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A refined theory of transversely isotropic piezoelectric plates

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Summary. A refined theory for transversely isotropic piezoelectric plates is derived from the general solution of three-dimensional transversely isotropic piezoelasticity by means of Lur'e operator method. As a special case, the governing differential equations for transversely isotropic elastic plates are obtained directly.

1 Introduction

As we all know, piezoelectric materials exhibit a coupling behavior between mechanical and electric fields, and are capable of converting mechanical energy into electric energy and vice versa. Just because of their intrinsic direct and converse piezoelectric effects, they are widely used as actuators and/or sensors in advanced intelligent structures such as civil, aeronautic and space structures.

As piezoelectric patches used as actuator or sensor are usually considered as plates because of their plate-like geometry, many scientists paid more attention to the study of piezoelectric plates. Two-dimensional theories for piezoelectric plates have been studied by many investigators. Tiersten [1] and Mindlin [2], [3] initiated the study based on power series expansions of the mechanical displacements and the electric potential along the thickness of the plate and the variational principle, while Bugdayci and Bogy [4] and Lee and Syngellakis et al. [5] applied trigonometric series representation to the research of piezoelectric plates. Thereafter numerous scholars carried out the study comprehensively following these ideas; extensive references on this topic can be found in Wang and Yang [6]. Besides, other scholars, such as Lee [7], Bisegna and Caruso [8], and Krommer [9], just to name a few, performed similar analyses based on classical plate theories or refined plate theories in combination with a gross linear, quadratic or biquadratic through-the-thickness distribution of the electric potential. But most of them are practically asymptotic theories derived after adopting certain *a priori* assumptions. As a consequence, the interest for exact solutions arises, for the purpose of verifying the accuracy of the results provided by approximate theories or computations. Ray and Rao et al. [10], Bisegna and Maceri [11], Heyliger [12], Ding and Chen [13], among others, presented contributions on this aspect based on three-dimensional analytical methods, but these models can only be solved analytically for some limited specific boundary conditions. Further, the state space approach [13], [14] was employed to solve laminated plates.

As for isotropic elastic thick plates, various theories were proposed by many authors with the help of some *ad hoc* assumptions. Reissner [15], [16], Hencky [17], and Kromm's [18] sixth-order plate theories are remarkable representatives. In addition, Chen and Archer [19] and Lo and Christensen [20] offered twelfth- and twenty-second- order theories, respectively. On the other side, Cheng [21] and Wang and Shi [22] published refined plate theories deduced directly from a Boussinesq-Garlerkin solution and a Papkovich-Neuber solution in a systematic manner, respectively. Parallel developments of Cheng's plate theory were obtained by Wang [23], and Yin and Wang [24] for transversely isotropic plates from the Lekhnitskii-Hu-Nowacki solution and the Elliot-Lodge solution, respectively.

The purpose of this paper is to extend our previous works [22], [24] to transversely isotropic piezoelectric plates. In the next section, we present the general solution of transversely isotropic piezoelectric media. In Sect. 3, we bring forth the problem, then with the help of the generalized Lur'e operator method we give the representations of displacement and stress components in terms of three generalized mechanical displacements and an electric potential in the mid-plane. In Sect. 4, the governing differential equations of the above-mentioned plates are derived by making use of the boundary conditions on the surfaces of the plate. In Sect. 5, the governing equations of transversely isotropic plates are obtained directly from the afore-mentioned equations by omitting the piezoelectric terms. Finally, a brief summary and some remarks are provided in Sect. 6.

2 General solution for transversely isotropic piezoelectric media

A Cartesian system (x, y, z) is introduced. Let the z -axis be perpendicular to the isotropic plane of the medium. The basic equations in terms of mechanical displacements u, v, w and electric potential ϕ in the absence of body forces and free charges can be written as

$$c_{11}u_{,xx} + \frac{1}{2}(c_{11} - c_{12})u_{,yy} + c_{44}u_{,zz} + \frac{1}{2}(c_{11} + c_{12})v_{,xy} + (c_{13} + c_{44})w_{,xz} + (e_{15} + e_{31})\phi_{,xz} = 0, \tag{1.1}$$

$$\frac{1}{2}(c_{11} - c_{12})v_{,xx} + c_{11}v_{,yy} + c_{44}v_{,zz} + \frac{1}{2}(c_{11} + c_{12})u_{,xy} + (c_{13} + c_{44})w_{,yz} + (e_{15} + e_{31})\phi_{,yz} = 0, \tag{1.2}$$

$$c_{44}(w_{,xx} + w_{,yy}) + c_{33}w_{,zz} + (c_{13} + c_{44})(u_{,xz} + v_{,yz}) + e_{15}(\phi_{,xx} + \phi_{,yy}) + e_{33}\phi_{,zz} = 0, \tag{1.3}$$

$$e_{15}(w_{,xx} + w_{,yy}) + e_{33}w_{,zz} + (e_{15} + e_{31})(u_{,xz} + v_{,yz}) - \epsilon_{11}(\phi_{,xx} + \phi_{,yy}) - \epsilon_{33}\phi_{,zz} = 0, \tag{1.4}$$

where $c_{ij}, e_{ij}, \epsilon_{ij}$ are the elastic stiffness coefficients, piezoelectric and dielectric constants, respectively. A comma followed by a variable denotes partial differentiation with respect to the variable. Its general solution takes the form [13], [25], [26]

$$u = \sum_{i=1}^{3} \Psi_{i,x} - \Psi_{4,y}, \tag{2.1}$$

$$v = \sum_{i=1}^{3} \Psi_{i,y} + \Psi_{4,x}, \tag{2.2}$$

$$w = \sum_{i=1}^{3} k_{1i} \Psi_{i,z}, \tag{2.3}$$

$$\phi = \sum_{i=1}^{3} k_{2i} \Psi_{i,z}, \tag{2.4}$$

where the potential functions $\Psi_i (i = 1, 2, 3, 4)$ satisfy the following quasi-harmonic equations:

$$\nabla_i^2 \Psi_i = \Psi_{i,xx} + \Psi_{i,yy} + \Psi_{i,z_i z_i} = 0,\ i = 1, 2, 3, 4, \tag{3}$$

in which $z_i = s_i z,\ 1/s_i^{\,2} = \lambda_i,\ (i = 1, 2, 3, 4)$, and

$$\begin{aligned} k_{1i} &= \frac{[e_{33}(e_{15}+e_{31}) + \epsilon_{33}(c_{13}+c_{44})]\lambda_i - [e_{15}(e_{15}+e_{31}) + \epsilon_{11}(c_{13}+c_{44})]\lambda_i^{\,2}}{\epsilon_{33}c_{33} + e_{33}^2 - (\epsilon_{11}c_{33} + 2e_{15}e_{33} + \epsilon_{33}c_{44})\lambda_i + (\epsilon_{11}c_{44} + e_{15}^2)\lambda_i^{\,2}}, \\ k_{2i} &= \frac{[e_{33}(c_{13}+c_{44}) - c_{33}(e_{15}+e_{31})]\lambda_i + (c_{44}e_{31} - c_{13}e_{15})\lambda_i^{\,2}}{\epsilon_{33}c_{33} + e_{33}^2 - (\epsilon_{11}c_{33} + 2e_{15}e_{33} + \epsilon_{33}c_{44})\lambda_i + (\epsilon_{11}c_{44} + e_{15}^{\,2})\lambda_i^{\,2}}, \quad i = 1, 2, 3, \end{aligned} \tag{4}$$

where λ_4 is defined as

$$\lambda_4 = \frac{2c_{44}}{c_{11} - c_{12}},$$

and $\lambda_i (i = 1, 2, 3)$ are the three roots of the algebraic equation

$$I_1\lambda^3 + I_2\lambda^2 + I_3\lambda + I_4 = 0. \tag{5}$$

Herein we assume they are distinct, and

$$\begin{aligned} I_1 &= (e_{15}^{\,2} + c_{44}\epsilon_{11})c_{11}, \\ I_2 &= 2e_{15}^2 c_{13} + 2e_{15}(c_{13}e_{31} - c_{11}e_{33}) - c_{44}e_{31}^{\,2} + \epsilon_{11}(c_{13}^{\,2} + 2c_{13}c_{44} - c_{11}c_{33}) - \epsilon_{33}c_{11}c_{44}, \\ I_3 &= e_{33}(c_{11}e_{33} + 2c_{44}e_{15}) - 2e_{33}(e_{15}+e_{31})(c_{13}+c_{44}) - \epsilon_{33}(c_{13}^{\,2} + 2c_{13}c_{44} - c_{11}c_{33}) \\ &\quad + \epsilon_{11}c_{33}c_{44} + c_{33}(e_{15}+e_{31})^2, \\ I_4 &= -(e_{33}^{\,2} + \epsilon_{33}c_{33})c_{44}. \end{aligned}$$

3 Displacement and stress expressions for transversely isotropic piezoelectric plates

Let us consider a homogeneous transversely isotropic piezoelectric plate occupying the domain

$$\Omega = \{(x, y, z) | (x, y) \in \Theta, |z| \le \frac{h}{2}\},$$

where Θ and h are the mid-plane and the thickness of the plate, respectively. Here we assume the plate is free of body forces, only subjected to surface loads. This problem can be divided into two categories, namely a symmetric one and an anti-symmetric one, which correspond to stretching and bending problems, respectively. Hereafter we only discuss the latter one.

On the supposition that the plate is only subjected to anti-symmetric load $q = q(x, y)$, both u and v are odd functions of z, and w is an even function of z . According to Eq. (2), Ψ_i are all odd functions of z. Generalizing the Lur'e operator method [27], we have the following solution of Eq. (3):

$$\Psi_i = \frac{\sin(s_i z\nabla)}{s_i\nabla} g_i(x,y),\ i = 1,2,3,4, \tag{6}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2,\ g_i(i = 1,2,3,4)$ are unknown functions of x and y yet to be determined, and

$$\begin{aligned}
\frac{\sin(s_i z\nabla)}{s_i\nabla} &= z - \frac{1}{3!}s_i^{\,2} z^3\nabla^2 + \frac{1}{5!}s_i^{\,4} z^5\nabla^4 - \ldots, \\
\cos(s_i z\nabla) &= 1 - \frac{1}{2!}s_i^{\,2} z^2\nabla^2 + \frac{1}{4!}s_i^{\,4} z^4\nabla^4 - \ldots.
\end{aligned} \tag{7}$$

Substituting Eq. (6) into Eq. (2), we obtain

$$\begin{aligned}
u &= \sum_{i=1}^{3}\frac{\sin(s_i z\nabla)}{s_i\nabla} g_{i,x} - \frac{\sin(s_4 z\nabla)}{s_4\nabla} g_{4,y}, \\
v &= \sum_{i=1}^{3}\frac{\sin(s_i z\nabla)}{s_i\nabla} g_{i,y} + \frac{\sin(s_4 z\nabla)}{s_4\nabla} g_{4,x}, \\
w &= \sum_{i=1}^{3} k_{1i}\cos(s_i z\nabla) g_i, \\
\phi &= \sum_{i=1}^{3} k_{2i}\cos(s_i z\nabla) g_i.
\end{aligned} \tag{8}$$

Then from Eq. (8) we can get the generalized mechanical displacements ψ_x, ψ_y, W and the electric potential Φ on the mid-plane of the plate,

$$\begin{aligned}
\psi_x &= -u_{,z}|_{z=0} = -(g_1 + g_2 + g_3)_{,x} + g_{4,y}, \\
\psi_y &= -v_{,z}|_{z=0} = -(g_1 + g_2 + g_3)_{,y} - g_{4,x}, \\
W &= w|_{z=0} = k_{11}g_1 + k_{12}g_2 + k_{13}g_3, \\
\Phi &= \phi|_{z=0} = k_{21}g_1 + k_{22}g_2 + k_{23}g_3.
\end{aligned} \tag{9}$$

From Eq. (9) it follows that

$$\begin{aligned}
g_1 &= \frac{1}{K}[(k_{12} - k_{13})\Phi + (k_{23} - k_{22})W + (k_{12}k_{23} - k_{13}k_{22})\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y})], \\
g_2 &= \frac{1}{K}[(k_{13} - k_{11})\Phi + (k_{21} - k_{23})W + (k_{13}k_{21} - k_{11}k_{23})\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y})], \\
g_3 &= \frac{1}{K}[(k_{11} - k_{12})\Phi + (k_{22} - k_{21})W + (k_{11}k_{22} - k_{12}k_{21})\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y})], \\
g_4 &= \frac{1}{\nabla^2}(\psi_{x,y} - \psi_{y,x}),
\end{aligned} \tag{10}$$

where $K = k_{11}(k_{23} - k_{22}) + k_{12}(k_{21} - k_{23}) + k_{13}(k_{22} - k_{21})$, and the term $\frac{1}{\nabla^2}(\bullet)$ denotes the logarithmic potential of $(\bullet)$. Then inserting Eq. (10) into Eq. (8) , we have

$$u = \frac{1}{K}\Big\{[(k_{12} - k_{13})\frac{\sin(s_1 z\nabla)}{s_1\nabla} + (k_{13} - k_{11})\frac{\sin(s_2 z\nabla)}{s_2\nabla} + (k_{11} - k_{12})\frac{\sin(s_3 z\nabla)}{s_3\nabla}]\Phi_{,x}$$

$$+\left[(k_{23}-k_{22})\frac{\sin(s_1z\nabla)}{s_1\nabla}+(k_{21}-k_{23})\frac{\sin(s_2z\nabla)}{s_2\nabla}+(k_{22}-k_{21})\frac{\sin(s_3z\nabla)}{s_3\nabla}\right]W_{,x}$$

$$+\left[(k_{12}k_{23}-k_{13}k_{22})\frac{\sin(s_1z\nabla)}{s_1\nabla}+(k_{13}k_{21}-k_{11}k_{23})\frac{\sin(s_2z\nabla)}{s_2\nabla}\right.$$

$$\left.+(k_{11}k_{22}-k_{12}k_{21})\frac{\sin(s_3z\nabla)}{s_3\nabla}\right]\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})_{,x}-K\frac{\sin(s_4z\nabla)}{s_4\nabla}\frac{1}{\nabla^2}(\psi_{x,y}-\psi_{y,x})_{,y}\Bigg\}, \quad (11)$$

$$v=\frac{1}{K}\Bigg\{\left[(k_{12}-k_{13})\frac{\sin(s_1z\nabla)}{s_1\nabla}+(k_{13}-k_{11})\frac{\sin(s_2z\nabla)}{s_2\nabla}+(k_{11}-k_{12})\frac{\sin(s_3z\nabla)}{s_3\nabla}\right]\Phi_{,y}$$

$$+\left[(k_{23}-k_{22})\frac{\sin(s_1z\nabla)}{s_1\nabla}+(k_{21}-k_{23})\frac{\sin(s_2z\nabla)}{s_2\nabla}+(k_{22}-k_{21})\frac{\sin(s_3z\nabla)}{s_3\nabla}\right]W_{,y}$$

$$+\left[(k_{12}k_{23}-k_{13}k_{22})\frac{\sin(s_1z\nabla)}{s_1\nabla}+(k_{13}k_{21}-k_{11}k_{23})\frac{\sin(s_2z\nabla)}{s_2\nabla}\right.$$

$$\left.+(k_{11}k_{22}-k_{12}k_{21})\frac{\sin(s_3z\nabla)}{s_3\nabla}\right]\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})_{,y}+K\frac{\sin(s_4z\nabla)}{s_4\nabla}\frac{1}{\nabla^2}(\psi_{x,y}-\psi_{y,x})_{,x}\Bigg\}, \quad (12)$$

$$w=\frac{1}{K}\Bigg\{[k_{11}(k_{12}-k_{13})\cos(s_1z\nabla)+k_{12}(k_{13}-k_{11})\cos(s_2z\nabla)+k_{13}(k_{11}-k_{12})\cos(s_3z\nabla)]\Phi$$

$$+[k_{11}(k_{23}-k_{22})\cos(s_1z\nabla)+k_{12}(k_{21}-k_{23})\cos(s_2z\nabla)+k_{13}(k_{22}-k_{21})\cos(s_3z\nabla)]W$$

$$+[k_{11}(k_{12}k_{23}-k_{13}k_{22})\cos(s_1z\nabla)+k_{12}(k_{13}k_{21}-k_{11}k_{23})\cos(s_2z\nabla)$$

$$+k_{13}(k_{11}k_{22}-k_{12}k_{21})\cos(s_3z\nabla)]\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})\Bigg\}, \quad (13)$$

$$\phi=\frac{1}{K}\Bigg\{[k_{21}(k_{12}-k_{13})\cos(s_1z\nabla)+k_{22}(k_{13}-k_{11})\cos(s_2z\nabla)+k_{23}(k_{11}-k_{12})\cos(s_3z\nabla)]\Phi$$

$$+[k_{21}(k_{23}-k_{22})\cos(s_1z\nabla)+k_{22}(k_{21}-k_{23})\cos(s_2z\nabla)+k_{23}(k_{22}-k_{21})\cos(s_3z\nabla)]W$$

$$+[k_{21}(k_{12}k_{23}-k_{13}k_{22})\cos(s_1z\nabla)+k_{22}(k_{13}k_{21}-k_{11}k_{23})\cos(s_2z\nabla)$$

$$+k_{23}(k_{11}k_{22}-k_{12}k_{21})\cos(s_3z\nabla)]\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})\Bigg\}. \quad (14)$$

Besides, by virtue of the constitutive equations

$$\begin{aligned}
\tau_{xz}&=c_{44}(u_{,z}+w_{,x})+e_{15}\phi_{,x},\\
\tau_{yz}&=c_{44}(v_{,z}+w_{,y})+e_{15}\phi_{,y},\\
\sigma_z&=c_{13}(u_{,x}+v_{,y})+c_{33}w_{,z}+e_{33}\phi_{,z},\\
D_z&=e_{31}(u_{,x}+v_{,y})+e_{33}w_{,z}-\epsilon_{33}\phi_{,z},
\end{aligned} \quad (15)$$

we can obtain part of stress and electric displacement components as follows:

$$K\tau_{xz}=D_1(z)\Phi_{,x}+D_2(z)W_{,x}+D_3(z)\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})_{,x}-D_4(z)\frac{1}{\nabla^2}(\psi_{x,y}-\psi_{y,x})_{,y}, \quad (16)$$

$$K\tau_{yz}=D_1(z)\Phi_{,y}+D_2(z)W_{,y}+D_3(z)\frac{1}{\nabla^2}(\psi_{x,x}+\psi_{y,y})_{,y}+D_4(z)\frac{1}{\nabla^2}(\psi_{x,y}-\psi_{y,x})_{,x}, \quad (17)$$

$$K\sigma_z = D_5(z)\Phi + D_6(z)W + D_7(z)\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y}), \tag{18}$$

$$KD_z = D_8(z)\Phi + D_9(z)W + D_{10}(z)\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y}), \tag{19}$$

where $D_i(z)(i = 1, \ldots, 10)$ are given in Appendix A.

4 Governing differential equations of transversely isotropic piezoelectric plates

The boundary conditions on the upper and lower surfaces of the plate read

$$\tau_{xz} = \tau_{yz} = D_z = 0, \ \sigma_z = \pm\frac{q}{2}, \ \text{at } z = \pm\frac{h}{2}. \tag{20}$$

Substituting Eqs. (16)–(19) into Eq. (20), we get the following equations in terms of $\psi_x, \ \psi_y, \ W$ and Φ:

$$\begin{aligned}
&D_1\Phi_{,x} + D_2W_{,x} + D_3\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y})_{,x} - D_4\frac{1}{\nabla^2}(\psi_{x,y} - \psi_{y,x})_{,y} = 0,\\
&D_1\Phi_{,y} + D_2W_{,y} + D_3\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y})_{,y} + D_4\frac{1}{\nabla^2}(\psi_{x,y} - \psi_{y,x})_{,x} = 0,\\
&D_5\Phi + D_6W + D_7\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y}) = \frac{Kq}{2},\\
&D_8\Phi + D_9W + D_{10}\frac{1}{\nabla^2}(\psi_{x,x} + \psi_{y,y}) = 0,
\end{aligned} \tag{21}$$

where $D_i = D_i(z)|_{z=h/2} \quad (i = 1, \ldots, 10)$.

As we can see, the generalized displacements $\psi_x, \ \psi_y$ may be expressed in terms of two functions F and f as

$$\psi_x = F_{,x} + f_{,y}, \quad \psi_y = F_{,y} - f_{,x}. \tag{22}$$

With this in mind, we may rewrite Eq. (21) as

$$(D_1\Phi + D_2W + D_3F)_{,x} - (D_4f)_{,y} = 0, \tag{23.1}$$

$$(D_1\Phi + D_2W + D_3F)_{,y} + (D_4f)_{,x} = 0, \tag{23.2}$$

$$D_5\Phi + D_6W + D_7F = \frac{Kq}{2}, \tag{23.3}$$

$$D_8\Phi + D_9W + D_{10}F = 0. \tag{23.4}$$

Since Eqs. (23.1) and (23.2) are Cauchy-Riemann equations, it can be assumed that

$$\begin{aligned}
&D_1\Phi + D_2W + D_3F = \alpha(x, y),\\
&D_4f = \beta(x, y),
\end{aligned} \tag{24}$$

where $\alpha(x, y)$ and $\beta(x, y)$ are to each other conjugate harmonic functions. In the light of the definitions of D_3 and D_4, we know Eq. (24) possesses the following particular solution:

$$\begin{aligned}
&F^* = \frac{\alpha}{l_{01}(k_{12}k_{23} - k_{13}k_{22}) + l_{02}(k_{13}k_{21} - k_{11}k_{23}) + l_{03}(k_{11}k_{22} - k_{12}k_{21})},\\
&f^* = \frac{\beta}{l_{04}}, \ \Phi^* = 0, \ \mathrm{W}^* = 0.
\end{aligned} \tag{25}$$

Because of $l_{01}(k_{12}k_{23}-k_{13}k_{22})+l_{02}(k_{13}k_{21}-k_{11}k_{23})+l_{03}(k_{11}k_{22}-k_{12}k_{21})=-l_{04}=-Kc_{44}$, the particular solution Eq. (25) affects neither ψ_x, ψ_y, W nor Φ, and therefore may be omitted. Thus Eq. (24) becomes

$$D_1\Phi+D_2W+D_3F=0, \tag{26.1}$$

$$D_4 f=0. \tag{26.2}$$

From Eqs. (23.1), (23.4) and (26.1), we can get

$$\frac{1}{\nabla^2}[D_3(D_5D_9-D_6D_8)+D_2(D_7D_8-D_5D_{10})+D_1(D_6D_{10}-D_7D_9)]W = \frac{K}{2}\frac{1}{\nabla^2}(D_1D_{10}-D_3D_8)q, \tag{27.1}$$

$$(D_1D_{10}-D_3D_8)\Phi=(D_3D_9-D_2D_{10})W, \tag{27.2}$$

$$\frac{1}{\nabla^2}(D_1D_{10}-D_3D_8)F=\frac{1}{\nabla^2}(D_2D_8-D_1D_9)W\ . \tag{27.3}$$

Substituting $z=h/2$ into Appendix A, and then the results into Eq. (27.1), we have

$$\begin{aligned}&\left[L_1\frac{\sin(\frac{s_1h\nabla}{2})\sin(\frac{s_2h\nabla}{2})\cos(\frac{s_3h\nabla}{2})}{s_1s_2\nabla^2}+L_2\frac{\sin(\frac{s_1h\nabla}{2})\cos(\frac{s_2h\nabla}{2})\sin(\frac{s_3h\nabla}{2})}{s_1s_3\nabla^2}\right.\\&\left.+L_3\frac{\cos(\frac{s_1h\nabla}{2})\sin(\frac{s_2h\nabla}{2})\sin(\frac{s_3h\nabla}{2})}{s_2s_3\nabla^2}\right]\nabla^2W\\&=\left[R_1\frac{\sin(\frac{s_1h\nabla}{2})\cos(\frac{s_2h\nabla}{2})}{s_1\nabla}-R_2\frac{\cos(\frac{s_1h\nabla}{2})\sin(\frac{s_2h\nabla}{2})}{s_2\nabla}+R_3\frac{\cos(\frac{s_1h\nabla}{2})\sin(\frac{s_3h\nabla}{2})}{s_3\nabla}\right.\\&\left.-R_4\frac{\sin(\frac{s_1h\nabla}{2})\cos(\frac{s_3h\nabla}{2})}{s_1\nabla}+R_5\frac{\sin(\frac{s_2h\nabla}{2})\cos(\frac{s_3h\nabla}{2})}{s_2\nabla}-R_6\frac{\cos(\frac{s_2h\nabla}{2})\sin(\frac{s_3h\nabla}{2})}{s_3\nabla}\right]\frac{q}{2},\end{aligned} \tag{28}$$

where the parameters $L_i(i=1,2,3)$ and $R_i(i=1,\ldots,6)$ can be found in Appendix B. There are the following relationships among them:

$$L_1+L_2+L_3=0,\ R_1-R_2+R_3-R_4+R_5-R_6=-K\left(c_{44}\epsilon_{11}+e_{15}^2\right). \tag{29}$$

Substituting Eq. (7) into Eq. (28) yields

$$D\left[1-\frac{1}{120}\xi_1h^2\nabla^2+\frac{1}{13440}\xi_2h^4\nabla^4+\ldots\right]\nabla^4W=\left[1-\frac{1}{24}\varsigma_1h^2\nabla^2-\frac{1}{1920}\varsigma_2h^4\nabla^4+\ldots\right]q, \tag{30}$$

where $\xi_i,\ \varsigma_i(i=1,2)$ can be found in Appendix C, and

$$D=\frac{h^3}{12K}\frac{L_3s_1^2+L_2s_2^2+L_1s_3^2}{c_{44}\epsilon_{11}+e_{15}^2} \tag{31}$$

is the bending rigidity of the plate. Taking the operator $1+\frac{\xi_1}{120}h^2\nabla^2+\frac{14\xi_1^2-15\xi_2}{201600}h^4\nabla^4$ on both sides of Eq. (30) and then omitting the higher-order terms, we obtain the governing equation of deflection,

$$D\nabla^4W=(1-A_1h^2\nabla^2-A_2h^4\nabla^4)q, \tag{32}$$

where

$$A_1 = \frac{5\varsigma_1 - \xi_1}{120},\ A_2 = \frac{15\xi_2 + 105\varsigma_2 + 70\xi_1\varsigma_1 - 14\xi_1^2}{201600}. \tag{33}$$

Moreover, based on Eq. (27.2) and Eq. (27.3), Φ and F can be expressed in terms of W as

$$\nabla^2\Phi = -\left[B_1 h^2\nabla^2 + B_2 h^4\nabla^4\right]\nabla^2 W, \tag{34}$$

$$F = \left[1 + C_1 h^2\nabla^2 + C_2 h^4\nabla^4\right]W, \tag{35}$$

where

$$B_1 = \frac{1}{24}\hat{\varsigma}_1,\ B_2 = \frac{10\varsigma_1\hat{\varsigma}_1 + 3\hat{\varsigma}_2}{5760}, \tag{36}$$

$$C_1 = \frac{1}{24}(\varsigma_1 + \overline{\varsigma}_1),\ C_2 = \frac{1}{5760}\left(10\varsigma_1^2 + 3\varsigma_2 + 3\overline{\varsigma}_2 + 10\varsigma_1\overline{\varsigma}_1\right). \tag{37}$$

Here $\hat{\varsigma}_1, \overline{\varsigma}_1$ and $\widehat{\varsigma}_2, \overline{\varsigma}_2$ are identical with ς_1, ς_2, respectively, only the corresponding $\hat{R}_i, \overline{R}_i (i = 1, \ldots, 6)$ differ slightly from $R_i (i = 1, \ldots, 6)$, detailed forms are available in Appendix D.

In addition, analogous to the method used by Wang and Shi [22], we can express f as

$$f = f_1 + f_2 + \ldots, \tag{38}$$

where f_i satisfies

$$\left[1 - \frac{s_4^2 h^2}{(2i-1)^2\pi^2}\nabla^2\right] f_i = 0,\ i = 1, 2, \ldots. \tag{39}$$

If terms with $i \geq 2$ are omitted, then we have

$$f = f_1, \tag{40}$$

$$f - \frac{s_4^2 h^2}{\pi^2}\nabla^2 f = 0. \tag{41}$$

Thus, Eqs. (32), (34), (35) and (38) together constitute the governing differential equations of transversely isotropic piezoelectric plates.

5 Governing differential equations of transversely isotropic elastic plates

In the above sections, transversely isotropic piezoelectric plates were studied, here we will discuss a special case, i.e., transversely isotropic elastic plates. In this case, $e_{15} = e_{31} = e_{33} = 0$, the cubic equation of λ, Eq. (5), can be reformulated as

$$\left[c_{11}c_{44}\lambda^2 + \left(c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33}\right)\lambda + c_{33}c_{44}\right](\lambda\varepsilon_{11} - \varepsilon_{33}) = 0. \tag{42}$$

Let λ_1, λ_2 be the roots of the first multiplier of Eq. (42), i.e.

$$c_{11}c_{44}\lambda^2 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})\lambda + c_{33}c_{44} = 0, \tag{43}$$

and let λ_3 be the root of the second part of Eq. (42), then it can be seen that λ_1 and λ_2 are only related to elastic constants, while λ_3 is only as related to dielectric coefficients. Hence from Eq. (4), it turns out that

$$k_{1i} = -\frac{c_{44}}{c_{13} + c_{44}} + \frac{c_{11}}{c_{13} + c_{44}}\frac{1}{s_i^2},\quad i = 1, 2,$$
$$k_{21} = k_{22} = 0, \tag{44}$$

while k_{13}, k_{23} correlate with λ_3.

Besides, when $e_{15} = e_{31} = e_{33} = 0$, Eqs. (1.1–3) are the equations of mechanical displacements u, v and w, and Eq. (1.4) is the equation that the electric potential ϕ must satisfy in dielectric problems, so from Eq. (2.3) we can reach $k_{13} = 0$. Substitution of Eq. (2.4) into Eq. (1.4) yields

$$k_{23} \neq 0. \tag{45}$$

From the following calculation we can see the results having nothing to do with k_{13} and k_{23} except the requirement $k_{23} \neq 0$.

It can be proved that the following identities based on Eqs. (43), (44) exist:

$$k_{11}k_{12} = 1, \ k_{11} - k_{12} = \frac{c_{33}}{c_{13} + c_{44}}(s_2^2 - s_1^2), \tag{46.1}$$

$$s_1^2 s_2^2 = \frac{c_{11}}{c_{33}}, \ s_1^2 + s_2^2 = -\frac{c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33}}{c_{33}c_{44}}. \tag{46.2}$$

Substituting Eq. (44) into the equations given in Appendices B and D, and taking into account Eq. (46.1), we obtain

$$\begin{aligned} &L_1 = R_1 = R_2 = R_4 = R_5 = 0, \ L_2 = -L_3 = c_{44}^2 \varepsilon_{11}(1 + k_{11})(1 + k_{12})k_{23}, \\ &R_3 = c_{44}\varepsilon_{11}(1 + k_{12})k_{23}, \ R_6 = c_{44}\varepsilon_{11}(1 + k_{11})k_{23}, \end{aligned} \tag{47}$$

$$\begin{aligned} &\widehat{R}_i = 0(i = 1, \ldots, 6), \ \overline{R}_1 = \overline{R}_2 = \overline{R}_4 = \overline{R}_5 = 0, \\ &\overline{R}_3 = c_{44}\varepsilon_{11}(1 + k_{11})k_{23}, \ \overline{R}_6 = c_{44}\varepsilon_{11}(1 + k_{12})k_{23}, \end{aligned} \tag{48}$$

and

$$K = (k_{11} - k_{12})k_{23}. \tag{49}$$

Substitution of Eqs. (47) and (49) into Eq. (31) results in

$$D = \frac{h^3}{12}c_{44}(1 + k_{11})(1 + k_{12})\frac{s_2^2 - s_1^2}{k_{11} - k_{12}}. \tag{50}$$

Similarly, substituting Eqs. (47) and (49) into the equations in Appendix C and then the results into Eq. (33), we get

$$\begin{aligned} A_1 &= \frac{(1 + k_{11})s_2^2 - (1 + k_{12})s_1^2}{8(k_{11} - k_{12})} - \frac{s_1^2 + s_2^2}{40}, \\ A_2 &= \frac{\left[(1 + k_{11})s_2^2 - (1 + k_{12})s_1^2\right]\left(s_1^2 + s_2^2\right)}{320(k_{11} - k_{12})} - \frac{27\left(s_1^2 + s_2^2\right)^2 - 20s_1^2 s_2^2}{67200} \\ &\quad - \frac{(1 + k_{11})s_2^4 - (1 + k_{12})s_1^4}{384(k_{11} - k_{12})}. \end{aligned} \tag{51}$$

In the same way, inserting Eqs. (48) and (49) into Eq. (36) one finds that

$$B_1 = B_2 = 0. \tag{52}$$

In view of Eqs. (48) and (49), after cumbersome calculations, Eq. (37) can be rewritten as

$$\begin{aligned} C_1 &= \frac{(2 + k_{11} + k_{12})(s_2^2 - s_1^2)}{8(k_{11} - k_{12})}, \\ C_2 &= \frac{90}{(k_{11} - k_{12})^2}\left\{\left[(1 + k_{11})^2 s_2^2 - (1 + k_{12})^2 s_1^2\right](s_2^2 - s_1^2) + (2 + k_{11} + k_{12})(s_2^2 - s_1^2)^2\right\} \\ &\quad + \frac{15}{k_{11} - k_{12}}(2 + k_{11} + k_{12})(s_1^4 - s_2^4). \end{aligned} \tag{53}$$

Hereupon we get the governing differential equations of transversely isotropic elastic plates,

$$
\begin{aligned}
&D\nabla^4 W = (1 - A_1 h^2 \nabla^2 - A_2 h^4 \nabla^4) q, \\
&\nabla^2 \Phi = 0, \\
&F = \left[1 + C_1 h^2 \nabla^2 + C_2 h^4 \nabla^4\right] W, \\
&f = f_1 + f_2 + \ldots,
\end{aligned}
\tag{54}
$$

in which $f_i (i = 1, 2, \ldots)$ satisfy Eq. (39). Noticeably Eq. (54) is consistent with the results deduced by Yin and Wang [24], which were from the Elliott [28]-Lodge [29] general solution.

Furthermore, using Eq. (46) the bending rigidity D and the coefficients A_i and $C_i (i = 1, 2)$ can be represented by elastic constants as follows:

$$
D = \frac{h^3}{12 c_{33}} [c_{11} c_{33} - c_{13}^2], \tag{55}
$$

$$
A_1 = \frac{4 c_{11} c_{33} - c_{13}(3 c_{44} + 4 c_{13})}{40 c_{33} c_{44}}, \tag{56}
$$

$$
A_2 = \frac{1}{67200 c_{33}^2} \left\{ 8 \left[\frac{c_{11} c_{33} - c_{13}(c_{13} + 2 c_{44})}{c_{44}} \right]^2 + 35 \frac{c_{13}}{c_{44}} [c_{11} c_{33} - c_{13}(c_{13} + 2 c_{44})] + 195 c_{11} c_{33} \right\} \tag{57}
$$

$$
C_1 = \frac{c_{11} c_{33} - c_{13}^2}{8 c_{33} c_{44}}, \tag{58}
$$

$$
C_2 = \frac{(c_{11} c_{33} - c_{13}^2)[5 c_{11} c_{33} - c_{13}(5 c_{13} + 4 c_{44})]}{384 c_{33}^2 c_{44}^2}. \tag{59}
$$

6 Conclusion

By introducing three generalized mechanical displacements and an electric potential on the mid-plane, the complete set of governing differential equations for transversely isotropic piezoelectric plates is constructed in a very straightforward way based on the general solution for transversely isotropic piezoelectric media along with the upper and lower surface conditions via the Lur'e operator method. It is found that the structure of the refined theory is quite similar to that of Reissner plate theory for purely elastic plates except for an additional electric term, namely Eq. (34). Not taking into account the piezoelectric effect, the governing differential equations for transversely isotropic elastic plates are obtained directly from the above-mentioned equations. It is interesting to note that the degenerated solution is consistent with the results gained by Yin and Wang from E-L general solution. Hence, the results obtained here are considered reliable as a basis for more general applications. For example, the bending of an infinitely large piezoelectric plate with a circular hole can be investigated, which will be presented in the near future.

Appendix A

$$D_1(z) = l_{01}(k_{12} - k_{13})\cos(s_1 z\nabla) + l_{02}(k_{13} - k_{11})\cos(s_2 z\nabla) + l_{03}(k_{11} - k_{12})\cos(s_3 z\nabla),$$

$$D_2(z) = l_{01}(k_{23} - k_{22})\cos(s_1 z\nabla) + l_{02}(k_{21} - k_{23})\cos(s_2 z\nabla) + l_{03}(k_{22} - k_{21})\cos(s_3 z\nabla),$$

$$D_3(z) = l_{01}(k_{12}k_{23} - k_{13}k_{22})\cos(s_1 z\nabla) + l_{02}(k_{13}k_{21} - k_{11}k_{23})\cos(s_2 z\nabla) + l_{03}(k_{11}k_{22} - k_{12}k_{21})\cos(s_3 z\nabla),$$

$$D_4(z) = l_{04}\cos(s_4 z\nabla),$$

$$D_5(z) = l_{05}(k_{12} - k_{13})\nabla\sin(s_1 z\nabla) + l_{06}(k_{13} - k_{11})\nabla\sin(s_2 z\nabla) + l_{07}(k_{11} - k_{12})\nabla\sin(s_3 z\nabla),$$

$$D_6(z) = l_{05}(k_{23} - k_{22})\nabla\sin(s_1 z\nabla) + l_{06}(k_{21} - k_{23})\nabla\sin(s_2 z\nabla) + l_{07}(k_{22} - k_{21})\nabla\sin(s_3 z\nabla),$$

$$D_7(z) = l_{05}(k_{12}k_{23} - k_{13}k_{22})\nabla\sin(s_1 z\nabla) + l_{06}(k_{13}k_{21} - k_{11}k_{23})\nabla\sin(s_2 z\nabla) + l_{07}(k_{11}k_{22} - k_{12}k_{21})\nabla\sin(s_3 z\nabla),$$

$$D_8(z) = l_{08}(k_{12} - k_{13})\nabla\sin(s_1 z\nabla) + l_{09}(k_{13} - k_{11})\nabla\sin(s_2 z\nabla) + l_{10}(k_{11} - k_{12})\nabla\sin(s_3 z\nabla),$$

$$D_9(z) = l_{08}(k_{23} - k_{22})\nabla\sin(s_1 z\nabla) + l_{09}(k_{21} - k_{23})\nabla\sin(s_2 z\nabla) + l_{10}(k_{22} - k_{21})\nabla\sin(s_3 z\nabla),$$

$$D_{10}(z) = l_{08}(k_{12}k_{23} - k_{13}k_{22})\nabla\sin(s_1 z\nabla) + l_{09}(k_{13}k_{21} - k_{11}k_{23})\nabla\sin(s_2 z\nabla) + l_{10}(k_{11}k_{22} - k_{12}k_{21})\nabla\sin(s_3 z\nabla),$$

where

$$l_{01} = c_{44}(1 + k_{11}) + e_{15}k_{21},\, l_{02} = c_{44}(1 + k_{12}) + e_{15}k_{22},\, l_{03} = c_{44}(1 + k_{13}) + e_{15}k_{23},\, l_{04} = Kc_{44},$$

$$l_{05} = (c_{13} - c_{33}k_{11}s_1^2 - e_{33}k_{21}s_1^2)/s_1,\, l_{06} = (c_{13} - c_{33}k_{12}s_2^2 - e_{33}k_{22}s_2^2)/s_2,$$

$$l_{07} = (c_{13} - c_{33}k_{13}s_3^2 - e_{33}k_{23}s_3^2)/s_3,\, l_{08} = (e_{31} - e_{33}k_{11}s_1^2 + \varepsilon_{33}k_{21}s_1^2)/s_1,$$

$$l_{09} = (e_{31} - e_{33}k_{12}s_2^2 + \varepsilon_{33}k_{22}s_2^2)/s_2,\, l_{10} = (e_{31} - e_{33}k_{13}s_3^2 + \varepsilon_{33}k_{23}s_3^2)/s_3.$$

Appendix B

$$L_1 = [c_{44}(1 + k_{13}) + e_{15}k_{23}]\left(c_{44}\varepsilon_{11} + e_{15}^2\right)(k_{12}k_{21} - k_{11}k_{22} + k_{21} - k_{22}),$$

$$L_2 = [c_{44}(1 + k_{12}) + e_{15}k_{22}]\left(c_{44}\varepsilon_{11} + e_{15}^2\right)(k_{11}k_{23} - k_{13}k_{21} + k_{23} - k_{21}),$$

$$L_3 = [c_{44}(1 + k_{11}) + e_{15}k_{21}]\left(c_{44}\varepsilon_{11} + e_{15}^2\right)(k_{13}k_{22} - k_{12}k_{23} + k_{22} - k_{23}),$$

$$R_1 = k_{13}[c_{44}(1 + k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{21} - e_{15}(1 + k_{11})],$$

$$R_2 = k_{13}[c_{44}(1 + k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{22} - e_{15}(1 + k_{12})],$$

$$R_3 = k_{12}[c_{44}(1 + k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{23} - e_{15}(1 + k_{13})],$$

$$R_4 = k_{12}[c_{44}(1 + k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{21} - e_{15}(1 + k_{11})],$$

$$R_5 = k_{11}[c_{44}(1 + k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{22} - e_{15}(1 + k_{12})],$$

$$R_6 = k_{11}[c_{44}(1 + k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{23} - e_{15}(1 + k_{13})].$$

Appendix C

$$\xi_1 = \frac{3\left(L_3 s_1^4 + L_2 s_2^4 + L_1 s_3^4\right) - 5\left(L_1 s_1^2 s_2^2 + L_3 s_2^2 s_3^2 + L_2 s_3^2 s_1^2\right)}{L_3 s_1^2 + L_2 s_2^2 + L_1 s_3^2},$$

$$\xi_2 = \frac{3\left(L_3 s_1^6 + L_2 s_2^6 + L_1 s_3^6\right) + 7\left[(2L_2 + L_3) s_1^2 s_2^4 + (L_2 + 2L_3) s_1^4 s_2^2 + (2L_1 + L_2) s_2^2 s_3^4\right]}{L_3 s_1^2 + L_2 s_2^2 + L_1 s_3^2}$$
$$+ \frac{7\left[(L_1 + 2L_2) s_2^4 s_3^2 + (2L_3 + L_1) s_3^2 s_1^4 + (L_3 + 2L_1) s_3^4 s_1^2\right]}{L_3 s_1^2 + L_2 s_2^2 + L_1 s_3^2},$$

$$\varsigma_1 = \frac{[-R_1 + 3R_2 - 3R_3 + R_4] s_1^2 + [-3R_1 + R_2 - R_5 + 3R_6] s_2^2 + [-R_3 + 3R_4 - 3R_5 + R_6] s_3^2}{K\left(c_{44}\varepsilon_{11} + e_{15}^2\right)},$$

$$\varsigma_2 = \frac{[R_1 - 5R_2 + 5R_3 - R_4] s_1^4 + [5R_1 - R_2 + R_5 - 5R_6] s_2^4 + [R_3 - 5R_4 + 5R_5 - R_6] s_3^4}{K\left(c_{44}\varepsilon_{11} + e_{15}^2\right)}$$
$$+ \frac{10\left[(R_1 - R_2) s_1^2 s_2^2 + (R_5 - R_6) s_2^2 s_3^2 + (R_3 - R_4) s_3^2 s_1^2\right]}{K\left(c_{44}\varepsilon_{11} + e_{15}^2\right)}.$$

Appendix D

$$\hat{R}_1 = k_{23}[c_{44}(1+k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{21} - e_{15}(1+k_{11})],$$

$$\hat{R}_2 = k_{23}[c_{44}(1+k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{22} - e_{15}(1+k_{12})],$$

$$\hat{R}_3 = k_{22}[c_{44}(1+k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{23} - e_{15}(1+k_{13})],$$

$$\hat{R}_4 = k_{22}[c_{44}(1+k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{21} - e_{15}(1+k_{11})],$$

$$\hat{R}_5 = k_{21}[c_{44}(1+k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{22} - e_{15}(1+k_{12})],$$

$$\hat{R}_6 = k_{21}[c_{44}(1+k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{23} - e_{15}(1+k_{13})],$$

$$\bar{R}_1 = [c_{44}(1+k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{21} - e_{15}(1+k_{11})], \bar{R}_2 = [c_{44}(1+k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{22} - e_{15}(1+k_{12})],$$

$$\bar{R}_3 = [c_{44}(1+k_{11}) + e_{15}k_{21}][\varepsilon_{11}k_{23} - e_{15}(1+k_{13})], \bar{R}_4 = [c_{44}(1+k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{21} - e_{15}(1+k_{11})],$$

$$\bar{R}_5 = [c_{44}(1+k_{13}) + e_{15}k_{23}][\varepsilon_{11}k_{22} - e_{15}(1+k_{12})], \bar{R}_6 = [c_{44}(1+k_{12}) + e_{15}k_{22}][\varepsilon_{11}k_{23} - e_{15}(1+k_{13})].$$

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