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The pressure of a punch in the form of an elliptic paraboloid on a thin elastic layer

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Summary. The problem of the indentation (without friction) of an absolutely rigid body into a thin elastic layer is investigated. It is assumed that the diameter of the contact area, which is unknown in advance, is large compared with the layer thickness. The inner asymptotic expansion for the contact pressure is obtained by means of the moment asymptotic expansion in the frame of the distributional theory of asymptotic expansions. The boundary layer is constructed using Alexandrov's method.

1 Introduction

Many papers have been published on contact problems of calculating the contact pressure under a punch indented into the plane edge of an elastic layer. The axisymmetrical non-classical contact problems of the theory of elasticity for an elastic layer of arbitrary thickness have been studied in detail by Vorovich et al. [1], and Alexandrov and Pozharskii [2].

An asymptotic solution of the unilateral (see, for example, Duvaut and Lions [3], Kalker [4] and Kravchuk [5]) contact problem for a punch in the form of an elliptic paraboloid was obtained first by Poroshin [6] in the case of a thick elastic layer by means of the so-called "large λ" method (see, for example, [7]). Another asymptotic solution of this problem was constructed by Alexandrov and Shmatkova [8] using Alexandrov's method [9] by assuming that the contact area is bounded by an ellipse. Independent of the earlier results [8], an analytic solution for the unilateral contact problem of indentation of the punch into the plane boundary of an elastic body of arbitrary geometry was found in [10] using the method of matched asymptotic expansions (Van Dyke [11], Ilyin [12] and others).

In a recent paper [13], we considered an elastic layer of finite thickness loaded by a punch in the form of a fourth-degree paraboloid. With the intent to round off the asymptotic investigation of this class of contact problems we will now consider the case of a thin layer. A two-dimensional contact problem similar to this was first considered by Alblas and Kuipers [14]. Without dwelling on the papers which have been published on the solution of two-dimensional contact problems for a thin elastic layer, we note that the results, that are satisfactory from the practical point of view, were obtained in [16], [15] and [14].

The complexity of the unilateral contact problem for analytical investigation is primarily due to the fact that the contact area is not known in advance and has to be found when solving the problem. If the thickness H of the elastic layer is small with respect to some characteristic length parameter of the contact area, then the contact problem under consideration becomes notoriously difficult. It is clear that for $H = 0$ a well defined problem does not exist so that a regular perturbation procedure cannot be applied. Various singular asymptotic methods have been developed to overcome this difficulty (Koiter [16], Alexandrov [17], Alblas and Kuipers [14], Babeshko [18]). The distributional asymptotic approach was proposed by Andrianov et al. [19].

The paper is divided into two parts. The first part deals with a distributional asymptotic derivation of the so-called [17] inner asymptotic expansion of the contact pressure, and the second one refers to a boundary layer in the neighborhood of the boundary of the contact area.

2 Formulation of the problem

We will assume that a smooth punch in the form of an elliptic paraboloid

$$x_3 = -\Phi(x_1, x_2), \qquad \Phi(x_1, x_2) = (2R_1)^{-1} x_1^2 + (2R_2)^{-1} x_2^2 \tag{1.1}$$

is impressed into an elastic layer of thickness H, fixed to a rigid base ($x_3 = H$), to a depth δ_0. Here R_1 and R_2 are the radii of curvature of the principal normal cross-sections of the surface of the punch at its vertex ($R_1 \geq R_2$).

It is natural to assume that the quantities δ_0 and H are small compared with the radii R_1 and R_2. Letting ε denote a small positive parameter, we put

$$H = \varepsilon H^*, \quad \delta_0 = \varepsilon \delta_0^*, \quad R_1 = \varepsilon^{-1} R_1^*, \quad R_2 = \varepsilon^{-1} R_2^*, \tag{1.2}$$

where the magnitudes of δ_0^* and R_1^*, R_2^* are comparable with H^*.

Within classical elastostatics the vector $\mathbf{u} = (u_1, u_2, u_3)$ of the displacements of points of the elastic layer satisfies the following system:

$$L(\nabla_x)\mathbf{u}(\varepsilon; \mathbf{x}) \equiv -\mu \nabla_x \cdot \nabla_x \mathbf{u}(\varepsilon; \mathbf{x}) - \frac{\mu}{1 - 2\nu} \nabla_x \nabla_x \cdot \mathbf{u}(\varepsilon; \mathbf{x}) = 0, \quad x_3 \in (0, H). \tag{1.3}$$

Here $L(\nabla_x)$ is the Lamé operator, μ is shear modulus, and ν is Poisson's ratio.

The boundary conditions on the upper edge ($x_3 = 0$) of the elastic layer are

$$\sigma_{31}(\mathbf{u}; \mathbf{x}', 0) = \sigma_{32}(\mathbf{u}; \mathbf{x}', 0) = 0, \quad \mathbf{x}' = (x_1, x_2) \in \mathbb{R}^2, \tag{1.4}$$

$$\begin{aligned} &u_3(\varepsilon; \mathbf{x}', 0) \geq \delta_0 - \Phi(\mathbf{x}'), && \sigma_{33}(\mathbf{u}; \mathbf{x}', 0) \leq 0, \\ &\left[u_3(\varepsilon; \mathbf{x}', 0) - \delta_0 + \Phi(\mathbf{x}')\right] \sigma_{33}(\mathbf{u}; \mathbf{x}', 0) = 0, && \mathbf{x}' \in \mathbb{R}^2, \end{aligned} \tag{1.5}$$

and on the lower edge ($x_3 = H$) of the elastic layer they are

$$u_3(\varepsilon; \mathbf{x}', H) = 0, \quad \mathbf{x}' \in \mathbb{R}^2. \tag{1.6}$$

Here $\sigma_{3j}(\mathbf{u})$ are the components of the stress tensor.

Finally, we have to add the regularity conditions at infinity (see, for example, [1]), i.e., functions $u_j(\varepsilon; \mathbf{x})$ ($i = 1, 2, 3$), together with their derivatives up to the first order, go to zero as $|\mathbf{x}'| = (x_1^2 + x_2^2)^{1/2} \to \infty$ for each fixed value of x_3 ($0 \leq x_3 \leq H$).

The punch is pressed into the layer by a force P so that contact extends along some contact area ω_ε. The contact area (where the equality sign holds in the first inequality of Eq. (1.5)) is not known in advance and is determined by the condition that the contact pressures

$$p_\varepsilon(x_1, x_2) = -\sigma_{33}(\mathbf{u}; x_1, x_2) \tag{1.7}$$

are positive.

We may consider any of the following two quantities: the total force P, defined by the formula

$$P = \iint_{\omega_\varepsilon} p_\varepsilon(x_1, x_2) dx_1 dx_2, \tag{1.8}$$

and the displacement δ_0, as known a priori. In view of what follows, however, it is convenient to regard the quantity δ_0 as such and to take P and parameters of the contact area ω_ε for unknown quantities, which have to be calculated yet.

3 Distributional derivation of the inner asymptotic expansion of the contact pressure

The strain problems for an elastic layer are conveniently treated with the aid of the two-dimensional Fourier transform. Thus, applying standard transform techniques (see [20], [2], etc.) we arrive at the expression

$$u_3(\mathbf{x}', 0) = -\frac{1}{4\pi^2\vartheta} \iint_{-\infty}^{+\infty} \hat{p}_\varepsilon(\alpha_1, \alpha_2) \frac{\mathscr{L}(\alpha H)}{\alpha} e^{-i(\alpha_1 x_1 + \alpha_2 x_2)} d\alpha_1 d\alpha_2, \tag{2.1}$$

in which $\vartheta = \mu(1-\nu)^{-1}$, $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2}$, and $\hat{p}_\varepsilon(\alpha_1, \alpha_2)$ denotes the transform of $-\sigma_{33}(\mathbf{u}; \mathbf{x}', 0)$, i.e.,

$$\hat{p}_\varepsilon(\alpha_1, \alpha_2) = \iint_{-\infty}^{+\infty} p_\varepsilon(y_1, y_2) e^{i(\alpha_1 y_1 + \alpha_2 y_2)} dy_1 dy_2. \tag{2.2}$$

In the case of an elastic layer which is fixed to a rigid base,

$$\mathscr{L}(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{2\varkappa \operatorname{ch} 2u + 1 + \varkappa^2 + 4u^2}, \qquad \varkappa = 3 - 4\nu. \tag{2.3}$$

The function $\mathscr{L}(u)$ is continuous and positive for $u \in (0, +\infty)$ and satisfies the asymptotic relations

$$\begin{aligned} &\mathscr{L}(u) = \mathscr{A} u + O(u^3), \quad u \to 0; \quad \mathscr{A} = \frac{4(\varkappa - 1)}{(\varkappa + 1)^2}; \\ &\mathscr{L}(u) = 1 + O(u^2 e^{-2u}), \quad u \to \infty. \end{aligned} \tag{2.4}$$

It can be shown (see [1], §22) that $\mathscr{L}(w)w^{-1}$ and $w[\mathscr{L}(w)]^{-1}$, being functions of the complex variable $w = u + iv$, are regular in strips $|v| \le \gamma_1$ and $|v| \le \delta_1$, respectively. Hence, it follows, in particular, that the kernel

$$K(x_1, x_2) = \iint_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos\frac{s_1 x_1}{H} \cos\frac{s_2 x_2}{H} ds_1 ds_2, \tag{2.5}$$

where $s = \sqrt{s_1^2 + s_2^2}$ decreases at infinity as rapidly as $\exp(-\gamma_1 H^{-1} |\mathbf{x}'|)$.

Using Eqs. (2.1), (2.2), (2.5), we obtain from the unilateral boundary condition (1.5) the following integral equation:

$$\iint_{\omega_\varepsilon} p_\varepsilon(\mathbf{y}) K(x_1 - y_1, x_2 - y_2) d\mathbf{y} = \pi^2 H \vartheta (\delta_0 - \Phi(x_1, x_2)). \tag{2.6}$$

In addition, we have the inequality

$$p_\varepsilon(x_1, x_2) > 0, \quad (x_1, x_2) \in \omega_\varepsilon. \tag{2.7}$$

A key point of the distributional asymptotic analysis is to make use of the small parameter ε contained in the kernel (2.5) in view of Eqs. (1.2). For this purpose, it is convenient to introduce the following new dimensionless variables:

$$x_j = H_* \xi_j \quad (j = 1,2), \qquad y_j = H_* \eta_j \quad (j = 1,2), \tag{2.8}$$

and the large parameter

$$\Lambda = \varepsilon^{-1}. \tag{2.9}$$

Substituting expressions (1.2), (2.8), (2.9) into Eq. (2.6), we readily obtain

$$\iint\limits_{\omega_\varepsilon^*} p_\varepsilon^*(\boldsymbol{\eta}) k\big(\Lambda(\xi_1 - \eta_1), \Lambda(\xi_2 - \eta_2)\big) d\boldsymbol{\eta} = \frac{\pi^2 \vartheta}{\Lambda^2 H_*} \big(\delta_0^* - \Phi^*(\boldsymbol{\eta})\big), \tag{2.10}$$

where

$$p_\varepsilon^*(\boldsymbol{\eta}) = p_\varepsilon(H_* \eta_1, H_* \eta_2), \tag{2.11}$$

$$k(\boldsymbol{\varepsilon}) = \iint\limits_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos s_1 \xi_1 \cos s_2 \xi_2 ds_1 ds_2, \tag{2.12}$$

$$\Phi^*(\boldsymbol{\xi}) = H_*^2 \big[(2R_1^*)^{-1} \xi_1^2 + (2R_2^*)^{-1} \xi_2^2\big]. \tag{2.13}$$

Now we recall the moment asymptotic expansion, the basic result in the distributional theory of asymptotic developments (see, for example, Estrada and Kanwal [21]). Before proceeding, we have to settle some points about the notation used.

If $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multi-index of nonnegative integers then we put $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$, $\boldsymbol{\xi}^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2}$, and

$$\mathbf{D}^\alpha f(\boldsymbol{\xi}) = \frac{\partial |\alpha| f(\xi_1, \xi_2)}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2}}, \qquad \mathbf{D}^0 f(\boldsymbol{\xi}) = f(\boldsymbol{\xi}).$$

We also use the standard notation $\boldsymbol{\alpha}! = \alpha_1! \alpha_2!$.

The moment asymptotic expansion can be written as [21]

$$k(\Lambda \boldsymbol{\xi}) \sim \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|} \mu_\alpha \mathbf{D}^\alpha \delta(\boldsymbol{\xi})}{\boldsymbol{\alpha}! \Lambda^{|\alpha|+2}}, \quad \Lambda \to \infty, \tag{2.14}$$

where $\mu_\alpha = \mu_{\alpha_1 \alpha_2}$ are the moments of the generalized function $k(\boldsymbol{\xi})$, given by

$$\mu_\alpha = \langle k(\boldsymbol{\xi}), \boldsymbol{\xi}^\alpha \rangle = \iint\limits_{-\infty}^{+\infty} k(\boldsymbol{\xi}) \xi_1^{\alpha_1} \xi_2^{\alpha_2} d\xi_1 d\xi_2. \tag{2.15}$$

The asymptotic expansion (2.14) is valid in several important spaces of distributions (see Wong [22], Estrada and Kanwal [21] etc.). In fact it holds for distributions of rapid decay at infinity.

The interpretation of Eq. (2.14) is in the distributional sense, namely

$$\langle k(\Lambda \boldsymbol{\xi}), \phi(\boldsymbol{\xi}) \rangle = \sum_{|\alpha|=0}^{N} \frac{\mu_\alpha \mathbf{D}^{\boldsymbol{\alpha}} \phi(\mathbf{0})}{\boldsymbol{\alpha}! \Lambda^{|\alpha|+2}} + O(\Lambda^{-N-3}), \quad \Lambda \to \infty, \tag{2.16}$$

for each $\phi(\boldsymbol{\xi})$ in the corresponding space of test functions.

Substituting expression (2.12) into Eq. (2.15), we obtain

$$\mu_\alpha = \iint\limits_{-\infty}^{+\infty} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \iint\limits_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos s_1 \xi_1 \cos s_2 \xi_2 \, ds_1 \, ds_2 \, d\xi_1 \, d\xi_2$$

$$=\frac{1}{4}\iint\limits_{-\infty}^{+\infty}\frac{\mathscr{L}(s)}{s}\prod_{j=1}^{2}\int\limits_{-\infty}^{+\infty}\xi_j^{\alpha_j}\cos s_j\xi_j\,d\xi_j\,ds_1\,ds_2. \tag{2.17}$$

Using the well-known representation for Dirac's delta function

$$\delta(s_j)=\frac{1}{\pi}\int\limits_{0}^{\infty}\cos s_j\xi_j\;d\xi_j, \tag{2.18}$$

we find from Eq. (2.17) by partial integration the following expressions:

$$\begin{aligned}
&\mu_\alpha=0,\quad |\alpha|=2n-1\quad (n\in\mathbb{N}),\\
&\mu_\alpha=0,\alpha_1=2k-1,\alpha_2=2n-2k+1,|\alpha|=2n\quad (k=1,2,\dots,n),\\
&\mu_{2k,2n-2k}=(-1)^n\pi^2\iint\limits_{-\infty}^{+\infty}\frac{\mathscr{L}(s)}{s}\delta^{(2k)}(s_1)\delta^{(2n-2k)}(s_2)ds_1ds_2,
\end{aligned} \tag{2.19}$$

where $k=0,1,\dots,n$ and $n\in\mathbb{N}\cup\{0\}$.

Next, by invoking the formulae $\delta(s_1,s_2)=\delta(s_1)\delta(s_2)$ and

$$\frac{\mathscr{L}(s)}{s}=\mathscr{A}(1+m_1s^2+m_2s^4+\dots), \tag{2.20}$$

we derive from Eq. (2.19)

$$\begin{aligned}
\mu_{2k,2n-2k}&=(-1)^n\pi^2\mathscr{A}m_n\iint\limits_{-\infty}^{+\infty}(s_1^2+s_2^2)^n\delta^{(2k)}(s_1)\delta^{(2n-2k)}(s_2)ds_1ds_2\\
&=(-1)^n\pi^2\mathscr{A}m_nC_n^k2^nk!(n-k)!\qquad\left(C_n^k=\frac{n!}{k!(n-k)!}\right).
\end{aligned} \tag{2.21}$$

Then, from the relations (2.14) and (2.21), where we may put $m_0=1$, we find

$$k(\Lambda\boldsymbol{\xi})\sim\sum_{n=0}^{\infty}(-1)^n\pi^2\mathscr{A}\frac{m_n}{\Lambda^{2n+2}}\sum_{k=0}^{n}C_n^k\frac{\partial^{2n}\delta(\boldsymbol{\xi})}{\partial\xi_1^{2k}\partial\xi_2^{2n-2k}}. \tag{2.22}$$

Substituting the moment asymptotic expansion (2.22) into the left-hand of Eq. (2.10) we obtain

$$\iint\limits_{\omega_\varepsilon^*}p_\varepsilon^*(\boldsymbol{\eta})k\big(\Lambda(\boldsymbol{\eta}-\boldsymbol{\xi})\big)d\boldsymbol{\eta}\sim\sum_{n=0}^{\infty}(-1)^n\pi^2\mathscr{A}\frac{m_n}{\Lambda^{2n+2}}\sum_{k=0}^{n}C_n^k\frac{\partial^{2n}p_\varepsilon^*(\boldsymbol{\xi})}{\partial\xi_1^{2k}\partial\xi_2^{2n-2k}}. \tag{2.23}$$

By the formula

$$\Delta^n\equiv\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}\right)^n=\sum_{k=0}^{n}C_n^k\frac{\partial^{2n}}{\partial\xi_1^{2k}\partial\xi_2^{2n-2k}},$$

where Δ is the Laplacian, according to Eqs. (2.10) and (2.23), we have

$$\sum_{n=0}^{\infty}(-1)^n\pi^2\mathscr{A}\frac{m_n}{\Lambda^{2n+2}}\Delta^np_\varepsilon^*(\boldsymbol{\xi})\sim\Lambda^{-2}\varphi^*(\boldsymbol{\xi}), \tag{2.24}$$

where

$$\varphi^*(\boldsymbol{\xi})=\frac{\pi^2\vartheta}{H_*}\big(\delta_0^*-\Phi^*(\boldsymbol{\xi})\big). \tag{2.25}$$

Equation (24) (like, however, the equations which arose earlier) contains a large parameter, and hence its solution can be expanded in an asymptotic series in inverse powers of Λ.

Let us put

$$p_\varepsilon^*(\boldsymbol{\xi}) \sim p_0^*(\boldsymbol{\xi}) + \Lambda^{-2} p_1^*(\boldsymbol{\xi}) + \Lambda^{-4} p_2^*(\boldsymbol{\xi}) + \dots \ . \tag{2.26}$$

Substituting (2.26) into (2.24), we obtain a system of equations for the successive determination of the functions $p_0^*(\boldsymbol{\xi}), p_1^*(\boldsymbol{\xi}), \dots$ of the type

$$\pi^2 \mathscr{A} m_0 p_0^*(\boldsymbol{\xi}) = \varphi^*(\boldsymbol{\xi}), \quad \sum_{j=0}^{k} (-1)^{k-j} m_{k-j} \Delta^{k-j} p_j^*(\boldsymbol{\xi}) = 0 \quad (k = 1, 2, \dots).$$

We then find

$$p_0^*(\boldsymbol{\xi}) = \frac{1}{\pi^2 \mathscr{A}} \varphi^*(\boldsymbol{\xi}), \tag{2.27}$$

$$p_k^*(\boldsymbol{\xi}) = -\sum_{j=0}^{k-1} (-1)^{k-j} m_{k-j} \Delta^{k-j} p_j^*(\boldsymbol{\xi}). \tag{2.28}$$

In the case (2.25) by simple calculations we obtain

$$p_0^*(\boldsymbol{\xi}) = \frac{\vartheta}{\mathscr{A} H_*} \left(\delta_0^* - \Phi^*(\boldsymbol{\xi})\right), \tag{2.29}$$

$$p_1^*(\boldsymbol{\xi}) = \frac{\vartheta m_1 H_*}{\mathscr{A}} \left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right), \tag{2.30}$$

$$p_k^*(\boldsymbol{\xi}) \equiv 0 \qquad (k = 2, 3, \dots).$$

Finally, using Eqs. (2.29), (2.30) we find from (2.26)

$$p_\varepsilon^*(\boldsymbol{\xi}) \sim \frac{\vartheta}{\mathscr{A} H_*} \left(\delta_0^* - H_*^2 \left(\frac{\xi_1^2}{2R_1^*} + \frac{\xi_2^2}{2R_2^*}\right) + \frac{2m_1}{\Lambda^2} \frac{H_*^2}{R^*}\right). \tag{2.31}$$

Here,

$$\frac{1}{R^*} = \frac{1}{2} \left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right).$$

Thus, the inner asymptotic representation of the contact pressure is expressed in the form (2.31).

4 A boundary layer

The contact pressure distribution density (2.29), which is the principal part of the inner asymptotic expansion (2.31), determines the main approximation ω_0^* to the required contact area ω_ε^* (in the coordinates (2.8)).

It is obvious from Eqs. (2.29) and (2.13) that the contact area ω_0^*, which corresponds to the density (2.29), is elliptic. The major semiaxis and the eccentricity of the contour Γ_0^* of the domain ω_0^* will be denoted by a^* and e. By simple calculations we find

$$a^* = \frac{1}{H^*} \sqrt{2\delta_0^* R_1}, \quad e^2 = 1 - \frac{R_2^*}{R_1^*}. \tag{3.1}$$

Let us now describe the behavior of the integral (2.6) and its density in the neighborhood of the contour Γ_ε^* of the domain ω_ε^*. We shall follow here the procedure introduced by Alexandrov [23].

In view of Eqs. (2.10) and (2.25) the integral equation (2.6) takes now the form

$$\iint\limits_{\omega_\varepsilon^*} p_\varepsilon^*(\boldsymbol{\eta})k\big(\Lambda(\boldsymbol{\xi}-\boldsymbol{\eta})\big)d\boldsymbol{\eta} = \varepsilon^2\varphi^*(\boldsymbol{\xi}). \tag{3.2}$$

Suppose $\xi_1 = f_1^*(s)$, $\xi_2 = f_2^*(s)$ is a natural parametrization of the contour Γ_0^*. We will assume that when going round Γ_0^* in the direction of increasing s coordinate, the region ω_0^* enclosed by Γ_0^* remains on the left. Then, the unit vector of the inward normal (with respect to the domain ω_0^*) to the contour Γ_0^* is

$$\mathbf{n}^0(s) = -f_2^{*\prime}(s)\mathbf{e}_1 + f_1^{*\prime}(s)\mathbf{e}_2, \tag{3.3}$$

where the prime denotes differentiation with respect to s.

In the neighborhood of the contour Γ_0^*, we introduce the local system of coordinates (s, n), associated with the Cartesian coordinates by the formulae

$$\xi_1 = f_1^*(s) + nn_1^0(s), \quad \xi_2 = f_2^*(s) + nn_2^0(s), \tag{3.4}$$

where n is the distance (taking the sign into account) along the inward normal to the contour Γ_0^*.

Further, let us assume that the contour Γ_ε^* of the required contact area ω_ε^* in the local coordinates is described by the equation

$$n = h_\varepsilon^*(s), \tag{3.5}$$

where $h_\varepsilon^*(s)$ is a function to be determined. We put

$$h_\varepsilon^*(s) = \varepsilon h^*(s). \tag{3.6}$$

In the small neighborhood $\Xi_\varepsilon^*(s)$ of the point s, where $|\boldsymbol{\eta}-\boldsymbol{\xi}(s)| = O\big(\sqrt{\varepsilon}\rho^*(s)\big)$ and $\rho^*(s) = \big[f_2^{*\prime\prime}(s)f_1^{*\prime}(s) - f_1^{*\prime\prime}(s)f_2^{*\prime}(s)\big]^{-1}$ is the radius of curvature of contour Γ_0^* at the point s, we make in the integral (3.2) the following change of variables:

$$\eta_1 = f_1^*(s') + n'n_1^0(s'), \quad \eta_2 = f_2^*(s') + n'n_2^0(s').$$

Next, we introduce the "fast" variables

$$\nu = \varepsilon^{-1}n; \quad \nu' = \varepsilon^{-1}n', \quad \sigma' = \varepsilon^{-1}(s'-s), \tag{3.7}$$

keeping the scale for the s-coordinate along Γ_0^* unchanged. From now on, the "slow" variable s is considered as fixed.

In the neighborhood $\Xi_\varepsilon^*(s)$, when $\varepsilon \to 0$ we have the following relations:

$$f_j^*(s') = f_j^*(s) + \varepsilon\sigma' f_j^{*\prime}(s) + O(\varepsilon^2) \quad (j = 1, 2),$$

$$n_j^0(s') = n_j^0(s) + O(\varepsilon) \quad (j = 1, 2),$$

$$|\boldsymbol{\eta}-\boldsymbol{\xi}(s)| = \varepsilon\sqrt{(\sigma')^2 + (\nu-\nu')^2} + O(\varepsilon^2),$$

$$\rho^*(s') = \rho^*(s) + O(\varepsilon), \quad h^*(s') = h^*(s) + O(\varepsilon),$$

$$\frac{D(\eta_1,\eta_2)}{D(s',n')} = 1 - \frac{\varepsilon\nu'}{\rho^*(s+\varepsilon\sigma')} = 1 + O(\varepsilon),$$

which hold for the smooth contour Γ_0^*.

Now, separating the principal asymptotic terms by virtue of the previous formulae, we take the limit

$$\lim_{\varepsilon\to 0} k\big(\varepsilon^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})\big) = k\big(\sigma' f_1^{*\prime}(s) + (v-v')n_1^0(s), \sigma' f_2^{*\prime}(s) + (v-v')n_2^0(s)\big). \tag{3.8}$$

By invoking the formulae (see Eq. (3.3))

$$f_1^{*\prime}(s) = \cos\psi,\ f_2^{*\prime}(s) = \sin\psi; \quad n_1^0(s) = -\sin\psi;\ n_2^0(s) = \cos\psi, \tag{3.9}$$

it can be shown directly that the right-hand side of the relation (3.8) equals to $k(\sigma', v'-v)$.

Actually, we have (see Eq. (2.12))

$$k(\boldsymbol{\xi}) = \frac{1}{4}\iint_{-\infty}^{+\infty} \frac{\mathscr{L}(s)}{s} e^{i(s_1\xi_1+s_2\xi_2)} ds_1 ds_2. \tag{3.10}$$

If we make the substitutions

$$s_1 = t_1\cos\psi - t_2\sin\psi, \quad s_2 = t_1\sin\psi + t_2\cos\psi;$$

$$\frac{D(s_1,s_2)}{D(t_1,t_2)} = 1, \quad s \equiv \sqrt{s_1^2+s_2^2} = \sqrt{t_1^2+t_2^2} \equiv t,$$

we find that the representation (3.10) can be written in the form

$$k(\boldsymbol{\xi}) = \frac{1}{4}\iint_{-\infty}^{+\infty} \frac{\mathscr{L}(t)}{t}\cos\big[\xi_1(t_1\cos\psi - t_2\sin\psi) + \xi_2(t_1\sin\psi + t_2\cos\psi)\big] dt_1 dt_2, \tag{3.11}$$

provided that $t^{-1}\mathscr{L}(t)$ is an even function for $t\in\mathbb{R}$.

Thus, from Eqs. (3.8), (3.9), and (3.11) it follows immediately that

$$\lim_{\varepsilon\to 0} k\big(\varepsilon^{-1}(\boldsymbol{\xi}-\boldsymbol{\eta})\big) = k(\sigma', v'-v). \tag{3.12}$$

On the other hand (see the right-hand side of Eq. (3.2)), we have

$$\varphi^*(\boldsymbol{\xi}) = \pi^2\vartheta H_*\big[\varepsilon v b_1^*(s) + \varepsilon^2 v^2 b_2^*(s)\big], \tag{3.13}$$

where

$$b_1^*(s) = -\frac{f_1^*(s)n_1^0(s)}{R_1^*} - \frac{f_2^*(s)n_2^0(s)}{R_2^*}, \quad 2b_2^*(s) = -\frac{n_1^0(s)^2}{R_1^*} - \frac{n_2^0(s)^2}{R_2^*}. \tag{3.14}$$

Hence, taking into account the fact that $p_\varepsilon^*(\boldsymbol{\eta}) \sim q_\varepsilon^*(s, v')$ in the neighborhood of the boundary of the contact area (see, for example, [17] and [18]) and letting $\varepsilon\to 0$, in view of Eqs. (3.2), (3.4) – (3.7), (3.12), and (3.13) we arrive at the following integral equation, in which the s coordinate occurs as a parameter:

$$\int_{h^*(s)}^{+\infty} q^{**}(s,v')N(v'-v)dv' = \pi^2\vartheta H_* b_1^*(s)v. \tag{3.15}$$

Here,

$$q_\varepsilon^*(s,v) = \varepsilon q^{**}(s,v),$$

$$N(t) = \int_{-\infty}^{+\infty} k(\sigma', t)d\sigma'. \tag{3.16}$$

Making use of formula (2.18) we find that the previous representation can be written in the form

$$N(t) = \pi \int_0^{+\infty} \frac{\mathscr{L}(s)}{s} \cos st\, ds. \tag{3.17}$$

Finally, in addition to Eq. (3.15), which the boundary layer $q^{**}(s, v)$ must satisfy, it is necessary to obey the condition (2.7). Hence, the density $q^{**}(s, v)$ is to be positive and satisfies the equation

$$q^{**}\big(s, h^*(s)\big) = 0. \tag{3.18}$$

Otherwise, we would contradict the assumption that contact must be made over the whole area ω_ε.

We note that the obtained Eq. (3.15) is of the Wiener–Hopf type and can be solved in closed form (see [15], [14], etc.). However, since a simple factorization for the function $w^{-1}\mathscr{L}(w)$ of the complex variable $w = u + iv$ is not possible, it is not possible to obtain a simple exact solution of Eq. (3.15). Thus, the approximate version of the Wiener–Hopf method has to be used. Confining our considerations to the first approximation, we replace the function $\mathscr{L}(s)$ by a simple algebraic function [17]

$$\tilde{\mathscr{L}}(s) = s\frac{\sqrt{s^2 + B^2}}{s^2 + C}. \tag{3.19}$$

It can easily be shown that the functions $s^{-1}\mathscr{L}(s)$ and $s^{-1}\tilde{\mathscr{L}}(s)$ satisfy the requirements stated by Koiter [16], i.e., have the same limits for s tending to 0 and ∞, provided that the following relation holds:

$$\frac{B}{C} = \mathscr{A}. \tag{3.20}$$

In addition, following Alexandrov [17] we put

$$\frac{1}{C} - \frac{1}{2B^2} = -2m_1,$$

where (see formula (2.20))

$$m_1 = \frac{1}{2\mathscr{A}} \lim_{s\to 0} \frac{d^2}{ds^2} \frac{\mathscr{L}(s)}{s}. \tag{3.21}$$

We replace $\mathscr{L}(s)$ in the integral (3.17) by $\tilde{\mathscr{L}}(s)$, substitute the kernel $\tilde{N}(t)$ into Eq. (3.15), and make in the obtained equation the change of variables

$$v = h^*(s) + \tau, \quad v' = h^*(s) + \tau'.$$

Thus, Eq. (3.15) can be transformed into the form

$$\int_0^{+\infty} \tilde{q}^{**}(s, h^*(s) + \tau')\tilde{N}(\tau' - \tau)d\tau' = \pi^2 \vartheta H_* b_1^*(s)\big(h^*(s) + \tau\big), \tag{3.22}$$

where

$$\tilde{N}(t) = \pi \int_0^{+\infty} \frac{\tilde{\mathscr{L}}(s)}{s} \cos st\, ds.$$

In order to avoid undue repetition in the solution of Eq. (3.22), we refer to [15] and [14] for a comprehensive account of the Wiener–Hopf techniques.

In the case (3.19), making use of Alexandrov's results [17], we arrive finally at

$$\frac{\tilde{q}^{**}(s,h^*(s)+\tau)}{\vartheta H_* b_1^*(s)} = \frac{\tau}{\mathscr{A}}\ \mathrm{erf}\ \sqrt{B\tau} - \frac{1}{\sqrt{\pi B\tau}}e^{-B\tau}\left(1-\frac{\sqrt{C}}{2B}-\frac{\tau}{\mathscr{A}}\right)+ + \frac{h^*(s)}{\mathscr{A}}\ \mathrm{erf}\ \sqrt{B\tau} + \frac{h^*(s)}{\sqrt{\mathscr{A}\pi\tau}}e^{-B\tau}, \tag{3.23}$$

where erf (x) is the error function, defined as

$$\mathrm{erf}\ (x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Since by Eq. (3.18) the function (3.23) must satisfy the equality $\tilde{q}^{**}(s,h^*(s))=0$, the boundary layer is taken to be the function

$$\tilde{q}^{**}(s,v) = \vartheta H_* b_1^*(s)\mathscr{A}^{-1}\left\{v\ \mathrm{erf}\sqrt{B\big(v-h^*(s)\big)}+ + \sqrt{\frac{v-h^*(s)}{\pi B}}\exp\Big(-B\big(v-h^*(s)\big)\Big)\right\}, \tag{3.24}$$

provided that

$$h^*(s) = \sqrt{\frac{\mathscr{A}}{B}} - \frac{1}{2B}, \tag{3.25}$$

when Eq. (3.18) is taken into account.

5 Concluding observations

First, it can be shown that we can rewrite formula (2.28) in the form

$$p_k^*(\boldsymbol{\xi}) = \frac{(-1)^k}{\pi^2\mathscr{A}}M_k\Delta^k f^*(\boldsymbol{\xi}) \quad (k=0,1,2,\ldots), \tag{4.1}$$

where in view of Eq. (2.27) we have $M_0 = 1$.

Now we observe the expansion (see also formula (2.20))

$$\frac{s}{\mathscr{L}(s)} = \frac{1}{\mathscr{A}}(1+M_1s^2+M_2s^4+\ldots).$$

It is easily proved by induction that for any positive integer k the recurrence relation that facilitates the calculation of the coefficients M_k from the expansion (2.20) has the form

$$M_k = -\sum_{j=0}^{k-1} m_{k-j}M_j \quad (k=1,2,\ldots). \tag{4.2}$$

On the other hand, we also readily arrive at Eq. (4.2) by substituting expression (4.1) into formula (2.28).

Thus, the constructed inner asymptotic expansion (2.26), (4.1), (2.3) is similar to the solution obtained earlier (see [1], §55).

Second, in the axisymmetric case the obtained boundary layer is essentially similar to the leading term of the asymptotics for the contact pressures in [1] (see formula (49.11)).

Third, we note that Alexandrov's condition (3.21) for the approximation (3.19) is exact in view of applications of Padé approximants for calculation of the coefficients in the approximation (3.19) from the Maclaurin series (2.20) (see also, Andrianov and Awrejcewicz [24]).

Further, we note that the approximate solution of the contact problem for a thin elastic layer which has been constructed also remains valid, for example, in the case of a punch in the form of a fourth-degree paraboloid. It is only necessary to calculate the values of the corresponding coefficient $b_1^*(s)$ in the expansion (3.13) for the corresponding contour Γ_0^* of the main approximation ω_0^* to the contact area ω_ε^*.

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