have

$$\mathscr{L}[u_k(y) - \chi_k u_{k-1}(y)] = hR_k(y) \tag{92}$$

subject to the boundary conditions

$$u_k(-d) = u_k(d) = 0. \tag{93}$$

Here

$$R_k(y) = u''_{k-1} + \frac{2(\beta_2+\beta_3)}{\mu}\sum_{j=0}^{k-1} u''_{k-1-j}\sum_{i=0}^{j} u'_i u'_{j-i} + \xi_k\frac{A}{\mu} \tag{94}$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k \geq 2, \end{cases} \tag{95}$$

$$\xi_k = \begin{cases} 1, & k \leq 1, \\ 0, & k \geq 2, \end{cases} \tag{96}$$

where primes denote the derivative with respect to y.

Now to obtain the first order homotopy solution, we set $k = 1$ and $q = 0$ in Eqs. (92) and (93). The solution of the resulting linear differential equation is given by

$$u_1(y) = \frac{h}{2}(\beta_2 + \beta_3)\frac{A^3}{\mu^4}\left[y^4 - d^4\right]. \tag{97}$$

Again when we put $k = 2$ and $q = 0$ for the second order solution of the problem, we come up with the second-order solution as under

$$u_2(y) = (1 + h)\left[4h\left((\beta_2 + \beta_3)\frac{A^3}{\mu^4}\left[y^4 - d^4\right]\right)\right] \tag{98}$$

$$+ 4h^2(\beta_2 + \beta_3)^2\frac{A^5}{\mu^7}\left[y^6 - d^6\right]. \tag{99}$$

Thus, the final expression for the homotopy analysis method solution up to second order is

$$u(y) = \frac{A}{2\mu}\left[y^2 - d^2\right] + \frac{h}{2}(\beta_2 + \beta_3)\frac{A^3}{\mu^4}\left[y^4 - d^4\right]$$

$$+ \frac{1}{2!}\left[(1 + h)\left\{4h\left((\beta_2 + \beta_3)\frac{A^3}{\mu^4}\left[y^4 - d^4\right]\right)\right\}\right.$$

$$\left. + 4h^2(\beta_2 + \beta_3)^2\frac{A^5}{\mu^7}\left[y^6 - d^6\right]\right]. \tag{100}$$

It can be observed from Eq. (100), that if we set $h = -1$, we recover the perturbation solution.

4.5 Generalized Couette flow problem

The zeroth-order deformation for the problem (47) with the boundary conditions (48) takes the form

$$(1 - q)\mathscr{L}[\widetilde{u}(y, q) - u_o(y)] = qh\left[\frac{\partial^2\widetilde{u}(y, q)}{\partial y^2} + \frac{6(\beta_2 + \beta_3)}{\mu}\left(\frac{\partial\widetilde{u}(y, q)}{\partial y}\right)^2\frac{\partial^2\widetilde{u}(y, q)}{\partial y^2} - \frac{A}{\mu}\right] \tag{101}$$

subject to the boundary conditions

$$\widetilde{u}(0, q) = 0, \quad \widetilde{u}(2d, q) = U, \tag{102}$$

where

$$\mathscr{L} = \frac{\partial^2}{\partial y^2} \tag{103}$$

is an auxiliary linear operator. The initial guess which satisfies the boundary condition (48) is

$$u_o(y) = \frac{U}{2d}y - \frac{A}{2\mu}(2dy - y^2). \tag{104}$$

As we applied homotopy analysis method in the previous sections, similarly in this case we follow the same steps to obtain the first-and second-order solutions:

$$u_1(y) = 6(\beta_2+\beta_3)h\left[\frac{1}{12}\frac{A^3}{\mu^4}y^4 - By^3 + Cy^2 - Dy\right], \tag{105}$$

where

$$B = \frac{1}{3}\frac{A^3}{\mu^4}d - \frac{1}{6}\frac{A^2}{\mu^3}\frac{U}{d},$$

$$C = \frac{1}{2}\frac{A^3}{\mu^4}d^2 - \frac{1}{2}\frac{A^2}{\mu^3}U + \frac{A}{8\mu^2}\frac{U^2}{d^2},$$

$$D = \frac{1}{3}\frac{A^3}{\mu^4}d^3 - \frac{1}{3}\frac{A^2}{\mu^3}Ud + \frac{1}{4}\frac{A}{\mu}\frac{U^2}{d},$$

and

$$\begin{aligned} u_2(y) = {} & 6(\beta_2+\beta_3)(1+h)h\Big[\frac{1}{12}\frac{A^3}{\mu^4}y^4 - \left(\frac{1}{3}\frac{A^3}{\mu^4}d - \frac{1}{6}\frac{A^2}{\mu^3}\frac{U}{d}\right)y^3 \\ & + \left(\frac{1}{2}\frac{A^3}{\mu^4}d^2 - \frac{1}{2}\frac{A^2}{\mu^3}U + \frac{1}{8}\frac{A}{\mu^2}\frac{U^2}{d^2}\right)y^2 - \Big(\frac{1}{3}\frac{A^3}{\mu^4}d^3 - \frac{1}{3}\frac{A^2}{\mu^3}Ud \\ & + \frac{1}{4}\frac{A}{\mu}\frac{U^2}{d}\Big)y\Big] + \frac{1}{2!}\Big\{36(\beta_2+\beta_3)^2h^2\Big[-\frac{1}{18}\frac{A^5}{\mu^7}y^6 - Oy^5 \\ & + Ry^4 - W_1y^3 + W_2y^2 - Xy\Big] - 6(\beta_2+\beta_3)(1+h)hTy\Big\}, \end{aligned} \tag{106}$$

where

$$O = \frac{2}{3}\frac{A^5}{\mu^7}d - \frac{1}{3}\frac{A^4}{\mu^6}\frac{U}{d},$$

$$R = \frac{5}{3}\frac{A^5}{\mu^7}d^2 - \frac{5}{3}\frac{A^4}{\mu^6}U + \frac{5}{12}\frac{A^3}{\mu^5}\frac{U^2}{d^2},$$

$$W_1 = \frac{22}{3}\frac{A^5}{\mu^7}d^3 - \frac{29}{9}\frac{A^4}{\mu^6}Ud + \frac{11}{6}\frac{A^3}{\mu^5}\frac{U^2}{d} - \frac{A^2}{4\mu^4}\frac{U^3}{d^3},$$

$$W_2 = 17\frac{A^5}{\mu^7}d^4 - \frac{8}{3}\frac{A^4}{\mu^6}Ud^2 + \frac{17}{6}\frac{A^3}{\mu^5}U^2 - \frac{A^2}{\mu^4}\frac{U^3}{d^2} + \frac{A}{16\mu^3}\frac{U^4}{d^4},$$

$$X = \frac{13}{12}\frac{A^5}{\mu^7}d^5 - \frac{148}{9}\frac{A^4}{\mu^6}Ud^3 + \frac{5}{3}\frac{A^3}{\mu^5}U^2d - \frac{A^2}{\mu^4}\frac{U^3}{d} + \frac{A}{8\mu^3}\frac{U^4}{d^3},$$

$$T = \frac{2}{3}\frac{A^3}{\mu^4}d^3 - \frac{2}{3}\frac{A^2}{\mu^3}Ud + \frac{A}{2\mu^2}\frac{U^2}{d}.$$

Therefore, the final homotopy solution of the problem up to second order becomes

$$u(y) = \frac{U}{2d}y - \frac{A}{\mu}\left(2dy - y^2\right) + 6(\beta_2+\beta_3)h\left[\frac{1}{12}\frac{A^3}{\mu^4}y^4 - By^3 + Cy^2 - Dy\right]$$

$$6(\beta_2+\beta_3)(1+h)h\Big[\frac{1}{12}\frac{A^3}{\mu^4}y^4 - \left(\frac{1}{3}\frac{A^3}{\mu^4}d - \frac{1}{6}\frac{A^2}{\mu^3}\frac{U}{d}\right)y^3$$

$$+\left(\frac{1}{2}\frac{A^3}{\mu^4}d^2 - \frac{1}{2}\frac{A^2}{\mu^3}U + \frac{1}{8}\frac{A}{\mu^2}\frac{U^2}{d^2}\right)y^2 - \Big(\frac{1}{3}\frac{A^3}{\mu^4}d^3 - \frac{1}{3}\frac{A^2}{\mu^3}Ud$$

$$+\frac{1}{4}\frac{A}{\mu}\frac{U^2}{d}\Big)y\Big] + \frac{1}{2!}\Big\{36(\beta_2+\beta_3)^2h^2\Big[-\frac{1}{18}\frac{A^5}{\mu^7}y^6 - Oy^5$$

$$+Ry^4 - W_1y^3 + W_2y^2 - Xy\Big] - 6(\beta_2+\beta_3)(1+h)hTy\Big\}. \tag{107}$$

If we set $h = -1$ in solution (107), the pertubation solution can be recovered.

The pressure distribution for the Poiseuille flow and the generalized plane Couette flow is given by

$$p = (2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2 + 4\left(\gamma_3+\gamma_4+\gamma_5+\frac{\gamma_6}{2}\right)\left(\frac{du}{dy}\right)^4. \tag{108}$$

The shearing viscosity coefficient μ, and the material constants $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_7$ and γ_8 do not play any role in the pressure distribution.

5 Conclusions

In this paper, plane Couette flows are studied using a fourth grade fluid. Depending on the relative motion of the plate's four problems, namely, plane Couette flow, plug flow, Poiseuille flow and generalized plane Couette flow are discussed. Perturbation technique as well as the homotopy analysis method are used to solve the four problems, and the results are compared.

Unlike the perturbation technique, homotopy analysis method does not require the presence of a parameter, small or large, in the equations governing the motion. However, the solution of the problem generally involves an auxiliary parameter h, which provides a family of solution expressions.

For plane Couette flow and plug flow problems, the two methods give the same solutions which do not depend on h. Also, our solution for plane Couette flow is the same as that for a Newtonian fluid and for the Oldroyd model with 6-constant obtained in [6].

For fully developed plane Poiseuille flow and generalized plane Couette flow problems, the solutions obtained by homotopy analysis method are more general than those given by the perturbation method, which be obtained by setting $h = -1$ in the solutions from the homotopy analysis method. Thus, homotopy analysis method is more efficient and flexible than perturbation method.

Acknowledgement

The authors are grateful for the constructive remarks of the reviewer which were helpful in preparing the revised version.

References

[1] Harris, J.: Rheology and non–Newtonian flow. London New York: Longman 1977
[2] Erdoĝau, E.: Steady pipe flow of fluid of fourth grade. ZAMM **61**, 466–469 (1981).
[3] Wang, Z. K., Kao, T. A.: An introduction to homotopy methods. Chongqing: Chongqing Publishing House 1991.
[4] Liao, S.: Beyond perturbation. Introduction to homotopy analysis method. Boca Raton, FL: Chapman & Hall/CRC 2004.
[5] Ayub, M., Rasheed, A., Hayat, T.: Exact flow of third-grade fluid past a porous plate using homotopy analysis method. Int. J. Engng. Sci. **41**(18), 2091–2103 (2003).
[6] Hayat, T., Khan, M., Ayub, M.: On the explicit analytic solutions of an Oldroyd 6-constant fluid. Int. J. Engng. Sci. **42**, 123–135 (2004).
[7] Joseph, D. D., Beavers, G. S., Fosdick, R. L.: The free surface on a liquid between cyliders rotating at different speeds. Arch. Rat. Mech. Anal. **49**, 321–401 (1973).
[8] Truesdell, C., Noll, W.: The nonlinear field theories of mechanics, 2nd ed. Berlin: Springer 1992.
[9] Beavers, G. S., Joseph, D. D.: The rotating rod viscometer. J. Fluid Mech. **69**, 475–511 (1975).

Authors' addresses: A. M. Siddiqui, Department of Mathematics, Pennsylvania State University, York Campus, 1031 Edgecomb Avenue, York, PA 17403, U.S.A.; M. Ahmed (E-mail: manshoor_post@yahoo.com), S. Islam and Q. K. Ghori, Department of Mathematics, COMSATS Institute of Information Technology, H-8/I, Islamabad, Pakistan