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The refined theory of magnetoelastic rectangular beams

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Summary. The problem of deducing a one-dimensional theory from a three-dimensional theory for a soft ferromagnetic elastic isotropic body is investigated. Based on the linear magnetoelasticity, the refined theory of magnetoelastic beams is presented by using the general solution for the soft ferromagnetic elastic solids and the Lur'e method. Based on the refined theory of magnetoelastic beams, the exact equations and solutions for the homogeneous beams are derived and the equations can be decomposed into three governing differential equations: the fourth-order equation, the transcendental equation and the magnetic equation. Moreover, the approximate equations and solutions for the beam under transverse loadings and magnetic field perturbations are derived directly from the refined beam theory. By omitting higher order terms and coupling effects, the refined beam theory can be degenerated into other well-known elastic and magnetoelastic theoretical models.

1 Introduction

A soft ferromagnetic material is characterized by small hysteretic losses and low remanent magnetization. The theoretical and experimental studies on the magnetoelastic interaction for ferromagnetic bodies or structures can be dated back to the 1960s. Brown [1] summarized his research work over the past 20 years on the interaction between magnetic and elastic processes in a ferromagnetic material, and developed a rigorous phenomenological theory of magnetoelasticity on the basis of the large deformation theory of elasticity and the classical theory of ferromagnetism. Since such a general nonlinear theory is rather complicated, a linearized version of Brown's theory has been developed by Pao and Yeh [2]. Based on Pao and Yeh's linear theory of magnetoelasticity, Huang and Wang [3] obtained a general solution for the soft ferromagnetic elastic solids.

Based on some assumptions, Moon and Pao [4] proposed a theoretical model (the magnetic couple model), and experimentally studied the magnetoelastic buckling of a ferromagnetic cantilevered beam-like plate in a uniform transverse magnetic field. Recently, Zhou and Zheng [5] established a theoretical model (the magnetic force model) to describe the magnetoelastic buckling phenomenon of ferromagnetic thin plates with geometrically nonlinear deformation, and given the governing equations of magnetoelastic plates in a nonuniform transverse magnetic field. Having found that almost every model fails in the simulation of the experimental phenomenon of Takagi et al. [6], Zhou and Zheng [7] derived a new theoretical model (the variational principle model) for a ferromagnetic body by a general variational principle.

Cheng [8] gave a refined plate theory from the Boussinesq-Galerkin elasticity solution and the Lur'e method [9] without ad hoc assumptions. The refined plate theory consists of three parts: the biharmonic equation, the shear equation and the transcendental equation. Zhao and Wang [10] also obtained Cheng's refined theory from a Papkovich-Neuber solution and strictly proved that it consists of the preceding three parts. A parallel development of Cheng's theory by Barrett and Ellis [11] has been obtained for the isotropic plates under transverse surface loadings (only homogeneous cases are considered in the previous works). Another parallel development of Cheng's plate theory has been obtained by Wang [12], [13] for the transversely isotropic plate problem and plane problem.

Wang and Shi [14] developed Cheng's theory by using a Papkovich-Neuber solution, and derived shear theory of plates from the refined plate theory. Yin and Wang [15] extended it for the transversely isotropic plates using an Elliott-Lodge solution. Recently, Gao and Wang [16], [17] extended [14] for the narrow rectangular isotropic elastic beams and thermoelastic beams, and derived the refined theory of beams. The exact equations for the beam without transverse surface loadings and the approximate equations for the beam under transverse loadings are derived from the refined beam theory, respectively.

In light of the character of large hysteretic losses and high remanent magnetization in a hard ferromagnetic material, Maugin [18] established a nonlinear continua theory for the magnetoelastic interactions to show the effect due to magnetization gradient and hysterisis. However, the general theory is rather complicated and it is difficult to apply it to the refined magnetoelastic beam theory. Based on the linear theory of Pao and Yeh [2], this paper presents the theory for a soft ferromagnetic elastic beam by using the method developed in [16]. In the next section, a general solution of the magnetoelastic equation is given in light of the work of [3]. In Sect. 3, the refined theory of a magnetoelastic beam is derived by using the general solution for the soft ferromagnetic elastic solids and the Lur'e method [9], then the displacements and stresses of the beam can be represented by the mid-plane displacements and the magnetic function. In Sect. 4, the exact equations for the homogeneous beam can be decomposed into three governing differential equations: the fourth-order equation, the transcendental equation and the magnetic equation. Finally, the approximate equations for the beam under transverse loadings and magnetic field perturbations are derived from the refined beam theory in Sect. 5.

2 The general solution of magnetoelastic equation

According to the linearized theory developed by Pao and Yeh [2], the magnetic quantities can be decomposed into two parts as

$$B_i = \bar{B}_i + b_i, \quad M_i = \bar{M}_i + m_i, \quad H_i = \bar{H}_i + h_i, \tag{2.1}$$

where B_i, M_i and H_i are magnetic induction, magnetization and magnetic intensity, respectively. The barred quantities are the magnetic fields in the rigid-body state with no mechanical singularities; the quantities in lower case represent singularities and are assumed as much smaller than in the undisturbed state.

In the absence of body force, the equilibrium equations of isotropic magnetoelasticity are expressed as

$$t_{ij,i} + \chi_1\mu_0(\bar{M}_i\bar{H}_{j,i} + \bar{M}_i h_{j,i} + m_i\bar{H}_{j,i}) = 0, \tag{2.2.1}$$

$$t_{ij} = \frac{\chi_2\mu_0}{\chi}\bar{M}_i\bar{M}_j + \chi_2\mu_0(\bar{H}_j m_i + \bar{H}_i m_j) + \sigma_{ij}, \tag{2.2.2}$$

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{2.2.3}$$

where t_{ij}, σ_{ij} and u_i are the components of magnetomechanical stress, elastic stress and displacement, respectively, λ and μ are Lame constants, μ_0 and χ are the magnetic permeability and the magnetic susceptibility, respectively, χ_1 and χ_2 are the parameters determined by some theoretical models for magnetoelastic interaction. δ_{ij} is the Kronecker delta symbol, the subscripts "," denote the partial derivative with respect to the spatial variables, and repeated indices imply summation.

According to magnetoelasticity, the basic equations of magnetic fields are of the form

$$\begin{aligned} &\bar{B}_i = \mu_0(1+\chi)\bar{H}_i, \quad \bar{M}_i = \chi\bar{H}_i, \quad b_i = \mu_0(1+\chi)h_i, \quad m_i = \chi h_i, \\ &e_{ijk}\bar{H}_{k,j} = 0, \quad \bar{B}_{i,i} = 0, \quad e_{ijk}h_{k,j} = 0, \quad b_{i,i} = 0, \end{aligned} \tag{2.3}$$

where e_{ijk} is the Levi-Civita permutation symbol.

We consider a soft ferromagnetic elastic straight beam of narrow rectangular cross-section as a plane stress problem. In a fixed rectangular coordinate system, z is the coordinate normal to the neutral surface (xy-plane) of the beam. We assume the beam length in x-direction is l, the beam width in y-direction is assumed 1, the beam height in z-direction is h, and $l >> h >> 1$. In a transverse uniform magnetic field $\bar{H}^+$ and in the absence of body force, the equilibrium equations of the plane stress problem are expressed as

$$\begin{aligned} &\nabla^2 u_x + \frac{1+v}{1-v}\frac{\partial e}{\partial x} + 2(1+v)\frac{\mu_0\chi(\chi_1+\chi_2)\bar{H}^+}{E}\frac{\partial h_x}{\partial z} = 0, \\ &\nabla^2 u_z + \frac{1+v}{1-v}\frac{\partial e}{\partial z} + 2(1+v)\frac{\mu_0\chi(\chi_1+\chi_2)\bar{H}^+}{E}\frac{\partial h_z}{\partial z} = 0, \end{aligned} \tag{2.4}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is the two-dimensional Laplacian operator, $e = \partial u_x/\partial x + \partial u_z/\partial z$, v and E are Poisson's ratio and Young's modulus, respectively.

Huang and Wang [3] obtained a general solution of the equations in the linearized theory of magnetoelasticity, so the solution of the governing equations (2.4) takes the same form

$$\begin{aligned} &u_x = P_1 - \frac{1+v}{4}\frac{\partial}{\partial x}(P_0 + xP_1 + zP_3), \quad u_z = \varphi + P_3 - \frac{1+v}{4}\frac{\partial}{\partial z}(P_0 + xP_1 + zP_3), \\ &Qh_x = \frac{\partial\varphi}{\partial x}, \quad Qh_z = \frac{\partial\varphi}{\partial z}, \end{aligned} \tag{2.5}$$

where

$$\nabla^2 P_0 = 0, \ \ \nabla^2 P_i = 0, \ \ \nabla^2\varphi = 0, \ \ Q = -2(1-v)\frac{\mu_0\chi(\chi_1+\chi_2)\bar{H}^+}{E} \quad (i = 1,3). \tag{2.6}$$

Moreover, they pointed out that the general solution is complete. The general solution looks like the famous Papkovich-Neuber solution for the magnetic term φ, so the refined theory of elastic beams [16] can be extended to magnetoelastic beams in this paper.

3 The refined theory of magnetoelastic beams

The problem of a magnetoelastic beam may be decomposed into two fundamental problems: the extension of a beam and the bending of a beam. In the case of the bending of a beam, the beam is subjected only to anti-symmetrical loadings, the perturbation of magnetic intensity and edge conditions, thus only odd functions of z are required for u_x and even functions of z for u_z.

For the Lur'e method [9], satisfying these requirements and treating Eqs. (2.6) as an ordinary differential equation in z with constant coefficients, one obtains the following symbolic solution of Eqs. (2.6):

$$P_0 = \frac{\sin(z\partial_x)}{\partial_x} g_0(x), \quad P_1 = \frac{\sin(z\partial_x)}{\partial_x} g_1(x),$$
$$P_3 = \cos(z\partial_x) g_3(x), \quad \varphi = \cos(z\partial_x) g_4(x), \tag{3.1}$$

where

$$\frac{\sin(z\partial_x)}{\partial_x} = z\left(1 - \frac{1}{3!}z^2\partial_x^2 + \frac{1}{5!}z^4\partial_x^4 - \dots\right), \quad \cos(z\partial_x) = 1 - \frac{1}{2!}z^2\partial_x^2 + \frac{1}{4!}z^4\partial_x^4 - \dots \ . \tag{3.2}$$

In Appendix A, it is proved that the harmonic function P_0 always can satisfy the following expression without loss in generality:

$$P_0 + xP_1 + zP_3 = -z\cos(z\partial_x) f(x), \tag{3.3}$$

where

$$f = \int_0^x g_1(t)\mathrm{d}t - g_3. \tag{3.4}$$

Substituting Eqs. (3.1) and (3.3) into Eqs. (2.5), one obtains

$$u_x = \frac{\sin(z\partial_x)}{\partial_x} g_1 + \frac{1+v}{4} z\cos(z\partial_x) f',$$
$$u_z = \cos(z\partial_x)(g_3 + g_4) + \frac{1+v}{4}[\cos(z\partial_x) - z\partial_x \sin(z\partial_x)] f, \tag{3.5}$$

where the differential symbol "′" denotes differentiation with respect to x. The angle of rotation and the deflection of the neutral surface can be found to be

$$\psi = -\left.\frac{\partial u_x}{\partial z}\right|_{z=0} = -\left(g_1 + \frac{1+v}{4}f'\right), \quad w = u_z|_{z=0} = g_3 + g_4 + \frac{1+v}{4}f. \tag{3.6}$$

From Eqs. (3.5) and (3.6), the final expressions for the displacements and the perturbations of the magnetic intensity are

$$u_x = -\frac{\sin(z\partial_x)}{\partial_x}\psi + \frac{1+v}{4}\left[z\cos(z\partial_x) - \frac{\sin(z\partial_x)}{\partial_x}\right] f', \tag{3.7.1}$$

$$u_z = \cos(z\partial_x) w - \frac{1+v}{4} z\partial_x \sin(z\partial_x) f, \tag{3.7.2}$$

$$Qh_x = \partial_x \cos(z\partial_x) g_4, \tag{3.7.3}$$

$$Qh_z = -\partial_x \sin(z\partial_x) g_4, \tag{3.7.4}$$

with the expression

$$f = -\int_0^x \psi(t)\mathrm{d}t - w + g_4. \tag{3.8}$$

Using Hooke's law, from Eqs. (3.7.1) and (3.7.2) the stress components σ_x, τ_{xz} and σ_z can be written as

$$\sigma_x = -\frac{E}{4}\left\{\left[\frac{1-v}{1+v}\frac{\sin(z\partial_x)}{\partial_x} - z\cos(z\partial_x)\right] f'' + \frac{4}{1+v}\frac{\sin(z\partial_x)}{\partial_x}\psi' + \frac{4v}{1-v^2}\frac{\sin(z\partial_x)}{\partial_x} g_4''\right\}, \tag{3.9.1}$$

$$\tau_{xz} = -\mu\left[\cos(z\partial_x)(\psi - w') + \frac{1+v}{2}z\partial_x \sin(z\partial_x) f'\right], \tag{3.9.2}$$

$$\sigma_z = -\frac{E}{4}\left\{\left[\frac{1-v}{1+v}\frac{\sin(z\partial_x)}{\partial_x} + z\cos(z\partial_x)\right]f'' + \frac{4}{1+v}\frac{\sin(z\partial_x)}{\partial_x}w'' + \frac{4v}{1-v^2}\frac{\sin(z\partial_x)}{\partial_x}g_4''\right\}. \tag{3.9.3}$$

4 Exact equations: no transverse surface loadings and magnetic field perturbations

In order to satisfy the homogeneous boundary conditions on the upper and lower surfaces of the magnetoelastic beam, we set

$$\tau_{xz} = 0, \quad \sigma_z = 0, \quad h_z = 0, \quad \text{at} \quad z = \pm h/2. \tag{4.1}$$

Substituting Eqs. (3.9.2), (3.9.3) and (3.7.4) into the boundary conditions (4.1) of the beam, we get the following equations:

$$\left(D_1 - D_2\partial_x^2\right)\psi - \left(D_1 + D_2\partial_x^2\right)w' + D_2\partial_x^2 g_4' = 0, \tag{4.2.1}$$

$$D_3\psi' - \left[\frac{4}{(1+v)h}D_2 - D_3\right]w'' - \left[\frac{4v}{(1-v^2)h}D_2 + D_3\right]g_4'' = 0, \tag{4.2.2}$$

$$D_2 g_4'' = 0. \tag{4.2.3}$$

The three differential operators $D_i (i = 1, 2, 3)$ are defined by

$$D_1 = \frac{4}{1+v}\cos\left(\frac{h\partial_x}{2}\right), \ D_2 = \frac{h}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right), \ D_3 = \frac{h}{2}\cos\left(\frac{h\partial_x}{2}\right) + \frac{1-v}{1+v}\frac{1}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right). \tag{4.3}$$

Taking the operator D_2 on both sides of Eq. (4.2.2) and using Eq. (4.2.3), then ignoring the magnetic function g_4, one obtains

$$\left(D_1 - D_2\partial_x^2\right)\psi - \left(D_1 + D_2\partial_x^2\right)w' = 0, \quad D_2D_3\psi' - D_2\left[\frac{4}{(1+v)h}D_2 - D_3\right]w'' = 0. \tag{4.4}$$

Equations (4.4) can be expressed by the following matrix equation:

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} \psi \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.5}$$

Let L_0 be the determinant of the 2×2 matrix of the preceding equation,

$$L_0 = \frac{4h^3}{1+v}\left\{\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]\right\}\partial_x^4, \tag{4.6}$$

and let $L_{ij}(i,j = 1, 2)$ be the factors of the matrix. The solutions of the preceding equation are

$$\begin{bmatrix} \psi \\ w \end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{4.7}$$

and ξ_i satisfies

$$L_0\xi_i = 0 \ (i = 1, 2). \tag{4.8}$$

In Appendix B, it is proved that the solutions of Eqs. (4.8) can be decomposed into two parts, so there are two functions $\xi_i^{(1)}$ and $\xi_i^{(2)}$,

$$\xi_i = \xi_i^{(1)} + \xi_i^{(2)} \ (i = 1, 2), \tag{4.9}$$

where the superscripts "(1)" and "(2)" indicate the fourth-order part and the transcendental part, respectively, and $\xi_i^{(1)}$ and $\xi_i^{(2)}$ satisfy the following two governing differential equations of the beam problem, respectively,

$$\partial_x^4 \xi_i^{(1)} = 0, \quad \frac{1}{h\partial_x^3} \sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]\xi_i^{(2)} = 0. \tag{4.10}$$

Then the angle of rotation and the deflection of the beam can be decomposed into two parts,

$$\psi = \psi^{(1)} + \psi^{(2)}, \quad w = w^{(1)} + w^{(2)}. \tag{4.11}$$

From Eq. (4.2.3), the magnetic function g_4 satisfies

$$\partial_x \sin\left(\frac{h\partial_x}{2}\right) g_4 = 0. \tag{4.12}$$

The solutions of Eqs. (4.11) and (4.12) will be investigated in the following three sections.

4.1 The fourth-order equation and the fourth-order solution

$\xi_i^{(1)}$ satisfies the fourth-order equation

$$\partial_x^4 \xi_i^{(1)} = 0, \tag{4.13}$$

and the solutions of $\psi^{(1)}, w^{(1)}$ and g_4 become

$$\begin{bmatrix} \psi^{(1)} \\ w^{(1)} \end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix}, \quad g_4 = 0. \tag{4.14}$$

By using Eqs. (4.13), (4.14) and Taylor series of the trigonometric functions (3.2), after tedious manipulation, the result turns out to be

$$\psi^{(1)} = \left(1 + \frac{1+v}{4} h^2 \partial_x^2\right) \partial_x w^{(1)}, \tag{4.15}$$

where

$$\partial_x^4 w^{(1)} = 0. \tag{4.16}$$

From Eqs. (3.7), the total displacements can be found to be

$$u_x^{(1)} = -z\partial_x\left[1 - \frac{1}{6}z^2\partial_x^2 + \frac{1+v}{12}(3h^2 - 2z^2)\partial_x^2\right] w^{(1)}, \quad u_z^{(1)} = \left(1 + \frac{v}{2} z^2 \partial_x^2\right) w^{(1)}, \quad h_z = 0, \tag{4.17}$$

and the normal stress and shear stress can be found to be

$$\sigma_x^{(1)} = -Ez\left(w^{(1)}\right)'', \quad \tau_{xz}^{(1)} = -\frac{E}{8}(h^2 - 4z^2)\left(w^{(1)}\right)''', \quad \sigma_z^{(1)} = 0. \tag{4.18}$$

Calculating moment and shear force for the present case yields

$$M_x^{(1)} = -D\left(w^{(1)}\right)'', \quad Q_x^{(1)} = -D\left(w^{(1)}\right)''', \tag{4.19}$$

where $D = Eh^3/12$ is the flexural rigidity of beams.

Equations (4.17)–(4.19) constitute the first-order theory of elastic beams, which coincide with the corresponding expressions of classical elasticity. Unlike in the customary beam theory, all the fundamental equations of the refined beam theory are deduced directly.

4.2 The transcendental equation and the transcendental solution

$\xi_i^{(2)}$ satisfies the transcendental equation

$$\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\xi_i^{(2)}=0, \tag{4.20}$$

and the solutions of $\psi^{(2)}$, $w^{(2)}$ and g_4 become

$$\begin{bmatrix}\psi^{(2)}\\ w^{(2)}\end{bmatrix}=\begin{bmatrix}L_{22} & -L_{12}\\ -L_{21} & L_{11}\end{bmatrix}\begin{bmatrix}\xi_1^{(2)}\\ \xi_2^{(2)}\end{bmatrix},\quad g_4=0. \tag{4.21}$$

Substituting Eqs. (4.21) into the displacement and stress expressions (3.7) and (3.9), respectively, one obtains the following expressions:

$$u_x^{(2)}=\frac{1}{E}\left[\frac{\partial^2 m}{\partial x^2}-(1+v)\frac{\partial^3\Phi}{\partial x^3}\right],\quad u_z^{(2)}=\frac{1}{E}\left[\frac{\partial^2 n}{\partial x^2}-(1+v)\frac{\partial^3\Phi}{\partial x^2\partial z}\right],\quad h_z=0, \tag{4.22}$$

$$\sigma_x^{(2)}=\frac{\partial^4\Phi}{\partial x^2\partial z^2},\quad \tau_{xz}^{(2)}=-\frac{\partial^4\Phi}{\partial x^3\partial z},\quad \sigma_z^{(2)}=\frac{\partial^4\Phi}{\partial x^4}. \tag{4.23}$$

Therefore, the moment and shear force are

$$M_x^{(2)}=0,\quad Q_x^{(2)}=0, \tag{4.24}$$

where the function $\Phi(x,z)$ has the expression

$$-\frac{1+v}{E}\Phi=h\sin\left(\frac{h\partial_x}{2}\right)\left[\frac{h}{2}\cos\left(\frac{h\partial_x}{2}\right)\frac{\sin(z\partial_x)}{\partial_x}-z\sin\left(\frac{h\partial_x}{2}\right)\frac{\cos(z\partial_x)}{\partial_x}\right]\frac{1}{\partial_x^2}\xi_1^{(2)}$$

$$+\left[-2\cos\left(\frac{h\partial_x}{2}\right)\frac{\sin(z\partial_x)}{\partial_x}+h\sin\left(\frac{h\partial_x}{2}\right)\sin(z\partial_x)+2z\cos\left(\frac{h\partial_x}{2}\right)\cos(z\partial_x)\right]\frac{1}{\partial_x^2}\xi_2^{(2)}, \tag{4.25}$$

and Φ satisfies the following equations:

$$\nabla^2\nabla^2\Phi=0, \tag{4.26}$$

$$\Phi=0,\quad \partial\Phi/\partial z=0\quad \text{at}\quad z=\pm h/2. \tag{4.27}$$

Furthermore, the functions $m(x,z)$ and $n(x,z)$ are conjugate harmonic functions, and satisfy

$$\frac{\partial m}{\partial x}=\frac{\partial n}{\partial z}=\nabla^2\Phi. \tag{4.28}$$

Equations (4.22)–(4.24) satisfy two edge conditions along the boundary of beams, and yet satisfy exactly all the fundamental equations in the theory of elasticity.

Combining the fourth-order solution (4.17)–(4.19) and the transcendental solution (4.22)–(4.24), we arrive at a second-order refined theory for bending elastic beams with the two governing differential equations (4.16) and (4.26). It is important to note that the equilibrium equations (2.4) are satisfied by any solution of the refined elastic beam theory. Therefore, omitting the magnetic terms, the governing equations of elastic beams are obtained directly from the magnetoelastic equations. It is interesting to note that the degenerated solution is consistent with the results gained by Gao and Wang [16].

4.3 The magnetic equation and the magnetic solution

The magnetic function g_4 satisfies the magnetic equation

$$\partial_x \sin\left(\frac{h\partial_x}{2}\right) g_4 = 0, \tag{4.29}$$

and the solutions of ψ and w become

$$\psi = 0, \quad w = 0. \tag{4.30}$$

From Eqs. (3.7) and (3.9), one obtains the following expressions in h_z:

$$u_x^M = \frac{1+v}{4}\frac{Q}{\partial_x}\left(h_z - z\frac{\partial h_z}{\partial z}\right), \quad u_z^M = \frac{1+v}{4} Q z h_z, \quad h_z = -\frac{1}{Q}\sin(z\partial_x) g_4', \tag{4.31}$$

$$\sigma_x^M = \frac{QE}{4}\left(\frac{1+v}{1-v}h_z - z\frac{\partial h_z}{\partial z}\right), \quad \tau_{xz}^M = \frac{QE}{4} z \frac{\partial h_z}{\partial x}, \quad \sigma_z^M = \frac{QE}{4}\left(\frac{1+v}{1-v}h_z + z\frac{\partial h_z}{\partial z}\right). \tag{4.32}$$

The moment and shear force are found to be

$$M_x^M = \frac{3-v}{2(1-v)}\frac{QE}{\partial_x^2}\left(h_z - z\frac{\partial h_z}{\partial z}\right)_{z=h/2}, \quad Q_x^M = \frac{1-v}{3-v}\partial_x M_x^M, \tag{4.33}$$

where the superscript "M" indicates the magnetic part, and h_z satisfies the following equations:

$$\nabla^2 h_z = 0, \tag{4.34}$$

$$h_z = 0 \quad \text{at} \quad z = \pm h/2. \tag{4.35}$$

Equations (4.19), (4.24) and (4.33) show that the transcendental solution does not yield moment and shear force which are found only from the fourth-order solution and the magnetic solution.

Combining the fourth-order solution, the transcendental solution and the magnetic solution just described, a refined theory for the bending of magnetoelastic beams can be established with the three governing differential equations (4.16), (4.26) and (4.34). An infinite number of boundary conditions at the edges of beams can be satisfied, and the only approximation in the theory is introduced by the approximate specification of the boundary conditions at the edges of the beam (i.e., the boundary conditions are specified in terms of the stress resultants or some combination of the angle of rotation and the deflection of the neutral surface, instead of the stress or displacement distribution over the thickness $-h/2 \le z \le h/2$). Therefore, in the cases where Saint-Venant's principle holds, the refined beam theory should be a very accurate one.

5 Approximate equations: transverse surface loadings and magnetic field perturbations

5.1 The governing equation and solution

Now let us consider the case that the beam is only subject to the transverse surface loadings and magnetic fields, i.e.,

$$\tau_{xz} = 0, \quad \sigma_z = \pm q/2, \quad h_z = \pm h_0 \quad \text{at} \quad z = \pm h/2. \tag{5.1}$$

Substituting Eqs. (3.9.2), (3.9.3) and (3.7.4) into the boundary conditions (5.1) of the beam, we get the following equations expressed by ψ, w and g_4:

$$(D_1 - D_2\partial_x^2)\psi - (D_1 + D_2\partial_x^2)w' + D_2\partial_x^2 g_4' = 0, \tag{5.2.1}$$

$$D_3\psi' - \left[\frac{4}{(1+v)h}D_2 - D_3\right]w'' - \left[\frac{4v}{(1-v^2)h}D_2 + D_3\right]\partial_x^2 g_4 = \frac{2}{E}q, \tag{5.2.2}$$

$$D_2\partial_x^2 g_4 = -Qhh_0. \tag{5.2.3}$$

Taking the operator $(D_1 - D_2\partial_x^2)D_2$ and $(D_1 + D_2\partial_x^2)D_2$ on both sides of Eq. (5.2.2), respectively, and then using Eqs. (5.2.1) and (5.2.3), one obtains

$$\frac{D_2}{h}\left[2hD_1D_3 - \frac{4D_2}{1+v}\left(D_1 - D_2\partial_x^2\right)\right]w'' = \frac{2D_2}{E}\left(D_1 - D_2\partial_x^2\right)q - Q\left[hD_1D_3 + \frac{4vD_2}{1-v^2}\left(D_1 - D_2\partial_x^2\right)\right]h_0. \tag{5.3}$$

$$\frac{D_2}{h}\left[2hD_1D_3 - \frac{4D_2}{1+v}\left(D_1 - D_2\partial_x^2\right)\right]\psi' = \frac{2D_2}{E}\left(D_1 + D_2\partial_x^2\right)q - Q\left[hD_1D_3 + \frac{4D_2}{1-v^2}\left(vD_1 + D_2\partial_x^2\right)\right]h_0. \tag{5.4}$$

Substitution of Eqs. (4.3) into Eqs. (5.3), (5.4) and (5.2.3) gives

$$\frac{Eh}{2\partial_x}\sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]w'' = \frac{1}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right)\left[\cos\left(\frac{h\partial_x}{2}\right) - \frac{1+v}{4}h\partial_x\sin\left(\frac{h\partial_x}{2}\right)\right]q$$
$$- \frac{QE}{4}\left[h + \frac{1+v}{1-v}\frac{\sin(h\partial_x)}{\partial_x} - \frac{1+v}{1-v}h\sin^2\left(\frac{h\partial_x}{2}\right)\right]h_0, \tag{5.5}$$

$$\frac{Eh}{2\partial_x}\sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]\psi' = \frac{1}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right)\left[\cos\left(\frac{h\partial_x}{2}\right) + \frac{1+v}{4}h\partial_x\sin\left(\frac{h\partial_x}{2}\right)\right]q$$
$$- \frac{QE}{4}\left[h + \frac{1+v}{1-v}\frac{\sin(h\partial_x)}{\partial_x} + \frac{1+v}{1-v}h\sin^2\left(\frac{h\partial_x}{2}\right)\right]h_0, \tag{5.6}$$

$$\frac{1}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right)g_4'' = -Qh_0. \tag{5.7}$$

Equations (5.5)–(5.7) are the exact governing equations for w, ψ and g_4 for the beam subject to the transverse surface loadings and magnetic field perturbations. Since these equations are of infinite order, however, it is not applicable in most cases. Using Taylor series of the trigonometric functions in Eqs. (3.2), and then dropping all the terms associated with h^4 or higher orders, we arrive at the following equations:

$$Dw'''' = \left(1 - \frac{8+5v}{40}h^2\partial_x^2\right)q - \frac{EQ}{1-v}\left(1 - \frac{14+25v}{120}h^2\partial_x^2\right)h_0, \tag{5.8}$$

$$D\psi''' = \left(1 + \frac{2+5v}{40}h^2\partial_x^2\right)q - \frac{EQ}{1-v}\left(1 + \frac{16+5v}{120}h^2\partial_x^2\right)h_0, \tag{5.9}$$

$$Dg_4'' = -\frac{EQh^2}{6}\left(1 + \frac{1}{24}h^2\partial_x^2\right)h_0. \tag{5.10}$$

Equations (5.8)–(5.10) form the basic equations for an approximate theory for the bending of magnetoelastic beam.

From Eqs. (5.8)–(5.10), the expressions for the displacements, stresses and stress resultants become

$$u_x = -\frac{z}{D\partial_x^3}\left(1+\frac{2+5v}{40}h^2\partial_x^2-\frac{2+v}{6}z^2\partial_x^2\right)q+\frac{EQ}{1-v}\frac{z}{D\partial_x^3}\left(1+\frac{16+5v}{120}h^2\partial_x^2-\frac{2+v}{6}z^2\partial_x^2\right)h_0,$$

$$u_z = \frac{1}{D\partial_x^4}\left(1-\frac{8+5v}{40}h^2\partial_x^2+\frac{v}{2}z^2\partial_x^2\right)q-\frac{EQ}{1-v}\frac{1}{D\partial_x^4}\left(1-\frac{14+25v}{120}h^2\partial_x^2+\frac{v}{2}z^2\partial_x^2\right)h_0, \quad (5.11)$$

$$h_z = \frac{2z}{h}\left(1+\frac{1}{24}h^2\partial_x^2-\frac{1}{6}z^2\partial_x^2\right)h_0,$$

$$\sigma_x = -\frac{12z}{h^3\partial_x^2}\left(1+\frac{1}{20}h^2\partial_x^2-\frac{1}{3}z^2\partial_x^2\right)q+\frac{EQ}{1-v}\frac{12z}{h^3\partial_x^2}\left(1+\frac{2}{15}h^2\partial_x^2-\frac{1}{3}z^2\partial_x^2\right)h_0, \quad (5.12)$$

$$\tau_{xz} = -\frac{3}{2h\partial_x}\left(1-4\frac{z^2}{h^2}\right)q+\frac{EQ}{1-v}\frac{3}{2h\partial_x}\left(1-4\frac{z^2}{h^2}\right)h_0,$$

$$\sigma_z = \frac{z}{h}\left(\frac{3}{2}-2\frac{z^2}{h^2}\right)q-\frac{EQ}{1-v}\frac{z}{8h}\left(1-4\frac{z^2}{h^2}\right)h_0, \quad (5.13)$$

$$M_x'' = -q+\frac{EQ}{1-v}\left(1+\frac{1}{12}h^2\partial_x^2\right)h_0, \quad Q_x' = -q+\frac{EQ}{1-v}h_0.$$

For the elastic beam, $h_0 = 0$, the results described above reduce to the corresponding results by Gao and Wang [16]. Equations (5.11) and (5.12) show that the boundary conditions at the two surfaces are satisfied completely. As in Barrett and Ellis [11], by adopting the works of Gregory and Wan [19], [20] into the case of the magnetoelastic beam, the similar discussion about the specification of the boundary conditions on the edges of the beam can be made. However, the issue will be discussed further and in detail in our other articles, but will not be addressed here.

5.2 Comparison with other theoretical models

As a special case, not taking into account the magnetic field effect, the governing differential equations for elastic beams are obtained directly from Eqs. (5.8) and (5.9), once again reduce to the corresponding equations by Gao and Wang [16], and are mostly the same as the governing equations of Timoshenko elastic beam theory [21].

According to the magnetic couple model [4], the magnetic force model [5] and the variational principle model [7], the governing equation for magnetoelastic beams is as follows:

$$Dw'''' = q_z^{em}, \quad (5.14)$$

where the equivalent transverse magnetic force q_z^{em} in a transverse magnetic field has the form

$$q_z^{em}(x) = \frac{\mu_0\chi(1+\chi)}{2}\left\{[H_z(x,h/2)]^2-[H_z(x,-h/2)]^2\right\}. \quad (5.15)$$

When the applied force is absent, i.e., let $q = 0$, namely, the beam is subject only to a uniform transverse magnetic field, the problem degenerates to the case described by [4], [5] and [7]. Dropping all the terms associated with h^2 from Eq. (5.8), we obtain the approximate governing equations for magnetoelastic beams in a uniform transverse magnetic field:

$$Dw'''' = 2\mu_0\chi(\chi_1+\chi_2)\bar{H}^+h_0. \quad (5.16)$$

Moon and Pao [4] assumed that the magnetic field in the ferromagnetic beam is approximately equal to the applied magnetic field when the ratio of the length to the thickness of the beam is very large, in which two parameters χ_1 and χ_2 fulfill

$$\chi_1 = 0, \quad \chi_2 = 1 + \chi.$$

Zhou and Zheng [5], [7] considered that the magnetic force system exerted on the ferromagnetic beams consists of a body magnetic force without the body magnetic couple, and obtained

$$\chi_1 = 1 + \chi, \quad \chi_2 = 0.$$

Hence, the three theoretical models entirely satisfy the expression

$$\chi_1 + \chi_2 = 1 + \chi. \tag{5.17}$$

Noticeably, the right term of Eq. (5.16) is identical to the equivalent transverse magnetic force q_z^{em} in Eq. (5.15), thus the degenerate solution in the uniform transverse magnetic field is consistent with the results gained by three theoretical models. A numerical example for the magnetoelastic buckling problem of a ferromagnetic beam in a uniform transverse magnetic field has been discussed by Zhou and Zheng [5]. According to the above comparison, the results for the magnetoelastic buckling problem obtained by the refined theory are the same as the corresponding results by Zhou and Zheng [5].

By omitting higher order terms and the coupling effect, the new magnetoelastic beam theory can be degenerated into other well-known elastic and magnetoelastic theories. Hence, the results obtained here are considered reliable as a basis for more general applications.

6 Conclusion

In the above sections, by using the general solution of the magnetoelastic equation and the Lur'e method, a refined theory for a magnetoelastic beam has been deduced systematically and directly from linear magnetoelasticity theory. In the case of homogenous boundary conditions, the refined beam theory is exact in the sense that a solution of the refined beam theory satisfies all the balance equations in the magnetoelasticity theory, and consists of three parts: the fourth-order equation, the transcendental equation and the magnetic equation. In the case of non-homogenous boundary conditions, the approximate governing equations and solutions are accurate up to the second-order terms with respect to beam thickness. For the above-mentioned two cases, the governing equations and solutions of elastic beams can be obtained directly from the corresponding magnetoelastic equations and solutions by omitting the magnetic fields effect. When the applied force is absent, the new magnetoelastic theory for the loading beam can still be justified by comparing its form with that of other well-known magnetoelastic theories. Therefore, in these cases the refined magnetoelastic beam theory should be a very accurate one.

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Appendix A

The method used in this appendix is obtained by extending previous work [14]. Next we will prove that when P_0 is defined according to Eq. (3.3), the general solution (2.5) is complete without loss in generality.

First, from the nonuniqueness of the Papkovich-Neuber solution, $\boldsymbol{P}$ and P_0 in Eqs. (2.5) can be changed to $\tilde{\boldsymbol{P}}$ and $\tilde{P}_0$, respectively, and

$$\tilde{\boldsymbol{P}} = \boldsymbol{P} + \nabla A, \quad \tilde{P}_0 = P_0 + \frac{4}{1+v}A - \boldsymbol{r} \cdot \nabla A, \tag{A.1}$$

where $\nabla = \boldsymbol{i}\partial/\partial x + \boldsymbol{k}\partial/\partial z$, $\boldsymbol{r} = (x, z)$, $\boldsymbol{P} = (P_1, P_3)$, in which $\boldsymbol{P}$ and P_0 have the form of expressions (3.1), and $A(x, z)$ is also a harmonic function. Therefore, we can set

$$A = \frac{\sin(z\partial_x)}{\partial_x} a(x). \tag{A.2}$$

Now we come to prove that it is always possible to choose a function a in Eq. (A.2) so that Eq. (3.3), i.e.,

$$\tilde{P}_0 + \boldsymbol{r} \cdot \tilde{\boldsymbol{P}} = -z\cos(z\partial_x)\tilde{f}, \tag{A.3}$$

may hold, in which

$$\tilde{f} = \frac{(g_1 + \partial_x a)}{\partial_x} - (g_3 + a) = f. \tag{A.4}$$

Substituting Eqs. (A.1) and (A.4) into (A.3), we get the expression

$$P_0 + \frac{4}{1+v}A + \boldsymbol{r} \cdot \boldsymbol{P} = -z\cos(z\partial_x)f. \tag{A.5}$$

Then inserting Eqs. (3.1) and (3.4) into (A.5), it is found that

$$A = -\frac{1+v}{4}\left[\frac{\sin(z\partial_x)}{\partial_x}g_0 + x\frac{\sin(z\partial_x)}{\partial_x}g_1 + z\frac{\cos(z\partial_x)}{\partial_x}g_1\right]. \tag{A.6}$$

Next the expression of a in Eq. (A.2) is to be given using the identity

$$\partial_x^{2n-2}\left(x\partial_x^2 g_1\right) = x\partial_x^{2n} g_1 + (2n-2)\left(\partial_x^{2n-2} g_1\right)_{,x}. \tag{A.7}$$

After tedious manipulation by using Eq. (A.7) and Taylor series of the trigonometric functions, the result turns out to be

$$\frac{\sin(z\partial_x)}{\partial_x}\left(x\partial_x^2 g_1 + 3\partial_x g_1\right) = \partial_x^2\left[x\frac{\sin(z\partial_x)}{\partial_x}g_1 + z\frac{\cos(z\partial_x)}{\partial_x}g_1\right]. \tag{A.8}$$

Substituting expression (A.8) into expression (A.6), we get

$$A = -\frac{1+v}{4}\frac{\sin(z\partial_x)}{\partial_x}\left[g_0 + \frac{1}{\partial_x^2}\left(x\partial_x^2 g_1 + 3\partial_x g_1\right)\right]. \tag{A.9}$$

From expression (A.9) we know Eq. (A.3) holds when

$$a(x) = -\frac{1+v}{4}\left[g_0 + \frac{1}{\partial_x^2}\left(x\partial_x^2 g_1 + 3\partial_x g_1\right)\right] \tag{A.10}$$

in Eq. (A.2).

For convenience, $\tilde{\boldsymbol{P}}$ and $\tilde{P}_0$ will still be written as $\boldsymbol{P}$ and P_0, respectively. Thus Eq. (3.3) holds. Consequently, if P_0 is taken according to Eq. (3.3), the general solutions (2.5) loose no generality.

Appendix B

The method used in this appendix is obtained by extending previous work [10]. Next, we will give and prove a lemma and a theorem.

B.1 The lemma

Supposing that $H(x)$ satisfies

$$\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]H=0, \tag{B.1}$$

then there exists function $B(x)$ which satisfies the following two equations,

$$\partial_x^4 B=H,\quad \frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B=0. \tag{B.2}$$

Proof: Assuming the function $C(x)$ can be found which satisfies the following equation:

$$\partial_x^2 C=H, \tag{B.3}$$

we can obtain the following equation:

$$\partial_x^2\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]C=\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]H=0. \tag{B.4}$$

Set

$$B_1=C-\frac{12}{h^2}\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]C. \tag{B.5}$$

Using Eqs. (B.3) and (B.4), we can get

$$\partial_x^2 B_1=\partial_x^2 C=H. \tag{B.6}$$

After tedious manipulation by using Eqs. (B.4) and (B.5), the result turns out to be

$$\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B_1=0. \tag{B.7}$$

Because $B_1(x)$ and $H(x)$ satisfy the same equation, $B_1(x)$ can be used instead of $H(x)$. Repeating Eqs. (B.3)–(B.7), we can obtain $B(x)$ such that

$$\partial_x^2 B=B_1,\quad \frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B=0. \tag{B.8}$$

From Eqs. (B.6) and (B.8), it is not difficult to verify that $B(x)$ satisfies Eqs. (B.2). So the proof of the lemma is finished.

B.2 The theorem

Supposing that ξ satisfies following equation

$$\left\{\frac{1}{h\partial_x^3}\sin\left(\frac{h\partial_x}{2}\right)\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\right\}\partial_x^4\xi=0, \tag{B.9}$$

then there exist $\xi^{(1)}$ and $\xi^{(2)}$ such that

$$\xi = \xi^{(1)} + \xi^{(2)}, \tag{B.10}$$

satisfying the following two equations:

$$\partial_x^4 \xi^{(1)} = 0, \quad \frac{1}{h\partial_x^3} \sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]\xi^{(2)} = 0. \tag{B.11}$$

Proof: Let

$$F = \partial_x^4 \xi, \tag{B.12}$$

then

$$\frac{1}{h\partial_x^3} \sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]F = 0. \tag{B.13}$$

According to the lemma, there exists $\xi^{(2)}$ such that

$$\partial_x^4 \xi^{(2)} = F, \quad \frac{1}{h\partial_x^3} \sin\left(\frac{h\partial_x}{2}\right)\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]\xi^{(2)} = 0. \tag{B.14}$$

From Eq. (B.12) and the first equation of Eqs. (B.14), we get

$$\partial_x^4 \xi^{(2)} = F = \partial_x^4 \xi, \tag{B.15}$$

namely,

$$\partial_x^4 \left(\xi - \xi^{(2)}\right) = 0. \tag{B.16}$$

Let

$$\xi^{(1)} = \xi - \xi^{(2)}, \tag{B.17}$$

then Eq. (B.16) becomes the first equation of Eqs. (B.11), so there are functions $\xi^{(1)}$ and $\xi^{(2)}$ which satisfy Eqs. (B.11). This completes the proof of the theorem.

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